J. Korean Math. Soc. **40** (2003), No. 1, pp. 29–40

ALMOST STABILITY OF THE MANN ITERATION METHOD WITH ERRORS FOR STRICTLY HEMI-CONTRACTIVE OPERATORS IN SMOOTH BANACH SPACES

Z. Liu, S. M. Kang, and S. H. Shim

Abstract. Let K be a nonempty closed bounded convex subset of an arbitrary smooth Banach space X and $T : K \to K$ be a strictly hemi-contractive operator. Under some conditions we obtain that the Mann iteration method with errors both converges strongly to a unique fixed point of T and is almost T-stable on K. The results presented in this paper generalize the corresponding results in [1]-[7], [20] and others.

1. Introduction

Let X be an arbitrary normed linear space, X^* be its dual space and $\langle x, f\rangle$ be the generalized duality pairing between $x \in X$ and $f \in X^*$. The mapping $J : X \to 2^{X^*}$ defined by

$$J(x) = \{f \in X^* : Re\langle x, f\rangle = \|x\|\|f\|, \|f\| = \|x\|\}, \quad x \in X$$

is called the *normalized duality mapping.* A Banach space X is called *smooth* if the norm of X is Gâteaux differentiable on $X - \{0\}$. Note that J is single-valued in smooth Banach spaces. A Banach space X is called *uniformly smooth* if X^* is uniformly convex. It is known that if X is uniformly smooth, then J is uniformly continuous on bounded subsets of X. The symbols $D(T)$ and $F(T)$ denote the domain and the set of fixed points of an operator T, respectively.

Received September 5, 2001. Revised August 16, 2002.

2000 Mathematics Subject Classification: 47H06, 47H10, 47H15, 47H17.

Key words and phrases: Mann iteration method with errors, strictly hemi-contractive operators, strongly pseudocontractive operators, local strongly pseudocontractive operators, smooth Banach spaces.

This work was supported by Korea Research Foundation Grant (KRF-99-005-D00003).

DEFINITION 1.1. ([6], [23]) Let X be an arbitrary normed linear space and $T : D(T) \subseteq X \to X$ be an operator.

(i) T is said to be *strongly pseudocontractive* if there exists $t > 1$ such that

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\| \tag{1.1}$$

for all $x, y \in D(T)$ and $r > 0$.

(ii) T is said to be *local strongly pseudocontractive* if for each $x \in D(T)$ there exists $t_x > 1$ such that

$$\|x - y\| \leq \|(1 + r)(x - y) - rt_x(Tx - Ty)\| \tag{1.2}$$

for all $y \in D(T)$ and $r > 0$.

(iii) T is said to be *strictly hemi-contractive* if $F(T) \neq \emptyset$ and if there exists $t > 1$ such that

$$\|x - q\| \leq \|(1 + r)(x - q) - rt(Tx - q)\| \tag{1.3}$$

for all $x \in D(T)$, $q \in F(T)$ and $r > 0$.

Clearly, each strongly pseudocontractive operator is local strongly pseudocontractive.

Let K be a nonempty convex subset of an arbitrary normed linear space X and $T : K \to K$ be an operator. Assume that $x_0 \in K$ and $x_{n+1} = f(T, x_n)$ defines an iteration scheme which produces a sequence $\{x_n\}_{n=0}^{\infty} \subset K$. Suppose, furthermore, that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $q \in F(T) \neq \emptyset$. Let $\{y_n\}_{n=0}^{\infty}$ be any bounded sequence in K and put $\varepsilon_n = \|y_{n+1} - f(T, y_n)\|$.

DEFINITION 1.2. (i) The iteration scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, x_n)$ is said to be *T-stable* on K if $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = q$.

(ii) The iteration scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, x_n)$ is said to be *almost T-stable* on K if $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = q$.

It is easy to verify that an iteration scheme $\{x_n\}_{n=0}^{\infty}$ which is T-stable on K is almost T-stable on K. Osilike [19] proved that an iteration scheme which is almost T-stable on X may fail to be T-stable on X.

Let us recall the following three iteration processes due to Ishikawa [11], Mann [16] and Xu [24], respectively.

Let K be a nonempty convex subset of an arbitrary normed linear space X and $T : K \to K$ be an operator.

(i) For any given $x_0 \in K$ the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_n T y_n, & n \geq 0, \\ y_n = (1 - b_n)x_n + b_n T x_n, & n \geq 0 \end{cases}$$

is called the *Ishikawa iteration sequence*, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1]$ satisfying appropriate conditions.

(ii) In particular, if $b_n = 0$ for all $n \geq 0$, then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_0 \in K, \quad x_{n+1} = (1 - a_n)x_n + a_n T x_n, \quad n \geq 0$$

is called the *Mann iteration sequence.*

(iii) For any given $x_0 \in K$ the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, & n \geq 0, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n, & n \geq 0, \end{cases}$$

where $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ are arbitrary bounded sequences in K and $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ and $\{c'_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1]$ such that $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$ for all $n \geq 0$ is called the *Ishikawa iteration sequence with errors.*

(iv) If, with the same notations and definitions as in (iii), $b'_n = c'_n = 0$ for all $n \geq 0$, then the sequence $\{x_n\}_{n=0}^{\infty}$ now defined by

$$x_0 \in K, \quad x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \quad n \geq 0$$

is called the *Mann iteration sequence with errors.*

It is clear that the Ishikawa and Mann iteration sequences are all special cases of the Ishikawa and Mann iteration sequences with errors, respectively.

In [3], Chidume proved that the Mann iteration method can be used to approximate fixed points of Lipschitz strongly pseudo-contractive mappings in L_p (or l_p) spaces for $p \in [2, \infty)$. Afterwards, several researchers extended the result in various aspects. Chidume [4] extended the result in [3] to both continuous strongly pseudo-contractive mappings and real uniformly smooth Banach spaces. Schu [22] generalized the result in [3] to both uniformly continuous strongly pseudo-contractive mappings and real smooth Banach spaces. Park [20] extended the result in [3] to both strongly pseudo-contractive mappings and certain smooth Banach spaces.

A few stability results for certain classes of nonlinear mappings have been established by several authors (see, e.g., [8]-[10], [12]-[15], [17]-[19]). Rhoades [21] proved that the Mann and Ishikawa iteration methods may exhibit different behaviors for different classes of nonlinear mappings. Harder and Hicks [10] revealed that the importance of investigating the stability of various iteration procedures for various classes of nonlinear mappings. Harder [8] established applications of stability results to first order differential equations. In [17], [18], Osilike established that certain Mann and Ishikawa iteration methods are T-stable on X when T is a Lipschitz strongly pseudocontractive operator in real q-uniformly smooth Banach spaces or real Banach spaces, respectively.

It is our purpose in this paper to prove that the Mann iteration method with errors can be used to approximate fixed points of strictly hemi-contractive operators in arbitrary nonempty closed bounded convex subsets of certain smooth Banach spaces. Meanwhile, we establish under suitable conditions the almost stability of the iteration method. The results presented in this paper generalize the corresponding results in [1]-[7], [20] and others.

2. Preliminaries

LEMMA 2.1. *Let $\{\alpha_n\}_{n=0}^\infty$, $\{\beta_n\}_{n=0}^\infty$ and $\{\gamma_n\}_{n=0}^\infty$ be nonnegative real sequences and let $\varepsilon > 0$ be a constant satisfying*

$$\beta_{n+1} \le (1-\alpha_n)\beta_n + \varepsilon\alpha_n + \gamma_n, \quad n \ge 0,$$

where $\sum_{n=0}^\infty \alpha_n = \infty$, $\alpha_n \le 1$ for all $n \ge 0$ and $\sum_{n=0}^\infty \gamma_n < \infty$. Then $\limsup_{n\to\infty} \beta_n \le \varepsilon$.

Proof. By a straightforward induction, we get that for $n \ge k \ge 0$,

$$\begin{aligned} \beta_{n+1} \le \beta_k \prod_{j=k}^{n}(1-\alpha_j) + \epsilon \sum_{j=k}^{n} \alpha_j \prod_{i=j+1}^{n} (1-\alpha_i) \\ + \sum_{j=k}^{n} \gamma_j \prod_{i=j+1}^{n} (1-\alpha_i), \end{aligned} \tag{2.1}$$

where $\prod_{i=n+1}^{n}(1-\alpha_i) \overset{\text{put}}{=} 1$. Note that $\sum_{j=k}^{n} \alpha_j \prod_{i=j+1}^{n}(1-\alpha_i) \le 1$. It follows from (2.1) that

$$\beta_{n+1} \leq \exp\left(-\sum_{j=k}^{n} \alpha_j\right)\beta_k + \epsilon + \sum_{j=k}^{n} \gamma_j. \tag{2.2}$$

For a given $\delta > 0$, there exists a positive integer k such that $\sum_{j=k}^{\infty} \gamma_j < \delta$. Thus (2.2) ensures that

$$\limsup_{n\to\infty} \beta_n \leq \epsilon + \delta.$$

Letting $\delta \to 0^+$, we consider that $\limsup_{n\to\infty} \beta_n \leq \epsilon$. This completes the proof. □

REMARK 2.1. In case $\gamma_n = 0$ for each $n \geq 0$, then Lemma 2.1 reduces to Lemma 1 of Park [20].

LEMMA 2.2. ([20]) *Let X be a smooth Banach space satisfying the following condition*:

($*$) *for any bounded subset D of X, there is $c : [0, \infty) \to [0, \infty)$ such that*

$$\lim_{t\to 0^+} \frac{c(t)}{t} = 0 \quad \text{and} \quad Re\,\langle x - y, J(x) - J(y)\rangle \leq c(\|x - y\|), \quad x, y \in D.$$

Then for any $\varepsilon > 0$ and any bounded subset K, there exists $\delta > 0$ such that

$$\|sx + (1 - s)y\|^2 \leq (1 - 2s)\|y\|^2 + 2s\, Re\,\langle x, J(y)\rangle + 2s\varepsilon \tag{2.3}$$

for all $x, y \in K$ and $s \in [0, \delta)$.

LEMMA 2.3. ([6]) *Let X be a Banach space and $T : D(T) \subseteq X \to X$ be an operator with $F(T) \neq \emptyset$. Then T is strictly hemi-contractive if and only if there exists $t > 1$ such that for all $x \in D(T)$ and $q \in F(T)$, there exist $j \in J(x - q)$ satisfying*

$$Re\,\langle x - Tx, j\rangle \geq \left(1 - \frac{1}{t}\right)\|x - q\|^2. \tag{2.4}$$

LEMMA 2.4. *Let X be an arbitrary normed linear space and $T : D(T) \subseteq X \to X$ be an operator.*

(i) *If T is a local strongly pseudocontractive operator and $F(T) \neq \emptyset$, then $F(T)$ is a singleton and T is strictly hemi-contractive.*

(ii) *If T is strictly hemi-contractive, then $F(T)$ is a singleton.*

Proof. Suppose that $F(T) \neq \emptyset$ and T is a local strongly pseudocontractive operator. We assert first of all that $F(T)$ is a singleton. Otherwise there exist distinct elements $p, q \in F(T)$. Since T is local strongly pseudocontractive, then there exists $t_p > 1$ such that for all $y \in D(T)$ and $r > 0$,

$$\|p - y\| \leq \|(1 + r)(p - y) - rt_p(Tp - Ty)\|. \tag{2.5}$$

Set $y = q \in F(T) \subseteq D(T)$ and $r = \frac{1}{2(t_p - 1)}$. It follows from (2.5) that

$$0 < \|p - q\| = |1 + r(1 - t_p)|\|p - q\| = \frac{1}{2}\|p - q\|,$$

which is a contradiction. Hence $F(T) = \{q\}$ for some $q \in D(T)$.

Next we show that T is strictly hemi-contractive. Note that T is a local strongly pseudocontractive operator and $F(T) = \{q\}$. Put $t = t_q$. Then (1.2) ensures that

$$\|q - y\| \leq \|(1 + r)(q - y) - rt(q - Ty)\|$$

for all $y \in D(T)$ and $r > 0$. That is, T is strictly hemi-contractive.

The proof of (ii) now follows exactly as in the first part of the proof of (i). This completes the proof. □

3. Main results

Our main results are as follows.

THEOREM 3.1. *Let X be a smooth Banach space satisfying the condition $(*)$. Let K be a nonempty closed bounded convex subset of X and $T : K \to K$ be a strictly hemi-contractive operator. Suppose that $\{u_n\}_{n=0}^{\infty}$ is an arbitrary sequence in K and $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$*

and $\{r_n\}_{n=0}^{\infty}$ *are any sequences in* $[0,1]$ *satisfying*

$$a_n + b_n + c_n = 1, \quad n \geq 0; \tag{3.1}$$

$$c_n(1 - r_n) = r_n b_n, \quad n \geq 0; \tag{3.2}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} r_n = 0; \tag{3.3}$$

$$\sum_{n=0}^{\infty}(b_n + c_n) = \infty. \tag{3.4}$$

Suppose that $\{x_n\}_{n=0}^{\infty}$ *is the sequence generated from an arbitrary* $x_0 \in K$ *by*

$$x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \quad n \geq 0. \tag{3.5}$$

Let $\{y_n\}_{n=0}^{\infty}$ *be any sequence in* K *and define* $\{\varepsilon_n\}_{n=0}^{\infty}$ *by*

$$\varepsilon_n = \|y_{n+1} - p_n\|, \quad n \geq 0, \tag{3.6}$$

where $p_n = a_n y_n + b_n T y_n + c_n u_n$. *Then*

(i) *the sequence* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point* q *of* T;

(ii) $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ *implies that* $\lim_{n\to\infty} y_n = q$, *so that* $\{x_n\}_{n=0}^{\infty}$ *is almost* T*-stable on* K;

(iii) $\lim_{n\to\infty} y_n = q$ *implies that* $\lim_{n\to\infty} \varepsilon_n = 0$.

Proof. It follows from Lemma 2.4 that $F(T)$ is a singleton. That is, $F(T) = \{q\}$ for some $q \in K$. Set $M = 1 + diam\, K$ and $d_n = b_n + c_n$ for all $n \geq 0$. It is easy to verify that

$$\max\{\|x_n - q\|, \|y_n - q\|, \|p_n - q\|, \|u_n - q\|, \varepsilon_n\} \leq M, \quad n \geq 0. \tag{3.7}$$

For given $\varepsilon > 0$ and the bounded subset K, there exists $\delta > 0$ satisfying (2.3). Note that (3.2) and (3.3) ensure that there exists N such that

$$d_n < \min\left\{\delta, \frac{1}{2(1-k)}\right\}, \quad r_n \leq \frac{2\varepsilon}{3M^2}, \quad n \geq N, \tag{3.8}$$

where $k = \frac{1}{t}$ and t satisfies (2.4). Using (3.1), (3.5), (3.7), (3.8), (2.3) and Lemma 2.2, we infer that

$$\begin{aligned}
&\|x_{n+1}-q\|^2\\
&=\|(1-d_n)(x_n-q)+d_n(Tx_n-q)+c_n(u_n-Tx_n)\|^2\\
&\le \big(\|(1-d_n)(x_n-q)+d_n(Tx_n-q)\|+Mc_n\big)^2\\
&\le \|(1-d_n)(x_n-q)+d_n(Tx_n-q)\|^2+3M^2c_n\\
&\le (1-2d_n)\|x_n-q\|^2+2d_n\, Re\,\langle Tx_n-q, J(x_n-q)\rangle \\
&\quad +2\varepsilon d_n+3M^2c_n \qquad (3.9)\\
&\le (1-2d_n)\|x_n-q\|^2+2kd_n\|x_n-q\|^2+2\varepsilon d_n+3M^2c_n\\
&=\{1-2(1-k)d_n\}\|x_n-q\|^2+2d_n\Big(\varepsilon+\frac{3}{2}M^2r_n\Big)\\
&\le \{1-2(1-k)d_n\}\|x_n-q\|^2+4d_n\varepsilon
\end{aligned}$$

for all $n \ge N$. It follows from Lemma 2.1, (3.1)-(3.4), (3.8) and (3.9) that

$$\limsup_{n\to\infty}\|x_n-q\|^2 \le \frac{2\varepsilon}{1-k}. \qquad (3.10)$$

Letting $\varepsilon \to 0^+$ in (3.9), we obtain that $\limsup_{n\to\infty}\|x_n-q\|^2=0$, which implies that $x_n \to q$ as $n\to\infty$.

Similarly we also have

$$\begin{aligned}
&\|p_n-q\|^2\\
&=\|(1-d_n)(y_n-q)+d_n(Ty_n-q)+c_n(u_n-Ty_n)\|^2\\
&\le \|(1-d_n)(y_n-q)+d_n(Ty_n-q)\|^2+3M^2c_n\\
&\le (1-2d_n)\|y_n-q\|^2+2d_n\, Re\,\langle Ty_n-q, J(y_n-q)\rangle\\
&\quad +2\varepsilon d_n+3M^2c_n \qquad (3.11)\\
&\le \{1-2(1-k)d_n\}\|y_n-q\|^2+2\varepsilon d_n+3M^2c_n\\
&\le \{1-2(1-k)d_n\}\|y_n-q\|^2+2d_n\Big(\varepsilon+\frac{3}{2}M^2r_n\Big)\\
&\le \{1-2(1-k)d_n\}\|y_n-q\|^2+4d_n\varepsilon
\end{aligned}$$

for all $n > N$.

Suppose that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$. In view of (3.11), (3.7) and (3.6), we infer that

$$\begin{aligned} \|y_{n+1} - q\|^2 &\le (\|p_n - q\| + \|y_{n+1} - p_n\|)^2 \\ &\le \|p_n - q\|^2 + 2M\varepsilon_n + \varepsilon_n^2 \\ &\le \{1 - 2(1-k)d_n\}\|y_n - q\|^2 + 4d_n\varepsilon + 3M\varepsilon_n \end{aligned} \tag{3.12}$$

for all $n \ge N$. By virtue of Lemma 2.1, (3.1)-(3.4), (3.8), (3.12), we conclude that for all $n \ge N$,

$$\limsup_{n\to\infty} \|y_n - q\|^2 \le \frac{2\varepsilon}{1-k},$$

which implies that $\limsup_{n\to\infty} \|y_n - q\|^2 = 0$ by the arbitrariness of ε. Hence $y_n \to q$ as $n \to \infty$.

Conversely, suppose that $\lim_{n\to\infty} y_n = q$. Then (3.11), (3.2) and (3.3) mean that

$$\begin{aligned} \varepsilon_n &\le \|y_{n+1} - q\| + \|p_n - q\| \\ &\le \|y_{n+1} - q\| + [(1 - 2(1-k)d_n)\|y_n - q\|^2 + 4d_n\varepsilon]^{1/2} \\ &\to 0 \end{aligned}$$

as $n \to \infty$. That is, $\varepsilon_n \to 0$ as $n \to \infty$. This completes the proof. □

Using the method of proof of Theorem 3.1, we can similarly prove the following.

THEOREM 3.2. *Let X, T, K and $\{u_n\}_{n=0}^{\infty}$ be as in Theorem 3.1. Suppose that $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ such that (3.1) and*

$$\lim_{n\to\infty} b_n = 0; \tag{3.13}$$

$$\sum_{n=0}^{\infty} b_n = \infty; \tag{3.14}$$

$$\sum_{n=0}^{\infty} c_n < \infty. \tag{3.15}$$

If $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{p_n\}_{n=0}^{\infty}$ and $\{\varepsilon_n\}_{n=0}^{\infty}$ are as in Theorem 3.1, then the conclusions of Theorem 3.1 hold.

REMARK 3.1. The convergence result in Theorem 3.2 extends Theorem 4.1 in [1], Theorem 4.1 in [2], the Theorem in [3], Theorem 1 in [4], Theorem 3 in [5], Theorem 1 in [6], Theorem 4 in [7] and the Theorem in [20] in the following ways:

(1) The Mann iteration methods in [1]-[7], [20] are replaced by the more general Mann iteration method with errors.

(2) The Lipschitz strongly pseudocontractive mappings in [3] and [7] and the continuous strongly pseudocontractive mappings in [1], [2], [4], [5] and [20] are weakened by the strictly hemi-contractive mappings, which are even not necessarily continuous.

(3) Theorem 3.2 holds in arbitrary smooth Banach spaces with either (i) or (ii) or (iii) in Lemma 2.2 whereas the results in [1]-[7] are satisfied in the restricted L_p (or l_p) spaces, q-uniformly smooth Banach spaces and real uniformly smooth Banach spaces, respectively.

(4) Conditions $\sum_{n=0}^{\infty} c_n^2 < \infty$ in [3] and $\sum_{n=0}^{\infty} c_n\, b(c_n) < \infty$ in [4], [5] and [6] are replaced by condition $\lim_{n\to\infty} b_n = 0$ in Theorem 3.2.

REMARK 3.2. The iteration parameters $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$ in Theorem 3.1 and the iteration parameters $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ in Theorem 3.2 are not dependent on either the geometry of the underlying Banach spaces or on any special property of the operators. The iteration parameters in Theorems 3.1 and 3.2 can be chosen at the start of the iteration process. Prototypes for the parameters in Theorems 3.1 and 3.2 are, respectively, as follows.

EXAMPLE 3.1. Let

$$a_n = 1 - (n+2)^{-1/4} - (n+2)^{-1/2}, \quad b_n = (n+2)^{-1/4},$$
$$c_n = (n+2)^{-1/2}, \quad r_n = \left[1 + (n+2)^{1/4}\right]^{-1}, \quad n \ge 0.$$

EXAMPLE 3.2. Let

$$a_n = 1 - b_n - c_n, \quad b_{2n} = (2n+2)^{-2}, \quad b_{2n+1} = (2n+3)^{-1},$$
$$c_n = (n+2)^{-2}, \quad n \ge 0.$$

REMARK 3.3. The above examples reveal that conditions (3.1)-(3.4) are different from conditions (3.1), (3.13)-(3.15). In fact, $\sum_{n=0}^{\infty} c_n = \infty$ in Example 3.1 and

$$\lim_{n\to\infty} r_{2n} = \lim_{n\to\infty} \frac{c_{2n}}{b_{2n} + c_{2n}} = \frac{1}{2} \neq 0$$

in Example 3.2.

QUESTION 3.1. Can the Ishikawa iteration method with errors be extended to Theorems 3.1 and 3.2?

ACKNOWLEDGEMENT. The authors sincerely thank the anonymous referee for his useful suggestions in revising this paper.

References

[1] S. S. Chang, *Some problems and results in the study of nonlinear analysis*, Nonlinear Anal. TMA **30** (1997), no. 7, 4197–4208.

[2] S. S. Chang, Y. J. Cho, B. S. Lee and S. M. Kang, *Iterative approximations of fixed points and solutions for strongly accretive and strongly pseudo-contractive mappings in Banach spaces*, J. Math. Anal. Appl. **224** (1998), 149–165.

[3] C. E. Chidume, *Iterative approximation of fixed points of Lipschitzian strictly pseudo-contractive mappings*, Proc. Amer. Math. Soc. **99** (1987), no. 2, 283–288.

[4] ———, *Approximation of fixed points of strongly pseudo-contractive mappings*, Proc. Amer. Math. Soc. **120** (1994), 545–551.

[5] ———, *Iterative solutions of nonlinear equations with strongly accretive operators*, J. Math. Anal. Appl. **192** (1995), 502–518.

[6] C. E. Chidume and M. O. Osilike, *Fixed point iterations for strictly hemi-contractive maps in uniformly smooth Banach spaces*, Numer. Funct. Anal. Optimiz. **15** (1994), 779–790.

[7] ———, *Ishikawa iteration process for nonlinear Lipschitz strongly accretive mappings*, J. Math. Anal. Appl. **192** (1995), 727–741.

[8] A. M. Harder, *Fixed point theory and stability results for fixed point iteration procedures, Ph. D. Thesis, University of Missouri-Rolla*, 1987.

[9] A. M. Harder and T. L. Hicks, *A stable iteration procedure for nonexpansive mappings*, Math. Japon. **33** (1988), 687–692.

[10] ———, *Stability results for fixed point iteration procedures*, Math. Japon. **33** (1988), 693–706.

[11] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147–150.

[12] Z. Liu and S. M. Kang, *Iterative approximation of fixed points for ϕ-hemicontractive operators in arbitrary Banach spaces*, Acta Sci. Math. (Szeged) **67** (2001), 821–831.

[13] ———, *Stability of Ishikawa iteration methods with errors for strong pseudo-contractions and nonlinear equations involving accretive operators in arbitrary real Banach spaces*, Math. and Computer Modell. **34** (2001), 319–330.

[14] ———, *Convergence theorems for ϕ-strongly accretive and ϕ-hemicontractive operators*, J. Math. Anal. Appl. **253** (2001), 35–49.

[15] ———, *Convergence and stability of the Ishikawa iteration procedures with errors for nonlinear equations of the ϕ-strongly accretive type*, Neural. Parallel & Sci. Comput. **9** (2001), 103–118.

[16] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[17] M. O. Osilike, *Stable iteration procedures for strong pseudocontractions and nonlinear operator equations of the accretive type*, J. Math. Anal. Appl. **204** (1996), 677–692.

[18] ———, *Stable iteration procedures for nonlinear pseudocontractive and accretive operator in arbitrary Banach spaces*, Indian J. pure appl. Math. **28** (1997), 1017–1029.

[19] ———, *Stability of the Mann and Ishikawa iteration procedures for ϕ-strongly pseudocontractions and nonlinear equations of the ϕ-strongly accretive type*, J. Math. Anal. Appl. **227** (1998), 319–334.

[20] J. A. Park, *Mann iteration process for the fixed point of strictly pseudocontractive mapping in some Banach spaces*, J. Korean Math. Soc. **31** (1994), 333–337.

[21] B. E. Rhoades, *Comments on two fixed point iteration methods*, J. Math. Anal. Appl. **56** (1976), 741–750.

[22] J. Schu, *Iterative construction of fixed points of strictly pseudocontractive mappings*, Applicable Anal. **40** (1991), 67–72.

[23] X. Weng, *Fixed point iteration for local strictly pseudo-contractive mapping*, Proc. Amer. Math. Soc. **113** (1991), no. 3, 727–731.

[24] Y. Xu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl. **224** (1998), 91–101.

Zeqing Liu
Department of Mathematics
Liaoning Normal University
Dalian, Liaoning 116029, P. R. China
E-mail: zeqingliu@sina.com.cn

Shin Min Kang and Soo Hak Shim
Department of Mathematics
Gyeongsang National University
Chinju 660-701, Korea
E-mail: smkang@nongae.gsnu.ac.kr