# Almost sure stability of discrete-time Markov Jump Linear Systems

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# Almost Sure Stability of Discrete-time Markov Jump Linear Systems

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**Abstract---** This paper deals with transient analysis and almost sure stability for discrete-time Markov Jump Linear System (MJLS). The expectation of sojourn time and activation number of any mode, and switching number between any two modes of discrete-time MJLS are presented firstly. Then a result on transient behavior analysis of discrete-time MJLS is given. Finally a new deterministically testable condition for the exponential almost sure stability of discrete-time MJLS is proposed.

**Keywords**: Markov Jump Linear Systems; Almost sure stability; Average dwell time

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## 1. Introduction

Markov Jump Linear Systems (MJLS) are composed of a set of linear subsystems (also called modes) and a switching sequence governed by a Markov stochastic process. MJLS are extensively used to model physical systems subject to abrupt changes or failures, e.g., fault tolerate systems[1], aerospace systems [2], networked control systems [3, 4], etc. Stability study for stochastic system is of fundamental importance. Several definitions of the stability have been proposed, such as $\delta$-moment stability, mean square stability (MS stability), almost sure stability (AS stability), etc[5-9]. The conservativeness of these definitions is quite different. $\delta$-moment stability requires that the expectation of $\delta$-th moment of state norm $E\left[\|x(t)\|^{\delta}\right]$ should converge to zero asymptotically. When $\delta = 2$, $\delta$-moment stability degenerates to be a special case called MS stability. AS stability, different from $\delta$-moment stability, requires the state trajectory converging to zero with probability one. From the application point of view, the convergence of system state trajectory with probability one is more relevant than that of moment behavior [10]. For Markov Jump Linear System, both MS stability and $\delta$-moment stability imply AS stability but not vice versa [5, 6, 11]. Most results on MJLS are given for the case that the modes are finite. For MJLS with infinite modes, the stability problems are studied in [12, 13]. In recent years, some researchers investigated the stability issue of MJLS for more complex scenarios, e.g. the transition probabilities are partial-known [14] or piecewise-constant [15], the switching between subsystems are governed by a Markovain process and a deterministic dwell time restriction jointly [16, 17].

For a MJLS system, the testabilities of such several stabilities are different. The results on MS stability are generally given in form of coupled Lyapunov equations [2, 6] [18, 19] which can solved effectively by Linear Matrix Inequality (LMI) toolbox. On AS stability, it is difficult to have a general checking technique. A necessary and sufficient condition for AS stability is that the top Lyapunov exponent defined over infinite time should be negative[20]. However, this condition is often hard to be assessed. Therefore, it is necessary to find some practically testable conditions. A sufficient condition for testing the AS stability is proposed for stochastic linear systems [10] [21] and discrete-time MJLS [22], which is based on the average norm contractivity of state transition matrix over a finite, yet unknown, time interval. The sufficient condition in [10] [22] [21] is non-deterministic, although it is less restrictive compared with top Lyapunov exponent method. Furthermore a solving technique based on Monte Carlo algorithm is developed to check this conditions [10] [21]. Recently, new deterministic sufficient conditions are presented on AS stability of continuous-time MJLS [7, 17, 23]. These sufficient conditions are obtained by investigating the statistics of switching actions and the total sojourn time of each mode in a MJLS. For the discrete-time MJLS, to the best of authors' knowledge, there is no equivalent result reported. This paper aims at bridging this gap. The main contributions of this paper consist in three aspects: firstly, the expectation of sojourn time and activation number of any mode, and the switching number between any two modes for discrete-time MJLS are given for the first time; secondly, a result on transient analysis of discrete-time MJLS is presented; thirdly, a new approach to test AS stability for discrete-time MJLS is obtained.

The paper is structured as follows. In section 2 some preliminaries and definitions are given. In section 3, the expectation of switching number between any two modes, and the total sojourn time and activation number of each mode are provided. In Section 4, a transient analysis result and an AS stability condition for discrete-time MJLS are proposed. Section 5 gives two examples and Section 6 concludes the paper.

## 2. Preliminaries and Definitions

Consider a discrete-time Markov Jump Linear System

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}_{\sigma(k)}\boldsymbol{x}_k \quad , \quad k \in Z^+ \tag{1}$$

where $Z^+$ is a positive integer set, state $\boldsymbol{x}_k \in \mathbb{R}^n$, the switching sequence $\left\{\sigma(k),\ k \in Z^+\right\}$ is a Markov chain, taking values on a finite set $\{1, 2, \cdots, N\}$, $N$ is the number of modes, $f_i = \Pr\{\sigma(0) = i\}$ is the initial provability distribution of MJLS. Markov chain $\{\sigma(k)\}$ in this paper is assumed to be irreducible and aperiodic, therefore it is ergodic and has a unique invariant distribution $\boldsymbol{\pi} = \left[\pi_1 \quad \pi_2 \quad \cdots \quad \pi_N\right]$, which can be calculated by

$$\begin{cases} \pi_j = \sum_{i=1}^{N} \pi_i p_{ij} \\ \sum_{j=1}^{N} \pi_j = 1 \end{cases} \tag{2}$$

**Definition 1** MJLS system (1) is said to be exponentially almost sure stable, if there exist $\rho > 0$ such that for any $\boldsymbol{x}_0 \in \mathbb{R}^n$ and any initial distribution $\sigma(0)$,

$$\Pr\left\{\limsup_{k\to\infty} \frac{1}{k} \ln \|\boldsymbol{x}_k\| \le -\rho\right\} = 1 \tag{3}$$

## 3. Sojourn Time and Switching Number of a Markov Chain

In this paper, the accumulate time of a MJLS sojourns in mode $j$ and the total number of switching from mode $i$ to mode $j$ in interval $(0,k]$ are denoted as $T_j^{(k)}$ and $n_{ij}^{(k)}$ respectively. The number of activations of mode $j$ in interval $(0,k]$ is denoted as $\overline{n}_j^{(k)}$, the probability that mode $j$ is active at instant $n$ is noted as $\overline{p}_j^{(n)}$. Stochastic variable $S_j^{(n)}$ is defined as

$$S_j^{(n)} = \begin{cases} 1, if \text{ mode } j \text{ is active at instant n} \\ 0, if \text{ any other mode is active at instant n} \end{cases}$$

The expectation $E\left[\cdot \,\middle|\, \Pr\{\sigma(0) = i\} = f_i, i = 1, 2, \cdots, N\right]$ is denoted as $E_{\boldsymbol{F}}[\cdot]$, similarly, $E_{\boldsymbol{\pi}}[\cdot] := E\left[\cdot \,\middle|\, \Pr\{\sigma(0) = i\} = \pi_i, i = 1, 2, \cdots, N\right]$.

**Proposition 1** Given a Markov chain $\{\sigma(k),\ k \in Z^+\}$ with $N$ modes, the following are satisfied for all modes $i$, $j$ ,

$$E_{\boldsymbol{F}}\left[T_j^{(k)}\right] = \pi_j k + \boldsymbol{c}^T\left(\sum_{i=1}^{k} \boldsymbol{P}^{i-1}\right)\boldsymbol{p_j} \tag{4}$$

$$E_{\boldsymbol{F}}\left[n_{ij}^{(k)}\right] = \pi_i p_{ij} k + p_{ij}\boldsymbol{c}^{\mathrm{T}}\left(\sum_{i=1}^{k} \boldsymbol{P}^{i-1}\right)\boldsymbol{p_i} + c_i p_{ij} \quad , \qquad i \neq j \tag{5}$$

$$E_{\boldsymbol{F}}\left[\overline{n}_j^{(k)}\right] = \left(1 - p_{jj}\right)\left[\pi_j k + \boldsymbol{c}^T\left(\sum_{i=1}^{k} \boldsymbol{P}^{i-1}\right)\boldsymbol{p_j}\right] + p_{jj}\left(\pi_j + \boldsymbol{c}^T \boldsymbol{P}^{k-1}\boldsymbol{p_j}\right) \tag{6}$$

where $\boldsymbol{c}^T = \left[c_1 \quad \cdots \quad c_n\right]$, $c_i = f_i - \pi_i$, $\boldsymbol{p_j}$ is the *j*-th column of transition matrix $\boldsymbol{P}$ .

**Proof:** Since $\overline{p}_j^{(n)} = \Pr\left\{S_j^{(n)} = 1\right\}$, this gives

$$E_{\boldsymbol{F}}\left[T_j^{(k)}\right] = E_F\left[S_j^{(1)} + S_j^{(2)} + \cdots + S_j^{(k)}\right] = \sum_{n=1}^{k} \overline{p}_j^{(n)} \tag{7}$$

and

$$\begin{aligned} \overline{p}_j^{(n)} &= \sum_{i=1}^{N} \overline{p}_i^{(n-1)} p_{ij}, \\ \overline{p}_j^{(n-1)} &= \sum_{i=1}^{N} \overline{p}_i^{(n-2)} p_{ij}, \\ &\vdots \\ \overline{p}_j^{(1)} &= \sum_{i=1}^{N} f_i p_{ij} = \sum_{i=1}^{N} \left(\pi_i + c_i\right) p_{ij} \end{aligned} \tag{8}$$

where $c_i = f_i - \pi_i$.

From Eq.(2), it follows that

$$\overline{p}_j^{(1)} = \sum_{i=1}^{N} \left(\pi_i + c_i\right) p_{ij} = \pi_j + \sum_{i=1}^{N} c_i p_{ij} = \pi_j + c^T \boldsymbol{p_j},$$

$$\overline{p}_j^{(2)} = \sum_{i=1}^{N} \overline{p}_i^{(1)} p_{ij} = \sum_{i=1}^{N} \left(\pi_i + c^T p_i\right) p_{ij} = \pi_j + \sum_{i=1}^{N} c^T p_i p_{ij} = \pi_j + \boldsymbol{c}^T \boldsymbol{P}\boldsymbol{p_j}$$

$$\overline{p}_j^{(3)} = \sum_{i=1}^{N} \overline{p}_i^{(2)} p_{ij} = \sum_{i=1}^{N} \left(\pi_i + \boldsymbol{c}^{\boldsymbol{T}} \boldsymbol{P} p_i\right) p_{ij} = \pi_j + \sum_{i=1}^{N} \boldsymbol{c}^{\boldsymbol{T}} \boldsymbol{P} p_i p_{ij} = \pi_j + \boldsymbol{c}^{\boldsymbol{T}} \boldsymbol{P}^{\boldsymbol{2}} \boldsymbol{p_j}$$

$$\vdots$$

$$\overline{p}_j^{(n)} = \sum_{i=1}^{N} \overline{p}_i^{(n-1)} p_{ij} = \pi_j + \boldsymbol{c}^{\boldsymbol{T}} \boldsymbol{P}^{\boldsymbol{n-1}} \boldsymbol{p_j}$$

where $\boldsymbol{c}^T = \begin{bmatrix} c_1 & c_2 & \cdots & c_N \end{bmatrix}$, $\boldsymbol{p}_j$ is the *j*-th column of transition matrix $P$.

Setting $\boldsymbol{P}^0 = I$, $\overline{p}_j^{(0)} = f_i$, it follows that

$$E_{\boldsymbol{F}}\left[T_j^{(k)}\right] = \sum_{n=1}^{k} \overline{p}_j^{(n)} = \pi_j k + \boldsymbol{c}^{\boldsymbol{T}} \left( \sum_{i=1}^{k} \boldsymbol{P}^{\boldsymbol{i-1}} \right) \boldsymbol{p}_{\boldsymbol{j}} \tag{9}$$

and the total number of switching from mode $i$ to mode $j$ in interval $(0, k]$ is

$$E_{\boldsymbol{F}}\left[n_{ij}^{(k)}\right] = \sum_{n=0}^{k-1} \overline{p}_i^{(n)} p_{ij} = f_i p_{ij} + \sum_{n=1}^{k-1} \overline{p}_i^{(n)} p_{ij}$$

$$= \pi_i p_{ij} k + p_{ij} \boldsymbol{c}^T \left( \sum_{i=1}^{k-1} \boldsymbol{P}^{i-1} \right) \boldsymbol{p}_{\boldsymbol{i}} + c_i p_{ij} \tag{10}$$

On the other hand, if $\sigma(n) = j, \sigma(n+1) \neq j$, then we call $n$ is a *switching-out point* of mode *j*. The total probability of *switching-out point* of mode *j* occurs during the interval $(0, k]$ is $\sum_{n=1}^{k-1} p_j^{(n)} \left(1 - p_{jj}\right)$. The probability that mode *j* is the latest mode being activated in $(0, k]$ is $p_j^{(k)}$. Therefore, the number of activations of mode *i* is

$$E_{\boldsymbol{F}}\left[\overline{n}_j^{(k)}\right] = \sum_{n=1}^{k-1} p_j^{(n)} \left(1 - p_{jj}\right) + p_j^{(k)}$$

$$= \left(1 - p_{jj}\right) \left[ \pi_j k + \boldsymbol{c}^T \left( \sum_{i=1}^{k} \boldsymbol{P}^{i-1} \right) \boldsymbol{p}_{\boldsymbol{j}} \right] + p_{jj} \left( \pi_j + \boldsymbol{c}^{\boldsymbol{T}} \boldsymbol{P}^{\boldsymbol{k-1}} \boldsymbol{p}_{\boldsymbol{j}} \right)$$

This completes the proof. □

**Remark 1** Proposition 1 gives the expectations of the sojourn time and switching number of MJLS along with time axis. It can be seen that these expectations are related with initial probability distribution. When initial probability distribution $\boldsymbol{F}$ equals with the unique invariant distribution, the following corollary can be obtained.

**Corollary 1** If the initial probability distribution equals to the unique invariant distribution of Markov chain, then

$$E_{\boldsymbol{\pi}}\left[T_j^{(k)}\right] = \pi_j k \tag{11}$$

$$E_{\boldsymbol{\pi}}\left[n_{ij}^{(k)}\right] = \pi_i p_{ij} k \tag{12}$$

$$E_{\boldsymbol{\pi}}\left[\overline{n}_j^{(k)}\right] = \pi_j \left(1 - p_{jj}\right) k + p_{jj} \pi_j \tag{13}$$

## 4. Almost Sure Stability of MJLS

**Lemma 1**[22] MJLS system (1) is exponential almost sure stable if and only if $E_{\boldsymbol{\pi}}[\lambda] < 0$, where the top Lyapunov exponent is defined as $\lambda = \lim_{k\to\infty} \frac{\ln\left\|\boldsymbol{A}_{\sigma(k)} \cdots \boldsymbol{A}_{\sigma(0)}\right\|}{k}$

We first show a result of the upper bound for $E\ln\|\boldsymbol{x}_{\boldsymbol{k}}\|$ along with time axis if the initial probability distribution $\boldsymbol{F}$ is known.

**Theorem 1** Consider MJLS system (1) with initial distribution $\boldsymbol{F}$, then expectation of $\ln\|\boldsymbol{x}_{\boldsymbol{k}}\|$ satisfies

$$E\ln\|\boldsymbol{x}_{\boldsymbol{k}}\| < kg_1 + g_2(k) + g_3 + E\ln\|\boldsymbol{x}_0\| \tag{14}$$

where

$$\begin{cases} g_1 = \dfrac{1}{2}\left(\displaystyle\sum_{\substack{i=1,j=1\\ i\neq j}}^{N} \pi_i p_{ij} \ln \mu_{ij} + \sum_{j=1}^{N} \pi_j \ln \lambda_j\right) \\ g_2(k) = \dfrac{1}{2}\displaystyle\sum_{\substack{i=1,j=1\\ i\neq j}}^{N} p_{ij} \boldsymbol{c}^{\mathrm{T}} \left(\sum_{i=1}^{k} \boldsymbol{P}^{i-1}\right) \boldsymbol{p}_i \ln \mu_{ij} + \frac{1}{2}\sum_{j=1}^{N} \boldsymbol{c}^{T} \left(\sum_{i=1}^{k} \boldsymbol{P}^{i-1}\right) \boldsymbol{p}_{\boldsymbol{j}} \ln \lambda_j \\ \qquad\qquad + \dfrac{1}{2}\ln\displaystyle\sum_{i=1}^{N} \frac{\pi_i + \boldsymbol{c}^{\boldsymbol{T}} \boldsymbol{P}^{\boldsymbol{k-1}} \boldsymbol{p}_{\boldsymbol{i}}}{\lambda_{\min}\left(Q_i\right)} \\ g_3 = \dfrac{1}{2}\ln\displaystyle\sum_{i=1}^{N} f_i \lambda_{\max}\left(Q_i\right) + \frac{1}{2}\sum_{\substack{i=1,j=1\\ i\neq j}}^{N} c_i p_{ij} \ln \mu_{ij} \end{cases} \tag{15}$$

And the parameters $\mu_{ij}$, $\lambda_j$ and $\boldsymbol{Q}_i > 0$ satisfies,

$$\begin{cases} \boldsymbol{A}_i^T \boldsymbol{Q}_i \boldsymbol{A}_i < \lambda_i \boldsymbol{Q}_i \\ \boldsymbol{Q}_j \le \mu_{ij} \boldsymbol{Q}_i \end{cases} \tag{16}$$

**Proof**: Choose Lyapunov function

$$V(k) = \boldsymbol{x}_k{}^T \boldsymbol{Q}_{\sigma(k)} \boldsymbol{x}_k \tag{17}$$

Consider a switching sequence as

$$\cdots \sigma(s_{l-1}-1) \ne \sigma(s_{l-1}) = \sigma(s_{l-1}+1) = \cdots = \sigma(s_l - 1) \ne \sigma(s_l) = \cdots = \sigma(k-1) = \sigma(k) \tag{18}$$

where $s_l, s_{l-1}, \cdots,$ denote switching instants, here, $s_l$ denotes the latest switching instant in $(0, k]$. Then it follows from Eq.(16) that

$$V(k) = \boldsymbol{x}_k{}^T \boldsymbol{Q}_{\sigma(k)} \boldsymbol{x}_k = \boldsymbol{x}_{k-1}^T \boldsymbol{A}_{\sigma(k-1)}^T \boldsymbol{Q}_{\sigma(k-1)} \boldsymbol{A}_{\sigma(k-1)} \boldsymbol{x}_{k-1} < \lambda_{\sigma(k-1)} V(k-1) \tag{19}$$

Due to $\sigma(k-1) = \sigma(k) = \sigma(s_l)$, hence $\lambda_{\sigma(k-1)} V(k-1)$ can be rewritten as $\lambda_{\sigma(s_l)} V(s_l)$.

Observing that there is no switching during $[s_l, k]$, therefore,

$$V(k) < \lambda_{\sigma(s_l)} V(k-1) < \lambda_{\sigma(s_l)}^2 V(k-2) < \cdots < \lambda_{\sigma(s_l)}^{k-s_l} V(s_l) \tag{20}$$

Then from $\sigma(s_{l-1}-1) \ne \sigma(s_{l-1}) = \sigma(s_{l-1}+1) = \cdots = \sigma(s_l-1) \ne \sigma(s_l)$ and Eq.(16), it follows

$$\begin{aligned} V(s_l) &= \boldsymbol{x}_{s_l}^T \boldsymbol{Q}_{\sigma(s_l)} \boldsymbol{x}_{s_l} \le \mu_{\sigma(s_l-1)\sigma(s_l)} \boldsymbol{x}_{s_l}^T \boldsymbol{Q}_{\sigma(s_l-1)} \boldsymbol{x}_{s_l} \\ &= \mu_{\sigma(s_l-1)\sigma(s_l)} \boldsymbol{x}_{s_l-1}^T \boldsymbol{A}_{\sigma(s_l-1)}^T \boldsymbol{Q}_{\sigma(s_l-1)} \boldsymbol{A}_{\sigma(s_l-1)} \boldsymbol{x}_{s_l-1} \\ &< \mu_{\sigma(s_l-1)\sigma(s_l)} \lambda_{\sigma(s_l-1)} V(s_l-1) \end{aligned}$$

Following a similar procedure, it leads to,

$$\begin{aligned} V(k) &< \lambda_{\sigma(s_l)}^{k-s_l} V(s_l) \\ &< \lambda_{\sigma(s_l)}^{k-s_l} \mu_{\sigma(s_l-1)\sigma(s_l)} \lambda_{\sigma(s_l-1)} V(s_l-1) \\ &< \lambda_{\sigma(s_l)}^{k-s_l} \mu_{\sigma(s_l-1)\sigma(s_l)} \lambda_{\sigma(s_{l-1})}^{s_l - s_{l-1}} V(s_{l-1}) \\ &\vdots \\ &\le \left( \prod_{\substack{i=1, j=1 \\ i \ne j}}^{N} \mu_{ij}^{n_{ij}^{(k)}} \right) \prod_{j=1}^{N} \left( \lambda_j^{T_j^{(k)}} \right) V(0) \end{aligned} \tag{21}$$

Denote $q(k)=\left(\prod_{\substack{i=1,j=1\\ i\neq j}}^{N}\mu_{ij}^{n_{ij}^{(k)}}\right)\prod_{j=1}^{N}\left(\lambda_j^{T_j^{(k)}}\right)$， then Eq.(21) can be simplified as

$$V\left(k\right)<q(k)V\left(0\right) \tag{22}$$

On the other hand, for any time instant k and any initial condition $x_0$, we have

$$\lambda_{\min}\left(\boldsymbol{Q}_{\sigma(k)}\right)\boldsymbol{x}_k^{\ T}\boldsymbol{x}_k<\boldsymbol{x}_k^{\ T}\boldsymbol{Q}_{\sigma(k)}\boldsymbol{x}_k=V\left(k\right)<q(k)V\left(0\right)\leq q(k)\lambda_{\max}\left(\boldsymbol{Q}_{\sigma(0)}\right)\boldsymbol{x}_0^{\ T}\boldsymbol{x}_0 \tag{23}$$

Then it can be seen that

$$\begin{aligned}E\left[\ln\frac{\|\boldsymbol{x}_k\|}{\|\boldsymbol{x}_0\|}\right]&<E\left[\ln\sqrt{q(k)\frac{\lambda_{\max}\left(\boldsymbol{Q}_{\sigma(0)}\right)}{\lambda_{\min}\left(\boldsymbol{Q}_{\sigma(k)}\right)}}\right]\\&<\frac{1}{2}\left(E\ln q(k)+E\ln\lambda_{\max}\left(\boldsymbol{Q}_{\sigma(0)}\right)+E\ln\frac{1}{\lambda_{\min}\left(\boldsymbol{Q}_{\sigma(k)}\right)}\right)\end{aligned} \tag{24}$$

Noticing that $f(x)=\ln(x)$ is a concave function, based on Jansen's inequality, it leads to

$$E\ln\lambda_{\max}\left(\boldsymbol{Q}_{\sigma(0)}\right)\leq\ln E\lambda_{\max}\left(\boldsymbol{Q}_{\sigma(0)}\right)=\ln\sum_{i=1}^{N}f_i\lambda_{\max}\left(\boldsymbol{Q}_i\right) \tag{25}$$

$$E\ln\frac{1}{\lambda_{\min}\left(\boldsymbol{Q}_{\sigma(k)}\right)}\leq\ln E\frac{1}{\lambda_{\min}\left(\boldsymbol{Q}_{\sigma(k)}\right)}=\ln\sum_{i=1}^{N}\frac{\overline{p}_i^{(k)}}{\lambda_{\min}\left(\boldsymbol{Q}_i\right)} \tag{26}$$

And

$$E\ln q(k)=\sum_{\substack{i=1,j=1\\ i\neq j}}^{N}\left(E_F\left[n_{ij}^{(k)}\right]\ln\mu_{ij}\right)+\sum_{j=1}^{N}\left(E_F\left[T_j^{(k)}\right]\ln\lambda_j\right) \tag{27}$$

where $\overline{p}_i^{(k)}$ is the probability of that mode *i* is active at instant *k*.

Applying $\overline{p}_i^{(k)}=\pi_i+\boldsymbol{c}^T\boldsymbol{P}^{k-1}\boldsymbol{p_i}$ into Eq.(26),

$$E\ln\frac{1}{\lambda_{\min}\left(\boldsymbol{Q}_{\sigma(k)}\right)}\leq\ln\sum_{i=1}^{N}\frac{\pi_i+\boldsymbol{c}^T\boldsymbol{P}^{k-1}\boldsymbol{p_i}}{\lambda_{\min}\left(\boldsymbol{Q}_i\right)} \tag{28}$$

Hence

$$E\left[\ln\frac{\|\boldsymbol{x}_k\|}{\|\boldsymbol{x}_0\|}\right] < \frac{1}{2}\left(\begin{array}{r}\sum_{\substack{i=1,j=1\\ i\neq j}}^{N}\left(E_{\boldsymbol{F}}\left[n_{ij}^{(k)}\right]\ln\mu_{ij}\right)+\sum_{j=1}^{N}\left(E_{\boldsymbol{F}}\left[T_j^{(k)}\right]\ln\lambda_j\right)\\ +\ln\sum_{i=1}^{N}f_i\lambda_{\max}\left(\boldsymbol{Q}_i\right)+\ln\sum_{i=1}^{N}\frac{\pi_i+\boldsymbol{c}^T\boldsymbol{P}^{k-1}\boldsymbol{p}_i}{\lambda_{\min}\left(\boldsymbol{Q}_i\right)}\end{array}\right) \tag{29}$$

Substituting Eq.(4) and (5) into Eq.(29) leads to Eq.(14).

This completes the proof. □

Based on theorem 1, the following result on AS stability of discrete time MJLS can be obtained.

**Theorem 2** Consider MJLS system (1), if there exist a set of $\boldsymbol{Q}_i > 0$, scalars $\lambda_i$, $\mu_{ij}$, such that Eq.(30)~(32) hold, then the system is exponential almost sure stable.

$$\boldsymbol{A}_i^T\boldsymbol{Q}_i\boldsymbol{A}_i - \lambda_i\boldsymbol{Q}_i < 0 \ , \tag{30}$$

$$\boldsymbol{Q}_j \le \mu_{ij}\boldsymbol{Q}_i \,, \tag{31}$$

$$\sum_{\substack{i=1,j=1\\ i\neq j}}^{N}\pi_i p_{ij}\ln\mu_{ij} + \sum_{i=1}^{N}\pi_i\ln\lambda_i < 0 \tag{32}$$

where $i,j=1,2,\cdots,N,\ i\neq j$.

**Proof:** Due to $\lambda_{\max}\left(\boldsymbol{Q}_{\sigma(0)}\right)$ and $\lambda_{\min}\left(\boldsymbol{Q}_{\sigma(k)}\right)$ are bounded, it follows from condition (33)( 34) and Eq. (243) that

$$E\left[\lim_{k\to\infty}\frac{\ln\max_{x_0\neq 0}\frac{\|\boldsymbol{x}_k\|}{\|\boldsymbol{x}_0\|}}{k}\right] < \frac{1}{2}E\left[\lim_{k\to\infty}\frac{\ln q(k)+\ln\frac{\lambda_{\max}\left(\boldsymbol{Q}_{\sigma(0)}\right)}{\lambda_{\min}\left(\boldsymbol{Q}_{\sigma(k)}\right)}}{k}\right] = \frac{1}{2}E\lim_{k\to\infty}\frac{\ln q(k)}{k} \tag{35}$$

Since $\left\|\boldsymbol{A}_{\sigma(k)}\cdots\boldsymbol{A}_{\sigma(0)}\right\| = \max_{x_0\neq 0}\frac{\left\|\boldsymbol{A}_{\sigma(k)}\cdots\boldsymbol{A}_{\sigma(0)}\boldsymbol{x}_0\right\|}{\|\boldsymbol{x}_0\|} = \max_{x_0\neq 0}\frac{\|\boldsymbol{x}_k\|}{\|\boldsymbol{x}_0\|}$ and MJLS (1) is ergodic, substitute the expression of $q(k)$ into Eq. (36),

$$E\left[\lim_{k\to\infty}\frac{\ln\left\|\boldsymbol{A}_{\sigma(k)}\cdots\boldsymbol{A}_{\sigma(0)}\right\|}{k}\right]=E\left[\lim_{k\to\infty}\frac{\ln\max\limits_{x_0\neq 0}\frac{\|\boldsymbol{x}_k\|}{\|\boldsymbol{x}_0\|}}{k}\right]<\frac{1}{2}E\lim_{k\to\infty}\frac{\ln q(k)}{k}$$

$$=\frac{1}{2}E\lim_{k\to\infty}\frac{\sum\limits_{\substack{i=1,j=1\\ i\neq j}}^{N}\left(E_\pi\left[n_{ij}^{(k)}\right]\ln\mu_{ij}\right)+\sum\limits_{j=1}^{N}\left(E_\pi\left[T_j^{(k)}\right]\ln\lambda_j\right)}{k} \quad (37)$$

Substituting Eq. (38) and (39) into Eq. (37) , and from condition (32) , it yields

$$E\left[\lim_{k\to\infty}\frac{\ln\left\|\boldsymbol{A}_{\sigma(k)}\cdots\boldsymbol{A}_{\sigma(0)}\right\|}{k}\right]<\frac{1}{2}\left(\sum_{\substack{i=1,j=1\\ i\neq j}}^{N}\pi_i p_{ij}\ln\mu_{ij}+\sum_{i=1}^{N}\pi_i\ln\lambda_i\right)<0$$

Based on Lemma 1, MJLS (1) is exponential almost sure stable.

This completes the proof. □

**Remark 2:** If $\boldsymbol{Q}_1=\boldsymbol{Q}_2=\cdots=\boldsymbol{Q}_N=\boldsymbol{I}$, then it is clear that we can choose $\mu_{ij}=1$ for any $i\neq j$. In the case Theorem 2 will degenerate to Theorem 2.1 in [24]. Therefore Theorem 2 can be regarded as a more general result based on a more flexible multiple Lyapunov Function method whereas Theorem 2.1 in [24] is based on a common Lyapunov Function method.

## 5. Numerical Example

Example 1: This example is to demonstrate Proposition 1.

Consider a two-mode Markov chain with transition matrix $\boldsymbol{P}=\begin{bmatrix}0.7 & 0.3\\ 0.6 & 0.4\end{bmatrix}$, the initial distribution is $\boldsymbol{F}=\begin{bmatrix}0.2 & 0.8\end{bmatrix}$. The unique invariant distribution is $\boldsymbol{\pi}=\left(\frac{p_{21}}{p_{12}+p_{21}},\frac{p_{12}}{p_{12}+p_{21}}\right)=\left(\frac{2}{3},\frac{1}{3}\right)$. Compute $E_{\boldsymbol{F}}\left[T_j^{(k)}\right]$ and $E_{\boldsymbol{F}}\left[n_{ij}^{(k)}\right]$ by using 3000 samples of the Markov chain. Table 1 and 2 shows the computation values as well as the theory value obtained from Proposition 1. As expected, the two sets of values are consistent.

Table .1 Expectations of sojourn time

| $E_F\left[T_1^{(k)}\right]$ | k=1 | k=2 | k=3 | k=4 | k=5 | k=6 | k=7 | k=8 | k=9 | k=10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Computation Value | 0.6267 | 1.2697 | 1.9150 | 2.5667 | 3.2527 | 3.8957 | 4.5323 | 5.2710 | 5.9660 | 6.6420 |
| Theory Value | 0.6200 | 1.2820 | 1.9482 | 2.6148 | 3.2815 | 3.9481 | 4.6148 | 5.2815 | 5.9481 | 6.6148 |

Table .2 Expectations of switching number

| $E_F\left[n_{12}^{(k)}\right]$ | k=2 | k=3 | k=4 | k=5 | k=6 | k=7 | k=8 | k=9 | k=10 |
|---|---|---|---|---|---|---|---|---|---|
| Computation Value | 0.1823 | 0.3860 | 0.5827 | 0.7823 | 0.9647 | 1.1937 | 1.3950 | 1.5750 | 1.9817 |
| Theory Value | 0.1860 | 0.3846 | 0.5845 | 0.7844 | 0.9844 | 1.1844 | 1.3844 | 1.5844 | 1.9844 |

Example 2: This example is to demonstrate Theorem 2. Consider a MJLS

$$\boldsymbol{x}_{k+1}=\boldsymbol{A}_{\sigma(k)}\boldsymbol{x}_k, \sigma(k)=1,2$$

where $\boldsymbol{A}_1=\begin{bmatrix}0.2 & 1\\ 0 & 0.2\end{bmatrix}$, $\boldsymbol{A}_2=\begin{bmatrix}0.9 & 0.4\\ 0.5 & 0.2\end{bmatrix}$, transition probability $\boldsymbol{P}=\begin{bmatrix}0.6 & 0.4\\ 0.1 & 0.9\end{bmatrix}$.

It is proved that this MJLS is not mean square stable [21].

By using Theorem 2.1 and Theorem 2.2 in [24], we can get

Theorem 2.1: $\left[\lambda_{\max}\left(A_1^T\times A_1\right)\right]^{p_1}\left[\lambda_{\max}\left(A_2^T\times A_2\right)\right]^{p_2}=1.0785^{0.2}\times 1.2597^{0.8}>1$

Theorem 2.2: $\|A_1\|_2^{p_1}\|A_2\|_2^{p_2}=1.1051>1$

It violates the condition of such two Theorems, therefore we cannot judge the AS stability of this MJLS by [24] .

By using Theorem 2 of this paper, we choose $\lambda_1$=0.2, $\lambda_2$=1.25 ; $\mu_{12}$=6.6, $\mu_{21}$=0.9 and

$$Q_1 = \begin{bmatrix} 18.9798 & -13.1455 \\ -13.1455 & 157.9363 \end{bmatrix},\ Q_2 = \begin{bmatrix} 124.8681 & -87.1653 \\ -87.1653 & 226.7459 \end{bmatrix}.$$

Then all conditions in Theorem 2 are satisfied. Therefore this MJLS is AS stable.

Fig.1 illustrates nine realizations of $\|x_k\|$ starting from the initial state $[3,-1]^{\mathrm{T}}$. One can see that the MJLS is exponential almost sure stable.

INSERT FIG.1 HERE

Fig.1 nine realizations of $\|x_k\|$ for example 2

## 6. Conclusions

In this paper, the transient and steady characteristics of discrete-time Markov jump linear systems are investigated. The expectations of sojourn time, activation number and switching number are given. Based on the above results, a theorem on transient analysis of discrete-time MJLS is then presented. After that a new deterministic sufficient condition on exponential almost sure stability of discrete-time MJLS is proposed in form of matrix inequalities. Finally, two numerical examples are provided to demonstrate the effectiveness of the proposed results.

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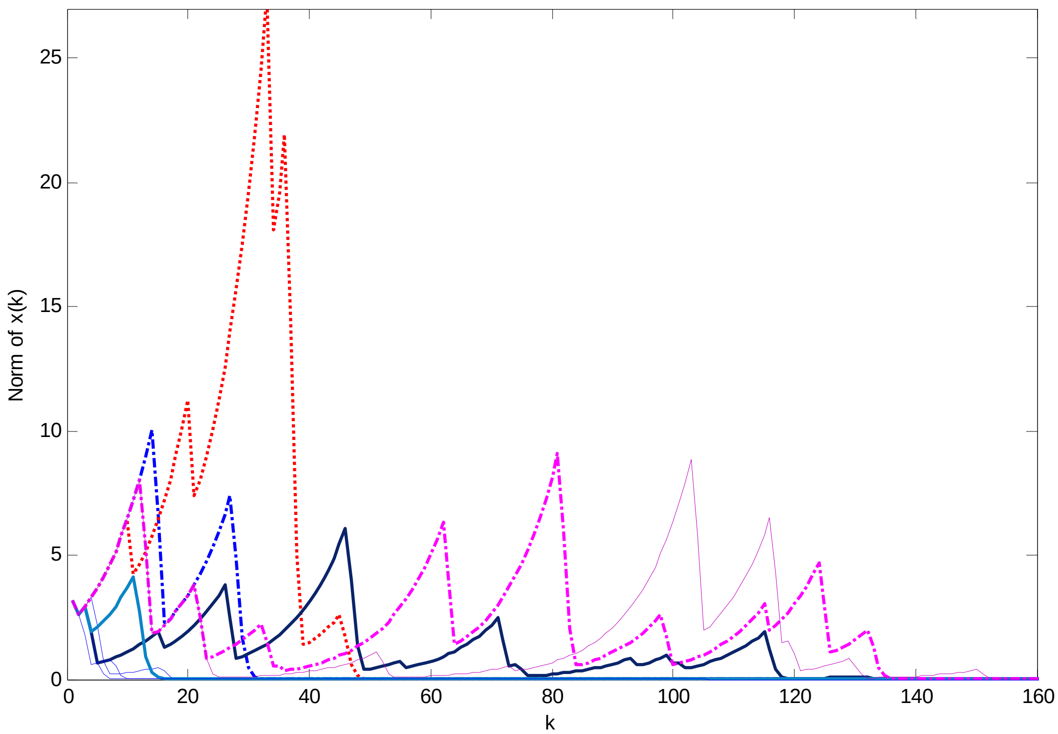


Fig.1 nine realizations of $\left\|\boldsymbol{x}_k\right\|$ for example 2