Almost Sure Exponential Stability of Stochastic Differential Delay Equations

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Abstract

This paper is concerned with the almost sure exponential stability of the multi-dimensional nonlinear stochastic differential delay equation (SDDE) with variable delays of the form $dx(t) = f(x(t-\delta_1(t)), t)dt + g(x(t-\delta_2(t)), t)dB(t)$, where $\delta_1, \ \delta_2 : \mathbb{R}_+ \to [0, \tau]$ stand for variable delays. We show that if the corresponding (non-delay) stochastic differential equation (SDE) $dy(t) = f(y(t), t)dt + g(y(t), t)dB(t)$ admits a Lyapunov function (which in particular implies the almost sure exponential stability of the SDE) then there exists a positive number τ^* such that the SDDE is also almost sure exponentially stable as long as the delay is bounded by τ^*. We provide an implicit lower bound for τ^* which can be computed numerically. Moreover, our new theory enables us to design stochastic delay feedback controls in order to stabilize unstable differential equations.

Key words: Almost sure exponential stability, stochastic differential delay equations, Itô formula, Brownian motion, stochastic stabilization.

AMS subject classifications: 60H10, 60J10, 93D15.

1 Introduction

It is very easy to show that the linear scalar stochastic differential equation (SDE)

$$dx(t) = \sigma x(t)dB(t) \tag{1.1}$$

is almost surely exponentially stable as long as $\sigma \neq 0$ (see, e.g., [2, 4, 11]). However, it is hard to show if the corresponding linear scalar stochastic differential delay equation (SDDE)

$$dx(t) = \sigma x(t-\tau)dB(t) \tag{1.2}$$

is almost surely exponentially stable when $\sigma \neq 0$. Mohammed and Scheutzow [19] showed that for a given fixed $\sigma \neq 0$, the SDDE (1.2) is almost surely exponentially stable provided the time delay τ is sufficiently small. Their proof for this was already hard. Scheutzow

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[21] considered a more general SDDE $dx(t) = \sigma f(x_t)dB(t)$ and generalised some of the results in [19] by the method of Lyapunov functionals.

This paper is concerned with the almost sure exponential stability of the multi-dimensional nonlinear SDDE with variable delays of the form

$$dx(t) = f(x(t - \delta_1(t)), t)dt + g(x(t - \delta_2(t)), t)dB(t), \tag{1.3}$$

where $\delta_1,\ \delta_2 : \mathbb{R}_+ \to [0, \tau]$ stand for variable delays. (For the general theory on SDDEs we refer the read to, for example, [6, 7, 8, 9, 16, 18].) This SDDE is in a much more general form than (1.2)—multi-dimension, nonlinearity and variable delays. Of course, Mohammed and Scheutzow [19] also treated scalar SDDEs with distributed delay which we do not in this paper due to the page limit here. (We will report the corresponding results elsewhere.) We show that if the corresponding (non-delay) stochastic differential equation (SDE)

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \tag{1.4}$$

admits a Lyapunov function, which in particular implies the almost sure exponential stability of the SDE, then there exists a positive number τ^* such that the SDDE (1.3) is also almost surely exponentially stable as long as the delay is bounded by τ^*. We provide an implicit lower bound for τ^* which can be computed numerically. We will see that the almost sure exponential stability of (1.2) for sufficiently small τ is a consequence of our new result.

More usefully, our new theory will open a new chapter in the area of stochastic stabilization—*the stabilization of unstable differential equations by stochastic delay feedback controls*. To explain this, we consider the scalar SDE

$$dx(t) = \alpha x(t)dt + \sigma x(t)dB(t), \tag{1.5}$$

where $0 < \alpha < \sigma^2/2$. It is known that this SDE is almost surely exponentially stable (see, e.g., [2, 4, 11]). If we regard this SDE as the stochastically controlled system of the unstable differential equation $\dot{x}(t) = \alpha x(t)$, we see that it is the stochastic feedback control $\sigma x(t)dB(t)$ that stabilizes the unstable system $\dot{x}(t) = \alpha x(t)$. Stochastic stabilization of linear systems was initiated by Khasminskii [5] and generalized by Arnold et al. [3]. Stochastic stabilization and destabilization of nonlinear systems in the plane were done by Scheutzow [20] and then were generalised to multi-dimensional nonlinear systems by Mao [10]. The theory was further developed by Appleby and Mao [1] to a class of functional differential equations and by Mao et al. [17] to hybrid differential equations. A common feature of these results is that the stochastic feedback control needs to depend on the current state $x(t)$ but not the delay state $x(t - \delta(t))$, even when the given system is a delay equation (see, e.g., [1]).

On the other hand, it is more realistic in practice if the control depends on a past state, say $x(t - \tau)$, due to a time lag τ (> 0) between the time when the observation of the state is made and the time when the feedback control reaches the system (see, e.g., [12, 15]). Accordingly, the stochastic control should depend on $x(t - \tau)$ but not $x(t)$, say in the form of $g(x(t - \tau), t)dB(t)$. Hence, the stochastically controlled system has the form

$$dx(t) = f(x(t), t)dt + g(x(t - \tau), t)dB(t). \tag{1.6}$$

The aim here is of course to design stochastic delay feedback control $g(x(t-\tau), t)dB(t)$ to make this controlled system become almost surely exponentially stable. However, there is so far no result on this stabilization problem, although there are some results when the delay feedback controls are in the drift part (see, e.g., [12, 15]). In this paper, we shall shed some light on this problem. We will show that when f is globally Lipschitz continuous, it is possible to design a linear stochastic delay feedback control $Ax(t-\tau)dB(t)$ to make the stochastically controlled system

$$dx(t) = f(x(t), t)dt + Ax(t-\tau)dB(t) \tag{1.7}$$

become almost surely exponentially stable.

It should also be pointed out that the almost sure exponential stability of discrete-time or partial discrete-time SDEs has recently been discussed by several authors. For example, Mao [13] discussed the almost sure exponential stability in the numerical simulation of SDEs (i.e., stability of discrete-time SDEs). You et al. [22] studied the stability of the controlled SDE $dx(t) = (f(x(t), t)+u(x([t/\tau]\tau), t))dt+g(x(t), t)dB(t)$, where $[t/\tau]$ denotes the integer part of t/τ and $u(x([t/\tau]\tau, t)$ is the feedback control based on the discrete-time state observations $x(0), x(\tau), x(2\tau)$ and so on. It is observed that the feedback control in [22] is in the drift part which is significantly different from the case where the control is in the diffusion part as in this paper (see equations (1.6) and (1.7)). Mao [14] investigated the almost sure exponential stability of the stochastically controlled system $dx(t) = f(x(t), t)dt + Ax([t/\tau]\tau)dB(t)$. This equation looks similar to equation (1.7) but they are in fact different in the sense that $x([t/\tau]\tau)$ in this equation is of discrete-time while $x(t-\tau)$ in (1.7) is of continuous-time. This equation is of course significantly different from our main SDDE (2.1) and our result on equation (1.7) (namely Corollary 5.2) is only a simple application of our general theory established in this paper.

All of the points made above do not only show the difficulty of our proposed problem but also highlight the differences between our current paper and the existing papers as well as the potential of our new theory in the area of stochastic stabilization. Let us begin to develop our new theory.

2 Problem Settings

Throughout this paper, unless otherwise specified, we will use the following notation. Let $|x|$ denote the Euclidean norm of vector $x \in \mathbb{R}^n$. For a matrix A, let $|A| = \sqrt{\text{trace}(A^T A)}$ be its trace norm and $\|A\| = \max\{|Ax| : |x| = 1\}$ be the operator norm. For a vector or matrix A, its transpose is denoted by A^T. If A is a symmetric real matrix ($A = A^T$), denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalue, respectively. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $\tau > 0$ and denote by $C([-\tau, 0]; \mathbb{R}^n)$ the family of continuous functions $\xi : [-\tau, 0] \to \mathbb{R}^n$ with the norm $\|\xi\| = \sup_{-\tau\leq u\leq 0} |\xi(u)|$. For $t \geq 0$, denote by $L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$ the family of $\mathcal{F}_t$-measurable $\mathbb{R}^n$-valued random variables ζ such that $\mathbb{E}|\zeta|^2 < \infty$, and by $L^2_{\mathcal{F}_t}(\Omega; C([-\tau, 0]; \mathbb{R}^n)$ the family of $\mathcal{F}_t$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables ξ such that $\mathbb{E}\|\xi\|^2 < \infty$. Let $\delta_1,\ \delta_2 : \mathbb{R}_+ \to [0, \tau]$ be Borel measurable functions.

Consider a nonlinear n-dimensional SDDE

$$dx(t) = f(x(t-\delta_1(t)), t)dt + g(x(t-\delta_2(t)), t)dB(t) \tag{2.1}$$

on $t \geq t_0$ with the initial data $x(t_0 + u) = \xi(u)$ for $u \in [-\tau, 0]$, where $t_0 \geq 0$, $\xi \in L^2_{\mathcal{F}_{t_0}}(\Omega; C([-\tau, 0]; \mathbb{R}^n)$ and

$$f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}.$$

We impose a standing hypothesis on f and g.

Assumption 2.1 *Assume that f and g are Borel measurable. Assume also that there exist two nonnegative constants K_1 and K_2 such that*

$$|f(x,t) - f(y,t)| \leq K_1|x-y| \quad \text{and} \quad |g(x,t) - g(y,t)| \leq K_2|x-y| \tag{2.2}$$

for all $x, y \in \mathbb{R}^n$ and $t \geq 0$. For the stability purpose of this paper, we moreover assume that $f(0,t) = 0$ and $g(0,t) = 0$ for all $t \geq 0$.

This assumption implies the linear growth condition

$$|f(x,t)| \leq K_1|x| \quad \text{and} \quad |g(x,t)| \leq K_2|x| \tag{2.3}$$

for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$. It is also known (see, e.g., [11, Theorem 3.2 on page 159]) that under Assumption 2.1, equation (2.1) has a unique solution on $t \geq t_0 - \tau$ and, moreover, the second moment of the solution is finite. We will denote the solution by $x(t; t_0, \xi)$ in order to emphasize the initial data ξ at time t_0, though we will often write it as $x(t)$. Moreover, we define $x_t = \{x(t+u) : u \in [-\tau, 0]\}$ for $t \geq t_0$ so x_t is an $\mathcal{F}_t$-adapted $C([-\tau, 0]; \mathbb{R}^n)$-valued stochastic process. We also have $x_t \in L^2_{\mathcal{F}_t}(\Omega; C([-\tau, 0]; \mathbb{R}^n)$. Furthermore, for any $t_0 \leq s \leq t < \infty$, we can regard $x(t)$ as the solution of the SDDE (2.1) on $t \geq s$ with the initial data x_s at time s. In other words, we have

$$x(t) = x(t; s, x_s), \quad t_0 \leq s \leq t < \infty. \tag{2.4}$$

This shows clearly that given x_s at time s, we can determine $x(t)$ for all $t \geq s$ by solving the SDDE (2.1) but the information on how the solution reaches x_s from x_{t_0} is of no further use. We should also point out that it would be sufficient to consider the initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n)$ for the purpose of the almost sure exponential stability in this paper. The reason why we consider the initial data ξ in a larger space $L^2_{\mathcal{F}_{t_0}}(\Omega; C([-\tau, 0]; \mathbb{R}^n))$ is because we find that it is more convenient to perform our stability analysis in the space $L^2_{\mathcal{F}_t}(\Omega; C([-\tau, 0]; \mathbb{R}^n))$.

The key technique used in this paper is to compare the SDDE (2.1) with the corresponding stochastic differential equation (SDE)

$$dy(t) = f(y(t), t)dt + g(y(t), t)dB(t) \tag{2.5}$$

for $t \geq t_0$ with the initial data $y(t_0) = y_0$, where $t_0 \geq 0$ and $y_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega; \mathbb{R}^n)$. For the general theory on SDEs we refer the reader to, for example, [2, 4, 5, 7, 11]. In particular, it is known (see, e.g., [11, Theorem 3.1 on page 51]) that under Assumption 2.1, equation (2.5) has a unique solution $y(t)$ on $t \geq t_0$ and, moreover, the second moment of the solution is finite. We will denote the solution by $y(t; t_0, y_0)$ when we need to emphasize the initial data y_0 at time t_0. We will choose t_0 and y_0 appropriately when we prove our theorems in this paper. Our stability problems are:

(a) If the SDE (2.5) is almost surely exponentially stable, is the SDDE (2.1) also almost surely exponentially stable provided τ is sufficiently small?

(b) If the answer to (a) is yes, can we obtain an upper bound, say τ^*, on τ such that the SDDE (2.1) is almost surely exponentially stable provided $\tau < \tau^*$?

It is therefore natural to assume that the SDE (2.5) is almost surely exponentially stable. There are many results on the almost sure exponential stability of the nonlinear SDE (2.5) (see, e.g., [2, 4, 5, 7, 11, 16]). We will cite one of the most useful criteria from Mao [11, Theorem 3.3 on page 121] for the use of this paper. For this purpose, we denote by $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ the family of non-negative functions $V(y,t)$ defined on $\mathbb{R}^n \times \mathbb{R}_+$ such that they are continuously twice differentiable in x and once in t. For $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, we define the function $LV : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ by

$$LV(y,t) = V_t(y,t) + V_y(y,t)f(y,t) + \frac{1}{2}\text{trace}(g^T(y,t)V_{yy}(y,t)g(y,t)),$$

where

$$V_t(y,t) = \frac{\partial V(y,t)}{\partial t},\ V_y(y,t) = \Big(\frac{\partial V(y,t)}{\partial y_i}\Big)_{1\times n},\ V_{yy}(y,t) = \Big(\frac{\partial^2 V(x,t)}{\partial y_i \partial y_j}\Big)_{n\times n}.$$

Let us now impose another assumption.

Assumption 2.2 *Assume that there exists a function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ and constants $q > 0,\ \bar{c}_1 \ge c_1 > 0,\ c_2 \in \mathbb{R},\ c_3 \ge 0$ with $c_3 > 2c_2$ such that for all $(y,t) \in \mathbb{R}^n \times \mathbb{R}_+$,*

$$c_1|y|^q \le V(y,t) \le \bar{c}_1|y|^q, \quad LV(y,t) \le c_2 V(y,t)$$

$$|V_y(y,t)g(y,t)|^2 \ge c_3(V(y,t))^2.$$

The theorem from Mao [11, Theorem 3.3 on page 121] states that *under Assumption 2.2, the SDE (2.5) is almost surely exponentially stable.* Our aim here is to establish the positive answers to Problems (a) and (b) listed above under this assumption.

3 Main Results

Our positive answers to the problems are stated in the following theorem.

Theorem 3.1 *Let Assumptions 2.1 and 2.2 hold. Then there is a positive number τ^* such that for any initial data $\xi \in L^2_{\mathcal{F}_{t_0}}(\Omega; C([-\tau,0];\mathbb{R}^n))$, the solution of equation (2.1) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t;t_0,\xi)|) < 0 \quad a.s. \tag{3.1}$$

provided $\tau < \tau^*$. In practice, we can first choose a constant $\theta \in (0,1)$ for (3.4) below to hold and set $p = \theta q$; and then choose another constant $\varepsilon \in (0,1)$ and set

$$T = \frac{1}{\gamma}\log\Big(\frac{2^{2.5p}M}{\varepsilon}\Big); \tag{3.2}$$

and finally let $\tau^* > 0$ be the unique root to the equation (in τ)

$$\varepsilon e^{(K_1+0.5K_2^2)p\tau} + 2^p H_1(\tau,p,\tau+T) + 4^p H_2(\tau,p,\tau+T) = 1, \tag{3.3}$$

where M, γ will be defined by (3.6) while H_1 and H_2 by (3.11) and (3.17), respectively.

In the statement above, we describe a way to determine τ^* by introducing two free parameters θ and ε. Unfortunately, we do not know how to determine these two parameters in order to get the optimal τ^* yet. Our bound on τ^* is therefore conservative and it is a challenge to get the optimal bound.

The proof of the theorem is very technical so we break it into a number of lemmas. Our first lemma shows that under Assumption 2.2, the SDE (2.5) is pth moment exponentially stable for all sufficiently small $p \in (0,1)$.

Lemma 3.2 *Let Assumption 2.2 hold. Let $\theta \in (0,1)$ be sufficiently small for which*

$$\theta q < 1 \quad \text{and} \quad 0.5(1-\theta)c_3 > c_2. \tag{3.4}$$

Set $p = \theta q$. Then the solution of the SDE (2.5) satisfies

$$\mathbb{E}|y(t;t_0,y_0)|^p \le M e^{-\gamma(t-t_0)}\mathbb{E}|y_0|^p, \quad \forall t \ge t_0 \tag{3.5}$$

for all $y_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega;\mathbb{R}^n)$, where

$$M = (\bar{c}_1/c_1)^\theta \quad \text{and} \quad \gamma = \theta(0.5(1-\theta)c_3 - c_2). \tag{3.6}$$

Proof. Fix any t_0 and y_0 and write $y(t;t_0,y_0) = y(t)$. Let us first consider the case when y_0 is deterministic, namely $y_0 \in \mathbb{R}^n$. Assertion (3.5) holds when $y_0 = 0$ so we need to show it for $y_0 \neq 0$. Fix any $y_0 \neq 0$. By Mao [11, Lemma 3.2 on page 120], we observe that $y(t) \neq 0$ for all $t \ge 0$ almost surely. Let $U(y,t) = (V(y,t))^\theta$. By the Itô formula, we can show that

$$e^{\gamma t}\mathbb{E}U(y(t),t) = e^{\gamma t_0}U(y_0,t_0) + \mathbb{E}\int_{t_0}^t \big[\gamma e^{\gamma s}U(y(s),s) + e^{\gamma s}LU(y(s),s)\big]ds$$

for $t \ge t_0$, where $LU : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ has the form

$$LU(y,t) = \theta(V(y,t))^{\theta-1}LV(y,t) - 0.5\theta(1-\theta)(V(y,t))^{\theta-2}|V_y(y,t)g(y,t)|^2.$$

But, by Assumption 2.2 and inequality (3.4), we have

$$LU(y,t) \le -\gamma U(y,t).$$

Consequently

$$e^{\gamma t}\mathbb{E}U(y(t),t) \le e^{\gamma t_0}U(y_0,t_0), \quad \forall t \ge t_0.$$

Noting that $c_1^\theta |y|^p \le U(y,t) \le \bar{c}_1^\theta |y|^p$, we then have

$$\mathbb{E}|y(t)|^p \le M|y_0|^p e^{-\gamma(t-t_0)}, \quad \forall t \ge t_0.$$

In other words, we have shown that the assertion holds when $y_0 \in \mathbb{R}^n$.

Now, for any $y_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega;\mathbb{R}^n)$, by the property of the conditional expectation, we derive that, for $t \ge t_0$,

$$\mathbb{E}|y(t)|^p = \mathbb{E}\Big(\mathbb{E}\big(|y(t)|^p\big|\mathcal{F}_{t_0}\big)\Big) \le \mathbb{E}\Big(M|y_0|^p e^{-\gamma(t-t_0)}\Big) = M e^{-\gamma(t-t_0)}\mathbb{E}|y_0|^p \tag{3.7}$$

as required. The proof is complete.

In the following lemma we give some estimates on the pth moment of the solution of the SDDE (2.1) for $p \in (0,1)$.

Lemma 3.3 *Let Assumption 2.1 hold and* $p \in (0,1)$. *Let* $\xi \in L^2_{\mathcal{F}_{t_0}}(\Omega; C([-\tau,0];\mathbb{R}^n)$ *be arbitrary and write* $x(t;t_0,\xi) = x(t)$. *Then, for all* $t \ge t_0$,

$$\mathbb{E}|x(t)|^p \le \Big(2e^{(2K_1+K_2^2)(t-t_0)}\Big)^{p/2}\mathbb{E}\|\xi\|^p, \tag{3.8}$$

$$\mathbb{E}\Big(\sup_{0\le u\le\tau}|x(t+u)-x(t)|^p\Big) \le H_1(\tau,p,t-t_0)\mathbb{E}\|\xi\|^p, \tag{3.9}$$

and

$$\mathbb{E}\Big(\sup_{t_0\le u\le t}|x(u)|^p\Big) \le \Big(3 + \frac{6((t-t_0)K_1^2+4K_2^2)}{2K_1+K_2^2}\,e^{(2K_1+K_2^2)(t-t_0)}\Big)^{p/2}\mathbb{E}\|\xi\|^p, \tag{3.10}$$

where (for $T\ge 0$*)*

$$H_1(\tau,p,T) = \Big(4\tau(\tau K_1^2+4K_2^2)e^{(2K_1+K_2^2)(T+\tau)}\Big)^{p/2}. \tag{3.11}$$

Proof. By the method of conditional expectation as we did in (3.7), we only need to show the lemma for $\xi \in C([-\tau,0];\mathbb{R}^n)$ (i.e., for deterministic initial data). By the Itô formula and Assumption 2.1, it is easy to show that, for $t\ge t_0$,

$$\mathbb{E}|x(t)|^2 \le |x(t_0)|^2 + (2K_1+K_2^2)\int_{t_0}^t\Big(\sup_{t_0-\tau\le u\le s}\mathbb{E}|x(u)|^2\Big)ds.$$

Noting that the right-hand-side term of the above inequality is increasing in $t\in[t_0,\infty)$, we hence have

$$\sup_{t_0\le u\le t}\mathbb{E}|x(u)|^2 \le |x(t_0)|^2 + (2K_1+K_2^2)\int_{t_0}^t\Big(\sup_{t_0-\tau\le u\le s}\mathbb{E}|x(u)|^2\Big)ds.$$

Consequently

$$\begin{aligned}\sup_{t_0-\tau\le u\le t}\mathbb{E}|x(u)|^2 &\le \|\xi\|^2 + \sup_{t_0\le u\le t}\mathbb{E}|x(u)|^2\\ &\le 2\|\xi\|^2 + (2K_1+K_2^2)\int_{t_0}^t\Big(\sup_{t_0-\tau\le u\le s}\mathbb{E}|x(u)|^2\Big)ds.\end{aligned}$$

The well-known Gronwall inequality yields

$$\sup_{t_0-\tau\le u\le t}\mathbb{E}|x(u)|^2 \le 2\|\xi\|^2e^{(2K_1+K_2^2)(t-t_0)}. \tag{3.12}$$

By the Hölder inequality, we then have

$$\mathbb{E}|x(t)|^p \le \big(\mathbb{E}|x(t)|^2\big)^{p/2} \le \Big(2e^{(2K_1+K_2^2)(t-t_0)}\Big)^{p/2}\|\xi\|^p. \tag{3.13}$$

Namely, assertion (3.8) holds. By the Hölder inequality, the Doob martingale inequality, Assumption 2.1 as well as (3.12), we further derive that

$$\begin{aligned}&\mathbb{E}\Big(\sup_{0\le u\le\tau}|x(t+u)-x(t)|^2\Big)\\ \le\;& 2\tau\mathbb{E}\int_t^{t+\tau}|f(x(s-\delta_1(s)),s)|^2ds + 8\mathbb{E}\int_t^{t+\tau}|g(x(s-\delta_2(s)),s)|^2ds\\ \le\;& 2\tau K_1^2\int_t^{t+\tau}\mathbb{E}|x(s-\delta_1(s))|^2ds + 8K_2^2\int_t^{t+\tau}\mathbb{E}|x(s-\delta_2(s))|^2ds\\ \le\;& 4\tau(\tau K_1^2+4K_2^2)\|\xi\|^2e^{(2K_1+K_2^2)(t+\tau-t_0)}.\end{aligned}$$

Once again, by the Hölder inequality, we have

$$\mathbb{E}\Big(\sup_{0\le u\le\tau}|x(t+u)-x(t)|^p\Big)\le H_1(\tau,p,t-t_0)\|\xi\|^p, \tag{3.14}$$

where $H_1(\tau,p,t-t_0)$ has been defined in the statement of the lemma. That is, another assertion (3.9) also holds. Similarly, we can show that

$$\begin{aligned}\mathbb{E}\Big(\sup_{t_0\le u\le t}|x(u)|^2\Big) &\le 3|x(t_0)|^2+3(t-t_0)K_1^2\int_{t_0}^t\mathbb{E}|x(s-\delta_1(s))|^2ds\\ &+ 12K_2^2\int_{t_0}^t\mathbb{E}|x(s-\delta_2(s))|^2ds\\ &\le \Big(3+\frac{6((t-t_0)K_1^2+4K_2^2)}{2K_1+K_2^2}\,e^{(2K_1+K_2^2)(t-t_0)}\Big)\|\xi\|^2.\end{aligned}$$

Consequently

$$\mathbb{E}\Big(\sup_{t_0\le u\le t}|x(u)|^p\Big)\le\Big(3+\frac{6((t-t_0)K_1^2+4K_2^2)}{2K_1+K_2^2}\,e^{(2K_1+K_2^2)(t-t_0)}\Big)^{p/2}\|\xi\|^p. \tag{3.15}$$

That is, the last assertion (3.10) holds too. The proof is complete.

The following lemma estimates the difference in the pth moment between the solution of the SDDE (2.1) and that of the SDE (2.5).

Lemma 3.4 *Let Assumption 2.1 hold and $p\in(0,1)$. Let $\xi\in L^2_{\mathcal{F}_{t_0}}(\Omega;C([-\tau,0];\mathbb{R}^n)$ be arbitrary and write $x(t;t_0,\xi)=x(t)$. Then, for all $t\ge t_0+\tau$,*

$$\mathbb{E}|y(t;t_0+\tau,x(t_0+\tau))-x(t)|^p\le H_2(\tau,p,t-t_0)\mathbb{E}\|\xi\|^p, \tag{3.16}$$

where (for $T\ge\tau$)

$$H_2(\tau,p,T)=\Big(\frac{8\tau(K_1+K_2^2)(\tau K_1^2+K_2^2)}{2K_1+K_2^2}\;e^{(3K_1+2K_2^2)(T-\tau)}\big[e^{(2K_1+K_2^2)T}-e^{(2K_1+K_2^2)\tau}\big]\Big)^{p/2}. \tag{3.17}$$

Proof. Once again, by the method of conditional expectation as we did in (3.7), we only need to show the lemma for $\xi\in C([-\tau,0];\mathbb{R}^n)$. Write $y(t;t_0+\tau,x(t_0+\tau))=y(t)$. By the Itô formula and Assumption 2.1, we can show that for $t\ge t_0+\tau$,

$$\begin{aligned}&\mathbb{E}|x(t)-y(t)|^2\\ \le&\mathbb{E}\int_{t_0+\tau}^t\Big[2K_1|x(s)-y(s)||x(s-\delta_1(s))-y(s)|+K_2^2|x(s-\delta_2(s))-y(s)|^2\Big]ds\\ \le&(3K_1+2K_2^2)\int_{t_0+\tau}^t\mathbb{E}|x(s)-y(s)|^2ds+2K_1\int_{t_0+\tau}^t\mathbb{E}|x(s)-x(s-\delta_1(s))|^2ds\\ &+2K_2^2\int_{t_0+\tau}^t\mathbb{E}|x(s)-x(s-\delta_2(s))|^2ds.\end{aligned}$$

The Gronwall inequality then implies

$$\begin{aligned}&\mathbb{E}|y(t)-x(t)|^2\le e^{(3K_1+2K_2^2)(t-t_0-\tau)}\\ \times&\Big(2K_1\int_{t_0+\tau}^t\mathbb{E}|x(s)-x(s-\delta_1(s))|^2ds+2K_2^2\int_{t_0+\tau}^t\mathbb{E}|x(s)-x(s-\delta_2(s))|^2ds\Big).\end{aligned} \tag{3.18}$$

On the other hand, by (3.12), we have that, for $s \in [t_0+\tau, t]$,

$$\begin{aligned}&\mathbb{E}|x(s)-x(s-\delta_i(s))|^2\\ \le&2\tau K_1^2\int_{s-\delta_i(s)}^s \mathbb{E}|x(u-\delta_1(u)|^2du + 2K_2^2\int_{s-\delta_i(s)}^s \mathbb{E}|x(u-\delta_2(u)|^2du\\ \le&4\tau(\tau K_1^2+K_2^2)\|\xi\|^2 e^{(2K_1+K_2^2)(s-t_0)},\end{aligned} \tag{3.19}$$

where $i=1$ or 2. Substituting this into (3.18) yields

$$\mathbb{E}|y(t)-x(t)|^2 \le (H_2(\tau,p,t-t_0))^{2/p}\|\xi\|^2, \tag{3.20}$$

where $H_2(\tau,p,t-t_0)$ has been defined in the statement of the lemma. A simple application of the Hölder inequality implies

$$\mathbb{E}|y(t)-x(t)|^p \le H_2(\tau,p,t-t_0)\|\xi\|^p, \tag{3.21}$$

which is the required assertion. The proof is complete.

We can now prove our main theorem in this paper.

Proof of Theorem 3.1. To make it clearer, we divide the proof into three steps.

Step 1. We first choose a constant $\theta \in (0,1)$ for (3.4) to hold and set $p=\theta q$ so $p\in(0,1)$. We then choose another constant $\varepsilon\in(0,1)$ and let

$$T=\frac{1}{\gamma}\log\Big(\frac{2^{2.5p}M}{\varepsilon}\Big), \quad \text{namely} \quad 2^{2.5p}Me^{-\gamma T}=\varepsilon. \tag{3.22}$$

Let τ^* be the unique root to equation (3.3). We first observe that once θ and ε are chosen, p and T are determined, while the left-hand-side term of equation (3.3) is a continuously increasing function of $\tau\ge 0$ and is equal to ε when $\tau=0$ so equation (3.3) must have a unique root $\tau^*>0$.

Fix $\tau\in(0,\tau^*)$ and $\xi\in L^2_{\mathcal{F}_{t_0}}(\Omega;C([-\tau,0];\mathbb{R}^n))$ arbitrarily and write $x(t;t_0,\xi)=x(t)$ for $t\ge t_0$. Write $y(t_0+\tau+T;t_0+\tau,x(t_0+\tau)=y(t_0+\tau+T)$. By Lemmas 3.2 and 3.3, we have

$$\mathbb{E}|y(t_0+\tau+T)|^p \le M\mathbb{E}|x(t_0+\tau)|^p e^{-\gamma T} \le Me^{-\gamma T}\Big(2e^{(2K_1+K_2^2)\tau}\Big)^{p/2}\mathbb{E}\|\xi\|^p. \tag{3.23}$$

By the elementary inequality $(a+b)^p\le 2^p(a^p+b^p)$ for any $a,b\ge 0$, we have

$$\mathbb{E}|x(t_0+\tau+T)|^p \le 2^p\mathbb{E}|y(t_0+\tau+T)|^p + 2^p\mathbb{E}|x(t_0+\tau+T)-y(t_0+\tau+T)|^p.$$

Using (3.23) as well as Lemma 3.4, we get

$$\mathbb{E}|x(t_0+\tau+T)|^p \le 2^p\Big(Me^{-\gamma T}\big(2e^{(2K_1+K_2^2)\tau}\big)^{p/2}+H_2(\tau,p,\tau+T)\Big)\mathbb{E}\|\xi\|^p. \tag{3.24}$$

On the other hand, by Lemma 3.3, we have

$$\begin{aligned}&\mathbb{E}\|x_{t_0+2\tau+T}\|^p\\ \le&2^p\mathbb{E}|x(t_0+\tau+T)|^p + 2^p\mathbb{E}\Big(\sup_{0\le u\le\tau}|x(t_0+\tau+T)-x(t_0+\tau+T+u)|^p\Big)\\ \le&2^p\mathbb{E}|x(t_0+\tau+T)|^p + 2^pH_1(\tau,p,\tau+T)\mathbb{E}\|\xi\|^p.\end{aligned} \tag{3.25}$$

Combining (3.25) with (3.24) along with (3.22), we get

$$\begin{aligned}&\mathbb{E}\|x_{t_0+2\tau+T}\|^p\\ \le&\Big(\varepsilon e^{(K_1+0.5K_2^2)p\tau}+4^pH_2(\tau,p,\tau+T)+2^pH_1(\tau,p,\tau+T)\Big)\mathbb{E}\|\xi\|^p.\end{aligned} \tag{3.26}$$

But, as $\tau<\tau^*$, we see from (3.3) that

$$\varepsilon e^{(K_1+0.5K_2^2)p\tau}+4^pH_2(\tau,p,\tau+T)+2^pH_1(\tau,p,\tau+T)<1.$$

We may therefore write

$$\varepsilon e^{(K_1+0.5K_2^2)p\tau}+4^pH_2(\tau,p,\tau+T)+2^pH_1(\tau,p,\tau+T)=e^{-\lambda(2\tau+T)}$$

for some $\lambda>0$. It then follows from (3.26) that

$$\mathbb{E}\|x_{t_0+2\tau+T}\|^p\le e^{-\lambda(2\tau+T)}\mathbb{E}\|\xi\|^p. \tag{3.27}$$

Step 2. Let us now consider the solution $x(t)$ on $t\ge t_0+2\tau+T$. By property (2.4), this can be regarded as the solution of the SDDE (2.1) with the initial data $x_{t_0+2\tau+T}$ at $t=t_0+2\tau+T$. By Step 1, we then have

$$\mathbb{E}\|x_{t_0+2(2\tau+T)}\|^p\le e^{-\lambda(2\tau+T)}\mathbb{E}\|x_{t_0+2\tau+T}\|^p.$$

This, together with (3.27), implies

$$\mathbb{E}\|x_{t_0+2(2\tau+T)}\|^p\le e^{-2\lambda(2\tau+T)}\mathbb{E}\|\xi\|^p.$$

Repeating this procedure, we have

$$\mathbb{E}\|x_{t_0+k(2\tau+T)}\|^p\le e^{-k\lambda(2\tau+T)}\mathbb{E}\|\xi\|^p \tag{3.28}$$

for all $k=1,2,\cdots$. But this holds for $k=0$ obviously so (3.28) holds for all $k=0,1,2,\cdots$. Now, by Lemma 3.3 (namely inequality (3.10)) as well as (3.28), we have

$$\mathbb{E}\Big(\sup_{t_0+k(2\tau+T)\le t\le t_0+(k+1)(2\tau+T)}|x(t)|^p\Big)\le C\mathbb{E}\|x_{t_0+k(2\tau+T)}\|^p\le Ce^{-k\lambda(2\tau+T)}\mathbb{E}\|\xi\|^p \tag{3.29}$$

for all $k=0,1,2,\cdots$, where

$$C=\Big(3+\frac{6((2\tau+T)K_1^2+4K_2^2)}{2K_1+K_2^2}\,e^{(2K_1+K_2^2)(2\tau+T)}\Big)^{p/2}.$$

Step 3. It now follows from (3.29) that

$$\mathbb{P}\Big(\sup_{t_0+k(2\tau+T)\le t\le t_0+(k+1)(2\tau+T)}|x(t)|^p\ge e^{-0.5k\lambda(2\tau+T)}\Big)\le Ce^{-0.5k\lambda(2\tau+T)}\mathbb{E}\|\xi\|^p$$

for all $k\ge 0$. By the Borel–Cantelli lemma (see, e.g., [11, Lemma 2.4 on page 7]), we obtain that for almost all $\omega\in\Omega$, there is an integer $k_0=k_0(\omega)$ such that

$$\sup_{t_0+k(2\tau+T)\le t\le t_0+(k+1)(2\tau+T)}|x(t)|^p<e^{-0.5k\lambda(2\tau+T)}\quad\forall k\ge k_0(\omega).$$

This implies easily that

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t,\omega)|)\le-\frac{\lambda}{2p}$$

for almost all $\omega\in\Omega$. The proof is hence complete.

4 Corollaries

To show the power and usefulness of our new Theorem 3.1, let us first consider the scalar linear SDDE

$$dx(t) = \alpha x(t-\tau)dt + \sigma x(t-\tau)dB(t) \tag{4.1}$$

on $t \ge t_0$ with the initial data $x_{t_0} = \xi \in L^2_{\mathcal{F}_{t_0}}(\Omega; C([-\tau, 0]; \mathbb{R})$. Here $x(t) \in \mathbb{R}$, $B(t)$ is a scalar Brownian motion, $\alpha \ge 0$ and $\sigma, \tau > 0$ are all constants and we assume that

$$2\alpha < \sigma^2. \tag{4.2}$$

It is obvious that Assumption 2.1 is satisfied with $K_1 = \alpha$ and $K_2 = \sigma$. To verify Assumption 2.2, let us consider the corresponding linear scalar SDE

$$dy(t) = \alpha y(t)dt + \sigma y(t)dB(t). \tag{4.3}$$

Define a $C^{2,1}$-function $V(y,t) = y^2$ for $(y,t) \in \mathbb{R} \times \mathbb{R}_+$. It is then easy to show that

$$LV(y,t) = (2\alpha + \sigma^2)y^2 \quad \text{and} \quad |V_y(y,t)\sigma y|^2 = 4\sigma^2 y^4.$$

That is, the parameters in Assumption 2.2 are

$$q = 2, \quad \bar{c}_1 = c_1 = 1, \quad c_2 = 2\alpha + \sigma^2, \quad c_3 = 4\sigma^2. \tag{4.4}$$

By (4.2), we have

$$c_3 - 2c_2 = 2\sigma^2 - 4\alpha > 0, \text{ namely } c_3 > 2c_2.$$

In other words, Assumption 2.2 is also satisfied with the parameters defined by (4.4). By Theorem 3.1, we have the following corollary.

Corollary 4.1 *Let condition (4.2) hold. Then there is a positive number τ^* such that for any initial data $\xi \in L^2_{\mathcal{F}_{t_0}}(\Omega; C([-\tau, 0]; \mathbb{R}))$, the solution of the SDDE (4.1) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t; t_0, \xi)|) < 0 \quad a.s. \tag{4.5}$$

provided $\tau < \tau^*$.

Our theory also enables us to obtain an estimate on τ^*. We can first choose a constant $\theta \in (0, 0.5)$ such that

$$(1-2\theta)\sigma^2 > 2\alpha \tag{4.6}$$

and set $\gamma = \theta((1-2\theta)\sigma^2 - 2\alpha)$. Choose another constant $\varepsilon \in (0,1)$ and set

$$T = \frac{1}{\gamma} \log\left(\frac{2^{5\theta}}{\varepsilon}\right). \tag{4.7}$$

Finally let $\tau^* > 0$ be the unique root to the equation (in τ)

$$\varepsilon e^{(2\alpha+\sigma^2)\theta\tau} + 4^\theta H_3(\tau, p, T) + 8^\theta H_4(\tau, p, T) = 1, \tag{4.8}$$

where

$$H_3(\tau, \theta, T) = \left(4\tau(\tau\alpha^2 + 4\sigma^2)e^{(2\alpha+\sigma^2)(T+2\tau)}\right)^\theta. \tag{4.9}$$

and

$$H_4(\tau,\theta,T)=\Big(\frac{4\tau(\alpha+2\sigma^2)(\tau\alpha^2+\sigma^2)}{2\alpha+\sigma^2}\; e^{(3\alpha+2\sigma^2)T}\big[e^{(2\alpha+\sigma^2)(T+\tau)}-e^{(2\alpha+\sigma^2)\tau}\big]\Big)^{\theta}. \tag{4.10}$$

An even simpler SDDE is equation (1.1), namely

$$dx(t)=\sigma x(t-\tau)dB(t), \tag{4.11}$$

which is the special case of equation (4.1) when $\alpha=0$. The following corollary is straightforward.

Corollary 4.2 *Let $\tau^*>0$ be the unique root to the equation*

$$\varepsilon e^{\sigma^2\theta\tau}+\Big(64\tau\sigma^2 e^{\sigma^2(T+2\tau)}\Big)^{\theta}+\Big(64\tau\sigma^2 e^{3\sigma^2T}\big[e^{\sigma^2T}-1\big]\Big)^{\theta}=1, \tag{4.12}$$

where $\theta\in(0,0.5)$ and $\varepsilon\in(0,1)$ are two free parameters and

$$\gamma=\theta(1-2\theta)\sigma^2,\quad T=\frac{1}{\gamma}\log\Big(\frac{2^{5\theta}}{\varepsilon}\Big).$$

Then for any initial data $\xi\in L^2_{\mathcal{F}_{t_0}}(\Omega;C([-\tau,0];\mathbb{R}))$, the solution of the SDDE (4.11) satisfies

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t;t_0,\xi)|)<0\quad a.s. \tag{4.13}$$

provided $\tau<\tau^*$.

Let us now consider a semi-linear SDDE

$$dx(t)=f(x(t-\delta_1(t)),t)dt+Ax(t-\delta_2(t))dB(t) \tag{4.14}$$

on $t\ge t_0$ with the initial data $x_{t_0}=\xi\in L^2_{\mathcal{F}_{t_0}}(\Omega;C([-\tau,0];\mathbb{R}^n)$, where $B(t)$ is a scalar Brownian motion, f, δ_1, δ_2 are the same as before and $A\in\mathbb{R}^{n\times n}$. We assume that f satisfies Assumption 2.1. The diffusion coefficient is linear so it satisfies Assumption 2.1 with $K_2=\|A\|$. For the square matrix A, we impose the following assumption.

Assumption 4.3 *There are two positive constants ρ_1 and ρ_2 such that*

$$\rho_2-0.5\rho_1>K_1 \tag{4.15}$$

and

$$|Ax|^2\le\rho_1|x|^2\quad and\quad |x^TAx|^2\ge\rho_2|x|^4 \tag{4.16}$$

for all $x\in\mathbb{R}^n$, where K_1 is the constant stated in Assumption 2.1.

We should also point out that there are many examples for the square matrix A that fulfils Assumption 4.3. For example, let $G\in\mathbb{R}^{n\times n}$ be a symmetric matrix such that

$$\lambda_{\min}(G)\ge\frac{\sqrt{3}}{2}\|G\|>0.$$

Let $A = \rho G$ for some positive constant ρ. Then

$$|Ax|^2 \le (\rho\|G\|)^2|x|^2 \quad \text{and} \quad |x^T Ax|^2 \ge (\rho\lambda_{\min}(G))^2|x|^4,$$

namely (4.16) holds with $\rho_1 = (\rho\|G\|)^2$ and $\rho_2 = (\rho\lambda_{\min}(G))^2$. Hence

$$\rho_2 - 0.5\rho_1 \ge 0.25\rho^2\|G\|^2.$$

So (4.15) holds as long as we choose $\rho^2 > 4K_1/\|G\|^2$.

To verify Assumption 2.2, let us consider the corresponding semi-linear SDE

$$dy(t) = f(y(t), t)dt + Ay(t)dB(t). \tag{4.17}$$

Define a $C^{2,1}$-function $V(y,t) = |y|^2$ for $(y,t) \in \mathbb{R}^n \times \mathbb{R}_+$. It is then easy to show that

$$LV(y,t) = 2y^T f(y,t) + |Ay|^2 \le (2K_1 + \rho_1)|y|^2$$

and

$$|V_y(y,t)Ay|^2 = 4|y^T Ay|^2 \ge 4\rho_2|y|^4.$$

We hence see that the parameters in Assumption 2.2 are

$$q = 2, \quad \bar{c}_1 = c_1 = 1, \quad c_2 = 2K_1 + \rho_1, \quad c_3 = 4\rho_2. \tag{4.18}$$

By (4.15), we have

$$c_3 - 2c_2 = 4\rho_2 - 2r_1 - 4K_1 = 4(\rho_2 - 0.5\rho_1 - K_1) > 0, \text{ namely } c_3 > 2c_2.$$

In other words, Assumption 2.2 is also satisfied with the parameters defined by (4.18). By Theorem 3.1, we have the following corollary.

Corollary 4.4 *Let Assumption 4.3 hold and f satisfy Assumption 2.1. Then there is a positive number τ^* such that for any initial data $\xi \in L^2_{\mathcal{F}_{t_0}}(\Omega; C([-\tau, 0]; \mathbb{R}^2))$, the solution of the semi-linear SDDE (4.14) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t; t_0, \xi)|) < 0 \quad a.s. \tag{4.19}$$

provided $\tau < \tau^*$. In practice, τ^* can be determined in the same way as described in Theorem 3.1 with $K_2 = \|A\|$ and other parameters being defined by (4.18).

5 Stabilization by Stochastic Delay Feedback Control

Let us first apply our theory to discuss the stabilization problem (1.6). Given an unstable differential equation

$$\dot{x}(t) = f(x(t), t), \tag{5.1}$$

we are required to design a stochastic delay feedback control $g(x(t-\tau), t)dB(t)$ to make the stochastically controlled system

$$dx(t) = f(x(t), t)dt + g(x(t-\tau), t)dB(t) \tag{5.2}$$

become almost surely exponentially stable. This SDDE is a special case of our main SDDE (2.1) with $\delta_1(t) = 0$ and $\delta_2(t) = \tau$. The following theorem on the stability of the SDDE (5.2) follows from Theorem 3.1 immediately.

Theorem 5.1 *Let Assumptions 2.1 and 2.2 hold. Then there is a positive number* τ^* *such that for any initial data* $\xi \in L^2_{\mathcal{F}_{t_0}}(\Omega; C([-\tau, 0]; \mathbb{R}^n))$*, the solution of equation (5.2) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t; t_0, \xi)|) < 0 \quad a.s. \tag{5.3}$$

provided $\tau < \tau^*$*. In practice,* τ^* *can be determined in the same way as described in Theorem 3.1*

Similarly, we can apply Corollary 4.2 to discuss the stabilization problem (1.7). We still consider the given unstable differential equation (5.1). We assume that its coefficient f satisfies Assumption 2.1 so it obeys the linear growth condition (2.3). We therefore look for a linear stochastic delay feedback control $Ax(t-\tau)dB(t)$ to stabilize equation (5.1), where $B(t)$ is a scalar Brownian motion and $A \in \mathbb{R}^{n\times n}$. That is, our stochastically controlled system has the form

$$dx(t) = f(x(t), t)dt + Ax(t-\tau)dB(t). \tag{5.4}$$

Our aim here is to design A and control τ sufficiently small in order for this controlled system to be almost surely exponentially stable. This SDDE is a special case of equation (4.14) with $\delta_1(t) = 0$ and $\delta_2(t) = \tau$. An application of Corollary 4.4 therefore gives the following result immediately.

Corollary 5.2 *Assume that* f *satisfies Assumption 2.1. Design the matrix* A *to satisfy Assumption 4.3. Determine* τ^* *in the same way as described in Theorem 3.1 with* $K_2 = \|A\|$ *and other parameters being defined by (4.18). Control the time lag* $\tau < \tau^*$*. Then the stochastically controlled system (5.4) is almost surely exponentially stable.*

6 Conclusions

In this paper we have investigated the almost sure exponential stability of the multi-dimensional nonlinear SDDEs with variable delays. In particular, our new theory has enabled us to design stochastic delay feedback controls to stabilize unstable differential equations. Although the stochastic stabilization of unstable differential equations have been studied by many authors, all results so far require the stochastic feedback controls depend on the current state $x(t)$. However, it is more realistic in practice if the control depends on a past state, say $x(t-\tau)$, due to a time lag τ between the time when the observation of the state is made and the time when the feedback control reaches the system. We have successfully shown that an unstable differential equation can be stabilized by a stochastic delay feedback control. Our new theory opens a new chapter in the area of stochastic stabilization—*the stabilization of unstable differential equations by stochastic delay feedback controls.*

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References

[1] Appleby, J. and Mao, X., Stochastic stabilization of functional differential equations, *Systems Control Lett.* 54(11) (2005), 1069–1081.

[2] Arnold, L., *Stochastic Differential Equations: Theory and Applications*, John Wiley and Sons, 1974.

[3] Arnold, L., Crauel, H. and Wihstutz, V., Stabilization of linear systems by noise, *SIAM J. Control Optim.* 21 (1983), 451–461.

[4] Friedman, A., *Stochastic Differential Equations and Their Applications*, Academic Press, 1976.

[5] Khasminskii, R.Z., *Stochastic Stability of Differential Equations*, Sijthoff and Noordhoff, 1981.

[6] Kolmanovskii, V.B. and Myshkis, A., *Applied Theory of Functional Differential Equations*, Kluwer Academic Publishers, 1992.

[7] Ladde, G.S. and Lakshmikantham, V., *Random Differential Inequalities*, Academic Press, 1980.

[8] Mao, X., *Stability of Stochastic Differential Equations with Respect to Semimartingales*, Longman Scientific and Technical, 1991.

[9] Mao, X., *Exponential Stability of Stochastic Differential Equations*, Marcel Dekker, 1994.

[10] Mao, X., Stochastic stabilisation and destabilisation, *Systems Control Lett.* 23 (1994), 279–290.

[11] Mao X., *Stochastic Differential Equations and Their Applications*, 2nd Edition, Elsevier, 2007.

[12] Mao, X., Stability and stabilization of stochastic differential delay equations, *IET Control Theory Appl.* 1(6) (2007), 1551–1566.

[13] Mao, X., Almost sure exponential stability in the numerical simulation of stochastic differential equations, *SIAM J. Numer. Anal.* 53(1) (2015), 370–389.

[14] Mao, X., Almost sure exponential stabilization by discrete-time stochastic feedback control, *IEEE Trans. Auto. Control* (2015), DOI: 10.1109/TAC.2015.2471696 (IEEE Early Access Articles).

[15] Mao, X., Lam, J. and Huang, L., Stabilisation of hybrid stochastic differential equations by delay feedback control, *Systems Control Lett.* 57 (2008), 927–935.

[16] Mao, X. and Yuan, C., *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, 2006.

[17] Mao, X., Yin, G. and Yuan, C., Stabilization and destabilization of hybrid systems of stochastic differential equations, *Automatica* 43 (2007), 264–273.

[18] Mohammed, S.-E.A., *Stochastic Functional Differential Equations*, Longman Scientific and Technical, 1986.

[19] Mohammed, S.-E.A. and Scheutzow, M.K.R., Lyapunov exponents of linear stochastic functional differential equations. Part II. Examples and case studies, *Ann. Prob.* 25(3) (1997), 1210–1240.

[20] Scheutzow, M.K.R., Stabilisation and destabilisation by noise in the plane, *Stoch. Anal. Appl.* 11(1) (1993), 97–113.

[21] Scheutzow, M.K.R., Exponential growth rates for stochastic delay differential equations, *Stoch. Dyn.* 5(2) (2005), 163–174.

[22] You, S., Liu, W., Lu, J., Mao, X. and Qiu, Q., Stabilization of hybrid systems by feedback control based on discrete-time state observations, *SIAM J. Control Optim.* 53(2) (2015), 905–925.