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Almost sure central limit theorem for exceedance point processes of stationary sequences

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Abstract. In this paper, we proved an almost sure central limit theorem for the exceedance point processes of a stationary sequence which satisfy some long range dependence conditions. As a by-product, we obtained the almost sure central limit theorem for the order statistics of the stationary sequence. The obtained results are also extended to the vector of point processes for some strong mixing random sequences.

1 Introduction

The almost sure central limit theorem (ASCLT) has been first introduced independently by Brosamler (1988) and Schatte (1988) for partial sum, and then it became an intensively studied subject. In its simplest form the ASCLT states that if $X_1, X_2, \dots$ is a i.i.d. sequence of random variables with mean 0 and variance 1, then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{t=1}^{n} \frac{1}{t} \mathbf{1}\big(t^{-1/2} S_t \le x\big) = \Phi(x) \qquad \text{a.s.}$$

for any $x \in \mathbf{R}$, where $S_n = \sum_{t=1}^{n} X_t$, $\mathbf{1}$ is indicator function and $\Phi(\cdot)$ stands for the standard normal distribution function.

Later on, Fahrner and Stadtmüller (1998) and independently Cheng et al. (1998) obtained the ASCLT for the maxima $M_t = \max_{k\le t} X_k$ of independent random variables. They proved that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{t=1}^{n} \frac{1}{t} \mathbf{1}\big(a_t(M_t - b_t) \le x\big) = G(x) \qquad \text{a.s.} \tag{1}$$

for any $x \in \mathbf{R}$ under the conditions that

$$\lim_{t\to\infty} P\big(a_t(M_t - b_t) \le x\big) = G(x) \tag{2}$$

with real sequences $a_t > 0$, $b_t \in \mathbf{R}$, $t \ge 1$ and a non-degenerate distribution $G(x)$.

One interesting direction is to extend (1) to dependent case. In this field, the first result was provided by Csáki and Gonchigdanzan (2002). Let $X_1, X_2, \dots$ be

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a stationary Gaussian random variable sequence with covariance function $r_t = EX_1X_{t+1}$ satisfying

$$r_t \log t (\log\log t)^{1+\varepsilon} = O(1) \tag{3}$$

for some $\varepsilon > 0$ or other conditions related to the convergence rate of the covariance function. Csáki and Gonchigdanzan (2002) and Chen and Lin (2007) proved that (1) still holds with some special constants a_t, b_t. We refer to Peng and Nadarajah (2011) for the non-stationary Gaussian case, Tan and Peng (2009) for more general dependent case and Tan (2013) for stationary Gaussian process. Recently, Choi (2010) and Tan and Wang (2014) have extended (1) to a stationary and non-stationary Gaussian random field. Other related results can be found in Tan and Wang (2012) and Hashorva and Weng (2013).

Another interesting direction is to extend (1) to order statistics. The pioneers in this field are Stadtmüller (2002) and Peng and Qi (2003) who studied the ASCLT for central order statistics of i.i.d. random variables. Especially for some fixed $k \in \mathbf{Z}^+$ they showed that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{t=1}^{n} \frac{1}{t} \mathbf{1}\big(a_t\big(M_t^{(k)} - b_t\big) \le x\big) = G(x) \sum_{s=0}^{k-1} \frac{(-\log G(x))^s}{s!} \qquad \text{a.s.} \tag{4}$$

for any $x \in \mathbf{R}$ provided that (2) held, where $M_t^{(k)}$ is the kth maximum among $X_1, X_2, \ldots, X_t$. For a more general result, we refer to Hörmann (2005). Dudziński (2009) extended (4) to some stationary Gaussian sequences provided that the covariance function of the sequence satisfies some very stronger conditions.

In this paper, we are interested in the similar questions for some stationary sequences. We prove the ASCLTs for exceedance point processes of some stationary sequences. As a by-product, we obtain the ASCLTs for order statistics of the stationary sequence. The above results are also extended to the vector of point processes for strong mixing random sequences. Now, let us introduce the dependence structure.

Let $\{X_n\}_{n\ge1}$ be a sequence of stationary random variables with common distribution function F and $\{u_n\}_{n\ge1}$ be a sequence of constants. For dealing with the limit properties of exceedance point processes, Hsing et al. (1988) introduced the following long range dependence condition $\triangle(u_n)$. Condition $\triangle(u_n)$ is said to be satisfied by $\{X_n\}_{n\ge1}$ if

$$\begin{aligned}\alpha_{n,l} = \sup\big\{\big|P(\mathbf{A}\cap\mathbf{B}) - P(\mathbf{A})P(\mathbf{B})\big| : \mathbf{A} \in \beta_1^k(u_n),\\ \mathbf{B} \in \beta_{k+l}^n(u_n), k = 1, 2, \ldots, n-l\big\}\end{aligned}$$

is such that $\alpha_{n,l} \to 0$, as $n \to \infty$, for some sequence $l_n = o(n)$. $\beta_i^j(u_n)$ denotes the σ-field generated by the events $\{X_s \le u_n\}, i \le s \le j$. Note that the condition $\triangle(u_n)$ is stronger than the distributional mixing condition $D(u_n)$ (see Leadbetter et al., 1983), but weaker than strong mixing.

Let N_n be the exceedance point process on $(0, 1]$ with points $(i/n\colon\ 1 \le i \le n$ for which $X_i > u_n)$, that is,

$$N_n(B) = \sum_{i/n \in B} \mathbf{1}(X_i > u_n),$$

for Borel set B on $(0, 1]$, which is a random measure on all Borel sets on $(0, 1]$. Under the condition $\triangle(u_n)$, Hsing et al. (1988) studied the limit of exceedance point process and showed that any limiting point process for exceedances is necessarily compound Poisson. More precisely, they proved the following result.

Theorem 1.1. *For every* $\tau \in (0, \infty)$, *let the constants* $\{u_n^\tau\}_{n\ge1}$ *be such that* $n[1 - F(u_n^\tau)] \to \tau$ *as* $n \to \infty$. *Suppose that for each* $\tau > 0$ *the stationary sequence* $\{X_n\}_{n\ge1}$ *satisfies the condition* $\triangle(u_n^\tau)$ *and for some* $\tau_1 > 0$, $N_n^{\tau_1}$ *converges in distribution to a point process* N^{τ_1}. *Then* N_n^τ *converges to a compound Poisson process* N^τ *for all* $\tau > 0$, *with Laplace transform given by*

$$L_{N^\tau}(f) = \exp\left\{-\theta\tau \int_0^1 \left[1 - \phi\bigl(f(t)\bigr)\right] dt\right\}, \tag{5}$$

where $\theta \in (0, 1)$ *and* $\phi(s) = \sum_{j=1}^\infty e^{-sj}\pi(j)$ *is the Laplace transform of a probability distribution* π *on* $\{1, 2, \dots\}$, θ *and* π *are independent of* τ. *Note that* π *is the limiting distribution of the cluster sizes. We also cited the definition of* π *from Hsing et al.* (1988).

Definition 1.1. Let $\{X_n\}_{n\ge1}$ be a stationary sequence satisfying the assumptions of Theorem 1.1. Separate $\{X_n\}_{n\ge1}$ into successive groups $(X_1, \dots, X_{r_n})$, $(X_{r_n+1}, \dots, X_{2r_n}), \dots$ of r_n consecutive terms (for appropriately chosen r_n). Then all exceedances of u_n, within a group are regarded as forming a cluster. Define the distribution π_n of cluster sizes on $\{1, 2, 3, \dots\}$ by

$$\pi_n(j) = P\left\{\sum_{i=1}^{r_n} \mathbf{1}(X_i > u_n) = j \,\middle|\, \sum_{i=1}^{r_n} \mathbf{1}(X_i > u_n) > 0\right\}, \qquad j = 1, 2, \dots.$$

Then

$$\pi(j) = \lim_{n\to\infty} \pi_n(j), \qquad j = 1, 2, \dots.$$

For other related results on point processes of i.i.d. and stationary random sequences, we refer to Chapters 2 and 5 of Leadbetter et al. (1983).

In this paper, we will consider the almost sure central limit theorem related Theorem 1.1. We will give main results in Section 2, and then give their proofs in Section 3.

2 Main results

In order to deal with the almost sure limit case, we need to strength the condition $\triangle(u_n)$ to slightly stronger case.

Definition 2.1. The sequence $\{X_n\}_{n\geq 1}$ satisfies condition $\nabla(u_n)$ for a given sequence $\{u_n\}_{n\geq 1}$, if

$$\beta_{n,l} = \sup\{|P(\mathbf{A}\cap\mathbf{B}) - P(\mathbf{A})P(\mathbf{B})| : \mathbf{A}\in\beta_1^k(u_k)\cup\beta_1^k(u_n),\\ \mathbf{B}\in\beta_{k+l}^n(u_n), k=1,2,\dots,n-l\}$$

is such that $\beta_{n,l}\to 0$, as $n\to\infty$, for some sequence $l_n = o(n)$.

The condition $\nabla(u_n)$ has been introduced by Ferreira (1995) for dealing with the extremes with a random number of variables from periodic sequences.

Now, we state our main results. As usual, $a_n \ll b_n$ means $\limsup_{n\to\infty}|a_n/b_n| < +\infty$.

Theorem 2.1. *Suppose that the constants $\{u_n^\tau\}_{n\geq 1}$ be such that for $\tau\in(0,\infty)$, $n[1-F(u_n^\tau)]\to\tau$ as $n\to\infty$. Assume that for each $\tau>0$ the stationary sequence $\{X_n\}_{n\geq 1}$ satisfies the condition $\nabla(u_n^\tau)$ with $\beta_{n,l}\ll(\log\log n)^{-(1+\varepsilon)}$ for some $\varepsilon>0$, and for some $\tau_1>0$, $N_n^{\tau_1}$ converges in distribution to a point process N^{τ_1}. Then for all $\tau>0$, each $r=1,2,\dots,$ and every Borel subset B of $(0,1]$*

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{i=1}^{n}\frac{1}{i}\mathbf{1}(N_i^\tau(B)\leq r) = P(N^\tau(B)\leq r) \qquad a.s., \tag{6}$$

where N^τ is a compound Poisson process with Laplace transform given by (5).

Applying Theorem 2.1, we can derive the almost sure limit theorem for the order statistics of dependent sequences. Let $M_n^{(k)}$ be the kth maximum among $X_1, X_2, \dots, X_n$.

Corollary 2.1. *Suppose that for each $\tau>0$ the stationary sequence $\{X_n\}_{n\geq 1}$ satisfies the condition $\nabla(u_n^\tau)$ with $\beta_{n,l}\ll(\log\log n)^{-(1+\varepsilon)}$ for some $\varepsilon>0$ and for some $\tau>0$, N_n^τ converges in distribution to a point process N^τ. Assume that $a_n>0, b_n$ are constants such that*

$$P(a_n(M_n^{(1)}-b_n)\leq x)\to G(x) \tag{7}$$

for some non-degenerate distribution function G (necessarily of extreme value type). Then for each $k=1,2,\dots,$

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{i=1}^{n}\frac{1}{i}\mathbf{1}(a_i(M_i^{(k)}-b_i)\leq x)\\ = G(x)\left[1+\sum_{j=1}^{k-1}\sum_{i=j}^{k-1}\frac{(-\log G(x))^j}{j!}\pi^{*j}(i)\right] \qquad a.s., \tag{8}$$

where for each j, π^{*j} is the j-fold convolution of the probability distribution π defined in Definition 1.1.

In the following corollary, we impose a local dependence condition $D'(u_n)$ from Leadbetter et al. (1983), which is to limit the possibility of clustering of more than one exceedance in a small interval and to obtain a simple Poisson limit for an exceedance point process formed by exceedances of high levels.

Corollary 2.2. *Suppose that the assumptions of Corollary 2.1 are satisfied. In addition, assume that condition $D'(u_n)$ holds with $u_n = a_n^{-1}x + b_n$, that is,*

$$\limsup_{n\to\infty} n \sum_{j=1}^{[n/l]} P(X_1 > u_n, X_j > u_n) \to 0 \qquad \text{as } l \to \infty. \tag{9}$$

Then for each $k = 1, 2, \ldots,$

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} \mathbf{1}\big(a_i\big(M_i^{(k)} - b_i\big) \le x\big) \\ = G(x) \sum_{j=0}^{k-1} \frac{(-\log G(x))^j}{j!} \qquad a.s.\end{aligned} \tag{10}$$

Especially, for i.i.d. case, we have the following result.

Corollary 2.3. *Suppose that $\{X_n\}_{n\ge1}$ is an i.i.d. random sequence and that the constants $\{u_n^\tau\}_{n\ge1}$ be such that for $\tau \in (0,\infty)$, $n[1 - F(u_n^\tau)] \to \tau$ as $n \to \infty$. Then for all $\tau > 0$, each $r = 1, 2, \ldots,$ and for every Borel subset B of $(0, 1]$*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} \mathbf{1}\big(N_i^\tau(B) \le r\big) = P\big(N^\tau(B) \le r\big) \qquad a.s., \tag{11}$$

and for each $k = 1, 2, \ldots,$

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} \mathbf{1}\big(M_i^{(k)} \le u_i^\tau\big) = e^{-\tau} \sum_{j=0}^{k-1} \frac{\tau^j}{j!} \qquad a.s., \tag{12}$$

where N^τ is a Poisson process with parameter τ.

Next, we deal with the almost sure limit theorem for vector of point processes.

Theorem 2.2. *Suppose that $\{X_n\}_{n\ge1}$ is a strong mixing random sequence with mixing rate $\alpha(n) \ll (\log\log n)^{-(1+\varepsilon)}$ and satisfies condition $D'(u_n^{\tau_i})$ and that the* *constants* $u_n^{\tau_1} > \cdots > u_n^{\tau_r}$ *be such that* $n[1 - F(u_n^{\tau_i})] \to \tau_i$ *as* $n \to \infty$, *where* $0 < \tau_1 \le \tau_2 \le \cdots \le \tau_r < \infty$. *Then, for* $k_1 > 0, \dots, k_r > 0$ *and for every Borel subset* B *of* $(0, 1]$

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} \mathbf{1}\big(N_i^{\tau_1}(B) = k_1, N_i^{\tau_2}(B) = k_1 + k_2, \dots, N_i^{\tau_r}(B) = k_1 + k_2 + \cdots + k_r\big) = \frac{\tau_1^{k_1}}{k_1!} \frac{(\tau_2 - \tau_1)^{k_2}}{k_2!} \cdots \frac{(\tau_r - \tau_{r-1})^{k_r}}{k_r!} e^{-\tau_r} \qquad a.s. \tag{13}$$

As a corollary, we obtain the following result which is an extension of the main result of Peng et al. (2009).

Corollary 2.4. *Suppose that* $\{X_n\}_{n\ge1}$ *is a strong mixing random sequence with mixing rate* $\alpha(n) \ll (\log\log n)^{-(1+\varepsilon)}$ *and satisfies condition* $D'(u_n(x_i))$ *with* $u_n(x_i) = a_n^{-1} x_i + b_n$, $i = 1, 2, \dots, r$ *and* $a_n > 0, b_n$ *are constants such that* (7) *holds. Let* $x_1 > x_2 > \cdots > x_r$. *Then, for* $1 \le k \le r$,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} \mathbf{1}\big(a_i\big(M_i^{(1)} - b_i\big) \le x_1, \dots, a_i\big(M_i^{(k)} - b_i\big) \le x_k\big) = \begin{cases} H(x_1, x_2, \dots, x_k), & x_1 > x_2 > \cdots > x_k; \\ 0, & \text{otherwise}, \end{cases} \qquad a.s., \tag{14}$$

where $H(x_1, x_2, \dots, x_k)$ *is defined by the marginal distribution as* $H_j(x) = G(x) \sum_{j=0}^{k-1} \frac{(-\log G(x))^j}{j!}$.

We end this section with several examples which illustrate our main results.

Example 2.1. The m-dependent stationary random sequences satisfy the condition $\nabla(u_n)$ with $\beta_{n,l} = 0$ for $l > m$. Thus, Theorem 2.1 holds for m-dependent stationary random sequences.

Example 2.2. The strong mixing condition implies the condition $\nabla(u_n)$. Thus, Theorem 2.1 holds for strong mixing random sequences with mixing rate $\alpha(n) \ll (\log\log n)^{-(1+\varepsilon)}$.

(1) The first example is from Matuła (1999).

Let $\{X_n\}_{n\ge1}$ be a sequence of square-integrable associated random variables. Let $u(n)$ denote the coefficient

$$u(n) = \sup_{k\in\mathbf{N}} \sum_{j:|j-k|\ge n} \operatorname{Cov}(X_j, X_k), \qquad n \in \mathbf{N} \cup \{0\}.$$

If $u(n) = O(n^{-\lambda})$, for some $\lambda > 1$, then $\{X_n\}_{n\geq 1}$ is strongly mixing with $\alpha(n) = O(n^{-\lambda+1})$.

(2) The second example is for stationary Gaussian sequence, which can be found in Doukhan (1994).

Assume that $\{X_n\}_{n\geq 1}$ is a stationary Gaussian sequence such that $r(n) = EX_0X_n = O(n^{-(1+\varepsilon)})$ with some $\varepsilon > 0$ and the spectral of $\{X_n\}_{n\geq 1}$ is bounded below, then $\{X_n\}_{n\geq 1}$ is a strong mixing random sequences with mixing rate $\alpha(n) = O(n^{-\varepsilon})$. Thus, the assertion of Theorem 2.1 holds.

The following example is from Hsing et al. (1988).

Example 2.3. Let $X_n = \max\{Y_n, Y_{n+1}\}$, where $\{Y_n\}_{n\geq 1}$ is an i.i.d. random sequence. Then for each $k = 1, 2, \ldots,$

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} \mathbf{1}\big(a_i\big(M_i^{(k)} - b_i\big) \leq x\big) \\ = G(x)\left[1 + \sum_{j=1}^{[k-1]/2} \frac{(-\log G(x))^j}{j!}\right] \qquad \text{a.s.,}\end{aligned} \tag{15}$$

where a_n, b_n are as in (7).

3 Proofs

Before proving the main results, we state a lemma which will be used in the proofs of our main results.

Lemma 3.1. *Let $(\xi_k)_{k=1}^{\infty}$ be a sequence of uniformly bounded random variables, that is, there exists some $M \in (0, \infty)$ such that $|\xi_k| \leq M$ a.s. for all $k \in \mathbf{N}$. If*

$$\operatorname{Var}\left(\sum_{k=1}^{n} \frac{1}{k}\xi_k\right) \ll \log^2 n (\log\log n)^{-(1+\varepsilon)}$$

for some $\varepsilon > 0$, then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}(\xi_k - E\xi_k) = 0 \qquad a.s.$$

Proof. See Lemma 3.1 of Csáki and Gonchigdanzan (2002). □

Proof of Theorem 2.1. Let $\eta_i = \mathbf{1}(N_i^{\tau}(B) \leq r) - P(N_i^{\tau}(B) \leq r)$. Notice that $(\eta_i)_{i=1}^{\infty}$ is a sequence of bounded random variables with $\operatorname{Var}(\eta_i) \leq 1$. We first show that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i}\eta_i = 0 \qquad \text{a.s.} \tag{16}$$

Using Lemma 3.1, we only need to show that

$$\mathrm{Var}\left(\sum_{i=1}^{n}\frac{1}{i}\eta_i\right)\ll\log^2 n(\log\log n)^{-(1+\varepsilon)}. \tag{17}$$

Now, we have

$$\begin{aligned}\mathrm{Var}\left(\sum_{i=1}^{n}\frac{1}{i}\eta_i\right) &= E\left(\sum_{i=1}^{n}\frac{1}{i}\eta_i\right)^2\\ &=\sum_{i=1}^{n}\frac{E\eta_i^2}{i^2}+2\sum_{1\le i<j\le n}\frac{E(\eta_i\eta_j)}{ij}\\ &=: L_{n,1}+2L_{n,2}.\end{aligned}$$

Clearly,

$$L_{n,1}=\sum_{i=1}^{n}\frac{1}{i^2}E\eta_i^2\le\sum_{i=1}^{n}\frac{1}{i^2}=O(1).$$

In the following part, we will use the following notation. Let $\mathbf{Z}$ denote the set of all integers. For any Borel set B on $(0,1]$ and any positive integer i, let

$$iB=\{xi:x\in B, xi\in\mathbf{Z}\}\quad\text{and}\quad x_0^i=\max\{x:x\in B, xi\in\mathbf{Z}\}\in(0,1].$$

For the set iB, let

$$i_B^M=\max\{x:x\in iB\}.$$

Let $l=l_{j_B^M}$ be such that $l_{j_B^M}\le\log j_B^M$ and $l_{j_B^M}\uparrow\infty$ as $j_B^M\to\infty$. For $i_B^M+l_{j_B^M}+1<j_B^M$, let

$$Q=\{x:x\in B, jx>l_{j_B^M}+i_B^M\},$$

consequently,

$$jB\setminus jQ=\{x:x\in jB, x\le l_{j_B^M}+i_B^M\}.$$

Now, for $i_B^M+l_{j_B^M}+1<j_B^M$, we have

$$\begin{aligned}|E(\eta_i\eta_j)| &= |\mathrm{Cov}(\mathbf{1}(N_i^\tau(B)\le r),\mathbf{1}(N_j^\tau(B)\le r))|\\ &=\left|\mathrm{Cov}\left(\mathbf{1}\left(\sum_{k\in iB}\mathbf{1}(X_k>u_i^\tau)\le r\right),\mathbf{1}\left(\sum_{k\in jB}\mathbf{1}(X_k>u_j^\tau)\le r\right)\right)\right|\\ &\le\left|\mathrm{Cov}\left(\mathbf{1}\left(\sum_{k\in iB}\mathbf{1}(X_k>u_i^\tau)\le r\right),\right.\right.\end{aligned}$$

$$\mathbf{1}\Big(\sum_{k\in jB}\mathbf{1}(X_k>u_j^\tau)\le r\Big)-\mathbf{1}\Big(\sum_{k\in jQ}\mathbf{1}(X_k>u_j^\tau)\le r\Big)\Big)\Big|$$

$$+\Big|\mathrm{Cov}\Big(\mathbf{1}\Big(\sum_{k\in iB}\mathbf{1}(X_k>u_i^\tau)\le r\Big),\mathbf{1}\Big(\sum_{k\in jQ}\mathbf{1}(X_k>u_j^\tau)\le r\Big)\Big)\Big|$$

$$\le E\Big|\mathbf{1}\Big(\sum_{k\in jB}\mathbf{1}(X_k>u_j^\tau)\le r\Big)-\mathbf{1}\Big(\sum_{k\in jQ}\mathbf{1}(X_k>u_j^\tau)\le r\Big)\Big|$$

$$+\Big|\mathrm{Cov}\Big(\mathbf{1}\Big(\sum_{k\in iB}\mathbf{1}(X_k>u_i^\tau)\le r\Big),\mathbf{1}\Big(\sum_{k\in jQ}\mathbf{1}(X_k>u_j^\tau)\le r\Big)\Big)\Big|$$

$$=: T_1+T_2.$$

Since $\{X_n\}_{n\ge1}$ satisfies the condition $\nabla(u_n)$ with $\beta_{n,l}\ll(\log\log n)^{-(1+\varepsilon)}$ for some $\varepsilon>0$, we have

$$T_2\ll(\log\log j)^{-(1+\varepsilon)}.$$

For the first term, we have

$$\begin{aligned}
T_1&=E\Big|\mathbf{1}\Big(\sum_{k\in jB}\mathbf{1}(X_k>u_j^\tau)\le r\Big)-\mathbf{1}\Big(\sum_{k\in jQ}\mathbf{1}(X_k>u_j^\tau)\le r\Big)\Big|\\
&=P\Big(\sum_{k\in jQ}\mathbf{1}(X_k>u_j^\tau)\le r\Big)-P\Big(\sum_{k\in jB}\mathbf{1}(X_k>u_j^\tau)\le r\Big)\\
&\le P\Big(\sum_{k\in jB\setminus jQ}\mathbf{1}(X_k>u_j^\tau)>0\Big)\\
&\le \sum_{k\in jB\setminus jQ}P(X_k>u_j^\tau)\\
&\le (i_B^M+l_{j_B^M})(1-F(u_j^\tau))\\
&=\frac{i_B^M+l_{j_B^M}}{j}\,j(1-F(u_j^\tau)).
\end{aligned}$$

It follows from the fact $B\subset(0,1]$ that

$$i_B^M\le i\quad\text{and}\quad l_{j_B^M}\le\log j.$$

Since the constants $\{u_n^\tau\}_{n\ge1}$ satisfy that $n[1-F(u_n^\tau)]\to\tau$ for $\tau\in(0,\infty)$, as $n\to\infty$, we have that $n[1-F(u_n^\tau)]$ is bounded. Thus,

$$T_1\ll\frac{i+\log j}{j}.$$

Now, for $L_{n,2}$, let C denote some constant, we have

$$\begin{aligned}
L_{n,2} &= \sum_{1\le i<j\le n} \frac{E(\eta_i\eta_j)}{ij} \\
&= \sum_{1\le i<j\le n, i_B^M + l_{j_B^M}+1<j_B^M} \frac{E(\eta_i\eta_j)}{ij} + \sum_{1\le i<j\le n, i_B^M + l_{j_B^M}+1\ge j_B^M} \frac{E(\eta_i\eta_j)}{ij} \\
&\ll \sum_{1\le i<j\le n, i_B^M + l_{j_B^M}+1<j_B^M} \frac{1}{ij}\left[\frac{i+\log j}{j} + \frac{1}{(\log\log j)^{1+\varepsilon}}\right] \\
&\quad + \sum_{1\le i<j\le n, i_B^M + l_{j_B^M}+1\ge j_B^M} \frac{1}{ij} \\
&\le \sum_{1\le i<j\le n} \frac{1}{ij}\frac{(i+\log j)}{j} + \sum_{1\le i<j\le n} \frac{1}{ij}\frac{1}{(\log\log j)^{1+\varepsilon}} \\
&\quad + \sum_{1\le i<j\le n, \log(j_B^M)+1\ge (j-i)_B^M} \frac{1}{ij} \\
&\ll \sum_{j=1}^{n}\frac{1}{j} + \sum_{j=1}^{n}\frac{\log j}{j^2}\sum_{i=1}^{j}\frac{1}{i} + \sum_{j=1}^{n}\frac{1}{j(\log\log j)^{1+\varepsilon}}\sum_{i=1}^{j}\frac{1}{i} \\
&\quad + \sum_{1\le i<j\le n, (\log j)+1\ge (j-i)x_0^{j-i}} \frac{1}{ij} \\
&\ll \log n + \sum_{j=1}^{n}\frac{(\log j)^2}{j^2} + \sum_{j=1}^{n}\frac{\log j}{j(\log\log j)^{1+\varepsilon}} + \sum_{j=1}^{n}\frac{1}{j}\sum_{i=j-C\log j}^{j}\frac{1}{i} \\
&\ll \log n + \log n + \log n \sum_{j=1}^{n}\frac{1}{j(\log\log j)^{1+\varepsilon}} + \log n \\
&\ll (\log n)^2(\log\log n)^{-(1+\varepsilon)}.
\end{aligned}$$

Thus, we have obtained (17). Note that Theorem 1.1 implies

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{i=1}^{n}\frac{1}{i}P\big(N_i^\tau(B)\le r\big) = P\big(N^\tau(B)\le r\big) \qquad \text{a.s.,} \tag{18}$$

and then the assertion of Theorem 2.1 follows from (16) and (18). □

Proof of Corollary 2.1. Let $\tau(x) = -\log G^{1/\theta}(x)$, where θ is defined in Theorem 1.1. From the proof of Theorem 6.1 of Hsing (1988), we have

$$P\big(N^{(\tau(x))} \le k-1\big) = e^{-\theta\tau(x)}\left[1 + \sum_{j=1}^{k-1}\sum_{i=j}^{k-1}\frac{(\theta\tau(x))^j}{j!}\pi^{*j}(i)\right].$$

Now, using Theorem 2.1, we have

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{\log n}\sum_{i=1}^{n}\frac{1}{i}\mathbf{1}\big(a_i\big(M_i^{(k)} - b_i\big) \le x\big)\\
&\quad= \lim_{n\to\infty}\frac{1}{\log n}\sum_{i=1}^{n}\frac{1}{i}\mathbf{1}\big(M_i^{(k)} \le u_n^{\tau(x)}\big)\\
&\quad= \lim_{n\to\infty}\frac{1}{\log n}\sum_{i=1}^{n}\frac{1}{i}\mathbf{1}\big(N_i^{(\tau(x))} \le k-1\big)\\
&\quad= P\big(N^{(\tau(x))} \le k-1\big) \qquad \text{a.s.}\\
&\quad= e^{-\theta\tau(x)}\left[1 + \sum_{j=1}^{k-1}\sum_{i=j}^{k-1}\frac{(\theta\tau(x))^j}{j!}\pi^{*j}(i)\right].
\end{aligned}$$

This completes the proof of Corollary 2.1. □

Proof of Corollary 2.2. Note that condition $\nabla(u_n)$ implies Condition $D(u_n)$, by Theorem 5.2.1 of Leadbetter et al. (1983), we have with $\tau(x) = -\log G(x)$

$$P\big(N^{(\tau(x))} \le k-1\big) = G(x)\sum_{j=0}^{k-1}\frac{(-\log G(x))^j}{j!}.$$

The remaining part of the proof is the same as that of Corollary 2.1. □

Proof of Corollary 2.3. Since the independence of $\{X_n\}_{n\ge1}$ implies that $\{X_n\}_{n\ge1}$ satisfies Conditions $D(u_n)$ and $D'(u_n)$, by Theorem 5.2.1 of Leadbetter et al. (1983), we have

$$P\big(N^{(\tau)} \le k-1\big) = e^{-\tau}\sum_{j=0}^{k-1}\frac{\tau^j}{j!}.$$

The remaining part of the proof is the same as that of Corollary 2.1. □

Proof of Theorem 2.2. Let $s_1 = k_1, s_2 = k_1 + k_2, \dots, s_r = k_1 + k_2 + \cdots + k_r$ and $\xi_i = \mathbf{1}(N_i^{\tau_1}(B) = s_1, N_i^{\tau_2}(B) = s_1, \dots, N_i^{\tau_r}(B) = s_r) - P(N_i^{\tau_1}(B) =$

$s_1, N_i^{\tau_2}(B) = s_2, \ldots, N_i^{\tau_r}(B) = s_r)$. Notice that $(\xi_i)_{i=1}^{\infty}$ is a sequence of bounded random variables with $\mathrm{Var}(\xi_i) \le 1$. We first show that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{i} \xi_i = 0 \qquad \text{a.s.} \tag{19}$$

Using Lemma 3.1, we only need to show that

$$\mathrm{Var}\left(\sum_{i=1}^{n} \frac{1}{i} \xi_i\right) \ll \log^2 n (\log\log n)^{-(1+\varepsilon)}. \tag{20}$$

Now, we have

$$\begin{aligned}\mathrm{Var}\left(\sum_{i=1}^{n} \frac{1}{i}\xi_i\right) &= E\left(\sum_{i=1}^{n} \frac{1}{i}\xi_i\right)^2 \\ &= \sum_{i=1}^{n} \frac{E\xi_i^2}{i^2} + 2 \sum_{1\le i<j\le n} \frac{E(\xi_i \xi_j)}{ij} \\ &=: T_{n,1} + 2T_{n,2}.\end{aligned}$$

Clearly,

$$T_{n,1} = \sum_{i=1}^{n} \frac{1}{i^2} E\xi_i^2 \le \sum_{i=1}^{n} \frac{1}{i^2} = O(1).$$

In the following part, we will use the notation introduced in the proof of Theorem 2.1. Let $N_j^{\tau_m}(Q) = \sum_{k\in jQ} \mathbf{1}(X_k > u_j^{\tau_m})$ for $m = 1, 2, \ldots, r$. Note that for $i_B^M + l_{j_B^M} + 1 < j_B^M$, we have

$$\begin{aligned}\left|E(\xi_i\xi_j)\right| &= \big|\mathrm{Cov}\big(\mathbf{1}\big(N_i^{\tau_1}(B) = s_1, \ldots, N_i^{\tau_r}(B) = s_r\big), \\ &\qquad \mathbf{1}\big(N_j^{\tau_1}(B) = s_1, \ldots, N_j^{\tau_r}(B) = s_r\big)\big)\big| \\ &\le \big|\mathrm{Cov}\big(\mathbf{1}\big(N_i^{\tau_1}(B) = s_1, \ldots, N_i^{\tau_r}(B) = s_r\big), \\ &\qquad \mathbf{1}\big(N_j^{\tau_1}(B) = s_1, \ldots, N_j^{\tau_r}(B) = s_r\big) \\ &\qquad - \mathbf{1}\big(N_j^{\tau_1}(Q) = s_1, \ldots, N_j^{\tau_r}(Q) = s_r\big)\big)\big| \\ &\quad + \big|\mathrm{Cov}\big(\mathbf{1}\big(N_i^{\tau_1}(B) = s_1, \ldots, N_i^{\tau_r}(B) = s_r\big), \\ &\qquad \mathbf{1}\big(N_j^{\tau_1}(Q) = s_1, \ldots, N_j^{\tau_r}(Q) = s_r\big)\big)\big| \\ &\le E\big|\mathbf{1}\big(N_j^{\tau_1}(B) = s_1, \ldots, N_j^{\tau_r}(B) = s_r\big) \\ &\qquad - \mathbf{1}\big(N_j^{\tau_1}(Q) = s_1, \ldots, N_j^{\tau_r}(Q) = s_r\big)\big| \\ &\quad + \big|\mathrm{Cov}\big(\mathbf{1}\big(N_i^{\tau_1}(B) = s_1, \ldots, N_i^{\tau_r}(B) = s_r\big),\end{aligned}$$

$$\mathbf{1}\big(N_j^{\tau_1}(Q)=s_1,\ldots,N_j^{\tau_r}(Q)=s_r\big)\big)\Big|$$
$$=: R_1+R_2.$$

Using the strong mixing of $\{X_n\}_{n\ge 1}$, we have

$$R_2 \ll (\log\log j)^{-(1+\varepsilon)}.$$

For the first term, we have

$$\begin{aligned}
R_1 &= P\big(N_j^{\tau_1}(Q)=s_1,\ldots,N_j^{\tau_r}(Q)=s_r\big)-P\big(N_j^{\tau_1}(B)=s_1,\ldots,N_j^{\tau_r}(B)=s_r\big)\\
&\le \sum_{m=1}^{r}\big[P\big(N_j^{\tau_m}(Q)=s_m\big)-P\big(N_j^{\tau_m}(B)=s_m\big)\big]\\
&\le \sum_{m=1}^{r}P\Big(\sum_{k\in jB\setminus jQ}\mathbf{1}\big(X_k>u_j^{\tau_m}\big)>0\Big)\\
&\le \sum_{m=1}^{r}\sum_{k\in jB\setminus jQ}P\big(X_k>u_j^{\tau_m}\big)\\
&\le \big(i_B^M+l_{j_B^M}\big)\sum_{m=1}^{r}\big(1-F\big(u_j^{\tau_m}\big)\big)\\
&= \frac{i_B^M+l_{j_B^M}}{j}\sum_{m=1}^{r}j\big(1-F\big(u_j^{\tau_m}\big)\big).
\end{aligned}$$

Since the constants $\{u_n^{\tau_m}\}_{n\ge 1}$ satisfy that $n[1-F(u_n^{\tau_m})]\to\tau_m$ for $\tau_m\in(0,\infty)$, as $n\to\infty$, we have that $n[1-F(u_n^{\tau_m})]$ is bounded uniformly for $m=1,2,\ldots,r$. Thus, for some constant C

$$R_1\le Cr\frac{i_B^M+l_{j_B^M}}{j}\ll\frac{i_B^M+l_{j_B^M}}{j}.$$

Now, similarly to the proof of Theorem 2.1, we conclude that

$$T_{n,2}\ll(\log n)^2(\log\log n)^{-(1+\varepsilon)}.$$

Thus, we have obtained (20). Note that strong mixing implies condition $D_r(u_n)$ from Leadbetter et al. (1983). So by Theorem 5.6.1 of Leadbetter et al. (1983), we have

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{\log n}\sum_{i=1}^{n}\frac{1}{i}P\big(N_i^{\tau_1}(B)=s_1,N_i^{\tau_2}(B)=s_1,\ldots,N_i^{\tau_r}(B)=s_r\big)\\
&\quad=\frac{\tau_1^{k_1}}{k_1!}\frac{(\tau_2-\tau_1)^{k_2}}{k_2!}\cdots\frac{(\tau_r-\tau_{r-1})^{k_r}}{k_r!}e^{-\tau_r}\qquad\text{a.s.,}
\end{aligned}\tag{21}$$

and then the assertion of Theorem 2.2 follows from (19) and (21). □

Proof of Corollary 2.4. Corollary 2.4 is a special case of Theorem 2.2. □

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