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ALMOST SURE ASYMPTOTIC BEHAVIOUR OF THE r-NEIGHBOURHOOD SURFACE AREA OF BROWNIAN PATHS

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Abstract. We show that whenever the q-dimensional Minkowski content of a subset $A \subset \mathbb{R}^d$ exists and is finite and positive, then the "S-content" defined analogously as the Minkowski content, but with volume replaced by surface area, exists as well and equals the Minkowski content. As a corollary, we obtain the almost sure asymptotic behaviour of the surface area of the Wiener sausage in $\mathbb{R}^d$, $d \geqslant 3$.

Keywords: Minkowski content, Kneser function, Brownian motion, Wiener sausage

MSC 2010: 28A75, 52A20, 60D05

1. Introduction

Let $X\colon [0,1] \to \mathbb{R}^d$ be the d-dimensional standard Brownian motion, i.e. its coordinates are independent Wiener processes starting at 0 and with variances $\operatorname{var}(X^{(i)}(s) - X^{(i)}(t)) = |s-t|$, $i = 1, \ldots, d$. Its r-parallel neighbourhoods

$$Z_r = \{x \in \mathbb{R}^d \colon \operatorname{dist}(x, X[0,1]) \leqslant r\}, \quad r > 0,$$

are sometimes called Wiener sausages, and are used to approximate the highly irregular and hardly tractable trajectory itself.

Exact formulas for the mean volume $\mathbb{E}\,\lambda^d(Z_r)$ were derived by Berezhkovski et al. [1], using the known distributions of exit times of balls for X. Recently it was shown that, as expected, the mean surface area of the Wiener sausage satisfies (see [4], [5])

$$\mathbb{E}\,\mathcal{H}^{d-1}(\partial Z_r) = \frac{\mathrm{d}}{\mathrm{d}r}\mathbb{E}\lambda^d(Z_r), \quad r > 0.$$

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More difficult is the study of almost sure properties. The almost sure asymptotic behaviour of the volume $\lambda^d(Z_r)$ was extensively studied (see, e.g., [6]) and almost sure asymptotic expansions as well as central limit theorems were obtained (see [2]). For the purpose of this note, we are interested only in the first order asymptotic terms ($r \to 0_+$):

$$\lambda^2(Z_r) = \frac{\pi}{|\log r|} + o\Big(\frac{1}{|\log r|}\Big), \quad d = 2, \tag{1.1}$$

$$\lambda^d(Z_r) = (d-2)\pi\kappa_{d-2}\, r^{d-2} + o(r^{d-2}), \quad d \geqslant 3, \tag{1.2}$$

where $\kappa_p = \pi^{p/2}/\Gamma(1+p/2)$. The aim of this note is to derive almost sure asymptotic behaviour for the surface area of the Wiener sausage. In particular, we show that in dimension $d \geqslant 3$, we have

$$\mathcal{H}^{d-1}(\partial Z_r) = (d-2)^2\pi\kappa_{d-2} r^{d-3} + o\left(r^{d-3}\right), \quad r \to 0_+, \quad d \geqslant 3. \tag{1.3}$$

This result, in fact, is not surprising since the leading expansion term is simply the differential of the leading term in the volume expansion. On the other hand, it has been shown in [5] that the asymptotic expansions of volume and surface area of parallel sets are not always so strictly related. It turned out that (1.3) does not follow from (1.2) in a simple way, and the proof of (1.2) using local times of Brownian motions cannot probably be easily adapted.

We use a different geometric approach. In fact, we show (Theorem 3.1) a general result saying that whenever A is a bounded set of $\mathbb{R}^d$ such that its Minkowski $(d-p)$-content

$$\mathcal{M}^{d-p}(A) = \lim_{r\to 0} \frac{\lambda^d(A_r)}{\kappa_p\, r^p}$$

exists for some $p \geqslant 1$ and equals a positive number a, then also

$$\lim_{r\to 0} \frac{\mathcal{H}^{d-1}(\partial A_r)}{p\kappa_p\, r^{p-1}} = a.$$

For the proof, we use the properties of the volume function derived by Stachó [7] and [5]. In fact, we show a result on the asymptotic behaviour of a Kneser function.

Finally, we demonstrate on an example that an analogous approach cannot be applied in the two-dimensional case where the area of the Wiener sausage is asymptotically proportional to $1/|\log r|$ as $r \to 0$. The asymptotic behaviour of the boundary length of the Wiener sausage in dimension 2 remains open, up to our knowledge.

2. Preliminaries

We start by listing some facts needed to prove our results. Throughout the paper we work in $\mathbb{R}^d$ with $d \geqslant 2$. We use the notation λ^d for the Lebesgue measure in $\mathbb{R}^d$ and $\mathcal{H}^s$ for the s-dimensional Hausdorff measure in $\mathbb{R}^d$, $0 \leqslant s \leqslant d$.

Definition 2.1. Let f be a continuous nonnegative real function defined on $[0, \infty)$. We say that f is a *Kneser function of order* d if for all $0 < a \leqslant b < \infty$ and $\lambda \geqslant 1$,

$$f(\lambda b) - f(\lambda a) \leqslant \lambda^d (f(b) - f(a)).$$

We list some properties of Kneser functions proved by Stachó.

Proposition 2.2 ([7, Lemma 2, Theorem 1]). *Let f be a Kneser function of order d. Then*

(i) *f is absolutely continuous,*
(ii) *$f'(t)$ exists for all $t > 0$ up to a countable set,*
(iii) *left and right hand side derivatives of f (f'_- and f'_+) exist for every $t > 0$, and $f'_- \geqslant f'_+$,*
(iv) *f'_- and f'_+ are continuous from the left and from the right, respectively,*
(v) *the function $f'_+(t)t^{1-d}$ is monotone decreasing, $t > 0$.*

Let A be a bounded subset of $\mathbb{R}^d$ and let $\operatorname{dist}_A(\cdot)$ denote the (Euclidean) distance function from the set A. Given $r > 0$, we denote by

$$A_r := \{z \in \mathbb{R}^d \colon \operatorname{dist}_A(z) \leqslant r\}$$

the r-parallel neighbourhood of A. Further, we denote by $V_A(r) := \lambda^d(A_r)$ and $S_A(r) := \mathcal{H}^{d-1}(\partial A_r)$ the volume and surface area of A_r, respectively.

The fact that the volume function V_A has the property from Definition 2.1 is due to Kneser.

Proposition 2.3 [3]. *For any bounded set $A \subset \mathbb{R}^d$, V_A is a Kneser function of order d.*

The volume and surface area of parallel sets are related as follows.

Proposition 2.4 ([5, Corollary 2.5]). *Let $A \subset \mathbb{R}^d$ be bounded. Then $(V_A)'_+(r) \leqslant S_A(r) \leqslant (V_A)'_-(r)$ for all $r > 0$. Moreover, $(V_A)'(r)$ exists and equals $S_A(r)$ for all $r > 0$ up to a countable set.*

The Minkowski q-dimensional content of a set $A \subset \mathbb{R}^d$ is defined as

$$\mathcal{M}^q(A) = \lim_{r\to 0} \frac{V_A(r)}{\kappa_{d-q}\, r^{d-q}},$$

whenever the limit exists. Analogously, if $q < d$, the q-dimensional *S-content* of A was introduced in [5] as

$$\mathcal{S}^q(A) = \lim_{r\to 0} \frac{S_A(r)}{(d-q)\kappa_{d-q}\, r^{d-q-1}},$$

whenever the limit exists. It was shown in [5] that if $\mathcal{S}^q(A)$ exists, then $\mathcal{M}^q(A)$ exists as well and they are equal. The oposite implication remains, however, open.

3. Main results

Theorem 3.1. *Let f be a Kneser function of order $d \geqslant 2$. If*

$$\lim_{r\to 0_+} \frac{f(r)}{r^p} = a \tag{3.1}$$

for some $p \in [1, d]$ and $0 < a < \infty$, then

$$\lim_{r\to 0_+} \frac{f'_+(r)}{p\, r^{p-1}} = a.$$

Corollary 3.1. *Let $A \subset \mathbb{R}^d$ be bounded. If the Minkowski q-dimensional content $\mathcal{M}^q(A)$ exists and is positive finite for some $q \leqslant d-1$, then the q-dimensional S-content exists as well and we have $\mathcal{S}^q(A) = \mathcal{M}^q(A)$.*

P r o o f. V_A is a Kneser function of order d by Proposition 2.3, hence we may apply Theorem 3.1 with $p = d - q$ and $a = \mathcal{M}^q(A)$. Since $S_A(r) \geqslant (V_A)'_+(r)$ by Proposition 2.4, we get

$$\liminf_{r\to 0} \frac{S_A(r)}{(d-q)\kappa_{d-q} r^{d-q-1}} \geqslant a.$$

For the opposite inequality, we use Propositions 2.2 and 2.4 to get

$$S_A(r) \leqslant (V_A)'_-(r) = \lim_{t\to r_-} (V_A)'_-(t) = \lim_{\mathcal{R}\ni t\to r_-} (V_A)'_+(t),$$

where $\mathcal{R}$ denotes the set of all $t > 0$ where the derivative $(V_A)'(t)$ exists. It follows now from Theorem 3.1 that

$$\limsup_{r\to 0} \frac{S_A(r)}{(d-q)\kappa_{d-q} r^{d-q-1}} \leqslant a,$$

and the proof of the corollary is completed. □

P r o o f of Theorem 3.1. First we prove that

$$\liminf_{r\to 0_+} \frac{f'_+(r)}{pr^{p-1}} \geqslant a. \tag{3.2}$$

Let $\delta > 0$ be given. We shall find an $\eta > 0$ such that

$$\frac{f'_+(r)}{pr^{p-1}} > a - \delta \tag{3.3}$$

whenever $0 < r < \eta$. Assume that (3.3) does not hold for some $r > 0$. Using Proposition 2.2 (v) we get $f'_+(t) \leqslant f'_+(r)t^{d-1}/r^{d-1}$, $t \geqslant r$, thus

$$f'_+(t) \leqslant h(t) := \frac{a-\delta}{r^{d-p}} \cdot pt^{d-1}.$$

We denote $s := \tau r$, where $\tau := (a/(a-\delta))^{1/(d-1)} > 1$.

Using the absolute continuity of f, we can obtain now an upper bound for the difference:

$$f(s) - f(r) = \int_r^s f'_+(t)\,\mathrm{d}t \leqslant \int_r^s h(t)\,\mathrm{d}t = (a-\delta)\frac{p}{d}\, r^p(\tau^d - 1). \tag{3.4}$$

On the other hand, the assumption (3.1) implies that for any $\varepsilon > 0$ there exists $r_0 > 0$ such that whenever $0 < r < r_0$ then

$$(a-\varepsilon)r^p < f(r) < (a+\varepsilon)r^p,$$

thus

$$f(s) - f(r) > a(s^p - r^p) - \varepsilon(s^p + r^p)$$

if $\tau r < r_0$, and using the above notation we get

$$f(s) - f(r) > ar^p(\tau^p - 1) - \varepsilon r^p(\tau^p + 1). \tag{3.5}$$

Putting (3.4) a (3.5) together we get the inequality

$$ar^p(\tau^p - 1) - \varepsilon r^p(\tau^p + 1) < (a-\delta)\frac{p}{d}\, r^p(\tau^d - 1),$$

hence

$$\varepsilon > \frac{1}{\tau^p+1}\Big(a(\tau^p-1)-(a-\delta)\frac{p}{d}(\tau^d-1)\Big) \tag{3.6}$$
$$= \frac{p(a-\delta)}{\tau^p+1}\Big(\frac{\tau^{d-1+p}-\tau^{d-1}}{p}-\frac{\tau^d-\tau^0}{d}\Big).$$

Since $\tau > 1$, the function $u \mapsto \tau^u$ is convex increasing and, as $d-1+p \geqslant d$ by assumption, the right-hand side of (3.6) is positive. Let us denote it by ε_0. Thus, if $0 < \varepsilon < \varepsilon_0$ then (3.3) must be true for all $r < \varepsilon_0/\tau$. This proves (3.2).

In the second part of the proof we show that

$$\limsup_{r\to 0_+} \frac{f(r)}{pr^{p-1}} \leqslant a. \tag{3.7}$$

The procedure is similar to the first part. Let

$$\frac{f'_+(r)}{pr^{p-1}} \geqslant a+\delta \tag{3.8}$$

for some fixed $\delta > 0$ and $r > 0$. We use Proposition 2.2 (v) to show that $f'_+(t) \geqslant g(t)$ for $0 < t < r$, with

$$g(t) = \frac{a+\delta}{r^{d-p}} \cdot pt^{d-1}.$$

We denote $v := \varrho r$ with $\varrho := (a/(a+\delta))^{1/(d-1)} < 1$. Similarly to (3.4) we obtain

$$f(r)-f(v) = \int_v^r f'_+(t)\,\mathrm{d}t \geqslant (a+\delta)\frac{p}{d}\cdot r^p\,(1-\varrho^d). \tag{3.9}$$

Let $\varepsilon > 0$ be given. Then by (3.1) we have

$$f(r)-f(p) < ar^p\,(1-\varrho^p)+\varepsilon\, r^p\,(1+\varrho^p) \tag{3.10}$$

for sufficiently small r. Putting (3.9) and (3.10) together, we get

$$\varepsilon > \frac{1}{1+\varrho^p}\Big((a+\delta)\frac{p}{d}(1-\varrho^p)-a(1-\varrho^p)\Big) = \frac{p(a+\delta)}{1+\varrho^p}\Big(\frac{1-\varrho^d}{d}-\frac{\varrho^{d-1}-\varrho^{d-1+p}}{p}\Big).$$

Now $\varrho < 1$, the function $u \mapsto \varrho^u$ is convex decreasing. Hence, the right hand side of the last equality is strictly positive again. This means, however, when choosing ε smaller, (3.8) cannot hold with r arbitrarily small. This proves (3.7) and the proof is complete. $\square$

4. Application for the Brownian path

Let X be the standard d-dimensional Brownain motion as in Introduction, and let $Z = X[0,1]$ be its random path up to time 1. It is known that Z has both the Hausdorff and Minkowski dimensions 2 and the Minkowski content is

$$\mathcal{M}^2(Z) = \lim_{r\to 0} \frac{\mathcal{H}^d(Z_r)}{\kappa_{d-2} r^{d-2}} = (d-2)\pi \quad \text{a.s.} \tag{4.1}$$

(see [2] and [6]).

We apply now Corollary 3.2 to obtain

Corollary 4.1. *If $d \geqslant 3$ we have almost surely*

$$\mathcal{S}^2(Z) = \lim_{r\to 0} \frac{S_Z(r)}{(d-2)\kappa_{d-2} r^{d-3}} = (d-2)\pi \quad \text{a.s.}$$

In dimension 2, the situation is more complicated. The Minkowski dimension of Z is 2 again, but the volume $V_Z(r)$ is asymptotically equivalent to $\pi/|\log r|$ a.s. We conjecture that

$$S_Z(r) \sim (\pi/|\log r|)' = \pi/(r \log^2 r), \quad r \to 0,$$

but a proof analogous to that of Theorem 3.1 fails, as shown by the following example.

Example 4.1. Given $a \in (0,\infty)$, there exists a Kneser function f of order 2 such that

$$\lim_{r\to 0} f(r)|\log r| = a, \tag{4.2}$$

but

$$\lim_{r\to 0} f'_+(r)\, r \log^2 r \quad \text{does not exist.} \tag{4.3}$$

P r o o f. We define a continuous function $f\colon (0,1] \to [0,\infty)$ by

$$f(x) = a_n x^2 + c_n, \quad x \in [2^{-n}, 2^{-(n-1)}],$$

where

$$a_n = \frac{a\, 2^{2n}}{3n(n-1)\log 2} \quad \text{and} \quad c_n = \frac{a(3n-4)}{3n(n-1)\log 2}.$$

Note that f can be extended to a Kneser function of order 2 and that

$$f(2^{-n}) = \frac{a}{n \log 2}, \quad n = 1, 2, \ldots.$$

We have $f_1(x) \leqslant f(x) \leqslant f_2(x)$ with

$$f_1(x) = f(2^{-n}), \quad x \in [2^{-n}, 2^{-(n-1)}),$$
$$f_2(x) = f(2^{-(n-1)}), \quad x \in (2^{-n}, 2^{-(n-1)}].$$

Since

$$\lim_{x \to 0_+} f_1(x)|\log x| = \lim_{n\to\infty} \frac{a}{n \log 2}\, n \log 2 = a,$$
$$\lim_{x \to 0_+} f_2(x)|\log x| = \lim_{n\to\infty} \frac{a}{(n-1) \log 2}\, n \log 2 = a,$$

we get (4.2).

The derivative of f fulfils

$$f'(x) = 2a_n x, \quad x \in (2^{-n}, 2^{-(n-1)})$$

and we have

$$\begin{aligned}\liminf_{x\to 0+} f'_+(x)x(\log x)^2 &= \lim_{n\to\infty} f'_+(2^{-n})\, 2^{-n} \left(\log 2^{-n}\right)^2 \\ &= \lim_{n\to\infty} \frac{2a\, 2^{2n}}{3n(n-1)\log 2}\, 2^{-n}\, 2^{-n} n^2 (\log 2)^2 \\ &= \frac{2}{3}\, a \log 2\end{aligned}$$

and, similarly,

$$\begin{aligned}\limsup_{x\to 0+} f'_-(x)x(\log x)^2 &= \lim_{n\to\infty} f'_-(2^{-(n-1)})\, 2^{-(n-1)} (\log 2^{-(n-1)})^2 \\ &= \lim_{n\to\infty} \frac{2a\, 2^{2n}}{3n(n-1)\log 2}\, 2^{-(n-1)}\, 2^{-(n-1)} (n-1)^2 (\log 2)^2 \\ &= \frac{8}{3}\, a \log 2.\end{aligned}$$

This shows that (4.3) is true. □

References

[1] *A. M. Berezhkovskii, Yu. A. Makhnovskii, R. A. Suris*: Wiener sausage volume moments. J. Stat. Phys. *57* (1989), 333–346.

[2] *J.-F. Le Gall*: Fluctuation results for the Wiener sausage. Ann. Probab. *16* (1988), 991–1018.

[3] *M. Kneser*: Über den Rand von Parallelkörpern. Math. Nachr. *5* (1951), 241–251. (In German.)

[4] *J. Rataj, V. Schmidt, E. Spodarev*: On the expected surface area of the Wiener sausage. Math. Nachr. *282* (2009), 591–603.

[5] *J. Rataj, S. Winter*: On volume and surface area of parallel sets. Indiana Univ. Math. J. *59* (2010), 1661–1685.
[6] *F. Spitzer*: Electrostatic capacity, heat-flow, and Brownian motion. Z. Wahrscheinlichkeitstheor. Verw. Geb. *3* (1964), 110–121.
[7] *L. L. Stachó*: On the volume function of parallel sets. Acta Sci. Math. *38* (1976), 365–374.

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