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ALMOST SURE CONVERGENCE OF MULTIPARAMETER MARTINGALES FOR MARKOV RANDOM FIELDS

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We prove that bounded multiparameter martingales converge almost surely if the underlying σ-fields are generated by a Markov random field which satisfies Dobrushin's uniqueness condition. An example shows that it is not enough to assume that the Markov field is uniquely determined by its conditional probabilities.

1. Introduction. Suppose that we have a two-dimensional random field $X_t (t \in \mathbb{Z}^2)$ with values in some state space. For a bounded random variable Z depending on the field and for a finite subset V of $\mathbb{Z}^2$, consider the conditional expectation

$$Z_V = E[Z \mid X_s (s \in V)]$$

of Z with respect to the random variables X_s with index $s \in V$. We are interested in the behavior of Z_V as V increases to $\mathbb{Z}^2$. It is well known that the martingale (Z_V) converges in L^p for any $p \geq 1$ along the filter of all finite subsets V of $\mathbb{Z}^2$; see [4] Chapter V, 44. But a classical counterexample of Dieudonné [5] shows that one cannot expect almost sure convergence even if the random variables X_t are independent and identically distributed. For almost sure convergence, it is therefore necessary to restrict V to some smaller class. Let us consider the class of rectangles $V_t = \{s \in \mathbb{Z}^2 \mid 0 \leq s \leq t\}$, where $s \leq t$ denotes the coordinatewise ordering of $\mathbb{Z}^2$, and let us put $\mathscr{F}_t = \sigma(X_s; 0 \leq s \leq t)$. Now we are dealing with two-parameter martingales

$$Z_t = E[Z \mid \mathscr{F}_t] \quad (t \geq 0).$$

Here again, a counterexample of Dubins and Pitman [8] shows that almost sure convergence does not hold in general. On the other hand we know, by a result of Cairoli [3], that bounded two-parameter martingales converge if the random field satisfies the following condition:

(1.1) For each $t = (t_1, t_2)$ the two σ-fields $\mathscr{F}^i_{t_i} = \sigma(X_s; s_i \leq t_i)$ $(i = 1, 2)$ are conditionally independent with respect to their intersection $\mathscr{F}_t$.

From the point of random fields, this condition is rather restrictive. In most cases there is some diagonal interaction between $\mathscr{F}^1_{t_1}$ and $\mathscr{F}^2_{t_2}$ which does not pass through $\mathscr{F}_t$. If this effect becomes too strong then one should expect almost sure convergence to fail. The breakdown of martingale convergence may thus be viewed as one of the various critical phenomena caused by strong spatial inter-

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action. On the other hand, it is natural to expect that martingales do converge for a reasonably large class of random fields where the interaction is not too strong.

In order to make this more precise, we consider random fields in the framework of Dobrushin [6]. This means that the random field is specified by the conditional probabilities

$$P[X_t \in A \mid X_s(s \neq t)] \quad (t \in T). \tag{1.2}$$

For a Markov random field, these conditional probabilities only depend on the values X_s for those sites s which are neighbors of t. In Section 3 we give an example of a Markov random field which shows that bounded martingales may not converge even if there is no phase transition, i.e., if the random field is uniquely determined by its conditional probabilities. But in Section 2 we prove that almost sure convergence does hold for any Markov field which satisfies Dobrushin's well-known uniqueness condition

$$c \equiv \sup_t \textstyle\sum_s C_{s,t} < 1. \tag{1.3}$$

The coefficient $C_{s,t}$ measures the influence of site s on the conditional probability (1.2) at site t; see (2.8) for the precise definition. The proof combines Dobrushin's contraction technique with an iterated application of "Hunt's lemma" (2.3).

Throughout this paper we consider the two-dimensional case, but this is only for notational convenience. The extension to higher dimensions is straightforward.

2. Convergence under Dobrushin's uniqueness condition. Let the random field be given by a probability measure P on $\Omega = S^T$, where S is some standard Borel space and T is the set of all couples $t = (t_1, t_2)$ of non-negative integers. The random variable X_t is defined by $X_t(\omega) = \omega(t)$. We write $s \leq t$ if $s_i \leq t_i (i = 1, 2)$ and put $|t - s| = |t_1 - s_1| + |t_2 - s_2|$. For $t = (t_1, t_2) \in T$ we define the σ-fields $\mathscr{F}^i_{t_i} = \sigma(X_s;\, s_i \leq t_i)$ $(i = 1, 2)$ and $\mathscr{F}_t = \mathscr{F}^1_{t_1} \cap \mathscr{F}^1_{t_2} = \sigma(X_s;\, s \leq t)$.

A bounded martingale with respect to the σ-fields $\mathscr{F}_t$ is of the form

$$Z_t = E[Z \mid \mathscr{F}_t] \quad (t \in T) \tag{2.1}$$

for some bounded measurable Z; see [4] Chapter V, 44. Our aim is to prove almost sure convergence of Z_t as $t_1, t_2 \uparrow \infty$. Let us first observe that the iterated conditional expectations

$$Z_t^{(1)} = E[Z \mid \mathscr{F}^1_{t_1} \mid \mathscr{F}^2_{t_2}] \quad (t \in T) \tag{2.2}$$

converge almost surely; note that condition (1.1) would imply $Z_t^{(1)} = Z_t$. In fact, the one-parameter martingale $Y_{t_1} = E[Z \mid \mathscr{F}^1_{t_1}]$ converges almost surely and satisfies $\sup |Y_{t_1}| \in L^1$ for any Z with $E[|Z| \log |Z|] < \infty$; see [4] Chapter V, 25. The convergence of $Z_t^{(1)} = E[Y_{t_1} \mid \mathscr{F}^2_{t_2}]$ thus follows from the following.

(2.3) LEMMA. *If* (Y_n) *converges almost surely and satisfies* $\sup |Y_n| \in L^1$ *then*

$$\lim_{n,m} E[Y_n \mid \mathscr{G}_m]$$

exists almost surely for any increasing or decreasing sequence of σ*-fields* $(\mathscr{G}_m)$.

(2.4) REMARKS. 1) The lemma is usually stated for the case $n = m$ and attributed to Hunt; see [4] Chapter V, 45. But it appears already, in the two-parameter form (2.3), in Blackwell and Dubins [2] Theorem 2. The proof shows that n may run through any partially ordered index set.

2) The observation that (2.3) implies the convergence of (2.2) is due to L. Sucheston; see [13] where the argument appears in a more general framework and with some other applications.

If we apply the lemma successively, with n ranging in T, then we obtain almost sure convergence of

$$Z_t^{(N+1)} = E[Z_t^{(N)} \mid \mathscr{F}_{t_1}^1 \mid \mathscr{F}_{t_2}^2] \quad (t \in T) \tag{2.5}$$

for each $N \geq 1$. This would imply almost sure convergence of the martingale (Z_t) if we could prove

$$\lim_N \sup_t |Z_t^{(N)} - Z_t| = 0 \quad \text{a.s.} \tag{2.6}$$

The purpose of this paper is to show that this uniform approximation holds for a large class of random fields.

From now on we assume that P is a Markov field specified by its conditional probabilities in the sense of [6]. More precisely, we assume that for each $t \in T$ we are given a conditional probability distribution $\pi_t(dx \mid \omega)$ of X_t with respect to $\sigma(X_s;\, s \neq t)$, which satisfies the following Markov property:

$$\pi_t(\cdot \mid \omega) = \pi_t(\cdot \mid \eta) \quad \text{if} \quad X_s(\omega) = X_s(\eta) \quad \text{for} \quad |s - t| = 1. \tag{2.7}$$

Let $C = (C_{s,t})$ denote Dobrushin's interaction matrix:

$$C_{s,t} = \sup\{\tfrac{1}{2}\| \pi_t(\cdot \mid \omega) - \pi_t(\cdot \mid \eta) \| : \omega = \eta \text{ off } s\} \tag{2.8}$$

where $\|\cdot\|$ denotes the total variation norm. Condition (1.3) implies that the Markov field P is uniquely determined by its conditional probabilities $(\pi_t)_{t \in T}$; this is Dobrushin's well-known uniqueness theorem [6].

(2.9). THEOREM. *If condition* (1.3) *holds then any bounded martingale* $E[Z \mid \mathscr{F}_t]$ *converges almost surely as* $t_1, t_2 \uparrow \infty$.

PROOF. Throughout the proof we fix $t = (t_1, t_2) \in T$ and put

$$T_i = \{s \in T \mid s_i > t_i\}, \quad \partial T_i = \{s \in T \mid s_i = t_i\}$$

for $i = 1, 2$. In view of (2.6) we have to see how the conditional expectation $E[Y \mid \mathscr{F}_{t_1}^1]$ of a bounded $\mathscr{F}_{t_2}^2$-measurable random variable Y depends on the sites s with $s_1 \leq t_1$ and $s_2 > t_2$. In part 1) we evaluate the conditional expectation as an integral with respect to an explicit conditional probability distribution $P[\cdot \mid \mathscr{F}_{t_1}^1]$, viewed as a random field on S^{T_1} which only depends on the sites s in the boundary ∂T_1. In part 2) we use Dobrushin's comparison theorem for random fields in order to get a quantitative estimate for this dependence on the boundary conditions. The iteration of this estimate in part 3) will lead to the uniform approximation (2.6).

1) The joint conditional distribution of the random variables $X_s (s \in T_1)$, given

the values $X_s(\omega)$ for $s_1 \leq t_1$, may be viewed as a Markov random field on S^{T_1}: the conditional probability distribution at a site $s \in T_1$ coincides with π_s for $s_1 > t_1 + 1$, and for $s = (t_1 + 1, s_2)$ it only involves the boundary value $X_{(t_1,s_2)}(\omega)$. It is clear that the collection of these conditional probabilities satisfies again the uniqueness condition (1.3). Thus the conditional distribution on S^{T_1} is uniquely determined by the boundary values $X_s(\omega)$ ($s \in T_1$). This is the so-called global Markov property; see [1], [9], or [12].

For the rest of the proof we fix the restriction ω_t of ω to $\{s \mid s \leq t\}$, i.e., the values $X_s(\omega)$ for $s \leq t$. A boundary condition on ∂T_1 can now be described by a point $x = (x_1, x_2, \cdots)$ in the space $B = S^{\{1,2,\cdots\}}$, putting $X_{(t_1,t_2+n)}(\omega) = x_n$. Let P^x denote the induced Markov random field on S^{T_1}, and let $(\pi_s^x)_{s\in T_1}$ denote the collection of its conditional probabilities. A bounded $\mathscr{F}_{t_2}^2$-measurable random variable Y is of the form $Y(\omega_t, \eta_t)$ with $\eta_t \in S^{T_1}$, and its conditional expectation $E[Y \mid \mathscr{F}_{t_1}^1](\omega)$ can now be calculated as the integral

$$\int Y(\omega_t, \cdot)\, dP^x.$$

2) Consider two boundary conditions $x, y \in B$. Applying Dobrushin's comparison theorem for random fields (see [7] Theorem 3 or [10] (2.4)), we obtain the estimate

$$\left| \int Y(\omega_t, \cdot)\, dP^x - \int Y(\omega_t, \cdot)\, dP^y \right| \leq \textstyle\sum_{u,v\in T_1} b_u D_{u,v}^{(1)} \delta_v(Y), \tag{2.10}$$

where $\delta_v(Y) = \sup\{|Y(\omega) - Y(\eta)| : \omega = \eta \text{ off } v\}$ denotes the maximal oscillation of Y at site v, where $D^{(1)}$ is the sum of the non-negative powers of the matrix $C^{(1)} = (C_{s,t})_{s,t\in T_1}$, and where we put

$$b_u = \sup\{\tfrac{1}{2}\| \pi_u^x(\cdot \mid \eta) - \pi_u^y(\cdot \mid \eta) \| : \eta \in S^{T_1}\}.$$

The local Markov property (2.7) implies $b_u = 0$ if $u_1 > t_1 + 1$ or $u_2 \leq t_2$, and also if $u = (t_1 + 1, t_2 + k)$ and $y_k = x_k$. For $u = (t_1 + 1, t_2 + k)$ and $y_k \neq x_k$ we have $b_u = C_{(t_1,t_2+k),(t_1+1,t_2+k)}$.

Let us now assume that Y depends only on ω_t and on the coordinates $s \in \partial T_2 \cap T_1$; by the global Markov property, this will be the case for the random variables which appear in (2.5). Having fixed ω_t, we can identify Y with a function f on B, writing

$$f(z) = Y(\omega) \quad \text{if} \quad X_{(t_1+n,t_2)}(\omega) = z_n \quad (n = 1, 2, \cdots)$$

for $z = (z_1, z_2, \cdots) \in B$. Let us also use the notation

$$E_1 f(x) = \int Y(\omega_t, \cdot)\, dP^x, \quad \delta_l(f) = \delta_{(t_1+l,t_2)}(Y)$$

$$A_{k,l}^{(1)} = C_{(t_1,t_2+k),(t_1+1,t_2+k)} D_{(t_1+1,t_2+k),(t_1+l,t_2)}^{(1)}$$

so that (2.10) becomes

$$|E_1 f(x) - E_1 f(y)| \leq \textstyle\sum_{k,l=1, x_k\neq y_k}^{\infty} A_{k,l}^{(1)} \delta_l(f). \tag{2.11}$$

The same estimate holds for the operator E_2 and the matrix $A^{(2)}$, obtained by interchanging the role of 1 and 2. So far we neglect the fact that (2.10) resp. (2.11) requires a certain continuity property for Y resp. f; this will be discussed in Section 4.

3) Iterating (2.11) we get

$$|(E_2E_1)^N f(x) - (E_2E_1)^N f(y)| \le \sum_{k,l=1}^{\infty} (A^{(2)}A^{(1)})_{k,l}^N \delta_l(f). \tag{2.12}$$

If we evaluate $(A^{(2)}A^{(1)})_{k,l}$ then we obtain a sum of terms $C_{u,v}C_{v,w}\cdots$ along paths $(u, v, w, \cdots)$ of the following form. The path starts at $u = (t_1 + k, t_2)$, goes to $(t_1 + k, t_2 + 1)$, and from there to some site $(t_1, t_2 + j)$ in ∂T_1, with steps of length 1 and staying inside T_2. Then it goes to $(t_1 + 1, t_2 + j)$, and from there to $(t_1 + l, t_2)$, with steps of length 1 and staying inside T_1. No path occurs twice, and each has at least length $k + l$. This implies

$$\sum_k (A^{(2)}A^{(1)})_{k,l} \le \sum_k \sum_{m\ge k+l} C^m_{(t_1+k,t_2+1),(t_1+l,t_2)} \le \sum_{m\ge l+1} c^m \le \frac{c^2}{1-c}$$

due to (1.3). In the same way we see that

$$\sum_k (A^{(2)}A^{(1)})_{k,l}^N \le \frac{c^{2N}}{1-c},$$

and so (2.12) implies

$$|(E_2E_1)^N f(x) - (E_2E_1)^N f(y)| \le \frac{c^{2N}}{1-c} \sum_{l=1}^{\infty} \delta_l(f). \tag{2.13}$$

We will show in part 4) that the function f on B which corresponds to the random variable $Y = Z_t^{(2)}$ in (2.5), satisfies

$$\sum_{l=1}^{\infty} \delta_l(f) \le 2\gamma \|Z\|_\infty \tag{2.14}$$

for some constant $\gamma < \infty$. With this f, the random variable $Z_t^{(N+2)}(\omega)$ is of the form $(E_2E_1)^N f(x)$ with $x_n = X_{(t_1+n,t_2)}(\omega)$ $(n = 1, 2, \cdot\cdot)$. Thus (2.13) and (2.14) imply

$$\sup |Z_t^{(N+2)}(\omega) - Z_t^{(N+2)}(\eta)| \le \frac{c^{2N}}{1-c} 2\gamma \|Z\|_\infty \tag{2.15}$$

where the sup is taken over all ω and η whose restrictions ω_t and η_t to $\{s \mid s \le t\}$ coincide. Evaluating $E[\cdot \mid \mathscr{F}_t](\omega)$ as an integral with respect to a conditional probability distribution $P[\cdot \mid \mathscr{F}_t](\omega)$ which is concentrated on the set of all η with $\eta_t = \omega_t$, we see that

$$|Z_t^{(N+2)}(\omega) - Z_t(\omega)| = |E[Z_t^{(N+2)}(\omega) - Z_t^{(N+2)}(\cdot) \mid \mathscr{F}_t](\omega)|$$

is also bounded by the right side of (2.15), and this is the desired uniform approximation (2.6).

4) We still have to clarify the following technical point. The estimate (2.11) is only valid under the assumption that f belongs to the class $C(B)$ of functions on B which can be approximated uniformly by bounded measurable functions depending only on finitely many coordinates. Let us therefore show that the

kernels $E_i (i = 1, 2)$ have the following strong Feller property:

(2.16) If f is a bounded measurable function on B then $E_i f$ belongs to $C(B)$ and satisfies $\sum_k \delta_k(Ef) < \infty$.

For $f \in C(B)$ the estimate (2.11) holds and implies

$$(2.17) \qquad |E_1 f(x) - E_1 f(y)| \le \sum_{k: x_k \ne y_k} \sum_l A^{(1)}_{k,l} 2 \| f \|_\infty .$$

But (2.17) clearly extends to any bounded measurable f. Since

$$\sum_{k,l} A^{(1)}_{k,l} \le \sum_{k,l} \sum_{m \ge k+l} C^m_{(t_1, t_2+k), (t_1+l, t_2)} \le \sum_l \sum_{m \ge l+1} c^m \equiv \gamma < \infty,$$

(2.17) implies $|E_1 f(x) - E_1 f(y)| \le \varepsilon$ whenever $x_k = y_k$ for $k \le k(\varepsilon)$, and this shows $E_1 f \in C(B)$. Moreover, (2.17) implies

$$\delta_k(E_1 f) \le \sum_l A^{(1)}_{k,l} 2 \| f \|_\infty$$

so that $\sum_k \delta_k(E_1 f) \le 2\gamma \| f \|_\infty < \infty$.

(2.18) Remarks. 1) Theorem (2.9) also holds in the case $T = \mathbb{Z}^2$ which was considered in the introduction. In this case, we might as well state it as a four-parameter result for the class of rectangles $V_{s,t} = \{u \in \mathbb{Z}^2 \mid s \le u \le t\} (s \le 0 \le t)$: Under condition (1.3), bounded martingales

$$Z_{s,t} = E[Z \mid \mathscr{F}_{s,t}] \quad (s \le 0 \le t),$$

where $\mathscr{F}_{s,t} = \sigma(X_u; s \le u \le t)$, converge almost surely as $t, t_2 \uparrow \infty$ and $s_1, s_2 \downarrow -\infty$, or as $t_1, t_2 \uparrow \infty$ with fixed s. The proof is essentially the same, except that the role of $Z^{(1)}_t$ is now played by

$$Z^{(1)}_{s,t} = E[Z \mid \mathscr{F}^1_{t_1} \mid \mathscr{F}^2_{t_2} \mid \mathscr{G}^1_{s_1} \mid \mathscr{G}^2_{s_2}],$$

where $\mathscr{G}^i_{s_i} = \sigma(X_u \mid u_i \ge s_i)$ for $i = 1, 2$.

2) The method also works if the Markov property is replaced by an exponential decay condition on the coefficients $C_{s,t}$. In particular, it is enough to require the Markov property (2.7) in terms of finite neighborhoods of the form $\{s \mid |s - t| \le r\}$ with some fixed range $r < \infty$.

3) We have assumed that the random variable Z is bounded in L_∞. Under hypothesis (1.1) it is enough to require boundedness in $L \log L$ because then we have to apply Lemma (3.1) only once. Since N iterations of the lemma would already require boundedness in $L \log^N L$, a natural assumption for the method above would seem to be boundedness in L_p for some $p > 1$. But then we would also need a reformulation of Dobrushin's contraction technique in terms of L^p-norms, and a corresponding strengthening of condition (1.3).

3. A counterexample. In this section we construct a Markov random field which admits no phase transition, but does admit a bounded martingale which fails to converge almost surely.

We take $T = \mathbb{Z}^2_+$ and $S = \{0, 1\}$. Put $T_n = \{t \in T \mid |t| = n\}$. The random field P on S^T will be defined by strictly positive measures on $S^{T_n} (n = 0, 1, \cdots)$, and by the assumption that the different strips T_n behave independently. This implies that P is uniquely determined by its conditional probabilities (1.2); see [6].

Consider a two-step Markov chain on $S = \{0, 1\}$ with transition probability

$$p(1 \mid (x, y)) = \begin{cases} \varepsilon & \text{if } y = 0 \\ \delta & \text{if } x = 0, \quad y = 1 \\ 1 - \delta & \text{if } x = 1, \quad y = 1 \end{cases}$$

whose initial distribution on $S \times S$ is strictly positive and puts weight ε on (1, 1). Let P_n be the corresponding distribution on $S^{\{0,\cdots,n\}}$. Denoting by ξ_i the ith projection, we have

$$P_n[\xi_i = 1 \ (0 \le i \le n)] = \varepsilon(1 - \delta)^{n-1} \tag{3.1}$$

$$P_n\left[\xi_i = 0 \left(\frac{n}{4} \le i \le \frac{3n}{4}\right)\right] \le (1 - \varepsilon)^{(n/2)-1}. \tag{3.2}$$

For $k \ge 2$ a simple estimate yields

$$P_n[\xi_k = 1] \le 2\varepsilon + (k - 2)\delta\varepsilon + \delta \tag{3.3}$$

so that

$$P_n[\xi_i = 1 (0 \le i \le n) \mid \xi_k = 1] \ge \frac{\varepsilon(1 - \delta)^{n-1}}{2\varepsilon + n\delta}. \tag{3.4}$$

Take $\varepsilon_r = r^{-2} (r = 2, 3, \cdot\cdot)$, n_r big enough so that (3.2) is $\le \frac{1}{2}$, and δ_r small enough so that (3.4) is $\ge \frac{1}{3}$. The random field P on S^T is now determined by the following conditions: (i) the different strips T_n behave independently; (ii) for $n = n_r (r \ge 2)$, the distribution on S^{T_n} is given by the Markov chain measure P_n above; (iii) for $n \ne n_r$, we take, e.g., a Bernoulli measure.

Let us now construct a bounded martingale of the form $P[A \mid \mathscr{F}_t] (t \in T)$ which does not converge almost surely. Put $A_r = \{x_t = 1 (t \in T_{n_r})\}$ and $A = \bigcup_{r=2}^{\infty} A_r$. For $0 < t \in T_{n_r}$ we have $P[A \mid \mathscr{F}_t] \ge P[A_r \mid \mathscr{F}_t] = P[A_r \mid X_t]$ since the different strips behave independently, and so (3.4) implies

$$P[A \mid \mathscr{F}_t] \ge \tfrac{1}{3} \quad \text{on} \quad \{X_t = 1\}. \tag{3.5}$$

For any $s > 0$, (3.2) and our choice of n_r imply

$$P[\textstyle\sum_{s \le t \in T_{n_r}} X_t > 0] \ge \tfrac{1}{2} \tag{3.6}$$

for sufficiently large r. It follows by Borel-Cantelli that there are, almost surely, infinitely many strips T_{n_r} which contain some $t \ge s$ with $X_t = 1$, and so (3.5) implies

$$\limsup P[A \mid \mathscr{F}_t] \ge \tfrac{1}{3} \quad \text{a.s.}$$

On the other hand,

$$\liminf P[A \mid \mathscr{F}_t] \le \lim P[A \mid \mathscr{F}_{(m,m)}] = 0 \quad \text{a.s. on} \quad A^c,$$

and $P[A^c] > 0$ since $\sum P[A_r]$ converges due to (3.1). This shows that $P[A \mid \mathscr{F}_t]$ $(t \in T)$ does not converge almost surely.

(3.7) REMARK. We have followed closely the construction of Dubins and Pitman [8], except that their daisies have been replaced by certain Markov chains. The point of the modification is that now the interaction becomes local in the sense of (2.18, 2), i.e., the resulting random field has the Markov property with range $r = 4$.

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