



# MIT Open Access Articles

*Almost sure well-posedness for the periodic 3D quintic nonlinear Schrödinger equation below the energy space*

The MIT Faculty has made this article openly available. **Please share** how this access benefits you. Your story matters.

<b>Citation</b>	Nahmod, Andrea and Gigliola Staffilani. "Almost Sure Well-Posedness for the Periodic 3D Quintic Nonlinear Schrödinger Equation Below the Energy Space." Journal of the European Mathematical Society 17, 7 (2015): 1687–1759 © 2015 European Mathematical Society
<b>As Published</b>	<a href="http://dx.doi.org/10.4171/JEMS/543">http://dx.doi.org/10.4171/JEMS/543</a>
<b>Publisher</b>	European Mathematical Publishing House
<b>Version</b>	Author's final manuscript
<b>Citable link</b>	<a href="http://hdl.handle.net/1721.1/115983">http://hdl.handle.net/1721.1/115983</a>
<b>Terms of Use</b>	Creative Commons Attribution-Noncommercial-Share Alike
<b>Detailed Terms</b>	<a href="http://creativecommons.org/licenses/by-nc-sa/4.0/">http://creativecommons.org/licenses/by-nc-sa/4.0/</a>

# ALMOST SURE WELL-POSEDNESS FOR THE PERIODIC 3D QUINTIC NONLINEAR SCHRÖDINGER EQUATION BELOW THE ENERGY SPACE

ANDREA R. NAHMOD\* AND GIGLIOLA STAFFILANI†

ABSTRACT. In this paper we prove an almost sure local well-posedness result for the periodic 3D quintic nonlinear Schrödinger equation in the supercritical regime, that is below the critical space  $H^1(\mathbb{T}^3)$ .

## 1. INTRODUCTION

In this paper we continue the study of almost sure well-posedness for certain dispersive equations in a supercritical regime. In the last two decades there has been a burst of activity -and significant progress- in the field of nonlinear dispersive equations and systems. These range from the development of analytic tools in nonlinear Fourier and harmonic analysis combined with geometric ideas to address nonlinear estimates, to related deep functional analytic methods and profile decompositions to study local and global well-posedness and singularity formation for these equations and systems. The thrust of this body of work has focused mostly on deterministic aspects of wave phenomena where sophisticated tools from nonlinear Fourier analysis, geometry and analytic number theory have played a crucial role in the methods employed. Building upon work by Bourgain [1, 2, 4] several works have appeared in which the well-posedness theory has been studied from a nondeterministic point of view relying on and adapting probabilistic ideas and tools as well (c.f. [11, 12, 34, 28, 29, 35, 26, 27, 32, 17, 25, 15, 18, 19] and references therein).

It is by now well understood that randomness plays a fundamental role in a variety of fields. Situations when such a point of view is desirable include: when there still remains a gap between local and global well-posedness when certain type of ill-posedness is present, and in the very important super-critical regime when a deterministic well-posedness theory remains, in general, a major open problem in the field. A set of important and tractable problems is concerned with those (scaling) equations for which global well posedness for *large data* is known at the *critical scaling level*. Of special interest is the case when the scale-invariant regularity  $s_c = 1$  (energy or Hamiltonian). A natural question then is that of understanding the *supercritical* (relative to scaling) long time dynamics for the nonlinear Schrödinger equation in the defocusing case. Whether blow up occurs from classical data in the defocussing case remains a difficult open problem in the subject. However, what seems within reach at this time is to investigate and seek an answer to these problems *from a nondeterministic viewpoint*; namely for *random data*.

In this paper we treat the energy-critical periodic quintic nonlinear Schrödinger equation (NLS), an especially important prototype in view of the results by Herr, Tzvetkov and Tataru [23] establishing small data global well posedness in  $H^1(\mathbb{T}^3)$  and of Ionescu and

---

\* The first author is funded in part by NSF DMS 1201443.

† The second author is funded in part by NSF DMS 1068815.

Pausader [24] proving *large data* global well posedness in  $H^1(\mathbb{T}^3)$  in the defocusing case, the first critical result for NLS on a compact manifold. Large data global well-posedness in  $\mathbb{R}^3$  for the energy-critical quintic NLS had been previously established by Colliander, Keel, Staffilani, Takaoka and Tao in [16].

Our interest in this paper is to establish a local almost sure well posedness for random data *below*  $H^1(\mathbb{T}^3)$ ; that is, in the supercritical regime relative to scaling<sup>1</sup> and without any kind of symmetry restriction on the data. In a seminal paper, Bourgain [4] obtained almost sure global well posedness for the 2D periodic defocusing (Wick ordered) cubic NLS with data *below*  $L^2(\mathbb{T}^2)$ , ie. in a supercritical regime ( $s_c = 0$ )<sup>2</sup>. Burq and Tzvetkov obtained similar results for the supercritical ( $s_c = \frac{1}{2}$ ) *radial* cubic NLW on compact Riemannian manifolds in 3D. Both global results rely on the existence and invariance of associated Gibbs measures. As it turns out, in many situations either the natural Gibbs or weighted Wiener construction does not produce an invariant measure or (and this is particularly so in higher dimensions) it is thought to be impossible to make any reasonable construction at all. In the case of the ball or the sphere, if one restricts to the radial case then constructions of invariant measures are possible as in [35, 12, 20, 21, 6, 7, 8]. Recently, a probabilistic method based on energy estimates has been used to obtain supercritical almost sure global results, thus circumventing the use of invariant measures and the restriction of radial symmetry. In this context Burq and Tzvetkov [13] and Burq, Thomann and Tzvetkov [14] considered the periodic cubic NLW, while Nahmod, Pavlovic and Staffilani [25] treated the periodic Navier-Stokes equations. Colliander and Oh [17] also proved almost sure global well-posedness for the *subcritical* 1D periodic cubic NLS below  $L^2$  in the absence of invariant measures by suitably adapting Bourgain's high-low method.

Extending the local solutions we obtain here to global ones is the next natural step; it is worth noting however that unlike the work of Bourgain [4] one would need to proceed in the absence of invariant measures; and unlike the work of Colliander and Oh [17] the smoother norm in our case, namely  $H^1(\mathbb{T}^3)$ , on which one would need to rest to extend the local theory to a global one is in fact *critical*. This forces the bounds on the Strichartz type norms to be of exponential type with respect to the energy, too large to be able to close the argument.

The problem we are considering here is the analogue of the supercritical local well-posedness result<sup>3</sup> proved by Bourgain in [4] for the periodic mass critical defocusing cubic NLS in 2D. Of course, Bourgain then constructed a 2D Gibbs measure and proved that for data in its statistical ensemble the local solutions obtained were in fact global, hence establishing almost sure global well posedness in  $H^{-\epsilon}(\mathbb{T}^2)$ ,  $\epsilon > 0$ .

There are several major complications in the work that we present below compared to the work of Bourgain: a quintic nonlinearity increases quite substantially the different cases that need to be treated when one analyzes the frequency interactions in the nonlinearity; the counting lemmata in a 3D lattice are much less favorable and the Wick ordering is not sufficient to remove certain resonant frequencies that need to be eliminated. The latter is not surprising, and in fact known within the context of quantum field renormalization (c.f. Salmhofer's book [30]). In particular, to overcome this last obstacle, we introduce an appropriate gauge transformation, we work on the gauged problem and then transfer the obtained result back to the original problem; which as a consequence is studied through

---

<sup>1</sup>i.e. for Cauchy data in  $H^s(\mathbb{T}^3)$ ,  $s < s_c = 1$  for the quintic NLS in 3D

<sup>2</sup>See Brydges and Slade [9] for a study of invariant measures associated to the 2D *focussing* cubic NLS.

<sup>3</sup>a.s for data in  $H^{-\beta}(\mathbb{T}^2)$ ,  $\beta > 0$

a contraction method applied in a certain metric space of functions. A similar approach was used by the second author in [31]. Finally our estimates take place in function spaces where we must be careful about working with the absolute value of the Fourier transform. In fact the norms of these spaces are not defined through the absolute value of the Fourier transform, a property of the  $X^{s,b}$  spaces in [4] which is quite useful, see for example Section 8.

In this work we consider the Cauchy initial value problem,

$$(1.1) \quad \begin{cases} iu_t + \Delta u = \lambda u|u|^4 & x \in \mathbb{T}^3 \\ u(0, x) = \phi(x) \end{cases}$$

where  $\lambda = \pm 1$ .

We randomize the data in the following sense,

**Definition 1.1.** Let  $(g_n(\omega))_{n \in \mathbb{Z}^3}$  be a sequence of complex i.i.d centered Gaussian random variables on a probability space  $(\Omega, A, \mathbb{P})$ . For  $\phi \in H^s(\mathbb{T}^3)$ , let  $(b_n)$  be its Fourier coefficients, that is

$$(1.2) \quad \phi(x) = \sum_{n \in \mathbb{Z}^3} b_n e^{in \cdot x}, \quad \sum_{n \in \mathbb{Z}^3} (1 + |n|)^{2s} |b_n|^2 < \infty$$

The map from  $(\Omega, A)$  to  $H^s(\mathbb{T}^3)$  equipped with the Borel sigma algebra, defined by

$$(1.3) \quad \omega \longrightarrow \phi^\omega, \quad \phi^\omega(x) = \sum_{n \in \mathbb{Z}^3} g_n(\omega) b_n e^{in \cdot x}$$

is called a map randomization.

**Remark 1.2.** The map (1.3) is measurable and  $\phi^\omega \in L^2(\Omega; H^s(\mathbb{T}^d))$ , is an  $H^s(\mathbb{T}^d)$ -valued random variable. We recall that such a randomization does not introduce any  $H^s$  regularization (see Lemma B.1 in [11] for a proof of this fact), indeed  $\|\phi^\omega\|_{H^s} \sim \|\phi\|_{H^s}$ . However randomization gives improved  $L^p$  estimates almost surely.

Our setting to show almost sure local well posedness is similar to that of Bourgain in [4]. More precisely, we consider data  $\phi \in H^{1-\alpha-\varepsilon}(\mathbb{T}^3)$  for any  $\varepsilon > 0$  of the form

$$(1.4) \quad \phi(x) = \sum_{n \in \mathbb{Z}^3} \frac{1}{\langle n \rangle^{\frac{5}{2}-\alpha}} e^{in \cdot x}$$

whose randomization is

$$(1.5) \quad \phi^\omega(x) = \sum_{n \in \mathbb{Z}^3} \frac{g_n(\omega)}{\langle n \rangle^{\frac{5}{2}-\alpha}} e^{in \cdot x}$$

Our main result can then be stated as follows,

**Theorem 1.3** (Main Theorem). Let  $0 < \alpha < \frac{1}{12}$ ,  $s \in (1 + 4\alpha, \frac{3}{2} - 2\alpha)$  and  $\phi$  as in (1.4). Then there exists  $0 < \delta_0 \ll 1$  and  $r = r(s, \alpha) > 0$  such that for any  $\delta < \delta_0$ , there exists  $\Omega_\delta \in A$  with

$$\mathbb{P}(\Omega_\delta^c) < e^{-\frac{1}{\delta^r}},$$

and for each  $\omega \in \Omega_\delta$  there exists a unique solution  $u$  of (1.1) in the space

$$S(t)\phi^\omega + X^s([0, \delta])_d,$$

where  $S(t)\phi^\omega$  is the linear evolution of the initial data  $\phi^\omega$  given by (1.5).

Here we denoted by  $X^s([0, \delta])_d$  the metric space  $(X^s([0, \delta]), d)$  where  $d$  is the metric defined by (2.21) in Section 2 and  $X^s([0, \delta])$  is the space introduced in Definition 4.4 below.

**Acknowledgement.** The authors would like to thank the Radcliffe Institute for Advanced Study at Harvard University for its hospitality during the final stages of this work. They also thank Luc Rey-Bellet for several helpful conversations.

## 2. REMOVING RESONANT FREQUENCIES: THE GAUGED EQUATION

The main idea in proving Theorem 1.3 goes back to Bourgain [4] and it consists on proving that if  $u$  solves (1.1), then  $w = u - S(t)\phi^\omega$  is smoother; see also [11, 17, 25]. In fact one reduces the problem to showing well-posedness for the initial value problem involving  $w$ , which is in fact treated as a deterministic function. The initial value problem that  $w$  solves does not become a subcritical one, but it is of a hybrid type involving also rougher but random terms, whose decay and moments play a fundamental role. For the NLS equation this argument can be carried out only after having removed certain resonant frequencies in the nonlinear part of the equation. In this section in fact we write the Fourier coefficients of the quintic expression  $|u|^4u$  and we identify the resonant part that needs to be removed in order to be able to take advantage of the moments coming from the randomized terms. We will go back to this concept more in details in Remark 2.1 below.

Let's start by assuming that  $\widehat{u}(n)(t) = a_n(t)$ . We introduce the notation

$$(2.1) \quad \Gamma(n)_{[i_1, i_2, \dots, i_r]} := \{(n_{i_1}, \dots, n_{i_r}) \in \mathbb{Z}^{3r} / n = n_{i_1} - n_{i_2} + \dots + (-1)^{r+1}n_{i_r}\}$$

to indicate various hyperplanes and  $\Gamma(n)_{[i_1, i_2, \dots, i_r]}^c$  is its complement.

Next, for fixed time  $t$ , we take  $\mathcal{F}$ , the Fourier transform in space, and write,

$$\begin{aligned} \mathcal{F}(|u(t)|^4u(t))(n) &= \sum_{\Gamma(n)_{[1, \dots, 5]}} a_{n_1}(t)\overline{a_{n_2}(t)}a_{n_3}(t)\overline{a_{n_4}(t)}a_{n_5}(t) \\ &= \sum_{\Gamma(n)_{[1, \dots, 5]} \cap \Gamma(0)_{[1, 2, 3, 4]}^c \cap \Gamma(0)_{[1, 2, 5, 4]}^c \cap \Gamma(0)_{[3, 2, 5, 4]}^c} a_{n_1}(t)\overline{a_{n_2}(t)}a_{n_3}(t)\overline{a_{n_4}(t)}a_{n_5}(t) \\ &+ \sum_{\Gamma(n)_{[1, \dots, 5]} \cap \Gamma(0)_{[1, 2, 3, 4]}} a_{n_1}(t)\overline{a_{n_2}(t)}a_{n_3}(t)\overline{a_{n_4}(t)}a_{n_5}(t) \\ &+ \sum_{\Gamma(n)_{[1, \dots, 5]} \cap \Gamma(0)_{[1, 2, 5, 4]}} a_{n_1}(t)\overline{a_{n_2}(t)}a_{n_3}(t)\overline{a_{n_4}(t)}a_{n_5}(t) \\ &+ \sum_{\Gamma(n)_{[1, \dots, 5]} \cap \Gamma(0)_{[3, 2, 5, 4]}} a_{n_1}(t)\overline{a_{n_2}(t)}a_{n_3}(t)\overline{a_{n_4}(t)}a_{n_5}(t) \\ &- \sum_{\Gamma(n)_{[1, \dots, 5]} \cap \Gamma(0)_{[1, 2, 3, 4]} \cap \Gamma(0)_{[1, 2, 5, 4]} \cap \Gamma(0)_{[3, 2, 5, 4]}^c} a_{n_1}(t)\overline{a_{n_2}(t)}a_{n_3}(t)\overline{a_{n_4}(t)}a_{n_5}(t) \\ &- \sum_{\Gamma(n)_{[1, \dots, 5]} \cap \Gamma(0)_{[1, 2, 3, 4]} \cap \Gamma(0)_{[3, 2, 5, 4]} \cap \Gamma(0)_{[1, 2, 5, 4]}^c} a_{n_1}(t)\overline{a_{n_2}(t)}a_{n_3}(t)\overline{a_{n_4}(t)}a_{n_5}(t) \\ &- \sum_{\Gamma(n)_{[1, \dots, 5]} \cap \Gamma(0)_{[3, 2, 5, 4]} \cap \Gamma(0)_{[1, 2, 5, 4]} \cap \Gamma(0)_{[1, 2, 3, 4]}^c} a_{n_1}(t)\overline{a_{n_2}(t)}a_{n_3}(t)\overline{a_{n_4}(t)}a_{n_5}(t) \end{aligned}$$

$$\begin{aligned}
& - 2 \sum_{\Gamma(n)_{[1, \dots, 5]} \cap \Gamma(0)_{[1, 2, 3, 4]} \cap \Gamma(0)_{[3, 2, 5, 4]} \cap \Gamma(0)_{[1, 2, 5, 4]}} a_{n_1}(t) \overline{a_{n_2}}(t) a_{n_3}(t) \overline{a_{n_4}}(t) a_{n_5}(t) \\
& = \sum_{k=1, \dots, 8} I_k,
\end{aligned}$$

We now rewrite each  $I_k$  using more explicitly the constraints in the hyperplanes.  $I_1$  is the most complicated and we start by rewriting it. To that effect we introduce the following notation:

$$(2.2) \quad \Lambda(n) := \Gamma(n)_{[1, \dots, 5]} \cap \Gamma(0)_{[1, 2, 3, 4]}^C \cap \Gamma(0)_{[1, 2, 5, 4]}^C \cap \Gamma(0)_{[3, 2, 5, 4]}^C$$

$$(2.3) \quad \Sigma(n) := \{(n_1, n_2, n_3, n_4, n_5) \in \Lambda(n) / n_1, n_3, n_5 \neq n_2, n_4\}.$$

We have

$$\begin{aligned}
(2.4) \quad I_1 & = \sum_{\Lambda(n)} a_{n_1}(t) \overline{a_{n_2}}(t) a_{n_3}(t) \overline{a_{n_4}}(t) a_{n_5}(t) \\
& = \sum_{\Sigma(n)} a_{n_1}(t) \overline{a_{n_2}}(t) a_{n_3}(t) \overline{a_{n_4}}(t) a_{n_5}(t) \\
& \quad + 6 \left( \sum_{n_2} |a_{n_2}|^2 \right) \sum_{\Gamma(n)_{[3, 4, 5]}, n_3, n_5 \neq n_4} a_{n_3}(t) \overline{a_{n_4}}(t) a_{n_5}(t) \\
& \quad - 6 |a_n|^2 \sum_{\Gamma(n)_{[3, 4, 5]}, n_3, n_5 \neq n_4} a_{n_3}(t) \overline{a_{n_4}}(t) a_{n_5}(t) \\
& \quad - 3 \sum_{\Gamma(n)_{[3, 1, 5]}, n_3, n_5 \neq n_1} |a_{n_1}(t)|^2 \overline{a_{n_1}}(t) a_{n_3}(t) a_{n_5}(t) \\
& \quad - 3 |a_n|^4 a_n(t) + 3 |a_n|^2 \overline{a_n}(t) \sum_{n_3 + n_5 = 2n} a_{n_3}(t) a_{n_5}(t) \\
& \quad - 6 \sum_{\Gamma(n)_{[2, 4, 5]}, n_2, n_5 \neq n_4} |a_{n_2}(t)|^2 a_{n_2}(t) \overline{a_{n_4}}(t) a_{n_5}(t) \\
& \quad + 2 \sum_{n=2n_2 - n_4, n_2 \neq n_4} |a_{n_2}(t)|^2 a_{n_2}^2(t) \overline{a_{n_4}}(t)
\end{aligned}$$

Note here that we can write

$$\begin{aligned}
(2.5) \quad |a_n|^2 \sum_{\Gamma(n)_{[3, 4, 5]}, n_3, n_5 \neq n_4} a_{n_3}(t) \overline{a_{n_4}}(t) a_{n_5}(t) & = -2 |a_n|^2 a_n \left( \sum_{n_2} |a_{n_2}|^2 \right) + |a_n|^4 a_n \\
& \quad + |a_n|^2 \sum_{\Gamma(n)_{[3, 4, 5]}} a_{n_3}(t) \overline{a_{n_4}}(t) a_{n_5}(t).
\end{aligned}$$

It is easier to see that for  $i = 2, 3, 4$

$$(2.6) \quad I_i = a_n(t) \sum_{\Gamma(0)_{[1, 2, 3, 4]}} a_{n_1}(t) \overline{a_{n_2}}(t) a_{n_3}(t) \overline{a_{n_4}}(t) = \hat{u}(n)(t) \int_{\mathbb{T}^3} |u|^4(x, t) dx,$$

while for  $j = 5, 6, 7$

$$(2.7) \quad I_j = -a_n^3(t) \sum_{n_2 + n_4 = 2n} \overline{a_{n_2}}(t) \overline{a_{n_4}}(t) + a_n^2 \sum_{n=n_2 + n_4 - n_1} \overline{a_{n_2}}(t) \overline{a_{n_4}}(t) a_{n_1}(t)$$

and for

$$(2.8) \quad I_8 = -2a_n^3(t) \sum_{n_2+n_4=2n} \overline{a_{n_2}(t)} \overline{a_{n_4}(t)}.$$

We summarize our findings from (2.4)- (2.8). In this part of the argument the time variable is not important, hence we will omit it for now. We write

$$(2.9) \quad \mathcal{F} \left( |u|^4 u - 3u \left( \int_{\mathbb{T}^3} |u|^4 dx \right) \right) (n) = \sum_{k=1}^7 J_k(a_n)$$

with

$$(2.10) \quad J_1 = \sum_{\Sigma(n)} a_{n_1} \overline{a_{n_2}} a_{n_3} \overline{a_{n_4}} a_{n_5}$$

$$(2.11) \quad J_2 = 6m \sum_{\Gamma(n)_{[1,2,3], n_3, n_1 \neq n_2}} a_{n_1} \overline{a_{n_2}} a_{n_3}$$

$$(2.12) \quad J_3 = -6 \sum_{\Gamma(n)_{[1,2,3], n_1, n_3 \neq n_2}} |a_{n_1}|^2 a_{n_1} \overline{a_{n_2}} a_{n_3} \\ -3 \sum_{\Gamma(n)_{[1,2,3], n_1, n_3, \neq n_2}} a_{n_1} |a_{n_2}|^2 \overline{a_{n_2}} a_{n_3}$$

$$(2.13) \quad J_4 = 2 \sum_{n=2n_1-n_2} |a_{n_1}|^2 a_{n_1}^2 \overline{a_{n_2}}$$

$$(2.14) \quad J_5 = -6|a_n|^2 \sum_{\Gamma(n)_{[123]}} a_{n_1} \overline{a_{n_2}} a_{n_3} + 3a_n^2 \sum_{\Gamma(n)_{[214]}} \overline{a_{n_2}} a_{n_1} \overline{a_{n_4}}$$

$$(2.15) \quad J_6 = -5a_n^3 \sum_{n=n_2+n_4} \overline{a_{n_2}} \overline{a_{n_4}} + 3|a_n|^2 \overline{a_n} \sum_{n=n_1+n_3} a_{n_1} a_{n_3}$$

$$(2.16) \quad J_7 = -11a_n |a_n|^4 + 12m |a_n|^2 a_n,$$

where  $m = \int_{\mathbb{T}^3} |u(t, x)|^2 dx$ , the conserved mass.

**Remark 2.1.** *In the calculations above we wrote the nonlinear terms in (1.1) in Fourier space, we isolated the term  $u \int_{\mathbb{T}^3} |u|^4 dx$  and we subtracted it from  $|u|^4 u$ , see (2.9). We show below that indeed in doing so we separated those terms that we claim are not suitable for smoother estimates from the ones that are. To understand this point let us replace  $a_n = \frac{g_n(\omega)}{\langle n \rangle^{\frac{3}{2}-\alpha}}$ , for  $\alpha$  small, whose anti-Fourier transform barely misses to be in  $H^1(\mathbb{T}^3)$ . We want to claim that the randomness coming from  $\{g_n(\omega)\}$  will increase the regularity of the nonlinearity in a certain sense, so that it can hold a bit more than one derivative. We realize immediately though that this claim cannot be true for the whole nonlinear term. For example the terms  $I_i$ ,  $i = 2, 3, 4$  have no chance to improve their regularity because they are simply linear with respect to  $a_n$ , hence they need to be removed. This same problem presented itself in the work of Bourgain [4] and Colliander-Oh [17] who considered the cubic NLS below  $L^2$ . In particular in their case the problematic term was of the type  $a_n \int_{\mathbb{T}^d} |u|^2 dx$  and the authors removed it by Wick ordering the Hamiltonian. An important ingredient in making this successful was that  $\int_{\mathbb{T}^d} |u|^2 dx$ , that is the mass, is independent of time. In our case Wick ordering the Hamiltonian is not helpful since it does not remove the terms  $I_i$ ,  $i = 2, 3, 4$ . As we mentioned before, the latter is not surprising, and in fact known within the context of quantum field renormalization (c.f. Salmhofer's book [30]).*

If we knew that  $\int_{\mathbb{T}^3} |u|^4 dx$  were constant in time, then we could simply relegate those terms to the linear part of the equation. Since this is obviously not the case relegating these expressions with the main linear part of the equation would prevent us from using the simple form of the solution for a Schrödinger equation with constant coefficients. A similar situation to the one just described presented itself in [31] where a gauge transformation was used to remove the time dependent linear terms. We are able to use the same idea in this context and this is the content of what follows in this section.

To prove Main Theorem 1.1 we proceed in two steps. First we consider the initial value problem

$$(2.17) \quad \begin{cases} iv_t + \Delta v = \mathcal{N}(v) & x \in \mathbb{T}^3 \\ v(0, x) = \phi(x), \end{cases}$$

where

$$(2.18) \quad \mathcal{N}(v) := \lambda \left( v|v|^4 - 3v \left( \int_{\mathbb{T}^3} |v|^4 dx \right) \right)$$

with  $\lambda = \pm 1$  and  $\phi(x)$  the initial datum as in (1.1). To make the notation simpler set

$$(2.19) \quad \beta_v(t) = 3 \int_{\mathbb{T}^3} |v|^4 dx$$

and define

$$(2.20) \quad u(t, x) := e^{i\lambda \int_0^t \beta_v(s) ds} v(t, x).$$

We observe that  $u$  solves the initial value problem (1.1). Now suppose that one obtains well-posedness for the initial value problem (2.17) in a certain Banach space  $(X, \|\cdot\|)$  then one can transfer those results to the initial value problem (1.1) by using a metric space  $X_d := (X, d)$  where

$$(2.21) \quad d(u, v) := \|e^{-i\lambda \int_0^t \beta_u(s) ds} u(t, x) - e^{-i\lambda \int_0^t \beta_v(s) ds} v(t, x)\|.$$

The fact that this is indeed a metric follows from using the properties of the norm  $\|\cdot\|$  and the fact that if

$$e^{-i\lambda \int_0^t \beta_u(s) ds} u(t, x) = e^{-i\lambda \int_0^t \beta_v(s) ds} v(t, x)$$

then  $\beta_v(t) = \beta_u(t)$  and hence  $u = v$ .

From this moment on we work exclusively with the initial value problem (2.17). In particular below we prove the following result:

**Theorem 2.2.** *Let  $0 < \alpha < \frac{1}{12}$ ,  $s \in (1 + 4\alpha, \frac{3}{2} - 2\alpha)$  and  $\phi$  as in (1.4). There exists  $0 < \delta_0 \ll 1$  and  $r = r(s, \alpha) > 0$  such that for any  $\delta < \delta_0$ , there exists  $\Omega_\delta \in A$  with*

$$\mathbb{P}(\Omega_\delta^c) < e^{-\frac{1}{\delta^r}},$$

and for each  $\omega \in \Omega_\delta$  there exists a unique solution  $u$  of (2.17) in the space

$$S(t)\phi^\omega + X^s([0, \delta]),$$

with initial condition  $\phi^\omega$  given by (1.5).

Here in the space  $X^s([0, \delta])$  is defined in Section 4.

Thanks to the transformation (2.20), Theorem 2.2 translates to Main Theorem 1.3.



## 3. PROBABILISTIC SET UP

We first recall a classical result that goes back to Kolmogorov, Paley and Zygmund.

**Lemma 3.1** (Lemma 3.1 [11]). *Let  $\{g_n(\omega)\}$  be a sequence of complex i.i.d. zero mean Gaussian random variables on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  and  $(c_n) \in \ell^2$ . Define*

$$(3.1) \quad F(\omega) := \sum_n c_n g_n(\omega)$$

Then, there exists  $C > 0$  such that for every  $\lambda > 0$  we have

$$(3.2) \quad \mathbb{P}(\{\omega : |F(\omega)| > \lambda\}) \leq \exp\left(\frac{-C\lambda^2}{\|F(\omega)\|_{L^2(\Omega)}^2}\right).$$

As a consequence there exists  $C > 0$  such that for every  $q \geq 2$  and every  $(c_n)_n \in \ell^2$ ,

$$\left\| \sum_n c_n g_n(\omega) \right\|_{L^q(\Omega)} \leq C\sqrt{q} \left( \sum_n c_n^2 \right)^{\frac{1}{2}}.$$

We also recall the following basic probability results:

**Lemma 3.2.** *Let  $1 \leq m_1 < m_2 \cdots < m_k = m$ ;  $f_1$  be a Borel measurable function of  $m_1$  variables,  $f_2$  one of  $m_2 - m_1$  variables,  $\dots$ ,  $f_k$  one of  $m_k - m_{k-1}$  variables. If  $\{X_1, X_2, \dots, X_m\}$  are real-valued independent random variables, then the  $k$  random variables  $f_1(X_1, \dots, X_{m_1}), f_2(X_{m_1+1}, \dots, X_{m_2}), \dots, f_k(X_{m_{k-1}+1}, \dots, X_{m_k})$  are independent random variables.*

**Lemma 3.3.** *Let  $k \geq 1$  and consider  $\{g_{n_j}\}_{1 \leq j \leq k}$  and  $\{g_{n'_j}\}_{1 \leq j \leq k} \in \mathcal{N}_{\mathbb{C}}(0, 1)$  complex  $L^2(\Omega)$ -normalized independent Gaussian random variables such that  $n_i \neq n_j$  and  $n'_i \neq n'_j$  for  $i \neq j$ .*

$$\left| \int_{\Omega} \prod_{j=1}^k g_{n_j}(\omega) \prod_{i=1}^k \bar{g}_{n'_i}(\omega) dp(\omega) \right| \leq \int_{\Omega} \prod_{\ell=1}^k |g_{n_\ell}(\omega)|^2 dp(\omega).$$

*Proof.* For every pair  $(n_\ell, n'_i)$  such that  $n_\ell = n'_i$  we write  $K_{n_j}(\omega) := |g_{n_j}(\omega)|^2$  and note that thanks to the independence and normalization of  $\{g_{n_j}\}$ , for  $n_j \neq n_i$ , we have that  $\mathbb{E}(K_{n_j} g_{n_i}) = 0$ . The latter together with Lemma 3.2 give the desired conclusion.  $\square$

More generally, in the next sections we will repeatedly use a classical Fernique or large deviation-type result related to the product of  $\{G_n\}_{1 \leq n \leq d} \in \mathcal{N}_{\mathbb{C}}(0, 1)$ , complex  $L^2$  normalized independent Gaussians. This result follows from the hyper-contractivity property of the Ornstein-Uhlenbeck semigroup (c.f. [35, 33] for a nice exposition) by writing  $G_n = H_n + iL_n$  where  $\{H_1, \dots, H_d, L_1, \dots, L_d\} \in \mathcal{N}_{\mathbb{R}}(0, 1)$  are real centered independent Gaussian random variables with the same variance. Note that  $\mathbb{E}(G_n^2) = \mathbb{E}(G_n) = 0$ .

**Remark 3.4.** *Note that for  $\{G_n(\omega)\}_n \in \mathcal{N}_{\mathbb{C}}(0, 1)$ , complex  $L^2$  normalized independent Gaussians, if we write  $|G_n(\omega)|^2 = (|G_n(\omega)|^2 - 1) + 1$ , then thanks to the independence and normalization of  $G_n$ ,  $Y_n(\omega) := |G_n(\omega)|^2 - 1$  is a centered real Gaussian random variable such that for  $n \neq n'$ ,  $\mathbb{E}(Y_n G_{n'}) = 0 = \mathbb{E}(Y_n Y_{n'})$ .*

**Proposition 3.5** (Propositions 2.4 in [33] and Lemma 4.5 in [35]). *Let  $d \geq 1$  and  $c(n_1, \dots, n_k) \in \mathbb{C}$ . Let  $\{G_n\}_{1 \leq n \leq d} \in \mathcal{N}_{\mathbb{C}}(0, 1)$  be complex centered  $L^2$  normalized independent Gaussians. For  $k \geq 1$  denote by  $A(k, d) := \{(n_1, \dots, n_k) \in \{1, \dots, d\}^k, n_1 \leq \dots \leq n_k\}$  and*

$$(3.3) \quad F_k(\omega) = \sum_{A(k,d)} c(n_1, \dots, n_k) G_{n_1}(\omega) \dots G_{n_k}(\omega).$$

Then for all  $d \geq 1$  and  $p \geq 2$

$$\|F_k\|_{L^p(\Omega)} \lesssim \sqrt{k+1} (p-1)^{\frac{k}{2}} \|F_k\|_{L^2(\Omega)}.$$

As a consequence from Chebyshev's inequality we have that for every  $\lambda > 0$

$$(3.4) \quad \mathbb{P}(\{\omega : |F_k(\omega)| > \lambda\}) \lesssim \exp\left(\frac{-C \lambda^{\frac{2}{k}}}{\|F(\omega)\|_{L^2(\Omega)}^{\frac{2}{k}}}\right).$$

**Remark 3.6.** *In Sections 7 and 8 we will rely repeatedly on Proposition 3.5, particularly (3.4), as well as Lemma 3.1, and (3.2). Indeed, in proving our estimates we will encounter expressions of the following type:*

Let  $\Sigma := \{(n_1, \dots, n_r, \ell_1, \dots, \ell_s) : |n_j| \sim N_j, |\ell_i| \sim L_i, n_j \neq \ell_i, 1 \leq j \leq r, 1 \leq i \leq s\}$  and

$$F(\omega) := \sum_{(n_1, \dots, n_r, \ell_1, \dots, \ell_s) \in \Sigma} c_{n_1} \dots c_{n_r} b_{\ell_1} \dots b_{\ell_s} g_{n_1}(\omega) \dots g_{n_r}(\omega) \bar{g}_{\ell_1}(\omega) \dots \bar{g}_{\ell_s}(\omega),$$

where  $\{g_{n_1}(\omega) \dots g_{n_r}, g_{\ell_1}(\omega) \dots g_{\ell_s}(\omega)\} \in \mathcal{N}_{\mathbb{C}}(0, 1)$  are complex centered  $L^2$  normalized independent Gaussians. Then by Proposition 3.5, there exist  $C > 0, \gamma = \gamma(r, s) > 0$  such that for every  $\lambda > 0$  we have

$$\mathbb{P}(\{\omega : |F(\omega)| > \lambda\}) \leq \exp\left(\frac{-C \lambda^{\frac{2}{\gamma}}}{\|F(\omega)\|_{L^2(\Omega)}^{\frac{2}{\gamma}}}\right).$$

We will also apply Proposition 3.5 in the context of Remark 3.4.

**Lemma 3.7.** *Let  $\{g_n(\omega)\}$  be a sequence of complex i.i.d zero mean Gaussian random variables on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . Then,*

- (1) For  $1 \leq p < \infty$  there exists  $c_p > 0$  (independent of  $n$ ) such that  $\|g_n\|_{L^p(\Omega)} \leq c_p$ .
- (2) Given  $\varepsilon, \delta > 0$ , for  $N$  large and  $\omega$  outside of a set of measure  $\delta$ ,

$$(3.5) \quad \sup_{|n| \geq N} |g_n(\omega)| \leq N^\varepsilon.$$

- (3) Given  $\varepsilon, \delta > 0$  and  $\omega$  outside of a set of measure  $\delta$ ,

$$(3.6) \quad |g_n(\omega)| \lesssim \langle n \rangle^\varepsilon$$

*Proof.* Part (1) follows from the fact that higher moments of  $\{g_n(\omega)\}$  are uniformly bounded.

For part (2) first recall that if  $\{X_j(\omega)\}_{j \geq 1}$  is a sequence of i.i.d random variables such that  $\mathbb{E}(|X_j|) = E < \infty$  then

$$(3.7) \quad \mathbb{P}(|X_j| \geq j) = \mathbb{P}(|X_1| \geq j)$$

and

$$\sum_j \mathbb{P}(|X_j| \geq j) = \sum_j \mathbb{P}(|X_1| \geq j) \leq \mathbb{E}(|X_1|) < \infty.$$

By Borel-Cantelli  $\mathbb{P}(|X_j| \geq j \text{ for infinitely many } j) = 0$  whence one can show that  $\lim_{j \rightarrow \infty} \frac{|X_j(\omega)|}{j} = 0$  almost surely in  $\omega$ . Egoroff's Theorem then ensures that given  $\delta > 0$

$$\lim_{j \rightarrow \infty} \frac{|X_j(\omega)|}{j} = 0$$

uniformly outside a set of measure  $\delta$ . Thus we have that for  $j_0$  sufficiently large

$$\frac{|X_j(\omega)|}{j} \leq 1 \quad j \geq j_0,$$

for  $\omega$  outside an exceptional set of  $\delta$  measure.

If  $\{g_n(\omega)\}$  are a sequence of i.i.d. complex Gaussian random variables then given  $\varepsilon > 0$ , if we choose  $r = \frac{1}{\varepsilon}$  then  $\mathbb{E}(|g_n|^r) < \infty$ . By applying the argument above with  $X_n(\omega) = |g_n(\omega)|^r$  we have the desired conclusion (cf. [28, 17])

For part (3) fix  $M \gg 1$  such that (2) holds for any  $|n| \geq M$ . By (3.7)

$$\mathbb{P}(|g_n(\omega)| \geq M^\varepsilon) = \mathbb{P}(|g_M(\omega)| \geq M^\varepsilon)$$

for all  $|n| \leq M$ . Let  $\mathcal{A} := \cup_{|n| \leq M-1} \{\omega / |g_M(\omega)| \geq M^\varepsilon\}$ , then by part (2)  $\mathbb{P}(\mathcal{A}) \leq C_M \delta$ . Hence by choosing a smaller  $\delta$  in part (2) we have the desired result.  $\square$

#### 4. FUNCTION SPACES

For the purpose of establishing our almost sure local well-posedness result, it suffices to work with  $X^s$  and  $Y^s$ , the atomic function spaces used by Herr, Tataru and Tzvetkov [23]. It is worth emphasizing that while working with these spaces, one should not rely on the notion of the norms depending on the absolute value of the Fourier transform, a feature that is quite useful when working within the context of  $X^{s,b}$  spaces.

In this section we recall their definition and summarize the main properties by following the presentation in [23] Section 2. In what follows,  $\mathcal{H}$  is a separable Hilbert space on  $\mathbb{C}$  and  $\mathcal{Z}$  denotes the set of finite partitions  $-\infty < t_0 < t_1 < \dots < t_K \leq \infty$  of the real line; with the convention that if  $t_k = \infty$  then  $v(t_K) := 0$  for any function  $v : \mathbb{R} \rightarrow \mathcal{H}$ .

**Definition 4.1** (Definition 2.1 in [23]). *Let  $1 \leq p < \infty$ . For  $\{t_k\}_{k=0}^K \in \mathcal{Z}$  and  $\{\phi_k\}_{k=0}^{K-1} \subset \mathcal{H}$  with  $\sum_{k=0}^{K-1} \|\phi_k\|_{\mathcal{H}}^p = 1$ . A  $U^p$ -atom is a piecewise defined function  $a : \mathbb{R} \rightarrow \mathcal{H}$  of the form*

$$a = \sum_{k=1}^K \chi_{[t_{k-1}, t_k)} \phi_{k-1}.$$

The atomic Banach space  $U^p(\mathbb{R}, \mathcal{H})$  is then defined to be the set of all functions  $u : \mathbb{R} \rightarrow \mathcal{H}$  such that

$$u = \sum_{j=1}^{\infty} \lambda_j a_j, \quad \text{for } U^p \text{ atoms } a_j, \quad \{\lambda_j\}_j \in \ell^1,$$

with the norm

$$\|u\|_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j, \quad \lambda_j \in \mathbb{C}, \text{ and } a_j \text{ an } U^p \text{ atom} \right\}$$

Here  $\chi_I$  denotes the indicator function over the set  $I$ . Note that for  $1 \leq p \leq q < \infty$ ,

$$(4.1) \quad U^p(\mathbb{R}, \mathcal{H}) \hookrightarrow U^q(\mathbb{R}, \mathcal{H}) \hookrightarrow L^\infty(\mathbb{R}, \mathcal{H}),$$

and functions in  $U^p(\mathbb{R}, \mathcal{H})$  are right continuous,  $\lim_{t \rightarrow -\infty} u(t) = 0$ .

**Definition 4.2** (Definition 2.2 in [23]). *Let  $1 \leq p < \infty$ . The Banach space  $V^p(\mathbb{R}, \mathcal{H})$  is defined to be the set of all functions  $v : \mathbb{R} \rightarrow \mathcal{H}$  such that*

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left( \sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{\mathcal{H}}^p \right)^{\frac{1}{p}} \quad \text{is finite.}$$

The Banach subspace of all right continuous functions  $v : \mathbb{R} \rightarrow \mathcal{H}$  such that  $\lim_{t \rightarrow -\infty} v(t) = 0$ , endowed with the same norm as above is denoted by  $V_{rc}^p(\mathbb{R}, \mathcal{H})$ . Note that

$$(4.2) \quad U^p(\mathbb{R}, \mathcal{H}) \hookrightarrow V_{rc}^p(\mathbb{R}, \mathcal{H}) \hookrightarrow L^\infty(\mathbb{R}, \mathcal{H}),$$

**Definition 4.3** (Definition 2.5 in [23]). *For  $s \in \mathbb{R}$  we let  $U_\Delta^p H^s$  - respectively  $V_\Delta^p H^s$  - be the space of all functions  $u : \mathbb{R} \rightarrow H^s(\mathbb{T}^3)$  such that  $t \rightarrow e^{-it\Delta} u(t)$  is in  $U^p(\mathbb{R}, H^s)$  -respectively in  $V_\Delta^p H^s$  - with norm*

$$\|u\|_{U_\Delta^p H^s} := \|e^{-it\Delta} u(t)\|_{U^p(\mathbb{R}, H^s)} \quad \|u\|_{V_\Delta^p H^s} := \|e^{-it\Delta} u(t)\|_{V^p(\mathbb{R}, H^s)}.$$

We will take  $\mathcal{H}$  to be  $H^s$ . We refer the reader to [22], [23], and references therein for detailed definitions and properties of the  $U^p$  and  $V^p$  spaces.

**Definition 4.4** (Definition 2.6 in [23]). *For  $s \in \mathbb{R}$  we define the space  $X^s$  as the space of all functions  $u : \mathbb{R} \rightarrow H^s(\mathbb{T}^3)$  such that for every  $n \in \mathbb{Z}^3$  the map  $t \rightarrow e^{it|n|^2} \widehat{u}(t)(n)$  is in  $U^2(\mathbb{R}, \mathbb{C})$ , and for which the norm*

$$(4.3) \quad \|u\|_{X^s} := \left( \sum_{n \in \mathbb{Z}^3} \langle n \rangle^{2s} \|e^{it|n|^2} \widehat{u}(t)(n)\|_{U_t^2}^2 \right)^{\frac{1}{2}} \quad \text{is finite.}$$

The  $X^s$  spaces are variations of the spaces  $U_\Delta^p H^s$  and  $V_\Delta^p H^s$  corresponding to the Schrödinger flow and defined as follows:

**Definition 4.5** (Definition 2.7 in [23]). *For  $s \in \mathbb{R}$  we define the space  $Y^s$  as the space of all functions  $u : \mathbb{R} \rightarrow H^s(\mathbb{T}^3)$  such that for every  $n \in \mathbb{Z}^3$  the map  $t \rightarrow e^{it|n|^2} \widehat{u}(t)(n)$  is in  $V_{rc}^2(\mathbb{R}, \mathbb{C})$ , and for which the norm*

$$(4.4) \quad \|u\|_{Y^s} := \left( \sum_{n \in \mathbb{Z}^3} \langle n \rangle^{2s} \|e^{it|n|^2} \widehat{u}(t)(n)\|_{V_t^2}^2 \right)^{\frac{1}{2}} \quad \text{is finite.}$$

Note that

$$(4.5) \quad U_\Delta^2 H^s \hookrightarrow X^s \hookrightarrow Y^s \hookrightarrow V_\Delta^2 H^s$$

whence one has that for any partition of  $\mathbb{Z}^3 := \cup_k C_k$ ,

$$\left( \sum_k \|P_{C_k} u\|_{V_\Delta^2 H^s}^2 \right)^{\frac{1}{2}} \lesssim \|u\|_{Y^s}$$

(cf. Section 2 in [23]).

Additionally, when  $s = 0$  by orthogonality we have

$$(4.6) \quad \left( \sum_k \|P_{C_k} u\|_{Y^0}^2 \right)^{\frac{1}{2}} = \|u\|_{Y^0}.$$

We also have the embedding

$$(4.7) \quad X^s \hookrightarrow Y^s \hookrightarrow L_t^\infty H_x^s$$

for  $s \geq 0$  (c.f. [24]).

**Remark 4.6** (Proposition 2.10 in [23]). *From the atomic structure of the  $U^2$  spaces one can immediately see that for  $s \geq 0$ ,  $T > 0$  and  $\phi \in H^s(\mathbb{T}^3)$ , the solution to the linear Schrödinger equation  $u := e^{it\Delta}\phi$  belongs to  $X^s([0, T])$  and  $\|u\|_{X^s([0, T])} \leq \|\phi\|_{H^s}$ .*

**Remark 4.7.** *Another important feature of the atomic structure of the  $U^2$  spaces is the fact that just like the  $X^{s,b}$  spaces they enjoy a ‘transfer principle’. We recall in our context the precise statement below for completeness.*

**Proposition 4.8** (Proposition 2.19 in [22]). *Let  $T_0 : L^2 \times \dots \times L^2 \rightarrow L_{loc}^1$  be a  $m$ -linear operator. Assume that for some  $1 \leq p, q \leq \infty$*

$$(4.8) \quad \|T_0(e^{it\Delta}\phi_1, \dots, e^{it\Delta}\phi_m)\|_{L^p(\mathbb{R}, L_x^q(\mathbb{T}^3))} \lesssim \prod_{i=1}^m \|\phi_i\|_{L^2(\mathbb{T}^3)}.$$

*Then, there exists an extension  $T : U_\Delta^p \times \dots \times U_\Delta^p \rightarrow L^p(\mathbb{R}, L^q(\mathbb{T}^3))$  satisfying*

$$(4.9) \quad \|T(u_1, \dots, u_m)\|_{L^p(\mathbb{R}, L_x^q(\mathbb{T}^3))} \lesssim \prod_{i=1}^m \|u_i\|_{U_\Delta^p};$$

*and such that  $T(u_1, \dots, u_m)(t, \cdot) = T_0(u_1(t), \dots, u_m(t))(\cdot)$ , a.e. In other words, one can reduce estimates for multilinear operators on functions in  $U_\Delta^p$  to similar estimates on solutions to the linear Schrödinger equation.*

We will use the following interpolation result at the end of Section 8 to obtain bounds in terms of the  $X^s$  spaces from those in  $U_\Delta^2 H^s$  and  $U_\Delta^p H^s$  just as in in [23] The proof relies solely on linear interpolation [22, 23].

**Proposition 4.9** (Proposition 2.20 in [22] and Lemma 2.4 [23]). *Let  $q_1, \dots, q_m > 2$  where  $m = 1, 2$ , or  $3$ ,  $E$  be a Banach space and  $T : U^{q_1} \times \dots \times U^{q_m} \rightarrow E$  be a bounded  $m$ -linear operator with*

$$(4.10) \quad \|T(u_1, \dots, u_m)\|_E \leq C \prod_{i=1}^m \|u_i\|_{U_\Delta^{q_i}}$$

*In addition assume there exists  $0 < C_2 \leq C$  such that the estimate*

$$(4.11) \quad \|T(u_1, \dots, u_m)\|_E \leq C_2 \prod_{i=1}^m \|u_i\|_{U_\Delta^2}$$

*holds true. Then,  $T$  satisfies the estimate*

$$(4.12) \quad \|T(u_1, \dots, u_m)\|_E \lesssim C_2 \left( \ln \frac{C}{C_2} + 1 \right)^m \prod_{i=1}^m \|u_i\|_{V^2}, \quad u_i \in V_{rc}^2, \quad i = 1, \dots, m,$$

where  $V_{rc}^2$  denotes the closed subspace of  $V^2$  of all right continuous functions of  $t$  such that  $\lim_{t \rightarrow -\infty} v(t) = 0$ .

Finally we state two results from [23] we rely on in the next sections. In what follows,  $\mathcal{I}$  denotes the Duhamel operator,

$$\mathcal{I}(f)(t) := \int_0^t e^{i(t-t')\Delta} f(t') dt', \quad t \geq 0,$$

defined for  $f \in L^1_{\text{loc}}([0, \infty), L^2(\mathbb{T}^3))$ .

**Proposition 4.10** (Proposition 2.11 in [23]). *Let  $s \geq 0$  and  $T > 0$ . For  $f \in L^1([0, T], H^s(\mathbb{T}^3))$  we have  $\mathcal{I}(f) \in X^s([0, T])$  and*

$$\|\mathcal{I}(f)\|_{X^s([0, T])} \leq \sup_{v \in Y^{-s}([0, T]): \|v\|_{Y^{-s}}=1} \left| \int_0^T \int_{\mathbb{T}^3} f(t, x) \overline{v(t, x)} dx dt \right|.$$

As a consequence, note we have

$$(4.13) \quad \|\mathcal{I}(f)\|_{X^s([0, T])} \lesssim \|f\|_{L^1([0, T], H^s(\mathbb{T}^3))}$$

**Proposition 4.11** (Proposition 4.1 in [23]). *Let  $s \geq 1$  be fixed. Then for all  $T \in (0, 2\pi]$  and  $u_k \in X^s([0, T])$ ,  $k = 1, \dots, 5$ , the estimate*

$$(4.14) \quad \|\mathcal{I}\left(\prod_{k=1}^5 \widetilde{u}_k\right)\|_{X^s([0, T])} \lesssim \sum_{j=1}^5 \|u_j\|_{X^s([0, T])} \prod_{k=1, k \neq j}^5 \|u_k\|_{X^1([0, T])}$$

holds true, where  $\widetilde{u}_k$  denotes either  $\overline{u_k}$  or  $u_k$ . In particular, (4.14) follows from the estimate for the multilinear form:

$$\left| \int_{[0, T] \times \mathbb{T}^3} \prod_{k=0}^5 \widetilde{u}_k dx dt \right| \lesssim \|u_0\|_{Y^{-s}([0, T])} \sum_{j=1}^5 \left( \|u_j\|_{X^s([0, T])} \prod_{k=1, k \neq j}^5 \|u_k\|_{X^1([0, T])} \right)$$

where  $u_0 := P_{\leq N} v$ .

Next, we recall the  $L^p(\mathbb{T} \times \mathbb{T}^3)$  Strichartz-type estimates of Bourgain's [5] in this context. First recall the usual Littlewood-Paley decomposition of periodic functions. For  $N \geq 1$  a dyadic number, we denote by  $P_{\leq N}$  the rectangular Fourier projection operator

$$P_{\leq N} f = \sum_{n=(n_1, n_2, n_3) \in \mathbb{Z}^3 : |n_i| \leq N} \widehat{f}(n) e^{in \cdot x}.$$

Then  $P_N = P_{\leq N} - P_{\leq N-1}$  so that  $P_{\leq N} = \sum_{M=1}^N P_M$  and  $P_N^\perp := I - P_N$ . We then have

$$\|f\|_{H^s(\mathbb{T}^3)} := \left( \sum_{n \in \mathbb{Z}^3} \langle n \rangle^{2s} |\widehat{f}(n)|^2 \right)^{\frac{1}{2}} \equiv \left( \sum_{N \geq 1} N^{2s} \|P_N(f)\|_{L^2(\mathbb{T}^3)}^2 \right)^{\frac{1}{2}}.$$

**Definition 4.12.** *For  $N \geq 1$ , we denote by  $\mathcal{C}_N$  the collection of cubes  $C$  in  $\mathbb{Z}^3$  with sides parallel to the axis of sidelength  $N$ .*

**Proposition 4.13.** [Proposition 3.1, Corollary 3.2 in [23] (cf. [5])] Let  $p > 4$ . For all  $N \geq 1$  we have

$$(4.15) \quad \|P_N e^{it\Delta} \phi\|_{L^p(\mathbb{T} \times \mathbb{T}^3)} \lesssim N^{\frac{3}{2} - \frac{5}{p}} \|P_N \phi\|_{L^2(\mathbb{T}^3)},$$

$$(4.16) \quad \|P_C e^{it\Delta} \phi\|_{L^p(\mathbb{T} \times \mathbb{T}^3)} \lesssim N^{\frac{3}{2} - \frac{5}{p}} \|P_C \phi\|_{L^2(\mathbb{T}^3)},$$

$$(4.17) \quad \|P_C u\|_{L^p(\mathbb{T} \times \mathbb{T}^3)} \lesssim N^{\frac{3}{2} - \frac{5}{p}} \|P_C u\|_{U_{\Delta}^p L^2},$$

where  $P_C$  is the Fourier projection operator onto  $C \in \mathcal{C}_N$  defined by the multiplier  $\chi_C$ , the characteristic function over  $C$ .

Finally we prove two propositions which will play an important role in Sections 7 and 8.

**Proposition 4.14.** Let  $u, v$  and  $w$  be functions of  $x$  and  $t$  such that,

$$\begin{aligned} \widehat{u}(n, t) &= a_n^1(t) a_n^2(t) a_n^3(t) \\ \widehat{v}(n, t) &= a_n^1(t) a_n^2(t) a_n^3(t) a_n^4(t) a_n^5(t) \\ \widehat{w}(n, t) &= a_n^1(t) a_n^2(t) a_n^3(t) \sum_m a_m^4 a_{n-m}^5. \end{aligned}$$

and  $|n| \sim N$ . Assume that  $J \subseteq \{1, 2, 3, 4, 5\}$  and if  $i \in J$  then

$$a_n^i(t) = \frac{g_n(\omega)}{|n|^{\frac{3}{2} + \varepsilon}} e^{it|n|^2}$$

while if  $i \notin J$  then there is a deterministic function  $f_i$  such that  $\widehat{f}_i(n, t) = a_n^i(t)$ . Then

$$(4.18) \quad \|P_N u\|_{L^p(\mathbb{T} \times \mathbb{T}^3)} \lesssim \prod_{i \notin J \cap \{1, 2, 3\}} \|P_N f_i\|_{Y^0}, \quad p > 4$$

$$(4.19) \quad \|P_N u\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim \prod_{i \notin J \cap \{1, 2, 3\}} \|P_N f_i\|_{Y^0}$$

$$(4.20) \quad \|P_N v\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim \prod_{i \notin J} \|P_N f_i\|_{Y^0}$$

$$(4.21) \quad \|P_N w\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim \prod_{i \notin J, i \neq 4, 5} \|P_N f_i\|_{Y^0} \prod_{j \notin J, j=4, 5} \|f_j\|_{Y^0}.$$

*Proof.* To prove (4.18) we write  $u = k_1 * k_2 * k_3$ , where the convolution is only with respect to the space variable. Then by Young's inequality in the space variable followed by Hölder's inequality and the embedding (4.7) we have the desired inequality.

To prove (4.19) we use Plancherel

$$\begin{aligned} \|P_N u\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} &\lesssim \left\| \|\chi_{|n| \sim N} a_n^1 a_n^2 a_n^3\|_{\ell^2} \right\|_{L^\infty(\mathbb{T})} \lesssim \left\| \prod_{i=1}^3 \|\chi_{|n| \sim N} a_n^i\|_{\ell^2} \right\|_{L^\infty(\mathbb{T})} \\ &\lesssim \left\| \prod_{i=1}^3 \|P_N f_i\|_{L_x^2} \right\|_{L^\infty(\mathbb{T})} \lesssim \prod_{i \notin J \cap \{1, 2, 3\}} \|P_N f_i\|_{L^\infty(\mathbb{T}, L^2(\mathbb{T}^3))} \end{aligned}$$

and the conclusion follows from the embedding (4.7).

To prove (4.20) we proceed in a similar manner.

To prove (4.21) we first write

$$\|P_N w\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \sim \|P_N(k_1 * k_2 * k_3 * (k_4 k_5))\|_{L^2(\mathbb{T} \times \mathbb{T}^3)},$$

and by Young's, Hölder's and Cauchy-Schwarz inequality we continue with

$$\begin{aligned} &\lesssim \left\| \prod_{i=1}^3 \|P_N k_i\|_{L^2} \|P_N(k_4 k_5)\|_{L^1} \right\|_{L^2(\mathbb{T})} \\ &\lesssim \left\| \prod_{i=1}^3 \|P_N k_i\|_{L^2} \|k_4\|_{L^2} \|k_5\|_{L^2} \right\|_{L^2(\mathbb{T})} \\ &\lesssim \prod_{i \notin J, i \neq 4,5} \|P_N f_i\|_{L^\infty(\mathbb{T}, L^2(\mathbb{T}^3))} \prod_{j \notin J, j=4,5} \|f_j\|_{L^\infty(\mathbb{T}, L^2(\mathbb{T}^3))}. \end{aligned}$$

□

We now state a trilinear  $L^2$  estimate that is similar to Proposition 3.5 in [23] but in which some of the functions may be linear evolution of random data.

**Proposition 4.15.** *Assume  $N_1 \geq N_2 \geq N_3$  and that  $C \in \mathcal{C}_{N_2}$ , a cube of sidelength  $N_2$ . Assume also that  $J \subseteq \{1, 2, 3\}$  and such that if  $j \in J$  then  $\widehat{u}_j(n) = e^{i|n|^{2t} \frac{g_n(\omega)}{|n|^{\frac{3}{2}+\varepsilon}}}$  for  $\varepsilon > 0$  small. Then*

$$(4.22) \quad \|P_C P_{N_1} \widetilde{u}_1 P_{N_2} \widetilde{u}_2 P_{N_3} \widetilde{u}_3\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim N_2 N_3 \prod_{j \notin J} \|P_{N_j} u_j\|_{U_\Delta^4 L^2}.$$

and

$$(4.23) \quad \|P_C P_{N_1} \widetilde{u}_1 P_{N_2} \widetilde{u}_2\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim N_2^{\frac{1}{2}+\varepsilon} \prod_{j \notin J} \|P_{N_j} u_j\|_{U_\Delta^4 L^2},$$

where  $\widetilde{u}_k$  denotes either  $\overline{u_k}$  or  $u_k$ .

Moreover (4.22) and (4.23) also hold with the  $Y^0$  norms in the right hand side.

*Proof.* To prove (4.22) we follow the proof of (24) in [23]. We write

$$\|P_C P_{N_1} \widetilde{u}_1 P_{N_2} \widetilde{u}_2 P_{N_3} \widetilde{u}_3\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim \|P_C P_{N_1} u_1\|_{L^p} \|P_{N_2} u_2\|_{L^p} \|P_{N_3} u_3\|_{L^q}$$

where  $\frac{2}{p} + \frac{1}{q} = \frac{1}{2}$  and  $4 < p < 5$ . Then we use (4.16) for the random linear functions and (4.17) for the deterministic functions to obtain

$$\|P_C P_{N_1} \widetilde{u}_1 P_{N_2} \widetilde{u}_2 P_{N_3} \widetilde{u}_3\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim N_2 N_3 \left(\frac{N_3}{N_2}\right)^{-2+\frac{10}{p}} \prod_{j \notin J} \|P_{N_j} u_j\|_{U_\Delta^4 L^2},$$

where we used the embedding (4.1).

To prove (4.23) we use Hölder's inequality to write

$$(4.24) \quad \|P_C P_{N_1} \widetilde{u}_1 P_{N_2} \widetilde{u}_2\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim \|P_C P_{N_1} u_1\|_{L^{4+\varepsilon}} \|P_{N_2} u_2\|_{L^{4+\varepsilon}}$$

we then we use (4.16), (4.17) and the embedding (4.1) to continue with

$$\lesssim N_2^{\frac{1}{2}+\varepsilon} \prod_{j \notin J} \|P_{N_j} u_j\|_{U_\Delta^4 L^2}.$$

To obtain the  $Y^0$  in the right hand side we use the interpolation Proposition 4.9 and the embedding (4.1). □



## 5. ALMOST SURE LOCAL WELL-POSEDNESS FOR THE INITIAL VALUE PROBLEM (2.17)

We define

$$(5.1) \quad v_0^\omega(t, x) = S(t)\phi^\omega(x)$$

where  $\phi^\omega(x)$  is as in (1.5) and instead of solving the initial value problem (2.17) we solve the one for  $w = v - v_0^\omega$ :

$$(5.2) \quad \begin{cases} iw_t + \Delta w = \mathcal{N}(w + v_0^\omega) & x \in \mathbb{T}^3 \\ w(0, x) = 0, \end{cases}$$

where  $\mathcal{N}(\cdot)$  was defined in (2.18). To understand the nonlinear term of (5.2) we express it in terms of its spatial Fourier transform. Let  $a_n := \hat{v}(n)$ ,  $\theta_n^\omega := \mathcal{F}(S(t)\phi^\omega)(n)$ , then  $b_n := \hat{w}(n) = a_n - \theta_n^\omega$ . Now we recall (2.9) and in it we replace  $a_n$  with  $b_n + \theta_n^\omega$ . Then

$$(5.3) \quad \mathcal{F}(\mathcal{N}(w + v_0^\omega))(n) = \sum_{k=1}^7 J_k(b_n + \theta_n^\omega),$$

where here  $J_k(b_n + \theta_n^\omega)$  means that the terms  $J_k$  defined in (2.10)–(2.16) are evaluated for the sequence  $(b_n + \theta_n^\omega)$  instead of  $a_n$ .

We are now ready to state the almost sure well-posedness result for the initial value problem (5.2).

**Theorem 5.1.** *Let  $0 < \alpha < \frac{1}{12}$ ,  $s \in (1 + 4\alpha, \frac{3}{2} - 2\alpha)$ . There exists  $0 < \delta_0 \ll 1$  and  $r = r(s, \alpha) > 0$  such that for any  $\delta < \delta_0$ , there exists  $\Omega_\delta \in A$  with*

$$\mathbb{P}(\Omega_\delta^c) < e^{-\frac{1}{\delta^r}},$$

and for each  $\omega \in \Omega_\delta$  there exists a unique solution  $w$  of (5.2) in the space  $X^s([0, \delta]) \cap C([0, \delta], H^s(\mathbb{T}^3))$ .

The proof of this theorem follows from the following two propositions via contraction mapping argument.

**Proposition 5.2.** *Let  $0 < \alpha < \frac{1}{12}$ ,  $s \in (1 + 4\alpha, \frac{3}{2} - 2\alpha)$ ,  $\delta \ll 1$  and  $R > 0$  be fixed. Assume  $N_i$ ,  $i = 0, \dots, 5$  are dyadic numbers and  $N_1 \geq N_2 \geq N_3 \geq N_4 \geq N_5$ . Then there exists  $\rho = \rho(s, \alpha) > 0$ ,  $\mu > 0$ , and  $\Omega_\delta \in A$  such that*

$$\mathbb{P}(\Omega_\delta^c) < e^{-\frac{1}{\delta^r}},$$

and such that for  $\omega \in \Omega_\delta$  we have:

If  $N_1 \gg N_0$  or  $P_{N_1}w = P_{N_1}v_0^\omega$

$$(5.4) \quad \left| \int_0^{2\pi} \int_{\mathbb{T}^3} D^s(\mathcal{N}(P_{N_i}(w + v_0^\omega))) \overline{P_{N_0}h} dx dt \right| \\ \lesssim \delta^{-\mu r} N_1^{-\rho} \|P_{N_0}h\|_{Y^{-s}} \left( 1 + \prod_{i \notin J} \|P_{N_i}w\|_{X^s} \right).$$

If  $N_1 \sim N_0$  and  $P_{N_1} w \neq P_{N_1} v_0^\omega$

$$(5.5) \quad \left| \int_0^{2\pi} \int_{\mathbb{T}^3} D^s (\mathcal{N}(P_{N_i}(w + v_0^\omega))) \overline{P_{N_0} h} dx dt \right| \\ \lesssim \delta^{-\mu r} N_2^{-\rho} \|P_{N_0} h\|_{Y^{-s}} \|P_{N_1} w\|_{X^s} \left( 1 + \prod_{i \notin J, i \neq 1} \|\psi_\delta P_{N_i} w\|_{X^s} \right),$$

where  $v_0^\omega$  is as in (5.1),  $w \in X^s([0, 2\pi])$ ,  $J \subseteq \{1, 2, 3, 4, 5\}$  denote those indices corresponding to random functions.

**Proposition 5.3.** *Let  $0 < \alpha < \frac{1}{12}$ ,  $s \in (1+4\alpha, \frac{3}{2}-2\alpha)$  and  $\delta \ll 1$  be fixed. Let  $v_0^\omega$  be defined as in (5.1) and assume  $w \in X^s([0, 2\pi])$ . Then there exist  $\theta = \theta(s, \alpha) > 0$ ,  $r = r(s, \alpha)$  and  $\Omega_\delta \in A$  such that*

$$\mathbb{P}(\Omega_\delta^c) < e^{-\frac{1}{\delta^r}},$$

and such that for  $\omega \in \Omega_\delta$

$$(5.6) \quad \|\mathcal{I}(\psi_\delta \mathcal{N}(w + v_0^\omega))\|_{X^s([0, 2\pi])} \lesssim \delta^\theta \left( 1 + \|\psi_\delta w\|_{X^s([0, 2\pi])}^5 \right)$$

where  $\mathcal{N}(\cdot)$  was defined in (2.18) and  $\psi_\delta$  is a smooth time cut-off of the interval  $[0, \delta]$ .

The proof of Proposition 5.2 is the content of Sections 7 and 8 while Proposition 5.3 is proved in Section 9.

## 6. AUXILIARY LEMMATA AND FURTHER NOTATION

We begin by recalling some counting estimates for integer lattice sets (c.f. Bourgain [5]).

**Lemma 6.1.** *Let  $S_R$  be a sphere of radius  $R$ ,  $B_r$  be a ball of radius  $r$  and  $\mathcal{P}$  be a plane in  $\mathbb{R}^3$ . Then we have*

$$(6.1) \quad \#Z^3 \cap S_R \lesssim R$$

$$(6.2) \quad \#Z^3 \cap B_r \cap S_R \lesssim \min(R, r^2)$$

$$(6.3) \quad \#Z^3 \cap B_r \cap \mathcal{P} \lesssim r^2.$$

Next, we state a result we will invoke when the the higher frequencies correspond to deterministic terms and one can afford to ignore the moments given by the lower frequency random terms as well as rely on Strichartz estimates.

**Lemma 6.2.** *Assume  $N_i$ ,  $i = 0, \dots, 5$  are dyadic numbers and  $N_1 \sim N_0$  and  $N_1 \geq N_2 \geq N_3 \geq N_4 \geq N_5$ . Let  $\{C\}$  be a partition of  $\mathbb{Z}^3$  by cubes  $C \in \mathcal{C}_{N_2}$ , and let  $\{Q\}$  be a partition of  $\mathbb{Z}^3$  by cubes  $Q \in \mathcal{C}_{N_3}$ . Then*

$$(6.4) \quad \sum_{N_i, i=0, \dots, 5} \left| \int_0^1 \int_{\mathbb{T}^3} P_{N_1} f_1 P_{N_2} f_2 P_{N_3} f_3 P_{N_4} f_4 P_{N_5} f_5 \overline{P_{N_0} h} dx dt \right| \lesssim \\ \sum_{N_i, i=0, \dots, 5} \left( \sup_{\tilde{C}} \|P_{\tilde{C}} P_{N_1} f_1 P_{N_2} f_2 P_{N_\ell} f_\ell\|_{L_{xt}^2}^2 \sum_{C, Q} \|P_Q P_C \overline{P_{N_0} h} P_{N_3} f_3 P_{N_r} f_r\|_{L_{xt}^2}^2 \right)^{\frac{1}{2}}$$

where  $\ell \neq r \in \{4, 5\}$  and  $\tilde{C}$  are cubes whose sidelength is  $10N_2$ .

*Proof.* The proof of (6.4) follows from orthogonality arguments.  $\square$

Just as Bourgain in [4], in the course of the proof we will use the following classical result about matrices, which we state as a lemma for convenience.

**Lemma 6.3.** *Let  $\mathcal{A} = (A_{ik})_{\substack{1 \leq i \leq N \\ 1 \leq k \leq M}}$  be an  $N \times M$  matrix with adjoint  $\mathcal{A}^* = (\overline{A}_{kj})_{\substack{1 \leq k \leq M \\ 1 \leq j \leq N}}$ . Then,*

$$(6.5) \quad \|\mathcal{A}\mathcal{A}^*\| \leq \max_{1 \leq j \leq N} \left( \sum_{k=1}^M |A_{jk}|^2 \right) + \left( \sum_{i \neq j} \left| \sum_{k=1}^M A_{ik} \overline{A}_{jk} \right|^2 \right)^{\frac{1}{2}}$$

where  $\|\cdot\|$  means the 2-norm.

*Proof.* Decompose  $\mathcal{A}\mathcal{A}^*$  into the sum of a diagonal matrix  $D$  plus an off-diagonal one  $F$ . Then note the 2-norm of  $D$  is bounded by the square root of the largest eigenvalue of  $DD^*$  which, since  $D$  is diagonal, is the maximum of the absolute value of the diagonal entries of  $D$ . This gives the first term in (6.5). Bounding the 2-norm of  $F$  by the Fröbenius norm of  $F$  gives the second term in (6.5).  $\square$

**Notation:** Given  $k$ -tuples  $(n_1, \dots, n_k) \in \mathbb{Z}^{3k}$ , a set of constraints  $\mathcal{C}$  on them, and a subset of indices  $\{i_1, \dots, i_h\} \subseteq \{1, \dots, k\}$ , we denote by  $S_{(n_{i_1}, \dots, n_{i_h})}$  the set of  $(k-h)$ -tuples  $(n_{j_1}, \dots, n_{j_{k-h}})$ ,  $\{j_1, \dots, j_{k-h}\} = \{1, \dots, k\} \setminus \{i_1, \dots, i_h\}$ , which satisfy the constraints  $\mathcal{C}$  for fixed  $(n_{i_1}, \dots, n_{i_h})$ . We also denote by  $|S_{(n_{i_1}, \dots, n_{i_h})}|$  its cardinality.

## 7. THE TRILINEAR AND BILINEAR BUILDING BLOCKS

In this section, we denote by  $D_j := e^{it\Delta} P_{N_j} \phi$  solutions to the linear equation for data  $\phi$  in  $L^2$  localized at frequency  $N_j$  and by  $R_k$  the function defined as,

$$(7.1) \quad \widehat{R}_k(n) = \chi_{\{|n| \sim N_k\}}(n) \frac{g_n(\omega)}{\langle n \rangle^{\frac{3}{2}}} e^{it|n|^2},$$

and representing the linear evolution of a random function of type (1.5), localized at frequency  $N_k$  and *almost*  $L^2$  normalized.

**7.1. Trilinear Estimates.** We prove certain trilinear estimates which serve as building blocks for the proof in Section 8. Their proofs are of the same flavor as those presented by Bourgain in [4]. For  $N_j$ ,  $j = 1, 2, 3$  dyadic numbers, let  $\alpha_j = 0$  or 1 for  $j = 1, 2, 3$  and define

$$(7.2) \quad \Upsilon(n, m) := \left\{ \begin{array}{l} (n_1, m_1; n_2, m_2; n_3, m_3) : \\ \left. \begin{array}{l} n = (-1)^{\alpha_1} n_1 + (-1)^{\alpha_2} n_2 + (-1)^{\alpha_3} n_3 \\ n_k \neq n_\ell \text{ whenever } \alpha_k \neq \alpha_\ell, \\ |n_j| \sim N_j, \quad j = 1, \dots, 3 \\ m = (-1)^{\alpha_1} m_1 + (-1)^{\alpha_2} m_2 + (-1)^{\alpha_3} m_3 \end{array} \right\} \end{array} \right.$$

and define  $T_\Upsilon$  to be the multilinear operator with multiplier  $\chi_\Upsilon$ .

**Proposition 7.1.** *Fix  $N_1 \geq N_2 \geq N_3$ ,  $r, \delta > 0$  and  $C \in \mathcal{C}_{N_2}$ . Then there exists  $\mu, \varepsilon > 0$ , a set  $\Omega_\delta \in A$  such that  $\mathbb{P}(\Omega_\delta^c) \leq e^{-\frac{1}{\delta^r}}$  and such that for any  $\omega \in \Omega_\delta$  we have the following estimates:*

$$(7.3) \quad \|T_Y(P_C \bar{R}_1, \tilde{D}_2, R_3)\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim \delta^{-\mu r} N_2^{\frac{5}{4}} N_1^{-\frac{1}{2}} \|P_{N_2} \phi\|_{L_x^2}.$$

$$(7.4) \quad \|T_Y(P_C \bar{R}_1, \tilde{D}_2, \bar{R}_3)\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim \delta^{-\mu r} N_2^{\frac{5}{4}} N_1^{-\frac{1}{2}} \|P_{N_2} \phi\|_{L_x^2}.$$

$$(7.5) \quad \|T_Y(P_C \tilde{D}_1, \bar{R}_2, R_3)\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim \delta^{-\mu r} N_2^{\frac{3}{4}} \|P_C P_{N_1} \phi\|_{L_x^2}.$$

$$(7.6) \quad \|T_Y(P_C \tilde{D}_1, R_2, R_3)\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim \delta^{-\mu r} N_2^{\frac{3}{4}} \|P_C P_{N_1} \phi\|_{L_x^2}.$$

$$(7.7) \quad \|T_Y(P_C \bar{R}_1, R_2, \tilde{D}_3)\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim \delta^{-\mu r} \left[ N_1^{-\frac{3}{4}} N_2^{\frac{1}{2}} N_3^{\frac{5}{4}} + N_1^{-\frac{1}{2}} N_2^{\frac{1}{2}} N_3^{\frac{3}{4}} \right] \|P_{N_3} \phi\|_{L_x^2}.$$

$$(7.8) \quad \|T_Y(P_C \bar{R}_1, \bar{R}_2, \tilde{D}_3)\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim \delta^{-\mu r} \left[ N_1^{-\frac{3}{4}} N_2^{\frac{1}{2}} N_3^{\frac{5}{4}} + N_1^{-\frac{1}{2}} N_2^{\frac{1}{2}} N_3^{\frac{3}{4}} \right] \|P_{N_3} \phi\|_{L_x^2}.$$

$$(7.9) \quad \|T_Y(P_C R_1, \tilde{D}_2, \tilde{D}_3)\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim \delta^{-\mu r} N_2^{\frac{1}{2} + \frac{3\theta}{4}} N_1^{-\frac{1}{2} + \varepsilon} \min(N_1, N_2^2)^{\frac{1-\theta}{2}} N_3^{\frac{3}{2}} \times \\ \times \|P_{N_2} \phi\|_{L_x^2} \|P_{N_3} \phi\|_{L_x^2}, \quad 0 \leq \theta \leq 1.$$

$$(7.10) \quad \|T_Y(P_C \tilde{D}_1, R_2, \tilde{D}_3)\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim \delta^{-\mu r} N_2^{\frac{1}{2} + \varepsilon} N_3^{\frac{3}{2}} \|P_{N_1} \phi\|_{L_x^2} \|P_{N_3} \phi\|_{L_x^2}.$$

$$(7.11) \quad \|T_Y(P_C \bar{R}_1, \bar{R}_2, R_3)\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim \delta^{-\mu r} N_1^{-\frac{1}{2}} N_2^{\frac{1}{2}}.$$

$$(7.12) \quad \|T_Y(P_C \bar{R}_1, R_2, \bar{R}_3)\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim \delta^{-\mu r} N_1^{-\frac{1}{2}} N_2^{\frac{1}{2}}.$$

$$(7.13) \quad \|T_Y(P_C \bar{R}_1, R_2, R_3)\|_{L^2(\mathbb{T} \times \mathbb{T}^3)} \lesssim \delta^{-\mu r} N_1^{-\frac{1}{2}} N_2^{\frac{1}{2}}.$$

Note that here the bar  $\bar{\phantom{x}}$  indicates complex conjugate while the tilde  $\tilde{\phantom{x}}$  indicates both complex conjugate or not. Also, without writing it explicitly, we always assume that if  $\bar{R}(n_1)$  and  $\tilde{R}(n_2)$  appear in the trilinear expressions in the left hand side, then  $n_1 \neq n_2$ .

**Remark 7.2.** In using the trilinear estimates above, sometimes it is convenient to interpret a random term as deterministic and choose the minimum estimate possible. For example, in considering  $\|P_C \bar{R}_1 \bar{R}_2 R_3\|_{L^2}$  we have a choice between (7.11) and (7.8) by thinking of  $R_3$  as an ‘almost’  $L^2$  normalized  $\tilde{D}_3$  function.

**Proposition 7.3.** Let  $D_j$  and  $R_k$  be as above and fix  $N_1 \geq N_2 \geq N_3$ ,  $r, \delta > 0$  and  $C \in \mathcal{C}_{N_2}$ . Then there exists  $\mu > 0$  and a set  $\Omega_\delta \in A$  such that  $\mathbb{P}(\Omega_\delta^c) \leq e^{-\frac{1}{\delta^r}}$  such that for any  $\omega \in \Omega_\delta$  we have (7.3) and (7.4).

*Proof.* As in [23] we will first assume that the deterministic functions  $D_i$  are localized linear solutions, that is  $D_i = P_{N_i} S(t)\psi$  and  $\hat{\psi}(n) = a_n$ . Once an estimate is proved with  $\|\chi_{N_i}(n) a_n\|_{\ell^2}$  in the right hand side we then invoke the transfer principle of Proposition 4.8 to complete the proof.

We start by estimating (7.3). Without any loss of generality we assume that  $\tilde{D}_2 = D_2$ . By using Fourier transform to write the left hand side we note that it is enough to estimate

$$(7.14) \quad \mathcal{T} := \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}^3} \left| \sum_{\substack{n = -n_1 + n_2 + n_3 \\ n_1 \neq n_2, n_3 \\ m = -|n_1|^2 + |n_2|^2 + |n_3|^2}} \chi_C(n_1) \frac{\bar{g}_{n_1}(\omega)}{|n_1|^{\frac{3}{2}}} a_{n_2} \frac{g_{n_3}(\omega)}{|n_3|^{\frac{3}{2}}} \right|^2,$$

where we recall that  $C$  is a cube of sidelength  $N_2$ . We are going to use duality and a change of variable since, as it will be apparent below, the counting with respect to the time frequency will be more favorable.

Using duality we have that

$$\mathcal{T} = \left[ \sup_{\|\gamma \otimes k\|_{\ell^2} \leq 1} \left| \sum_{m,n} k(n) \gamma(m) \sum_{\substack{n=-n_1+n_2+n_3 \\ n_1 \neq n_2, n_3 \\ m=-|n_1|^2+|n_2|^2+|n_3|^2}} \chi_C(n_1) \frac{\bar{g}_{n_1}(\omega)}{|n_1|^{\frac{3}{2}}} a_{n_2} \frac{g_{n_3}(\omega)}{|n_3|^{\frac{3}{2}}} \right| \right]^2$$

Let  $\zeta := m - |n_2|^2 = -|n_1|^2 + |n_3|^2$ , then we continue with

$$\begin{aligned} \mathcal{T} &= \left[ \sup_{\|\gamma \otimes k\|_{\ell^2} \leq 1} \left| \sum_{n_2} a_{n_2} \sum_{\zeta} \gamma(\zeta + |n_2|^2) \sum_{\substack{n=-n_1+n_2+n_3 \\ n_1 \neq n_2, n_3 \\ \zeta=-|n_1|^2+|n_3|^2}} \chi_C(n_1) \frac{\bar{g}_{n_1}(\omega)}{|n_1|^{\frac{3}{2}}} \frac{g_{n_3}(\omega)}{|n_3|^{\frac{3}{2}}} k_n \right| \right]^2 \\ &\lesssim \sup_{\|\gamma \otimes k\|_{\ell^2} \leq 1} \|a_{n_2}\|_{\ell_{n_2}^2}^2 \|\gamma\|_{\ell_{\zeta}^2}^2 \sum_{n_2, \zeta} \left| \sum_{\substack{n=-n_1+n_2+n_3 \\ n_1 \neq n_2, n_3 \\ \zeta=-|n_1|^2+|n_3|^2}} \chi_C(n_1) \frac{\bar{g}_{n_1}(\omega)}{|n_1|^{\frac{3}{2}}} \frac{g_{n_3}(\omega)}{|n_3|^{\frac{3}{2}}} k_n \right|^2. \end{aligned}$$

All in all, we then have to estimate uniformly for  $\|\gamma \otimes k\|_{\ell^2} \leq 1$ ,

$$(7.15) \quad \|a_{n_2}\|_{\ell^2}^2 \|\gamma\|_{\ell^2}^2 \sum_{n_2} \sum_{|\zeta| \leq N_1 N_2} \left| \sum_n \sigma_{n_2, n} k_n \right|^2,$$

where

$$\sigma_{n_2, n} = \sum_{\substack{n_2=n_1+n-n_3, n_1 \neq n_2, n_3 \\ \zeta=-|n_1|^2+|n_3|^2}} \chi_C(n_1) \frac{\bar{g}_{n_1}(\omega)}{|n_1|^{\frac{3}{2}}} \frac{g_{n_3}(\omega)}{|n_3|^{\frac{3}{2}}}.$$

Note that  $\sigma_{n, n_2}$  also depends on  $\zeta$  but we estimate it independently of  $\zeta$ . If we denote by  $\mathcal{G}$  the matrix of entries  $\sigma_{n_2, n}$ , and we recall that the variation in  $\zeta$  is at most  $N_1 N_2$ , we are then reduced to estimating

$$\|a_{n_2}\|_{\ell^2}^2 N_1 N_2 \|\mathcal{G} \mathcal{G}^*\|.$$

We note that by Lemma 6.3

$$\|\mathcal{G} \mathcal{G}^*\| \lesssim \max_{n_2} \sum_n |\sigma_{n_2, n}|^2 + \left( \sum_{n_2 \neq n'_2} \left| \sum_{n \in \tilde{C}} \sigma_{n_2, n} \bar{\sigma}_{n'_2, n} \right|^2 \right)^{\frac{1}{2}} =: M_1 + M_2,$$

where  $\tilde{C}$  is a cube of sidelength approximately  $N_2$ . To estimate  $M_1$  we first define the set

$$\mathcal{S}_{(\zeta, n_2)} = \{(n_1, n, n_3) : n_2 = n_1 + n - n_3, n_1 \neq n_2, n_3, \zeta = -|n_1|^2 + |n_3|^2\}$$

with  $|S_{(\zeta, n_2)}| \lesssim N_3^3 N_1$ , where we use (6.1) for fixed  $n_3$ . Then we have

$$M_1 \lesssim \sup_{(n_2, \zeta)} \left| \sum_{\substack{n_2 = n_1 + n - n_3, n_1 \neq n_2, n_3 \\ \zeta = -|n_1|^2 + |n_3|^2}} \chi_C(n_1) \frac{\bar{g}_{n_1}(\omega) g_{n_3}(\omega)}{|n_1|^{\frac{3}{2}} |n_3|^{\frac{3}{2}}} \right|^2.$$

Now we use (3.4) with  $\lambda = \delta^{-r} \|F_2(\omega)\|_{L^2}$  and Lemma 3.3 to obtain for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$  the bound

$$(7.16) \lesssim \sup_{(n_2, \zeta)} \delta^{-2r} \sum_{S(\zeta, n_2)} \sum_{S(\zeta, n_2)} \frac{1}{|n_1|^{\frac{3}{2}}} \frac{1}{|n_3|^{\frac{3}{2}}} \frac{1}{|\xi_1|^{\frac{3}{2}}} \frac{1}{|\xi_3|^{\frac{3}{2}}} \left| \int_{\Omega} \bar{g}_{n_1}(\omega) g_{n_3}(\omega) g_{\xi_1}(\omega) \bar{g}_{\xi_3}(\omega) dp(\omega) \right|^2$$

$$\lesssim \sup_{(n_2, \zeta)} \delta^{-2r} \sum_{S(\zeta, n_2)} \frac{1}{|n_1|^3} \frac{1}{|n_3|^3} \lesssim \delta^{-2r} N_1^{-3} N_3^{-3} N_3^3 N_1 \sim \delta^{-2r} N_1^{-2}.$$

To estimate  $M_2$  we first write

$$M_2^2 = \sum_{n_2 \neq n'_2} \left| \sum_{n \in \tilde{C}} \sigma_{n_2, n} \bar{\sigma}_{n'_2, n} \right|^2 \sim \sum_{n_2 \neq n'_2} \left| \sum_{S(n_2, n'_2)} \frac{\bar{g}_{n_1}(\omega) g_{n_3}(\omega) g_{n'_1}(\omega) \bar{g}_{n'_3}(\omega)}{|n_1|^{\frac{3}{2}} |n_3|^{\frac{3}{2}} |n'_1|^{\frac{3}{2}} |n'_3|^{\frac{3}{2}}} \right|^2$$

where

$$S_{(n_2, n'_2, \zeta)} = \left\{ (n, n_1, n_3, n'_1, n'_3) : \begin{array}{l} n_2 = n_1 + n - n_3, \quad n'_2 = n'_1 + n - n'_3, \\ n_1 \neq n_2, n_3, \quad n'_1 \neq n'_2, n'_3, \quad n \in \tilde{C} \\ \zeta = -|n_1|^2 + |n_3|^2, \quad \zeta = -|n'_1|^2 + |n'_3|^2 \end{array} \right\}.$$

We need to organize the estimates according to whether some frequencies are the same or not, in all we have six cases:

- **Case  $\beta_1$ :**  $n_1, n'_1, n_3, n'_3$  are all different.
- **Case  $\beta_2$ :**  $n_1 = n'_1$ ;  $n_3 \neq n'_3$ .
- **Case  $\beta_3$ :**  $n_1 \neq n'_1$ ;  $n_3 = n'_3$ .
- **Case  $\beta_4$ :**  $n_1 \neq n'_3$ ;  $n_3 = n'_1$ .
- **Case  $\beta_5$ :**  $n_1 = n'_3$ ;  $n_3 \neq n'_1$ .
- **Case  $\beta_6$ :**  $n_1 = n'_3$ ;  $n_3 = n'_1$ .

**Case  $\beta_1$ :** We define the set

$$S_{(\zeta)} = \left\{ (n_2, n'_2, n, n_1, n_3, n'_1, n'_3) : \begin{array}{l} n_2 = n_1 + n - n_3, \quad n'_2 = n'_1 + n - n'_3, \\ n_1 \neq n_2, n_3, \quad n'_1 \neq n'_2, n'_3, \quad n_1, n'_1 \in C \\ \zeta = -|n_1|^2 + |n_3|^2, \quad \zeta = -|n'_1|^2 + |n'_3|^2 \end{array} \right\}.$$

and we note that  $|S_{(\zeta)}| \lesssim N_1^2 N_3^6 N_2^3$  since  $n \in \tilde{C}$  and for fixed  $n_3$  and  $n'_3$  we use (6.1) to count  $n_1$  and  $n'_1$ . Using (3.4) with  $\lambda = \delta^{-2r} \|F_4(\omega)\|_{L^2}$  and again Lemma 3.3 we can write for  $\omega$  as above

$$M_2^2 \lesssim \delta^{-4r} \sum_{n_2 \neq n'_2} \sum_{S(n_2, n'_2, \zeta)} \frac{1}{|n_1|^3} \frac{1}{|n_3|^3} \frac{1}{|n'_1|^3} \frac{1}{|n'_3|^3} \lesssim \delta^{-4r} N_1^{-6} N_3^{-6} N_1^2 N_3^6 N_2^3 \sim \delta^{-4r} N_1^{-4} N_2^3.$$

**Case  $\beta_2$ :** First define the set,

$$S_{(n_2, n'_2, n_3, n'_3, \zeta)} = \left\{ \begin{array}{l} n_2 = n_1 + n - n_3, \quad n'_2 = n_1 + n - n'_3, \\ (n, n_1) : n_1 \neq n_2, n'_2, n_3, n'_3, \quad n \in \tilde{C} \\ \zeta = -|n_1|^2 + |n_3|^2, \quad \zeta = -|n_1|^2 + |n'_3|^2 \end{array} \right\}.$$

To compute  $|S_{(n_2, n'_2, n_3, n'_3, \zeta)}|$  we count  $n_1$ , then  $n$  is determined. Since  $n_1$  sits on a sphere then by (6.1) we have  $|S_{(n_2, n'_2, n_3, n'_3, \zeta)}| \lesssim N_1$ . Then we set

$$S_{(\zeta)} = \left\{ \begin{array}{l} n_2 = n_1 + n - n_3, \quad n'_2 = n_1 + n - n'_3, \\ (n_2, n'_2, n, n_1, n_3, n'_3) : n_1 \neq n_2, n'_2, n_3, n'_3, \quad n \in \tilde{C} \\ \zeta = -|n_1|^2 + |n_3|^2, \quad \zeta = -|n_1|^2 + |n'_3|^2 \end{array} \right\}.$$

with  $|S_{(\zeta)}| \lesssim N_1 N_3^6 N_2^3$ , where we used again that  $n \in \tilde{C}$  and (6.1). Now, we have that

$$(7.17) \quad M_2^2 \sim \sum_{n_2 \neq n'_2} \left| \sum_{S_{(n_2, n'_2, \zeta)}} \frac{|g_{n_1}(\omega)|^2}{|n_1|^3} \frac{g_{n_3}(\omega)}{|n_3|^{\frac{3}{2}}} \frac{\bar{g}_{n'_3}(\omega)}{|n'_3|^{\frac{3}{2}}} \right|^2 \lesssim Q_1 + Q_2,$$

where

$$(7.18) \quad Q_1 := \sum_{n_2 \neq n'_2} \left| \sum_{S_{(n_2, n'_2, \zeta)}} \frac{(|g_{n_1}(\omega)|^2 - 1)}{|n_1|^3} \frac{g_{n_3}(\omega)}{|n_3|^{\frac{3}{2}}} \frac{\bar{g}_{n'_3}(\omega)}{|n'_3|^{\frac{3}{2}}} \right|^2$$

$$(7.19) \quad Q_2 := \sum_{n_2 \neq n'_2} \left| \sum_{S_{(n_2, n'_2, \zeta)}} \frac{1}{|n_1|^3} \frac{g_{n_3}(\omega)}{|n_3|^{\frac{3}{2}}} \frac{\bar{g}_{n'_3}(\omega)}{|n'_3|^{\frac{3}{2}}} \right|^2$$

We estimate first  $Q_2$ . We rewrite,

$$(7.20) \quad Q_2 \sim \sum_{n_2 \neq n'_2} \left| \sum_{n_3, n'_3} \left[ \sum_{S_{(n_2, n'_2, n_3, n'_3, \zeta)}} \frac{1}{|n_1|^3} \frac{1}{|n_3|^{\frac{3}{2}}} \frac{1}{|n'_3|^{\frac{3}{2}}} \right] g_{n_3}(\omega) \bar{g}_{n'_3}(\omega) \right|^2.$$

We now proceed as in the argument presented in (7.16) above. We use (3.4) with  $\lambda = \delta^{-r} \|F_2(\omega)\|_{L^2}$ , Lemma 3.3 and then use (3.6), to obtain that for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$  one has,

$$(7.20) \lesssim \delta^{-2r} \sum_{n_2 \neq n'_2} \sum_{n_3, n'_3} \left[ \sum_{S_{(n_2, n'_2, n_3, n'_3, \zeta)}} \frac{1}{|n_1|^3} \frac{1}{|n_3|^{\frac{3}{2}}} \frac{1}{|n'_3|^{\frac{3}{2}}} \right]^2$$

$$\lesssim \delta^{-2r} N_1^{-6} N_3^{-6} \sum_{n_2 \neq n'_2} \sum_{n_3, n'_3} |S_{(n_2, n'_2, n_3, n'_3, \zeta)}|^2$$

$$\lesssim \delta^{-2r} N_1^{-6} N_3^{-6} N_1 \sum_{n_2 \neq n'_2} \sum_{n_3, n'_3} |S_{(n_2, n'_2, n_3, n'_3, \zeta)}|$$

$$(7.21) \lesssim \delta^{-2r} N_1^{-6} N_3^{-6} N_1 |S_{(\zeta)}| \sim \delta^{-2r} N_1^{-4} N_2^3.$$

To estimate  $\mathcal{Q}_1$  we let

$$(7.22) \quad S_{(n_2, n'_2, n_1, n_3, n'_3, \zeta)} := \left\{ n : \begin{array}{l} n_2 = n_1 + n - n_3, \quad n'_2 = n_1 + n - n'_3, \\ n_1 \neq n_2, n'_2, n_3, n'_3, \quad n \in \tilde{C} \\ \zeta = -|n_1|^2 + |n_3|^2, \quad \zeta = -|n_1|^2 + |n'_3|^2 \end{array} \right\},$$

and note that its cardinality is 1 since  $n$  is determined for fixed  $(n_2, n'_2, n_1, n_3, n'_3)$ . We have,

$$\mathcal{Q}_1 \sim \sum_{n_2 \neq n'_2} \left| \sum_{n_1 \neq n_2, n'_2, n_3, n_3 \neq n'_3} \left[ \sum_{S_{(n_2, n'_2, n_1, n_3, n'_3)}^2} \frac{1}{|n_1|^3} \frac{1}{|n_3|^{\frac{3}{2}}} \frac{1}{|n'_3|^{\frac{3}{2}}} \right] (|g_{n_1}(\omega)|^2 - 1) g_{n_3}(\omega) \bar{g}_{n'_3}(\omega) \right|^2.$$

Proceeding like above, we obtain in this case that for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ ,

$$\mathcal{Q}_1 \lesssim \delta^{-2r} N_1^{-6} N_3^{-6} |S_{(\zeta)}| \sim \delta^{-2r} N_1^{-5} N_2^3,$$

which is a better estimate. Hence all in all we obtain in this case that

$$(7.23) \quad M_2^2 \lesssim \delta^{-2r} N_1^{-4} N_2^3.$$

**Case  $\beta_3$ :** In this case we define first

$$S_{(n_2, n'_2, n_1, n'_1, \zeta)} = \left\{ (n, n_3) : \begin{array}{l} n_2 = -n_1 + n - n_3, \quad n'_2 = n'_1 + n - n_3, \\ n_3, n_2, n'_2 \neq n_1, n'_1, \quad n \in \tilde{C} \\ \zeta = -|n_1|^2 + |n_3|^2, \quad \zeta = -|n'_1|^2 + |n_3|^2 \end{array} \right\},$$

with  $|S_{(n_2, n'_2, n_1, n'_1, \zeta)}| \lesssim N_3^2$  by (6.2) since  $n$  is determined by  $n_3$  and these ones lies on a sphere of radius at most  $N_1$  intersection a ball of radius  $N_3$ . If now we define

$$S_{(\zeta)} = \left\{ (n_2, n'_2, n, n_1, n'_1, n_3) : \begin{array}{l} n_2 = -n_1 + n - n_3, \quad n'_2 = n'_1 + n - n_3, \\ n_3, n_2, n'_2 \neq n_1, n'_1, \quad n \in \tilde{C} \\ \zeta = -|n_1|^2 + |n_3|^2, \quad \zeta = -|n'_1|^2 + |n_3|^2 \end{array} \right\},$$

then  $|S_{(\zeta)}| \lesssim N_1^2 N_3^3 N_2^3$ , since again  $n$  ranges in a cube of size  $N_2$  and we use (6.1) to count  $n_1$  and  $n'_1$ . We follow the argument used above in (7.17)-(7.23) to bound  $M_2^2$  but now with the couple  $(n_1, n'_1)$  instead and corresponding sums  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$ . Just as in Case  $\beta_2$  above, the bound for  $\mathcal{Q}_2$  is larger. We then obtain for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ ,

$$\begin{aligned} M_2^2 &\lesssim \delta^{-2r} N_1^{-6} N_3^{-6} \sum_{n_2 \neq n'_2} \sum_{n_1, n'_1} |S_{(n_2, n'_2, n_1, n'_1, \zeta)}|^2 \\ &\lesssim \delta^{-2r} N_1^{-6} N_3^{-6} N_3^2 \sum_{n_2 \neq n'_2} \sum_{n_1, n'_1} |S_{(n_2, n'_2, n_1, n'_1, \zeta)}| \\ &\lesssim \delta^{-2r} N_1^{-6} N_3^{-6} N_3^2 |S_{(\zeta)}| \sim \delta^{-2r} N_1^{-4} N_3^{-1} N_2^3. \end{aligned}$$

**Case  $\beta_4$ :** In this case note that  $N_1 \sim N_3 \sim N_2$ . We define the two sets. First

$$S_{(n_2, n'_2, n_1, n'_3, \zeta)} = \left\{ (n, n_3) : \begin{array}{l} n_2 = n_1 + n - n_3, \quad n'_2 = n_3 + n - n'_3, \\ n_2, n'_2, n_3, n'_3 \neq n_1, \quad n \in \tilde{C} \\ \zeta = -|n_1|^2 + |n_3|^2, \quad \zeta = |n_3|^2 + |n'_3|^2 \end{array} \right\}$$



and since  $n_3$  lives on a sphere of radius at most  $N_1$ , from (6.1) we have  $|S_{(n_2, n'_2, n_1, n'_3, \zeta)}| \lesssim N_1$  and then

$$S_{(\zeta)} = \left\{ (n_2, n'_2, n, n_1, n'_3, n_3) : \begin{array}{l} n_2 = n_1 + n - n_3, \quad n'_2 = n_3 + n - n'_3, \\ n_2, n'_2, n_3, n'_3 \neq n_1, \quad n \in \tilde{C} \\ \zeta = -|n_1|^2 + |n_3|^2, \quad \zeta = -|n_3|^2 + |n'_3|^2 \end{array} \right\},$$

with  $|S_{(\zeta)}| \lesssim N_1 N_2^3 N_3^6$ . Just as in case  $\beta_3$  and following the argument in (7.17)-(7.23) but with the couple  $(n_1, n'_3)$  instead we obtain that for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ ,

$$\begin{aligned} M_2^2 &\lesssim \delta^{-2r} N_1^{-6} N_3^{-6} \sum_{n_2 \neq n'_2} \sum_{n_1, n'_3} |S_{(n_2, n'_2, n_1, n'_3, \zeta)}|^2 \\ &\lesssim \delta^{-2r} N_1^{-6} N_3^{-6} N_1 \sum_{n_2 \neq n'_2} \sum_{n_1, n'_3} |S_{(n_2, n'_2, n_1, n'_3, \zeta)}| \\ &\lesssim \delta^{-2r} N_1^{-6} N_3^{-6} N_1 |S_{(\zeta)}| \sim \delta^{-2r} N_1^{-4} N_2^{-3}. \end{aligned}$$

**Case  $\beta_5$ :** By symmetry this case is exactly the same as Case  $\beta_4$ .

We are now ready to put all the estimates above together and bound  $\mathcal{T}$  in cases  $\beta_1 - \beta_5$ :

$$\begin{aligned} \mathcal{T} &\lesssim \|a_{n_2}\|_{\ell^2}^2 N_1 N_2 \| \mathcal{G} \mathcal{G}^* \| \lesssim \|a_{n_2}\|_{\ell^2}^2 N_1 N_2 (M_1 + M_2) \\ &\lesssim \|a_{n_2}\|_{\ell^2}^2 \delta^{-2r} N_1 N_2 N_1^{-2} N_2^{\frac{3}{2}} \lesssim \delta^{-2r} N_2^{\frac{5}{2}} N_1^{-1} \|a_{n_2}\|_{\ell^2}^2. \end{aligned}$$

**Case  $\beta_6$ :** In this case

$$S_{(n_2, n'_2, \zeta)} = \left\{ (n, n_1, n_3) : \begin{array}{l} n_2 = n_1 + n - n_3, \quad n'_2 = n_3 + n - n_1, \\ n_1 \neq n_2, n'_2, n_3, \quad |n_1|^2 = |n_3|^2, \quad n \in \tilde{C} \end{array} \right\}.$$

At this point notice that the summation on  $\zeta$  is eliminated and that in this case  $N_1 \sim N_2 \sim N_3$ . We have  $S_{(n_2, n'_2, \zeta)} \sim N_3^4$ . Using (3.6) we have, for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ , that

$$\begin{aligned} (7.24) \quad M_2^2 &= \sum_{n_2 \neq n'_2} \left| \sum_{n \in \tilde{C}} \sigma_{n_2, n} \bar{\sigma}_{n'_2, n} \right|^2 \sim \sum_{n_2 \neq n'_2} \left| \sum_{S_{(n_2, n'_2, \zeta)}} \frac{|g_{n_1}(\omega)|^2 |g_{n_3}(\omega)|^2}{|n_1|^3 |n_3|^3} \right|^2 \\ &\lesssim \sum_{n_2 \neq n'_2} N_1^{-6+\varepsilon} N_3^{-6} |S_{(n_2, n'_2, \zeta)}|^2 \lesssim N_1^{-6+\varepsilon} N_3^{-6} N_3^4 |S_{(\zeta)}|, \end{aligned}$$

where

$$S_{(\zeta)} = \left\{ (n_2, n'_2, n, n_1, n_3) : \begin{array}{l} n_2 = n_1 + n - n_3, \quad n'_2 = n_3 + n - n_1, \\ n_1 \neq n_3, n_2, n'_2 \quad |n_1|^2 = |n_3|^2, \quad n \in \tilde{C} \end{array} \right\}$$

and  $|S_{(\zeta)}| \lesssim N_2^3 N_3^4$ . Hence  $M_2 \lesssim N_1^{-3+\varepsilon} N_2^{\frac{5}{2}}$  and as a consequence

$$\mathcal{T} \lesssim \|a_{n_2}\|_{\ell^2}^2 N_1^{-3+\varepsilon} N_2^{\frac{5}{2}}.$$

We now notice that to prove (7.4) we first have to consider the case when  $n_1 = n_3$ , which here it is not excluded, and then we can use exactly the same argument as above since a plus or minus sign in front of  $n_3$  does not change any of the counting.

Consider now (7.4) with  $n_1 = n_3$ . Note that  $N_1 \sim N_2 \sim N_3$ . We now have

$$(7.25) \quad \mathcal{T} := \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}^3} \left| \sum_{\substack{n = -2n_1 + n_2 \\ m = -2|n_1|^2 + |n_2|^2}} \frac{(\bar{g}_{n_1}(\omega))^2}{|n_1|^3} a_{n_2} \right|^2.$$

Let  $S_{(m,n)} = \{(n_1, n_2) / n = -2n_1 + n_2, m = -2|n_1|^2 + |n_2|^2\}$ , and note that  $|S_{(m,n)}| \lesssim N_1$ . Then

$$\mathcal{T} \lesssim N_1 \sum_{m,n} \sum_{S_{(m,n)}} \frac{|\bar{g}_{n_1}(\omega)|^4}{|n_1|^6} |a_{n_2}|^2 \sim N_1 \sum_{n, n_1 \in \mathbb{Z}^3} \frac{|\bar{g}_{n_1}(\omega)|^4}{|n_1|^6} |a_{n+2n_1}|^2 \lesssim N_1^{-2+\varepsilon} \|a_{n_2}\|_{\ell^2}^2,$$

where we use (3.6) for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ .  $\square$

**Proposition 7.4.** *Let  $D_j$  and  $R_k$  be as above and fix  $N_1 \geq N_2 \geq N_3$ ,  $r, \delta > 0$  and  $C \in \mathcal{C}_{N_2}$ . Then there exists  $\mu > 0$  and a set  $\Omega_\delta \in A$  such that  $\mathbb{P}(\Omega_\delta^c) \leq e^{-\frac{1}{\delta^r}}$  such that for any  $\omega \in \Omega_\delta$  we have (7.5) and (7.6).*

*Proof.* We start by estimating (7.5) where without any loss of generality we assume that  $\tilde{D}_1 = D_1$ . We now have,

$$(7.26) \quad \mathcal{T} := \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}^3} \left| \sum_{\substack{n = n_1 - n_2 + n_3 \\ n_2 \neq n_3 \\ m = |n_1|^2 - |n_2|^2 + |n_3|^2}} \chi_C(n_1) a_{n_1} \frac{\bar{g}_{n_2}(\omega)}{|n_2|^{\frac{3}{2}}} \frac{g_{n_3}(\omega)}{|n_3|^{\frac{3}{2}}} \right|^2.$$

We are going to use duality and change of variables with  $\zeta := m - |n_1|^2 = -|n_2|^2 + |n_3|^2$  again. Note though that if  $n_1$  is in a cube  $C$  of size  $N_2$  then also  $n$  will be in a cube  $\tilde{C}$  of approximately the same size. Then just as in (7.15) we need to estimate

$$\|\chi_C a_{n_1}\|_{\ell^2}^2 \|\gamma\|_{\ell^2}^2 \sum_{n_1} \sum_{|\zeta| \leq N_2^2} \left| \sum_n \sigma_{n_1, n} \chi_{\tilde{C}}(n) k_n \right|^2,$$

where

$$\sigma_{n_1, n} = \sum_{\substack{n_1 = n_2 + n - n_3, n_2 \neq n_1, n_3 \\ \zeta = -|n_2|^2 + |n_3|^2}} \frac{\bar{g}_{n_2}(\omega)}{|n_2|^{\frac{3}{2}}} \frac{g_{n_3}(\omega)}{|n_3|^{\frac{3}{2}}}.$$

If we denote by  $\mathcal{G}$  the matrix of entries  $\sigma_{n_1, n}$ , and we recall that the variation in  $\zeta$  is at most  $N_2^2$ , we are then reduced to estimating

$$\|\chi_C a_{n_1}\|_{\ell^2}^2 N_2^2 \|\mathcal{G} \mathcal{G}^*\|.$$

We note that by Lemma 6.3,

$$\|\mathcal{G} \mathcal{G}^*\| \lesssim \max_{n_1} \sum_n |\sigma_{n_1, n}|^2 + \left( \sum_{n_1 \neq n'_1} \left| \sum_{n \in \tilde{C}} \sigma_{n_1, n} \bar{\sigma}_{n'_1, n} \right|^2 \right)^{\frac{1}{2}} =: M_1 + M_2,$$

where  $\tilde{C}$  is a cube of side length approximately  $N_2$ . From this point on the proof is similar to the one already provided for (7.3) where  $n_2$  is replaced by  $n_1$ . We still go through the argument though, since the size of  $n_1$  and  $n_2$  are different.

To estimate  $M_1$  we first define the set

$$S_{(\zeta, n_1)} = \{(n_2, n, n_3) : n_2 \neq n_1, n_3, \quad n_2 = n_1 - n + n_3, \quad \zeta = -|n_2|^2 + |n_3|^2\}.$$

Applying (6.1) for each fixed  $n_3$ , we have that  $|S_{(\zeta, n_1)}| \lesssim N_3^3 N_2$  since  $n_2$  sits on a sphere of radius approximately  $N_2$ . Then we proceed as in (7.16) to obtain for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ , the bound

$$M_1 \lesssim \delta^{-r} N_2^{-3} N_3^{-3} N_3^3 N_2 \sim \delta^{-2r} N_2^{-2}.$$

To estimate  $M_2$  we first write

$$M_2^2 = \sum_{n_1 \neq n'_1} \left| \sum_{n \in \tilde{C}} \sigma_{n_1, n} \bar{\sigma}_{n'_1, n} \right|^2 \sim \sum_{n_1 \neq n'_1} \left| \sum_{S_{(n_1, n'_1, \zeta)}} \frac{\bar{g}_{n_2}(\omega) g_{n_3}(\omega) g_{n'_2}(\omega) \bar{g}_{n'_3}(\omega)}{|n_2|^{\frac{3}{2}} |n_3|^{\frac{3}{2}} |n'_2|^{\frac{3}{2}} |n'_3|^{\frac{3}{2}}} \right|^2$$

where

$$S_{(n_1, n'_1, \zeta)} = \left\{ \begin{array}{l} n_2 = n_1 - n + n_3, \quad n'_2 = n'_1 - n + n'_3, \\ (n, n_2, n_3, n'_2, n'_3) : n_2 \neq n_1, n_3, \quad n'_2 \neq n'_1, n'_3, \quad n \in \tilde{C} \\ \zeta = -|n_2|^2 + |n_3|^2, \quad \zeta = -|n'_2|^2 + |n'_3|^2 \end{array} \right\}.$$

We organize once again the estimates according to whether some frequencies are the same or not. As before, all in all we have six cases:

- **Case  $\beta_1$ :**  $n_2, n'_2, n_3, n'_3$  are all different.
- **Case  $\beta_2$ :**  $n_2 = n'_2$ ;  $n_3 \neq n'_3$ .
- **Case  $\beta_3$ :**  $n_2 \neq n'_2$ ;  $n_3 = n'_3$ .
- **Case  $\beta_4$ :**  $n_2 \neq n'_2$ ;  $n_3 = n'_2$ .
- **Case  $\beta_5$ :**  $n_2 = n'_3$ ;  $n_3 \neq n'_2$ .
- **Case  $\beta_6$ :**  $n_2 = n'_3$ ;  $n_3 = n'_2$ .

**Case  $\beta_1$ :** We define the set

$$S_{(\zeta)} = \left\{ \begin{array}{l} n_2 = n_1 - n + n_3, \quad n'_2 = n'_1 - n + n'_3 \\ (n_1, n'_1, n, n_2, n_3, n'_2, n'_3) : n_2 \neq n_1, n_3, \quad n'_2 \neq n'_1, n'_3, \quad n_1, n'_1 \in C \\ \zeta = -|n_2|^2 + |n_3|^2, \quad \zeta = -|n'_2|^2 + |n'_3|^2 \end{array} \right\}.$$

and note that  $|S_{(\zeta)}| \lesssim N_2^2 N_3^6 N_2^3$  by Lemma 6.1 since for  $n_3$  fixed,  $n_2$  and  $n'_2$  sit on sphere of radius  $\sim N_2$  and  $n \in \tilde{C}$  a cube of side length approximately  $N_2$ . Hence, for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ , we obtain in this case,

$$M_2^2 \lesssim \delta^{-4r} N_2^{-6} N_3^{-6} N_2^2 N_3^6 N_2^3 \sim \delta^{-4r} N_2^{-1}.$$

**Case  $\beta_2$ :** In this case we define two sets. We start with

$$S_{(n_1, n'_1, n_3, n'_3, \zeta)} = \left\{ \begin{array}{l} n_2 = n_1 - n + n_3, \quad n_2 = n'_1 - n + n'_3, \\ (n, n_2) : n_2 \neq n_1, n'_1, n_3, n'_3, \quad n \in \tilde{C} \\ \zeta = -|n_2|^2 + |n_3|^2, \quad \zeta = -|n_2|^2 + |n'_3|^2 \end{array} \right\}.$$

To compute  $|S_{(n_1, n'_1, n_3, n'_3, \zeta)}|$ , it is enough to count  $n_2$ , then  $n$  is determined. Since  $n_2$  sits on a sphere of radius  $\sim N_2$  we have by (6.1) that  $|S_{(n_1, n'_1, n_3, n'_3, \zeta)}| \lesssim N_2$ . Then we set

$$S_{(\zeta)} = \left\{ \begin{array}{l} n_2 = n_1 - n + n_3, \quad n_2 = n'_1 - n + n'_3, \\ (n_1, n'_1, n, n_2, n_3, n'_3) : n_2 \neq n_1, n'_1, n_3, n'_3, \quad n \in \tilde{C} \\ \zeta = -|n_2|^2 + |n_3|^2, \quad \zeta = -|n_2|^2 + |n'_3|^2 \end{array} \right\}$$

for which  $|S_{(\zeta)}| \lesssim N_2 N_3^6 N_2^3$ , where we used again that  $n \in \tilde{C}$ . Arguing as in (7.17)- (7.23), we then have for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$  that

$$\begin{aligned} M_2^2 &\lesssim \delta^{-2r} N_2^{-6} N_3^{-6} \sum_{n \neq n'} \sum_{n_3, n'_3} |S_{(n_1, n'_1, n_3, n'_3, \zeta)}|^2 \\ &\lesssim \delta^{-2r} N_2^{-6} N_3^{-6} N_2 \sum_{n_1 \neq n'_1} \sum_{n_3, n'_3} |S_{(n_1, n'_1, n_3, n'_3, \zeta)}| \\ &\lesssim \delta^{-2r} N_2^{-6} N_3^{-6} N_2 |S_{(\zeta)}| \sim \delta^{-2r} N_2^{-1}. \end{aligned}$$

**Case  $\beta_3$ :** In this case we define first

$$S_{(n_2, n'_2, n_1, n'_1, \zeta)} = \left\{ \begin{array}{l} n_2 = n_1 - n + n_3, \quad n'_2 = n'_1 - n + n_3, \\ (n, n_3) : n_2, n'_2 \neq n_3, n_1, n'_1, \quad n \in \tilde{C} \\ \zeta = -|n_2|^2 + |n_3|^2, \quad \zeta = -|n'_2|^2 + |n_3|^2 \end{array} \right\}$$

for which  $|S_{(n_2, n'_2, n_1, n'_1, \zeta)}| \lesssim N_3^2$  since  $n$  is determined by  $n_3$  and this one lies on a sphere of radius at most  $N_1$  intersection a ball of radius  $N_3$  (see Lemma 6.1). Then we define

$$S_{(\zeta)} = \left\{ \begin{array}{l} n_2 = n_1 - n + n_3, \quad n'_2 = n'_1 - n + n_3, \\ (n_2, n'_2, n, n_1, n'_1, n_3) : n_2, n'_2 \neq n_3, n_1, n'_1, \quad n \in \tilde{C} \\ \zeta = -|n_2|^2 + |n_3|^2, \quad \zeta = -|n'_2|^2 + |n_3|^2 \end{array} \right\}$$

for which  $|S_{(\zeta)}| \lesssim N_2^2 N_3^3 N_2^3$ , since again  $n$  ranges in a cube of size  $N_2$ . We then have, as usual using (3.4) and (3.6) as above that for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ ,

$$\begin{aligned} M_2^2 &\lesssim \delta^{-2r} N_2^{-6} N_3^{-6} \sum_{n_1 \neq n'_1} \sum_{n_2, n'_2} |S_{(n_2, n'_2, n_1, n'_1, \zeta)}|^2 \\ &\lesssim \delta^{-2r} N_2^{-6} N_3^{-6} N_3^2 \sum_{n_1 \neq n'_1} \sum_{n_2, n'_2} |S_{(n_2, n'_2, n_1, n'_1, \zeta)}| \\ &\lesssim \delta^{-2r} N_2^{-6+\varepsilon} N_3^{-6} N_3^2 |S_{(\zeta)}| \sim \delta^{-2r} N_2^{-1+\varepsilon} N_3^{-1}. \end{aligned}$$

**Case  $\beta_4$ :** In this case note that  $N_3 \sim N_2$ . We define the two sets

$$S_{(n_1, n'_1, n_2, n'_2, \zeta)} = \left\{ \begin{array}{l} n_2 = n_1 - n + n_3, \quad n_3 = n'_1 - n + n'_3, \\ (n, n_3) : n_2 \neq n_1, n_3; n_3 \neq n'_3, n'_1, \quad n \in \tilde{C} \\ \zeta = -|n_2|^2 + |n_3|^2, \quad \zeta = -|n_3|^2 + |n'_3|^2 \end{array} \right\}$$

with  $|S_{(n_1, n'_1, n_2, n'_3, \zeta)}| \lesssim N_2$  since  $n_3$  lives on a sphere of radius at most  $N_2$ ; and

$$S_{(\zeta)} = \left\{ (n_1, n'_1, n, n_2, n'_3, n_3) : \begin{array}{l} n_2 = n_1 - n + n_3, \quad n_3 = n'_1 - n + n'_3, \\ n_2 \neq n_1, n_3; n_3 \neq n'_3, n'_1, \quad n \in \tilde{C} \\ \zeta = -|n_2|^2 + |n_3|^2, \quad \zeta = -|n_3|^2 + |n'_3|^2 \end{array} \right\}$$

with  $|S_{(\zeta)}| \lesssim N_2 N_2^3 N_3^6$  since for fixed  $n_3, n'_3$ , the frequencies  $n_2$  sit on a sphere of radius at most  $N_2$  and  $n \in \tilde{C}$  (see Lemma 6.1). We then have as above that for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ ,

$$\begin{aligned} M_2^2 &\lesssim \delta^{-2r} N_2^{-6} N_3^{-6} \sum_{n \neq n'} \sum_{n_2, n'_3} |S_{(n_1, n'_1, n_2, n'_3, \zeta)}|^2 \\ &\lesssim \delta^{-2r} N_2^{-6} N_3^{-6} N_2 \sum_{n \neq n'} \sum_{n_2, n'_3} |S_{(n_1, n'_1, n_2, n'_3, \zeta)}| \\ &\lesssim \delta^{-2r} N_2^{-6} N_3^{-6} N_2 |S_{(\zeta)}| \sim \delta^{-2r} N_2^{-1}. \end{aligned}$$

**Case  $\beta_5$ :** By symmetry this case is exactly the same as Case  $\beta_4$ .

We are now ready put all the estimates together and bound  $\mathcal{T}$  in cases  $\beta_1 - \beta_5$ :

$$\begin{aligned} \mathcal{T} &\lesssim \|\chi_C a_{n_1}\|_{\ell^2}^2 N_2^2 \|\mathcal{G}\mathcal{G}^*\| \lesssim \|a_{n_1}\|_{\ell^2}^2 N_2^2 (M_1 + M_2) \\ &\lesssim \|\chi_C a_{n_1}\|_{\ell^2}^2 \delta^{-2r} N_2^2 N_2^{-\frac{1}{2}} \sim \|\chi_C a_{n_1}\|_{\ell^2}^2 \delta^{-2r} N_2^{\frac{3}{2}}. \end{aligned}$$

**Case  $\beta_6$ :** In this case

$$S_{(n_1, n'_1, \zeta)} = \left\{ (n, n_2, n_3) : \begin{array}{l} n_2 = n_1 - n + n_3, \quad n_3 = n'_1 - n + n_2, \\ n_2 \neq n_3, n_1, \quad |n_2|^2 = |n_3|^2, \quad n \in \tilde{C} \end{array} \right\}.$$

At this point notice that  $\Delta\zeta = 1$  and that in this case  $N_2 \sim N_3$ . We have  $S_{(n_1, n'_1, \zeta)} \sim N_3^4$ . We then have as in (7.24)

$$M_2^2 \lesssim N_2^{-6+\epsilon} N_3^{-6} N_3^4 |S_{(\zeta)}|$$

where

$$S_{(\zeta)} = \left\{ (n_1, n'_1, n, n_2, n_3) : \begin{array}{l} n_2 = n_1 - n + n_3, \quad n_3 = n'_1 - n + n_2, \\ n_2 \neq n_3, n_1, \quad |n_2|^2 = |n_3|^2, \quad n \in \tilde{C} \end{array} \right\}$$

and  $|S_{(\zeta)}| \lesssim N_2^3 N_3^4$ . Hence, all in all we have that for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ ,  $M_2 \lesssim N_2^{-\frac{1}{2}+\epsilon}$  and as a consequence,

$$\mathcal{T} \lesssim \|\chi_C a_{n_1}\|_{\ell^2}^2 N_2^{-\frac{1}{2}+\epsilon}$$

in this case, which is a better bound.

To prove (7.6) we write

$$(7.27) \quad \mathcal{T} := \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}^3} \left| \sum_{\substack{n=n_1+n_2+n_3 \\ m=|n_1|^2+|n_2|^2+|n_3|^2}} \chi_C(n_1) a_{n_1} \frac{g_{n_2}(\omega)}{|n_2|^{\frac{3}{2}}} \frac{g_{n_3}(\omega)}{|n_3|^{\frac{3}{2}}} \right|^2.$$

We can repeat the argument above after checking the case  $n_2 = n_3$ . In this case (7.27) becomes

$$\mathcal{T} = \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}^3} \left| \sum_{\substack{n=n_1+2n_2 \\ m=|n_1|^2+2|n_2|^2}} \chi_C(n_1) a_{n_1} \frac{(g_{n_2}(\omega))^2}{|n_2|^3} \right|^2.$$

Let  $S_{(m,n)} = \{(n_1, n_2) / n = n_1 + 2n_2, m = |n_1|^2 + 2|n_2|^2\}$ , and note that by Lemma 6.1,  $|S_{(m,n)}| \lesssim \min(N_1, N_2^2)$ . Then

$$\begin{aligned} \mathcal{T} &\lesssim \min(N_1, N_2^2) \sum_{m,n} \sum_{S_{(m,n)}} \frac{|g_{n_2}(\omega)|^4}{|n_2|^6} |\chi_C a_{n_1}|^2 \\ &\sim \min(N_1, N_2^2) \sum_{n, n_1} \frac{|g_{\frac{n-n_1}{2}}(\omega)|^4}{|\frac{n-n_1}{2}|^6} |\chi_C a_{n_1}|^2 \lesssim \min(N_1, N_2^2) N_2^{-3+\varepsilon} \|\chi_C a_{n_1}\|_{l^2}^2, \end{aligned}$$

where we used (3.6) for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ .  $\square$

**Proposition 7.5.** *Let  $D_j$  and  $R_k$  be as above and fix  $N_1 \geq N_2 \geq N_3$ ,  $r, \delta > 0$  and  $C \in \mathcal{C}_{N_2}$ . Then there exists  $\mu > 0$  and a set  $\Omega_\delta \in A$  such that  $\mathbb{P}(\Omega_\delta^c) \leq e^{-\frac{1}{\delta^r}}$  such that for any  $\omega \in \Omega_\delta$  we have (7.7) and (7.8).*

*Proof.* Without loss of generality we assume that  $\tilde{D}_3 = D_3$ . We write,

$$(7.28) \quad \mathcal{T} := \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}^3} \left| \sum_{\substack{n=-n_1+n_2+n_3, \\ n_1 \neq n_2, n_3 \\ m=-|n_1|^2+|n_2|^2+|n_3|^2}} \chi_C(n_1) \frac{\bar{g}_{n_1}(\omega)}{|n_1|^{\frac{3}{2}}} \frac{g_{n_2}(\omega)}{|n_2|^{\frac{3}{2}}} a_{n_3} \right|^2,$$

where  $C \in \mathcal{C}_{N_2}$ . Let us now define

$$\sigma_{n, n_3} = \sum_{\substack{n=-n_1+n_2+n_3, \\ n_1 \neq n_2, n_3 \\ m=-|n_1|^2+|n_2|^2+|n_3|^2}} \chi_C(n_1) \frac{\bar{g}_{n_1}(\omega)}{|n_1|^{\frac{3}{2}}} \frac{g_{n_2}(\omega)}{|n_2|^{\frac{3}{2}}}.$$

If we denote by  $\mathcal{G}$  the matrix of entries  $\sigma_{n, n_3}$ , by using that the variation in  $m$  is at most  $N_1 N_2$  we can then continue the estimate of  $\mathcal{T}$  in (7.28) by

$$\mathcal{T} \lesssim \|a_{n_3}\|_{\ell^2}^2 N_1 N_2 \|\mathcal{G} \mathcal{G}^*\|.$$

Once again by Lemma 6.3,

$$\|\mathcal{G} \mathcal{G}^*\| \lesssim \max_n \sum_{n_3} |\sigma_{n, n_3}|^2 + \left( \sum_{n \neq n'} \left| \sum_{n_3} \sigma_{n, n_3} \bar{\sigma}_{n', n_3} \right|^2 \right)^{\frac{1}{2}} =: M_1 + M_2.$$

To estimate  $M_1$  we first define the set

$$S_{(m,n)} = \{(n_1, n_2, n_3) : n_1 \neq n_2, n_3, \quad n = -n_1 + n_2 + n_3, \quad m = -|n_1|^2 + |n_2|^2 + |n_3|^2\}.$$

By (6.3) we have that  $|S_{(m,n)}| \lesssim N_3^3 N_2^2$  since once fixed  $n_3$  we use  $m = -|n_2 + n_3 - n|^2 + |n_2|^2 + |n_3|^2$  to count  $n_2$  which lives on the intersection of a plane with a ball of radius  $N_2$ . Then as in (7.16) we have, for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ , that

$$\begin{aligned} M_1 &\lesssim \sup_{n,m} \sum_{n_3} \left| \sum_{\substack{n=-n_1+n_2+n_3, n_1 \neq n_2, n_3 \\ m=-|n_1|^2+|n_2|^2+|n_3|^2}} \frac{\bar{g}_{n_1}(\omega)}{|n_1|^{\frac{3}{2}}} \chi_C(n_1) \frac{g_{n_2}(\omega)}{|n_2|^{\frac{3}{2}}} \right|^2 \\ &\lesssim \sup_{n,m} \delta^{-r} N_1^{-3} N_2^{-3} |S_{(m,n)}| \lesssim \delta^{-2r} N_1^{-3} N_2^{-3} N_3^3 N_2^2 \sim \delta^{-2r} N_1^{-3} N_2^{-1} N_3^3. \end{aligned}$$

To estimate  $M_2$  we first write

$$M_2^2 = \sum_{n \neq n'} \left| \sum_{n_3} \sigma_{n,n_3} \bar{\sigma}_{n',n_3} \right|^2 \sim \sum_{n \neq n'} \left| \sum_{S_{(n,n',m)}} \frac{\bar{g}_{n_1}(\omega)}{|n_1|^{\frac{3}{2}}} \frac{g_{n_2}(\omega)}{|n_2|^{\frac{3}{2}}} \frac{g_{n'_1}(\omega)}{|n'_1|^{\frac{3}{2}}} \frac{\bar{g}_{n'_2}(\omega)}{|n'_2|^{\frac{3}{2}}} \right|^2,$$

where

$$S_{(n,n',m)} = \left\{ \begin{array}{l} n = -n_1 + n_2 + n_3, \quad n' = -n'_1 + n'_2 + n_3, \\ (n_3, n_1, n_2, n'_1, n'_2) : n_1 \neq n_2, n_3, \quad n'_1 \neq n'_2, n_3, \quad n_1, n'_1 \in C \\ m = -|n_1|^2 + |n_2|^2 + |n_3|^2, \quad m = -|n'_1|^2 + |n'_2|^2 + |n_3|^2 \end{array} \right\}.$$

Just like for the proof of (7.3) we need to organize the estimates according to whether some frequencies are the same or not, in all we have six cases.

- **Case  $\beta_1$ :**  $n_1, n'_1, n_2, n'_2$  are all different.
- **Case  $\beta_2$ :**  $n_1 = n'_1$ ;  $n_2 \neq n'_2$ .
- **Case  $\beta_3$ :**  $n_1 \neq n'_1$ ;  $n_2 = n'_2$ .
- **Case  $\beta_4$ :**  $n_1 \neq n'_2$ ;  $n_2 = n'_1$ .
- **Case  $\beta_5$ :**  $n_1 = n'_2$ ;  $n_2 \neq n'_1$ .
- **Case  $\beta_6$ :**  $n_1 = n'_2$ ;  $n_2 = n'_1$ .

**Case  $\beta_1$ :** In this case we let

$$S_{(m)} = \left\{ \begin{array}{l} n = -n_1 + n_2 + n_3, \quad n' = -n'_1 + n'_2 - n_3, \\ (n, n', n_3, n_1, n_2, n'_1, n'_2) : n_1 \neq n_2, n_3, \quad n'_1 \neq n'_2, n_3, \quad n_1, n'_1 \in C \\ m = -|n_1|^2 + |n_2|^2 + |n_3|^2, \quad m = -|n'_1|^2 + |n'_2|^2 + |n_3|^2 \end{array} \right\}$$

with  $|S_{(m)}| \lesssim N_1^2 N_2^6 N_3^3$ . As in the argument we use for (7.16), this gives that for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ ,

$$M_2^2 \lesssim \delta^{-4r} N_1^{-6} N_2^{-6} N_1^2 N_2^6 N_3^3 \sim \delta^{-4r} N_1^{-4} N_3^3.$$

**Case  $\beta_2$ :** In this case we define two sets. We start with

$$S_{(n,n',n_2,n'_2,m)} = \left\{ \begin{array}{l} n = -n_1 + n_2 + n_3, \quad n' = -n_1 + n_3 + n'_2, \\ (n_3, n_1) : n_1 \neq n_2, n'_2, n_3 \\ m = -|n_1|^2 + |n_2|^2 + |n_3|^2, \quad m = -|n_1|^2 + |n_3|^2 + |n'_2|^2 \end{array} \right\}.$$

To compute  $|S_{(n,n',n_2,n'_2,m)}|$  we count  $n_3$  then  $n_1$  is determined. Since  $n_3$  sit on a plane we have by (6.3) that  $|S_{(n,n',n_2,n'_2,m)}| \lesssim N_3^2$ . Then we set

$$S_{(m)} = \left\{ \begin{array}{l} n = -n_1 + n_2 + n_3, \quad n' = -n_1 + n_3 + n'_2, \\ (n, n', n_3, n_1, n_2, n'_2) : n_1 \neq n_2, n'_2, n_3 \\ m = -|n_1|^2 + |n_2|^2 + |n_3|^2, \quad m = -|n_1|^2 + |n_3|^2 + |n'_2|^2 \end{array} \right\}$$

for which  $|S_{(m)}| \lesssim N_1 N_2^6 N_3^3$ . We then have following the argument in (7.17)-(7.23) that for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ ,

$$\begin{aligned} M_2^2 &\lesssim \delta^{-2r} N_1^{-6} N_2^{-6} \sum_{n \neq n'} \sum_{n_2, n'_2} |S_{(n,n',n_2,n'_2,m)}|^2 \\ &\lesssim \delta^{-2r} N_1^{-6} N_2^{-6} N_3^2 \sum_{n \neq n'} \sum_{n_2, n'_2} |S_{(n,n',n_2,n'_2,m)}| \\ &\lesssim \delta^{-2r} N_1^{-6} N_2^{-6} N_3^2 |S_{(m)}| \sim \delta^{-2r} N_1^{-5} N_3^5. \end{aligned}$$

**Case  $\beta_3$ :** In this case we define first

$$S_{(n,n',n_1,n'_1,m)} = \left\{ \begin{array}{l} n = -n_1 + n_2 + n_3, \quad n' = -n'_1 + n_2 + n_3, \\ (n_2, n_3) : n_2, n_3 \neq n_1, n'_1, \quad n_1, n'_1 \in C \\ m = -|n_1|^2 + |n_2|^2 + |n_3|^2, \quad m = -|n'_1|^2 + |n_2|^2 + |n_3|^2 \end{array} \right\}$$

with  $|S_{(n,n',n_1,n'_1,m)}| \lesssim N_3^2$  since  $n_2$  is determined by  $n_3$  and this one lines on a sphere of radius at most  $N_1$ . On the other hand

$$S_{(m)} = \left\{ \begin{array}{l} n = -n_1 + n_2 + n_3, \quad n' = -n'_1 + n_2 + n_3, \\ (n, n', n_2, n_1, n'_1, n_3) : n_2, n_3 \neq n_1, n'_1, \quad n_1, n'_1 \in C \\ m = -|n_1|^2 + |n_2|^2 + |n_3|^2, \quad m = -|n'_1|^2 + |n_2|^2 + |n_3|^2 \end{array} \right\}$$

with  $|S_{(m)}| \lesssim N_1^2 N_3^3 N_3^3$ . Hence arguing as above we have

$$\begin{aligned} M_2^2 &\lesssim \delta^{-2r} N_1^{-6} N_2^{-6} \sum_{n \neq n'} \sum_{n_1, n'_1} |S_{(n,n',n_1,n'_1,m)}|^2 \\ &\lesssim \delta^{-2r} N_1^{-6} N_2^{-6} N_3^2 \sum_{n \neq n'} \sum_{n_1, n'_1} |S_{(n,n',n_1,n'_1,m)}| \\ &\lesssim \delta^{-2r} N_1^{-6} N_2^{-6} N_3^2 |S_{(m)}| \sim \delta^{-2r} N_1^{-4} N_2^{-3} N_3^5, \end{aligned}$$

for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ .

**Case  $\beta_4$ :** We define the two sets

$$S_{(n,n',n_1,n'_2,m)} = \left\{ \begin{array}{l} n = -n_1 + n_2 + n_3, \quad n' = -n_2 + n'_2 + n_3, \\ (n_2, n_3) : n_2, n_3 \neq n_1, n'_2 \\ m = -|n_1|^2 + |n_2|^2 + |n_3|^2, \quad m = -|n_2|^2 + |n'_2|^2 + |n_3|^2 \end{array} \right\}$$



for which, since  $n_3$  lives on a sphere of radius at most  $N_1$ , we have  $|S_{(n,n',n_1,n'_2,m)}| \lesssim \min(N_1, N_3^2)$  and

$$S_{(m)} = \left\{ \begin{array}{l} n = -n_1 + n_2 + n_3, \quad n' = -n_2 + n'_2 + n_3, \\ (n, n', n_3, n_1, n'_2, n_2) : n_2, n_3 \neq n_1, n'_2 \\ m = -|n_1|^2 + |n_2|^2 + |n_3|^2, \quad m = -|n_2|^2 + |n'_2|^2 + |n_3|^2 \end{array} \right\}$$

with  $|S_{(m)}| \lesssim N_1 N_3^3 N_2^6$ . We then have in this case that

$$\begin{aligned} M_2^2 &\lesssim \delta^{-2r} N_1^{-6} N_2^{-6} \sum_{n \neq n'} \sum_{n_1, n'_2} |S_{(n,n',n_1,n'_2,m)}|^2 \\ &\lesssim \delta^{-2r} N_1^{-6} N_2^{-6} \min(N_1, N_3^2) \sum_{n \neq n'} \sum_{n_1, n'_2} |S_{(n,n',n_1,n'_2,m)}| \\ &\lesssim \delta^{-2r} N_1^{-6} N_2^{-6} \min(N_1, N_3^2) |S_{(m)}| \sim \delta^{-2r} N_1^{-4} N_3^3, \end{aligned}$$

for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ .

**Case  $\beta_5$ :** This case is exactly the same as Case  $\beta_4$ .

We are now ready to estimate  $\mathcal{T}$  in cases  $\beta_1 - \beta_5$ .

$$\begin{aligned} \mathcal{T} &\lesssim \|a_{n_2}\|_{\ell^2}^2 N_1 N_2 \|\mathcal{G}\mathcal{G}^*\| \lesssim \|a_{n_2}\|_{\ell^2}^2 N_1 N_2 (M_1 + M_2) \\ &\lesssim \|a_{n_2}\|_{\ell^2}^2 \delta^{-4r} [N_1 N_2 (N_1^{-\frac{5}{2}} N_3^{\frac{5}{2}} + N_1^{-2} N_3^{\frac{3}{2}})] \lesssim \delta^{-4r} [N_1^{-\frac{3}{2}} N_2 N_3^{\frac{5}{2}} + N_1^{-1} N_2 N_3^{\frac{3}{2}}] \|a_{n_2}\|_{\ell^2}^2. \end{aligned}$$

**Case  $\beta_6$ :** In this case

$$S_{(n,n')} = \left\{ \begin{array}{l} n = -n_1 + n_2 + n_3, \quad n' = -n_2 + n_1 + n_3, \\ (n_3, n_1, n_2) : n_1 \neq n_2, n_3, \quad |n_1|^2 = |n_2|^2, \quad m = |n_3|^2 \end{array} \right\}$$

so  $N_1 \sim N_2$  and  $\Delta m \lesssim N_3^2$ . We have  $S_{(n,n',m)} \lesssim N_2^3 N_3$  since  $n_3$  sits on a sphere of radius at most  $N_3$ . We then have as in (7.24) that for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ ,

$$M_2^2 \lesssim N_1^{-6+\varepsilon} N_2^{-6} N_3^3 N_3 |S_{(m)}|,$$

where

$$S_{(m)} = \left\{ \begin{array}{l} n = -n_1 + n_2 + n_3, \quad n' = -n_2 + n_1 + n_3, \\ (n, n', n_3, n_1, n_2) : n_1 \neq n_2, n_3, \quad |n_1|^2 = |n_2|^2, \quad m = |n_3|^2 \end{array} \right\}$$

and  $|S_{(m)}| \lesssim N_3 N_2^3 N_2$  since again  $n_3$  sits on a sphere of radius at most  $N_3$  and for fixed  $n_2$  we have that  $n_1$  sits on a sphere of radius at most  $N_2$ . Hence  $M_2 \lesssim N_1^{-\frac{5}{2}+\varepsilon} N_3$  and as a consequence

$$\mathcal{T} \lesssim \|a_{n_3}\|_{\ell^2}^2 N_3^3 N_1^{-\frac{5}{2}+\varepsilon}.$$

The proof of (7.8) proceeds very much like the one we just presented. Actually when  $n_1 = n_2$  the estimates may be made better since we will not have planes, but spheres involved in the counting. On the other hand here  $n_1 = n_2$  could be a possibility. In this case we have

$$\mathcal{T} := \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}^3} \left| \sum_{\substack{n = -2n_1 + n_3 \\ m = -2|n_1|^2 + |n_3|^2}} \frac{(\bar{g}_{n_1}(\omega))^2}{|n_1|^3} a_{n_3} \right|^2$$

Let  $S_{(m,n)} = \{(n_1, n_3) / n = -2n_1 + n_3, n_3 \neq n_1, m = -2|n_1|^2 + |n_3|^2\}$ , and note that by Lemma 6.1,  $|S_{(m,n)}| \lesssim \min(N_1, N_3^2)$ . Then using (3.6) for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ , we have that

$$\begin{aligned} \mathcal{T} &\lesssim \min(N_1, N_3^2) \sum_{m,n} \sum_{S_{(m,n)}} \frac{|g_{n_1}(\omega)|^4}{|n_1|^6} |a_{n-2n_1}|^2 \\ &\lesssim \min(N_1, N_3^2) N_1^{-3+\varepsilon} \|a_{n_3}\|_{\ell^2}^2. \end{aligned}$$

□

**Proposition 7.6.** *Let  $D_j$  and  $R_k$  be as above and fix  $N_1 \geq N_2 \geq N_3$ ,  $r, \delta > 0$  and  $C \in \mathcal{C}_{N_2}$ . Then there exists  $\mu, \varepsilon > 0$  and a set  $\Omega_\delta \in A$  such that  $\mathbb{P}(\Omega_\delta^c) \leq e^{-\frac{1}{\delta^r}}$  such that for any  $\omega \in \Omega_\delta$  we have (7.9) for any  $0 \leq \theta \leq 1$ , and (7.10) .*

*Proof.* We now move to (7.9). Without loss of generality we assume that  $\tilde{D}_i = D_i$ ,  $i = 2, 3$ . We have that

$$\mathcal{T} := \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}^3} \left| \sum_{\substack{n=n_1+n_2+n_3 \\ m=|n_1|^2+|n_2|^2+|n_3|^2}} \chi_C(n_1) \frac{g_{n_1}(\omega)}{|n_1|^{\frac{3}{2}}} a_{n_2} a_{n_3} \right|^2.$$

Then

$$\mathcal{T} \lesssim \sum_{m \in \mathbb{Z}, n \in \tilde{C}} \left| \sum_{n_2, n_3} \sigma_{n, n_2} a_{n_2} a_{n_3} \right|^2,$$

where  $\tilde{C}$  is again a cube of sidelength approximately  $N_2$  and

$$\sigma_{n, n_2} = \begin{cases} \frac{g_{n-n_2-n_3}(\omega)}{|n-n_2-n_3|^{\frac{3}{2}}} & \text{if } m = |n-n_2-n_3|^2 + |n_2|^2 + |n_3|^2 \\ 0 & \text{otherwise} \end{cases}.$$

Note that  $\sigma_{n, n_2}$  also depends on  $m$  and  $n_3$  but we estimate it independently of  $m$  and  $n_3$  and take the supremum on them. By Cauchy-Schwarz in  $n_3$ , the fact that  $\Delta m \lesssim N_1 N_2$  and Lemma 6.3 we have that

$$\mathcal{T} \lesssim \|a_{n_3}\|_{\ell^2}^2 \|a_{n_2}\|_{\ell^2}^2 N_1 N_2 N_3^3 \|\mathcal{G}\mathcal{G}^*\|;$$

and as usual by Lemma 6.3 we have that

$$\|\mathcal{G}\mathcal{G}^*\| \lesssim \max_n \sum_{n_2} |\sigma_{n, n_2}|^2 + \left( \sum_{n \neq n' \in \tilde{C}} \left| \sum_{n_2} \sigma_{n, n_2} \bar{\sigma}_{n', n_2} \right|^2 \right)^{\frac{1}{2}} =: M_1 + M_2.$$

To estimate  $M_1$  we will use the set  $S(n, n_3, m) := \{n_2 : m = |n-n_2-n_3|^2 + |n_2|^2 + |n_3|^2\}$  with cardinality  $|S(n, n_3, m)| \lesssim N_1$  since this set describes a sphere whose radius is at most  $N_1$ . Using (3.6) we estimate

$$(7.29) \quad M_1 \lesssim \sum_{n_2 \in S(n, n_3, m)} N_1^{-3+\varepsilon} \lesssim N_1^{-2+\varepsilon},$$

for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ .

To estimate  $M_2$  we first define the set

$$S_{(n_3, m)} := \{(n, n', n_2) : m = |n - n_2 - n_3|^2 + |n_2|^2 + |n_3|^2, m = |n' - n_2 - n_3|^2 + |n_2|^2 + |n_3|^2\}$$

and note that  $|S_{(n_3, m)}| \lesssim N_2^3 N_1^2$ . Then using (3.4) and proceeding using arguments similar to those presented for (7.17)-(7.23) we have

$$(7.30) \quad M_2^2 \lesssim \delta^{-2r} N_1^{-6} |S_{(n_3, m)}| \lesssim \delta^{-2r} N_1^{-6} N_2^3 N_1^2.$$

By using the estimates of  $M_1$  and  $M_2$  we obtain that for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ ,

$$(7.31) \quad \mathcal{T} \lesssim \|a_{n_3}\|_{\ell^2}^2 \|a_{n_2}\|_{\ell^2}^2 \delta^{-r} N_1 N_2 N_3^3 N_1^{-2} N_2^{\frac{3}{2}} \lesssim \|a_{n_3}\|_{\ell^2}^2 \|a_{n_2}\|_{\ell^2}^2 \delta^{-r} N_1^{-1} N_2^{\frac{5}{2}} N_3^3.$$

We will interpolate this estimate with the one we obtain below.

$$(7.32) \quad \begin{aligned} \mathcal{T} &\lesssim N_1 N_2 \sup_m \sum_{n \in \mathbb{Z}^3} \left| \sum_{\substack{n=n_1+n_2+n_3 \\ m=|n_1|^2+|n_2|^2+|n_3|^2}} \chi_C(n_1) \frac{g_{n_1}(\omega)}{|n_1|^{\frac{3}{2}}} a_{n_2} a_{n_3} \right|^2 \\ &\lesssim N_1 N_2 \|a_{n_3}\|_{\ell^2}^2 \sup_m \sum_{n, n_3 \in \mathbb{Z}^3} \left| \sum_{\substack{n=n_1+n_2+n_3 \\ m=|n_1|^2+|n_2|^2+|n_3|^2}} \chi_C(n_1) \frac{g_{n_1}(\omega)}{|n_1|^{\frac{3}{2}}} a_{n_2} \right|^2 \\ &\lesssim N_1 N_2 \|a_{n_3}\|_{\ell^2}^2 N_1^{-3+\varepsilon} \sup_m \sum_{n, n_3 \in \mathbb{Z}^3} |S_{(n, n_3, m)}| \sum_{S_{(n, n_3, m)}} |a_{n_2}|^2 \\ &\lesssim N_1 N_2 \|a_{n_3}\|_{\ell^2}^2 N_1^{-3+\varepsilon} \min(N_2^2, N_1) \sup_m \sum_{n_2} \sum_{S_{(n_2, m)}} |a_{n_2}|^2 \\ &\lesssim N_1 N_2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n_2}\|_{\ell^2}^2 N_1^{-3+\varepsilon} \min(N_2^2, N_1) N_1 N_3^3 \\ &\sim N_1^{-1+\varepsilon} N_3^3 N_2 \min(N_2^2, N_1) \|a_{n_2}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2, \end{aligned}$$

where  $S_{(n, n_3, m)} = \{(n_1, n_2) / n = n_1 + n_2 + n_3, n_1 \in C, m = |n_1|^2 + |n_2|^2 + |n_3|^2\}$ , with  $|S_{(n, n_3, m)}| \lesssim \min(N_2^2, N_1)$ ,  $S_{(n_2, m)} = \{(n, n_1, n_3) / n = n_1 + n_2 + n_3, n_1 \in C, m = |n_1|^2 + |n_2|^2 + |n_3|^2\}$  with  $|S_{(n_2, m)}| \lesssim N_1 N_3^3$  and we used (3.6) for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ .

The estimate of (7.9) now follows by interpolating (7.32) with (7.31).

We now move to (7.10). Again without loss of generality we assume that  $\tilde{D}_i = D_1$ ,  $i = 1, 3$ . We use duality and the change of variables  $\zeta = m - |n_1|^2 = |n_2|^2 + |n_3|^2$  as in the proof of Proposition 7.3. We note that the variation of  $\zeta$  is at most  $N_2^2$  and that  $n \in \tilde{C}$ , a cube of side length approximately  $N_2$ . We use (3.6) for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$

and Lemma 6.1 to reduce the bound for  $\mathcal{T}$  to estimating,

$$\begin{aligned}
& N_2^2 \sup_{\zeta} \sum_{n_1 \in \mathbb{Z}^3} \left| \sum_{\substack{n_1 = n - n_2 - n_3 \\ \zeta = |n_2|^2 + |n_3|^2}} \chi_{\tilde{C}}(n) k_n \frac{g_{n_2}(\omega)}{|n_2|^{\frac{3}{2}}} a_{n_3} \right|^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \\
& \lesssim N_2^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \sup_{\zeta} \sum_{n_1, n_3 \in \mathbb{Z}^3} \left| \sum_{\substack{n_1 = n - n_2 - n_3 \\ \zeta = |n_2|^2 + |n_3|^2}} \chi_{\tilde{C}}(n) k_n \frac{g_{n_2}(\omega)}{|n_2|^{\frac{3}{2}}} \right|^2 \\
& \lesssim N_2^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 N_2^{-3+\varepsilon} \sup_{\zeta} \sum_{n_1, n_3 \in \mathbb{Z}^3} |S_{(n_1, n_3, \zeta)}| \sum_{S_{(n_1, n_3, \zeta)}} \chi_{\tilde{C}}(n) |k_n|^2 \\
& \lesssim N_2^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 N_2^{-3+\varepsilon} N_2 \sup_{\zeta} \sum_n \sum_{S_{(n, \zeta)}} |k_n|^2 \\
& \lesssim N_2^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 N_2^{-3+\varepsilon} N_2 N_2 N_3^3 \|k_n\|_{\ell^2}^2 \\
& \sim N_2^{1+\varepsilon} N_3^3 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|k_n\|_{\ell^2}^2,
\end{aligned}$$

where  $S_{(n_1, n_3, \zeta)} = \{(n, n_2) / n_1 = n - n_2 - n_3, n \in \tilde{C}, \zeta = |n_2|^2 + |n_3|^2\}$ , with  $|S_{(n_1, n_3, \zeta)}| \lesssim N_2$ ,  $S_{(n, \zeta)} = \{(n_1, n_2, n_3) / n_1 = n - n_2 - n_3, n_1 \in C, \zeta = |n_2|^2 + |n_3|^2\}$  with  $|S_{(n, \zeta)}| \lesssim N_2 N_3^3$ .  $\square$

**Proposition 7.7.** *Let  $R_k$  be as above and fix  $N_1 \geq N_2 \geq N_3$ ,  $r, \delta > 0$  and  $C \in \mathcal{C}_{N_2}$ . Then there exists  $\mu > 0$  and a set  $\Omega_\delta \in A$  such that  $\mathbb{P}(\Omega_\delta^c) \leq e^{-\frac{1}{\delta^r}}$  such that for any  $\omega \in \Omega_\delta$  we have (7.11), (7.12) and (7.13).*

*Proof.* We start by estimating (7.11). We consider

$$(7.33) \quad \mathcal{T} := \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}^3} \left| \sum_{\substack{n = n_1 + n_2 - n_3 \\ n_1, n_2 \neq n_3 \\ m = |n_1|^2 + |n_2|^2 - |n_3|^2}} \chi_C(n_1) \frac{\bar{g}_{n_1}(\omega)}{|n_1|^{\frac{3}{2}}} \frac{\bar{g}_{n_2}(\omega)}{|n_2|^{\frac{3}{2}}} \frac{g_{n_3}(\omega)}{|n_3|^{\frac{3}{2}}} \right|^2.$$

Note that if  $n_1 = n_2$  we get -say-  $(\bar{g}_{n_1}(\omega))^2$  which are still independent and mean zero since the  $g_{n_i}(\omega)$  are complex Gaussian random variables. Hence we are still within the framework of Lemma 3.4 for  $\omega$  and so this case does not require a separate argument.

We first remark that the variation  $\Delta m \sim N_1 N_2$ . Then we use Lemma 3.4 to obtain, for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ , that

$$\mathcal{T} \lesssim \delta^{-\frac{3}{2}r} N_1 N_2 N_1^{-3} N_2^{-3} N_3^{-3} \sup_m |S_{(m)}| \lesssim \delta^{-\frac{3}{2}r} N_1^{-1} N_2$$

where  $S_{(m)} := \{(n, n_1, n_2, n_3) / n = n_1 + n_2 - n_3, n_1 \in C; m = |n_1|^2 + |n_2|^2 - |n_3|^2\}$  and  $|S_{(m)}| \lesssim N_3^3 N_2^3 N_1$ .

To estimate (7.12) and (7.13) we proceed just like above.  $\square$

**7.2. Bilinear Estimate.** We prove the following bilinear estimate which will be used in the next Section 8. We use the same notation as in the previous subsection 7.1.

**Proposition 7.8.** *Fix  $N_1 \geq N_2 \geq N_3$  and  $r, \delta > 0$ . Assume also that  $C$  is a cube of sidelength  $N_2$ . Then there exists  $\mu, \varepsilon > 0$  and a set  $\Omega_\delta \in A$  such that  $\mathbb{P}(\Omega_\delta^c) \leq e^{-\frac{1}{\delta^r}}$  and such that for any  $\omega \in \Omega_\delta$  and  $0 \leq \theta \leq 1$  we have the following estimate:*

$$(7.34) \quad \|P_C R_1 D_2\|_{L^2([0,1] \times \mathbb{T}^3)} \lesssim \delta^{-\mu r} N_1^{-\frac{1}{2} + \varepsilon} \min(N_1, N_2^2)^{\frac{1-\theta}{2}} N_2^{\frac{1}{2} + \frac{3\theta}{4}} \|D_2\|_{U_\Delta^2 L_x^2}.$$

*Proof.* To prove (7.34) we follow the argument presented for (7.9) after performing a Cauchy-Schwarz. In fact we have

$$\|P_C R_1 D_2\|_{L^2}^2 = \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}^3} \left| \sum_{\substack{n=n_1+n_2 \\ m=|n_1|^2+|n_2|^2}} \chi_C(n_1) \frac{g_{n_1}(\omega)}{|n_1|^{\frac{3}{2}}} a_{n_2} \right|^2.$$

Then

$$\|P_C R_1 D_2\|_{L^2}^2 \lesssim \sum_{m; n \in \tilde{C}} \left| \sum_{n_2} \sigma_{n, n_2} a_{n_2} \right|^2,$$

where  $\tilde{C}$  is a cube of side length approximately  $N_2$  and

$$\sigma_{n, n_2} = \begin{cases} \frac{g_{n-n_2}(\omega)}{|n-n_2|^{\frac{3}{2}}} & \text{if } m = |n-n_2|^2 + |n_2|^2 \\ 0 & \text{otherwise} \end{cases}.$$

We then have

$$\|P_C R_1 D_2\|_{L^2}^2 \lesssim \|a_{n_2}\|_{\ell^2}^2 N_1 N_2 \|\mathcal{G}\mathcal{G}^*\|.$$

Then using the estimates (7.29) and (7.30), we obtain for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ ,

$$(7.35) \quad \|P_C R_1 D_2\|_{L^2} \lesssim \|a_{n_2}\|_{\ell^2} \delta^{-r} N_1^{-\frac{1}{2} + \varepsilon} N_2^{\frac{5}{4}}.$$

We also use (7.32) to estimate (7.9). By repeating the argument to prove (7.32) in our bilinear setting, we obtain, for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ , that

$$\begin{aligned} \|P_C R_1 D_2\|_{L^2}^2 &\lesssim N_1 N_2 \sup_m \sum_{n \in \mathbb{Z}^3} \left| \sum_{\substack{n=n_1+n_2 \\ m=|n_1|^2+|n_2|^2}} \chi_C(n_1) \frac{g_{n_1}(\omega)}{|n_1|^{\frac{3}{2}}} a_{n_2} \right|^2 \\ &\lesssim N_1 N_2 \|a_{n_3}\|_{\ell^2}^2 \sup_m \sum_{n \in \mathbb{Z}^3} \left| \sum_{\substack{n=n_1+n_2 \\ m=|n_1|^2+|n_2|^2}} \chi_C(n_1) \frac{g_{n_1}(\omega)}{|n_1|^{\frac{3}{2}}} a_{n_2} \right|^2 \\ &\lesssim N_1 N_2 N_1^{-3+\varepsilon} \sup_m \sum_{n \in \mathbb{Z}^3} |S_{(n,m)}| \sum_{S_{(n,m)}} |a_{n_2}|^2 \\ &\lesssim N_1 N_2 N_1^{-3+\varepsilon} \min(N_2^2, N_1) \sup_m \sum_{n_2} \sum_{S_{(n_2,m)}} |a_{n_2}|^2 \\ &\lesssim N_1 N_2 \|a_{n_2}\|_{\ell^2}^2 N_1^{-3+\varepsilon} \min(N_2^2, N_1) N_1 \end{aligned}$$

$$\sim N_1^{-1+\varepsilon} N_2 \min(N_2^2, N_1) \|a_{n_2}\|_{\ell^2}^2,$$

where  $S_{(n,m)} = \{(n_1, n_2) / n = n_1 + n_2, n_1 \in C, m = |n_1|^2 + |n_2|^2\}$ , with  $|S_{(n,m)}| \lesssim \min(N_2^2, N_1)$ ,  $S_{(n_2,m)} = \{(n, n_1) / n = n_1 + n_2, n_1 \in C, m = |n_1|^2 + |n_2|^2\}$  with  $|S_{(n_2,m)}| \lesssim N_1$  and we used (3.6). Hence we also have

$$(7.36) \quad \|P_C R_2 D_1\|_{L^2} \lesssim \|a_{n_2}\|_{\ell^2} N_1^{-\frac{1}{2}+\varepsilon} N_2^{\frac{1}{2}} \min(N_2^2, N_1)^{\frac{1}{2}}.$$

By interpolating (7.35) and (7.36) we finally have the estimate (7.34).  $\square$

**Remark 7.9.** *Later we only use (7.34) with  $\theta = 1$  while estimating in the next section the term  $J_4$  defined (2.13).*

## 8. PROOF OF PROPOSITION 5.2

In this section we use a notation similar to the one introduced at the beginning of Section 7 to indicate deterministic and random functions. The reader should pay attention though to the fact that the new functions we define in this section have a different normalization than the ones in Section 7, hence the slight change of notation.

If  $u_i$  is random, then we write

$$\widehat{P_{N_i} u_i}(n) = \chi_{\{|n| \sim N_i\}}(n) \frac{g_n(\omega)}{|n|^{\frac{5}{2}-\alpha}} e^{i|n|^2 t} \sim \widehat{\mathcal{R}_i}(n);$$

while if  $u_i$  is deterministic we write

$$\widehat{P_{N_i} u_i}(n) \sim \widehat{\mathcal{D}_i}(n)$$

where  $\widehat{\mathcal{D}_i}(n)$  is supported in  $\{|n| \sim N_i\}$ . Below we will make a heavy use of Proposition 7.1 when functions  $\mathcal{R}_i$  are involved instead of  $R_i$ . This will not be explicitly mentioned every time, but the reader will notice that a normalization will take place in the appropriate places.

We first estimate the terms  $J_2 - J_7$  then we move to  $J_1$ .

**8.1. Estimates Involving the Term  $J_2$ .** We start by estimating the term  $J_2$  as in (2.11). This reduces to analyzing the sum over  $N_0, N_1, \dots, N_3$  of quadri-linear forms:

$$(8.1) \quad \left| \int_{\mathbb{T}} \int_{\mathbb{T}^3} T_{\Upsilon} (P_{N_1} u_1, P_{N_2} \bar{u}_2, P_{N_3} u_3) P_{N_0} \bar{h} \, dx dt \right|$$

where  $T_{\Upsilon}$  is the multilinear operator defined in (7.2).

The general outline of the proof involves the use of Cauchy-Schwarz, cutting the top frequency window if necessary, the transfer principle Proposition 4.8 and suitably applying the trilinear estimates in subsection 7.1. Without any loss of generality, we then fix the relative ordering  $N_1 \geq N_2 \geq N_3$  above and consider the following cases where  $T_{\Upsilon}$  acts on :

- **Case 1:**    a)  $(\bar{\mathcal{R}}_1, \mathcal{R}_2, \mathcal{R}_3)$     b)  $(\mathcal{R}_1, \bar{\mathcal{R}}_2, \mathcal{R}_3)$     c)  $(\mathcal{R}_1, \mathcal{R}_2, \bar{\mathcal{R}}_3)$
- **Case 2:**    a)  $(\bar{\mathcal{D}}_1, \mathcal{R}_2, \mathcal{R}_3)$     b)  $(\mathcal{D}_1, \bar{\mathcal{R}}_2, \mathcal{R}_3)$     c)  $(\mathcal{D}_1, \mathcal{R}_2, \bar{\mathcal{R}}_3)$
- **Case 3:**    a)  $(\bar{\mathcal{R}}_1, \mathcal{R}_2, \mathcal{D}_3)$     b)  $(\mathcal{R}_1, \bar{\mathcal{R}}_2, \mathcal{D}_3)$     c)  $(\mathcal{R}_1, \mathcal{R}_2, \bar{\mathcal{D}}_3)$
- **Case 4:**    a)  $(\bar{\mathcal{R}}_1, \mathcal{D}_2, \mathcal{R}_3)$     b)  $(\mathcal{R}_1, \bar{\mathcal{D}}_2, \mathcal{R}_3)$     c)  $(\mathcal{R}_1, \mathcal{D}_2, \bar{\mathcal{R}}_3)$ ,
- **Case 5:**    a)  $(\bar{\mathcal{D}}_1, \mathcal{R}_2, \mathcal{D}_3)$     b)  $(\mathcal{D}_1, \bar{\mathcal{R}}_2, \mathcal{D}_3)$     c)  $(\mathcal{D}_1, \mathcal{R}_2, \bar{\mathcal{D}}_3)$
- **Case 6:**    a)  $(\bar{\mathcal{R}}_1, \mathcal{D}_2, \mathcal{D}_3)$     b)  $(\mathcal{R}_1, \bar{\mathcal{D}}_2, \mathcal{D}_3)$     c)  $(\mathcal{R}_1, \mathcal{D}_2, \bar{\mathcal{D}}_3)$
- **Case 7:**    a)  $(\bar{\mathcal{D}}_1, \mathcal{D}_2, \mathcal{R}_3)$     b)  $(\mathcal{D}_1, \bar{\mathcal{D}}_2, \mathcal{R}_3)$     c)  $(\mathcal{D}_1, \mathcal{D}_2, \bar{\mathcal{R}}_3)$

- **Case 8:**    **a)**  $(\bar{\mathcal{D}}_1, \mathcal{D}_2, \mathcal{D}_3)$     **b)**  $(\mathcal{D}_1, \bar{\mathcal{D}}_2, \mathcal{D}_3)$     **c)**  $(\mathcal{D}_1, \mathcal{D}_2, \bar{\mathcal{D}}_3)$

**Case 1, a):** If  $N_1 \sim N_0$  we cut the support of  $\hat{h}$  and hence that of  $\hat{\mathcal{R}}_1$  with respect to cubes  $C$  of side length  $N_2$  and use Cauchy-Schwarz to get

$$(8.1) \lesssim \|P_C P_{N_0} h\|_{L^2_{x,t}} \|T_\Upsilon(P_C \bar{\mathcal{R}}_1, \mathcal{R}_2, \mathcal{R}_3)\|_{L^2_{x,t}}.$$

To estimate the second factor we use (7.13) and normalization we obtain the bound

$$(8.1) \lesssim \delta^{-\mu r} N_1^{-\frac{3}{2}+\alpha} N_2^{-\frac{1}{2}+\alpha} N_3^{1-\alpha} \|P_C P_{N_0} h\|_{L^2_{x,t}}.$$

Renormalizing  $h$  and using the embedding (4.7) we obtain

$$(8.2) \quad \left| \int_0^\pi \int_{\mathbb{T}^3} T_\Upsilon(P_C P_{N_1} u_1, P_{N_2} \bar{u}_2, P_{N_3} u_3) P_C P_{N_0} \bar{h} \, dx dt \right| \\ \lesssim \delta^{-\mu r} N_0^s N_1^{-\frac{3}{2}+\alpha} N_2^{-\frac{1}{2}+\alpha} N_3^{-1+\alpha} \|P_C P_{N_0} h\|_{Y^{-s}} \lesssim \delta^{-\mu r} N_1^{s+\alpha-\frac{3}{2}} \|P_C P_{N_0} h\|_{Y^{-s}},$$

which suffices provided  $s + \alpha < \frac{3}{2}$  and  $\alpha < \frac{1}{2}$ .

If  $N_1 \sim N_2$  the cut with the cubes  $C$  above is not needed and the argument proceeds as above. The condition here is  $s < 2 - 2\alpha$ .

**Case 1, b), c)** are treated similarly replacing (7.13) respectively by (7.12) and (7.11).

**Case 2, a):** Let us assume that  $N_0 \sim N_1$ . We use the argument above. To estimate

$$\|T_\Upsilon(P_C \bar{\mathcal{D}}_1, \mathcal{R}_2, \mathcal{R}_3)\|_{L^2_{x,t}}.$$

we use (7.6) and after taking derivatives and normalizing we obtain the bound

$$(8.2) \lesssim \delta^{-\mu r} N_2^{\alpha-\frac{1}{4}} \|P_{N_1} u_1\|_{U^2_{\Delta} H^s} \|P_C P_{N_0} h\|_{Y^{-s}}$$

which suffices provided  $\alpha < \frac{1}{4}$ . A similar bound holds when  $N_1 \sim N_2$  without cutting with cubes  $C$ .

**Case 2, b), c)** are treated similarly replacing (7.6) by (7.5).

**Case 3, a), b) c):** We use the argument above with (7.7) and (7.8) accordingly. If  $N_1 \sim N_0$  we obtain a bound of the form

$$(8.2) \lesssim \delta^{-\mu r} N_0^s [N_1^{\alpha-\frac{7}{4}} N_2^{-\frac{1}{2}+\alpha} N_3^{\frac{5}{4}-s} + N_1^{\alpha-\frac{3}{2}} N_2^{-\frac{1}{2}+\alpha} N_3^{\frac{3}{4}-s}] \|P_{N_3} u_3\|_{U^2_{\Delta} H^s} \|P_C P_{N_0} h\|_{Y^{-s}} \\ \lesssim \delta^{-\mu r} N_1^{-\beta(s,\alpha)} \|P_{N_3} u_3\|_{U^2_{\Delta} H^s} \|P_C P_{N_0} h\|_{Y^{-s}}$$

provided  $\alpha < \frac{1}{4}$  and  $s + \alpha < \frac{3}{2}$ . A similar bound holds when  $N_1 \sim N_2$  without cutting with cubes  $C$ .

**Case 4, a), b), c):** We use the argument above with (7.3) and (7.4) accordingly. If  $N_1 \sim N_0$  we obtain a bound of the form

$$(8.2) \lesssim \delta^{-\mu r} N_1^{-\beta(s,\alpha)} \|P_{N_2} u_2\|_{U^2_{\Delta} H^s} \|P_C P_{N_0} h\|_{Y^{-s}}$$

provided  $\alpha < \frac{1}{4}$  and  $s + \alpha < \frac{3}{2}$ . A similar bound holds when  $N_1 \sim N_2$  without cutting with cubes  $C$ .

**Case 5, a), b), c):** We use the argument above with (7.10). If  $N_1 \sim N_0$  we obtain a bound of the form

$$(8.2) \lesssim \delta^{-\mu r} N_2^{1+\alpha-s} \|P_{N_2} u_2\|_{U^2_{\Delta} H^s} \|P_{N_3} u_3\|_{U^2_{\Delta} H^s} \|P_C P_{N_0} h\|_{Y^{-s}}$$

which suffices provided  $s > 1 + \alpha$ . A similar bound holds when  $N_1 \sim N_2$  without cutting with cubes  $C$ .

**Case 6 a):** If  $N_1 \sim N_0$  we proceed as above and we bound

$$\begin{aligned} & \left| \int_{\mathbb{T}} \int_{\mathbb{T}^3} T_{\Upsilon} (P_C P_{N_1} \bar{\mathcal{R}}_1, P_{N_2} \mathcal{D}_2, P_{N_3} \mathcal{D}_3) P_C P_{N_0} \bar{h} \, dx dt \right| \\ & \lesssim \|T_{\Upsilon} (P_C P_{N_1} \bar{\mathcal{R}}_1, P_{N_2} \mathcal{D}_2, P_{N_3} \mathcal{D}_3)\|_{L_{xt}^2} \|P_C P_{N_0} h\|_{L_{xt}^2}. \end{aligned}$$

Then we use (7.9), normalization and the embedding (4.7) to obtain the bound

$$(8.1) \lesssim \delta^{-\mu r} N_1^{s-\frac{3}{2}+\alpha+\varepsilon} N_2^{\frac{1}{2}+\frac{3\theta}{4}-s} \min(N_1, N_2)^{\frac{1-\theta}{2}} N_3^{-s+\frac{3}{2}} \\ \times \|P_{N_3} u_3\|_{U_{\Delta}^2 H^s} \|P_{N_2} u_2\|_{U_{\Delta}^2 H^s} \|P_C P_{N_0} h\|_{Y^{-s}}.$$

If  $N_1 \geq N_2^2$  then

$$N_1^{s-\frac{3}{2}+\alpha+\varepsilon} N_2^{\frac{1}{2}+\frac{3\theta}{4}-s} \min(N_1, N_2)^{\frac{1-\theta}{2}} N_3^{-s+\frac{3}{2}} \leq N_1^{s-\frac{3}{2}+\alpha+\varepsilon} N_2^{3-2s-\frac{\theta}{4}} \leq N_1^{\alpha+\varepsilon-\frac{\theta}{8}}$$

provided that  $s < \frac{3}{2} - \frac{\theta}{8}$  which forces  $\alpha < \frac{\theta}{8}$ .

On the other hand if  $N_1 < N_2^2$  we have that,

$$N_1^{s-\frac{3}{2}+\alpha+\varepsilon} N_2^{\frac{1}{2}+\frac{3\theta}{4}-s} \min(N_1, N_2)^{\frac{1-\theta}{2}} N_3^{-s+\frac{3}{2}} \leq N_1^{s-1+\alpha+\varepsilon-\frac{\theta}{2}} N_2^{2-2s+\frac{3\theta}{4}} \leq N_2^{2\alpha+\varepsilon-\frac{\theta}{4}}$$

provided  $s > 1 + \frac{\theta}{2} - \alpha$ . By letting, for example,  $\theta = 10\alpha$  we obtain that  $1 + 4\alpha < s < \frac{3}{2} - 2\alpha$  in this case, while still satisfying the requirement that  $\alpha < \frac{\theta}{8}$  from Case a).

If  $N_1 \sim N_2$  the argument is similar and easier. For Case 6 b) and c) we repeat the argument since (7.9) is not sensitive to conjugation on the random function.

**Case 7 a):** In this case we would like to use the Strichartz estimate (4.22). But since

$$T_{\Upsilon}(\bar{\mathcal{D}}_1, \mathcal{D}_2, \mathcal{R}_3) \neq \bar{\mathcal{D}}_1 \mathcal{D}_2 \mathcal{R}_3$$

we need to add back the frequencies that have been removed, i.e. allow for  $n_2$  or  $n_3$  to be equal to  $n_1$ . If we were working with spaces whose norms are based on the absolute value of the time-space Fourier coefficients, like the  $X^{s,b}$  space, this would not be an issue, but since we are using  $U^p L^2$  spaces we need to put back those missing frequencies. We show below that reintroducing these frequencies will not bring back the whole linear term that we have gauged away but only a part that has sufficient regularity to be controlled.

We start by assuming that the Fourier coefficient associated to  $\mathcal{D}_1(t)$  is  $a_{n_1}(t)$ , to  $\mathcal{D}_2(t)$  is  $b_{n_2}(t)$  and to  $\mathcal{R}_3(t)$  is  $c_{n_3}(t)$ . Then we write

$$\begin{aligned} & \sum_{n=-n_1+n_2+n_3, n_2, n_3 \neq n_1} \chi_{N_1} a_{n_1} \chi_{N_2} b_{n_2} \chi_{N_3} c_{n_3} = -\chi_{N_3} c_n \left( \sum_{n_1} \chi_{N_1} a_{n_1} \chi_{N_2} b_{n_1} \right) \\ & -\chi_{N_2} b_n \left( \sum_{n_3} \chi_{N_1} a_{n_3} \chi_{N_3} c_{n_3} \right) + \chi_{N_1} a_n \chi_{N_2} b_n \chi_{N_3} c_n \\ & + \sum_{n=-n_1+n_2+n_3} \chi_{N_1} a_{n_1} \chi_{N_2} b_{n_2} \chi_{N_3} c_{n_3} = A_1(n) + A_2(n) + A_3(n) + A_4(n). \end{aligned}$$

Then we have that

$$(8.1) \lesssim \sum_{i=1}^4 \left| \int_{\mathbb{T}} \int_{\mathbb{T}^3} \mathcal{F}^{-1}(A_i)(x, t) P_{N_0} \bar{h}(x, t) \, dx dt \right|.$$



We now start with the estimate of  $A_1$ . Using Plancherel and Cauchy-Schwarz we have

$$\left| \int_{\mathbb{T}} \int_{\mathbb{T}^3} \mathcal{F}^{-1}(A_1)(x, t) P_{N_0} \bar{h}(x, t) dx dt \right| \lesssim \|A_1(n)\|_{L^2(\mathbb{T}, \ell^2)} \|P_{N_0} h(x, t)\|_{L^2_{x,t}}.$$

We first notice that  $A_1$  is not zero only if  $N_3 \sim N_1$ . Then

$$\|A_1(n)\|_{L^2(\mathbb{T}, \ell^2)} \lesssim \|\mathcal{D}_1\|_{L_t^\infty L_x^2} \|\mathcal{D}_2\|_{L_t^\infty L_x^2} \|\mathcal{R}_3\|_{L^2(\mathbb{T}, L^2(\mathbb{T}^3))}.$$

By renormalizing and using the embedding (4.7) we obtain that

$$\left| \int_{\mathbb{T}} \int_{\mathbb{T}^3} \mathcal{F}^{-1}(A_1)(x, t) P_{N_0} \bar{h}(x, t) dx dt \right| \lesssim N_2^{-s-1+\alpha} \|P_{N_1} u_1\|_{U_\Delta^2 H^s} \|P_{N_2} u_2\|_{U_\Delta^2 H^s} \|P_{N_0} h\|_{Y^{-s}}.$$

We now note that  $A_2 = 0$  unless  $N_0 \sim N_1 \sim N_2$  and

$$\|A_2(n)\|_{L^2([0, \pi], \ell^2)} \lesssim \|\mathcal{D}_2\|_{L^2(\mathbb{T}, L^2(\mathbb{T}^3))} \|\mathcal{D}_1\|_{L_t^\infty L_x^2} \|\mathcal{R}_3\|_{L_t^\infty L_x^2}.$$

Also in this case we then have

$$\left| \int_{\mathbb{T}} \int_{\mathbb{T}^3} \mathcal{F}^{-1}(A_2)(x, t) P_{N_0} \bar{h}(x, t) dx dt \right| \lesssim N_2^{-s-1+\alpha} \|P_{N_1} u_1\|_{U_\Delta^2 H^s} \|P_{N_2} u_2\|_{U_\Delta^2 H^s} \|P_{N_0} h\|_{Y^{-s}}.$$

Now we note that  $A_3 = 0$  unless  $N_1 \sim N_2 \sim N_3$ . Then

$$\|A_3(n)\|_{L^2(\mathbb{T}, \ell^2)} \lesssim \|\mathcal{D}_1\|_{L_t^\infty L_x^2} \|\mathcal{D}_2\|_{L_t^\infty L_x^2} \|\mathcal{R}_3\|_{L^2(\mathbb{T}, L^2(\mathbb{T}^3))},$$

where we used that  $\|a_n\|_{\ell^\infty} \lesssim \|a_n\|_{\ell^2}$ . Hence also in this case

$$\left| \int_{\mathbb{T}} \int_{\mathbb{T}^3} \mathcal{F}^{-1}(A_3)(x, t) P_{N_0} \bar{h}(x, t) dx dt \right| \lesssim N_2^{-s-1+\alpha} \|P_{N_1} u_1\|_{U_\Delta^2 H^s} \|P_{N_2} u_2\|_{U_\Delta^2 H^s} \|P_{N_0} h\|_{Y^{-s}}.$$

Finally we estimate the term involving  $A_4$ . Assume first that  $N_0 \sim N_1$ . Then we need to estimate

$$(8.3) \quad \left| \int_{\mathbb{T}} \int_{\mathbb{T}^3} \mathcal{F}^{-1}(A_4)(x, t) P_C P_{N_0} \bar{h}(x, t) dx dt \right|$$

where we cut by cubes  $C$  of size length  $N_2$ . We use Cauchy-Schwarz, (4.22), embedding (4.7) and normalization to obtain that

$$\begin{aligned} (8.3) &\lesssim N_0^s N_1^{-s} N_2^{1-s} N_3^\alpha \|P_C P_{N_1} u_1\|_{U_\Delta^4 H^s} \|P_{N_2} u_2\|_{U_\Delta^4 H^s} \|P_C P_{N_0} h\|_{Y^{-s}} \\ &\lesssim N_2^{1-s+\alpha} \|P_C P_{N_1} u_1\|_{U_\Delta^4 H^s} \|P_{N_2} u_2\|_{U_\Delta^4 H^s} \|P_C P_{N_0} h\|_{Y^{-s}}. \end{aligned}$$

If  $N_1 \sim N_2$  then the cutting with cubes  $C$  is automatic and a similar bound holds.

Cases b) and c) are similar since the argument presented above is not effected by complex conjugation.

**Case 8:** This case is similar and better than Case 7.

**8.2. Estimates Involving the Term  $J_3$ .** We start by noting that  $\mathcal{F}(J_3)(n)$  is comprised by terms of the form

$$(8.4) \quad \sum_{\Gamma(n)_{[123]}, n_1, n_3 \neq n_2} \hat{w}_{n_1}(t) \bar{a}_{n_2}(t) b_{n_3}(t),$$

where  $\hat{w}_{n_1}(t) = c_{n_1}(t) d_{n_1}(t) r_{n_1}(t)$ . We note that in the worse case, i.e. when the three factors of  $\hat{w}$  correspond to random functions,  $w(t) \in H^{3-3\alpha}$ , hence  $w$  can always be thought of as a deterministic function. We estimate  $J_3$  using the arguments presented for the estimate of  $J_2$  in subsection 8.1, but for the reason just explained we do not have to consider Case 1) of that section. For Cases 2)-6) we proceed by first applying the transfer principle to the *quintilinear* expression associated to (8.4) and then regroup into as single *deterministic* function those with the same frequency  $n_1$ . Then we apply the appropriate trilinear estimates in Proposition 7.1. The term involving the  $\ell^2$  norm of the product of the three coefficients in  $n_1$  can be bounded by the product of the  $\ell^2$  norm of each. We transfer and normalize back as usual.

This same argument is also used to estimate the  $A_i(x, t)$ ,  $i = 1, 2, 3$  of Case 7). To estimate  $A_4$  we use again Strichartz inequality in Proposition 4.13 placing  $w$  in  $L^p$  with  $p > 4$ . Then we use (4.18).

**8.3. Estimates Involving the Term  $J_4$ .** Let  $w$  now be such that  $\hat{w}_{n_2}(t) = a_{n_2}(t) c_{n_2}(t) d_{n_2}(t) r_{n_2}(t)$  and  $v$  such that,  $\hat{v}(n_1) = b_{n_1}$ . To estimate the contribution of  $J_4$  we need to estimate a term such as

$$\int_{\mathbb{T}} \int_{\mathbb{T}^3} P_{N_0}(wv) \overline{P_{N_0} h} dx dt = \int_{\mathbb{T}} \int_{\mathbb{T}^3} P_{N_0} \left( \sum_{N_1, N_2} P_{N_1} v P_{N_2} w \right) \overline{P_{N_0} h} dx dt.$$

Since  $w \in H^{4-4\alpha}$ , hence much smoother than  $v$ , the less advantageous situation is when  $N_1 \sim N_0$  and  $N_2 \ll N_1$  and this is the one we consider below. We cut the frequency support of  $P_{N_0} h$  with cubes  $C$  of size  $N_2$  and we write

$$\begin{aligned} & \left( \int_{\mathbb{T}} \int_{\mathbb{T}^3} P_{N_0} P_{N_1} v P_{N_2} w \overline{P_{N_0} h} dx dt \right)^2 \\ & \lesssim \left( \sum_C \|P_C P_{N_0} h\|_{L_t^2 L_x^2}^2 \sup \|P_C P_{N_1} v P_{N_2} w\|_{L_t^2 L_x^2} \right)^2. \end{aligned}$$

We assume first that  $v$  is random. Then the remarks in Subsection 8.2 combined with the transfer principle and the bilinear estimate (7.34) with  $\theta = 1$  give

$$\|P_C P_{N_1} v P_{N_2} w\|_{L_t^2 L_x^2} \lesssim \delta^{-\mu r} N_1^{-\frac{1}{2} + \varepsilon} N_2^{\frac{5}{4}} \prod_{i \notin J} \|\mathcal{D}_i\|_{U_{\Delta}^2 L^2}.$$

After normalizing we obtain the bound

$$N_1^{-\frac{3}{2} + s + \varepsilon + \alpha} N_2^{-\frac{11}{4} + 4\alpha}$$

which entails  $s + \alpha < \frac{3}{2}$

If  $v$  is deterministic then we use the bilinear estimate (4.23) and after normalization we obtain the bound

$$N_2^{-\frac{7}{2} + 4\alpha}.$$

**8.4. Estimates Involving the Terms  $J_5, J_6$  and  $J_7$ .** We work with the first term of  $J_5$ , the second term being analogous. Given a dual function  $h$  we define a new function  $k$  such that

$$\widehat{k}(n, t) = \chi_{N_0} a_n^1(t) a_n^2(t) \widehat{h}(n, t)$$

where  $a_n^i(t)$  are the Fourier coefficient of either a random or a deterministic function. Assume that  $N_1 \sim N_0$ . Then we cut the support of  $\widehat{h}$  with cubes  $C$  of sidelength  $N_2$ . By Plancherel and Cauchy-Schwarz we need to bound

$$\|P_C k\|_{L_{xt}^2} \quad \text{and} \quad \left\| \sum_{\Gamma(n)_{[1,2,3]}} \chi_C \chi_{N_1} b_{n_1} \chi_{N_2} \bar{c}_{n_2} \chi_{N_3} d_{n_3} \right\|_{L_t^2 \ell^2}.$$

Clearly

$$\|P_C k\|_{L_t^2 L_x^2} \lesssim \|P_C P_{N_0} h\|_{L_t^\infty L_x^2} \prod_{i=1}^2 \|\chi_{N_i} a_n^i\|_{L_t^\infty \ell^2}^2.$$

On the other hand by (4.22) we have that

$$\left\| \sum_{\Gamma(n)_{[1,2,3]}} \chi_C \chi_{N_1} b_{n_1} \chi_{N_2} \bar{c}_{n_2} \chi_{N_3} d_{n_3} \right\|_{L_t^2 \ell^2}$$

has a bound of  $N_2 N_3$ . By normalizing, assuming at worst that all functions are random, we obtain

$$N_0^{s-2+2\alpha} N_1^{-1+3\alpha}.$$

If  $N_1 \sim N_2$  the situation is similar.

To estimate  $J_6$  we use Cauchy-Schwarz and (4.21), while for the two terms in  $J_7$  we use respectively (4.20) and (4.19).

**8.5. Estimates Involving the Term  $J_1$ .** The term  $J_1$  in (2.10) can be written as the sum over  $N_0, N_1, \dots, N_5$  - dyadic numbers- of

$$(8.5) \quad \left| \int_{\mathbb{T}} \int_{\mathbb{T}^3} P_{N_0} T_{\Upsilon} (P_{N_1} \tilde{u}_1, P_{N_2} \tilde{u}_2, P_{N_3} \tilde{u}_3, P_{N_4} \tilde{u}_4, P_{N_5} \tilde{u}_5) P_{N_0} \bar{h} dx dt \right|$$

where  $T_{\Upsilon}$  is the multilinear operator associated to the multiplier  $\chi_{\Upsilon}$ , the indicator function over the set  $\Upsilon$  now defined by

$$(8.6) \quad \Upsilon(n, m) := \left\{ (n_1, m_1; \dots, n_5, m_5) : \begin{array}{l} n = (-1)^{\alpha_1} n_1 + \dots + (-1)^{\alpha_5} n_5 \\ n_k \neq n_\ell \text{ whenever } \alpha_k \neq \alpha_\ell, \\ |n_j| \sim N_j, \quad j = 1, \dots, 5 \\ m = (-1)^{\alpha_1} m_1 + \dots + (-1)^{\alpha_5} m_5 \end{array} \right\}$$

where  $\alpha_j$  are 0 or 1 for  $j = 1, \dots, 5$ .

8.5.1. *The all deterministic case DDDDD.* Without loss of generality we assume that  $u_2$  and  $u_4$  are conjugated. Our goal is to use Strichartz estimates as in (4.22), but the operator  $T_\Upsilon(P_{N_1}u_1, P_{N_2}\bar{u}_2, P_{N_3}u_3, P_{N_4}\bar{u}_4, P_{N_5}u_5)$  is not a product of the functions involved since in the convolution of the Fourier coefficient some frequencies have been removed. We need to add back the frequencies that have been removed, i.e. allow for  $n_2$  or  $n_3$  to be equal  $n_1$ . If we were working with spaces whose norms are based on the absolute value of the time-space Fourier coefficients, like the  $X^{s,b}$  space, this would not be an issue, but since we are using  $U^pL^2$  spaces we need to put back those missing frequencies. We show below that reintroducing these frequencies will not bring back the whole linear term that we have gauged away but only a part that has sufficient regularity to be controlled. See also Subsection 8.1.

From (2.9) we see that

$$\begin{aligned}
(8.7) \quad P_{N_0}(\mathcal{F}^{-1}J_1)(x, t) &= P_{N_0}T_\Upsilon(P_{N_1}u_1, P_{N_2}\bar{u}_2, P_{N_3}u_3, P_{N_4}\bar{u}_4, P_{N_5}u_5)(x, t) \\
&= P_{N_0}(P_{N_1}u_1P_{N_2}\bar{u}_2P_{N_3}u_3P_{N_4}\bar{u}_4P_{N_5}u_5)(x, t) \\
&- \sum_{i=1}^5 P_{N_0}P_{N_i}u_i(x, t) \int_{\mathbb{T}^3} \prod_{j \neq i, j \in \{1,2,3,4,5\}} P_{N_j}\tilde{u}_j(x, t) dx \\
&- \sum_{i=2}^7 c_i P_{N_0}\mathcal{F}^{-1}J_i(P_{N_1}u_1, P_{N_2}\bar{u}_2, P_{N_3}u_3, P_{N_4}\bar{u}_4, P_{N_5}u_5)(x, t),
\end{aligned}$$

where  $c_i$  are constants and we specified as an argument of  $\mathcal{F}^{-1}J_1$  the functions involved in its definition. The last sum involving  $J_2$ - $J_7$  has been already estimated in Subsections 8.1-8.4 above. On the other hand the first term, which is now a product of functions can be estimated as in Proposition 4.11. Finally we estimate

$$(8.8) \quad \left\| \sum_{i=1}^5 P_{N_0}P_{N_i}\tilde{u}_i(x, t) \int_{\mathbb{T}^3} \prod_{j \neq i, j \in \{1,2,3,4,5\}} P_{N_j}\tilde{u}_j(y, t) dy \right\|_{L_{x,t}^2}.$$

We first note that each term of the sum is zero unless  $N_i \sim N_0$  and that

$$(8.9) \quad (8.8) \lesssim \sum_{i=1}^5 \|P_{N_0}P_{N_i}u_i\|_{L_t^\infty L_x^2} \prod_{j \neq i, j \in \{1,2,3,4,5\}} N_j^{\frac{1}{4}} \|P_{N_j}u_j(x, t)\|_{U_\Delta^4 L_x^2}.$$

and this is enough since all are deterministic.

8.5.2. *The case DDDDR.* In (8.5) we assume without any loss of generality that  $u_5$  is random and that  $N_1 \geq N_2 \geq N_3 \geq N_4$ . Also in the argument below one can check that the location of the complex conjugates does not affect the proof hence here we assume that  $u_2$  and  $u_4$  are complex conjugate.

We consider the following cases:

- **Case a):**  $N_5 \sim N_0$  and  $N_1 \leq N_5$ .
- **Case b):**  $N_1 \sim N_0$  and  $N_2 \leq N_5 \leq N_1$ .
- **Case c):**  $N_1 \sim N_5$  and  $N_0 \leq N_1$ .
- **Case d):**  $N_1 \sim N_0$  and  $N_5 \leq N_2$ .
- **Case e):**  $N_1 \sim N_2$  and  $N_5 \leq N_1$ .

**Case a):** Proceeding as in the trilinear estimates we first decompose the support of  $\chi_{N_0} \hat{h}$  with cubes  $C$  of sidelength  $N_1$  in (8.5). By Cauchy-Schwarz, the transfer principle and Plancherel we are reduced to estimating

$$(8.10) \quad \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z}^3} \left| \sum_{\substack{n=n_5-n_2+n_3-n_4+n_1, \\ n_1, n_3, n_5 \neq n_2, n_4 \\ m=|n_5|^2-|n_2|^2+|n_3|^2-|n_4|^2+|n_1|^2}} \chi_C(n_5) \frac{g_{n_5}(\omega)}{|n_5|^{\frac{3}{2}}} a_{n_1} \bar{a}_{n_2} a_{n_3} \bar{a}_{n_4} \right|^2.$$

We define the set

$$S_{(n_5, n, m)} = \left\{ (n_1, n_2, n_3, n_4) : \begin{array}{l} n = n_5 - n_2 + n_3 - n_4 + n_1, \\ n_1, n_3, n_5 \neq n_2, n_4, \quad n_5 \in C \\ m = |n_5|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_1|^2 \end{array} \right\}$$

and note that that  $|S_{(n_5, n, m)}| \lesssim N_4^3 N_3^3 N_2^2$ . Also note that the variation of  $m$ ,  $\Delta m \sim N_5 N_1$ , therefore by Lemma 3.4, for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$  we have

$$(8.10) \lesssim \delta^{-2\mu r} N_5 N_1 N_5^{-3} \sum_m \sum_{n_5} \left| \sum_{S_{(n_5, n, m)}} a_{n_1} \bar{a}_{n_2} a_{n_3} \bar{a}_{n_4} \right|^2 \\ \lesssim \delta^{-2\mu r} N_5^{-2} N_1 \sup_m \sum_{n_5} \sum_{S_{(n_5, n, m)}} |S_{(n_5, n, m)}| |a_{n_1}|^2 |a_{n_2}|^2 |a_{n_3}|^2 |a_{n_4}|^2 \\ \lesssim \delta^{-2\mu r} N_5^{-2} N_1 N_4^3 N_3^3 N_2^2 \sum_{n_1, n_2, n_3, n_4} |a_{n_1}|^2 |a_{n_2}|^2 |a_{n_3}|^2 |a_{n_4}|^2 |S_{(n_1, n_2, n_3, n_4, m)}| \\ \lesssim \delta^{-2\mu r} N_5^{-1} N_1 N_4^3 N_3^3 N_2^2 \prod_{i=1}^4 \|a_{n_i}\|_{\ell^2}^2$$

where

$$S_{(n_1, n_2, n_3, n_4, m)} = \left\{ (n_1, n_5) : \begin{array}{l} n = n_5 - n_2 + n_3 - n_4 + n_1, \quad n_5 \in C \\ m = |n_5|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_1|^2 \end{array} \right\}$$

and in the last inequality we used that  $|S_{(n_1, n_2, n_3, n_4, m)}| \leq N_5$ . After renormalizing and taking square root we obtain the bound of  $N_5^{-3s+\alpha+3}$  which entails  $s > 1 + \frac{\alpha}{3}$ .

**Case b):** We will proceed by duality and a change of variables  $\zeta = m - |n_1|^2$  as in the proof of Proposition 7.3 in particular see (7.15). We also cut the window  $N_1$  by cubes  $C$  of sidelength  $N_5$ . We have to bound

$$(8.11) \quad \|\gamma\|_{\ell^2_\zeta}^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \sum_{(\zeta, n_1) \in \mathbb{Z} \times \mathbb{Z}^3} \left| \sum_{\substack{n=n_5-n_2+n_3-n_4+n_1, \\ n_1, n_3, n_5 \neq n_2, n_4 \\ \zeta=|n_5|^2-|n_2|^2+|n_3|^2-|n_4|^2}} \frac{g_{n_5}(\omega)}{|n_5|^{\frac{3}{2}}} \chi_{\bar{C}}(n) k_n \bar{a}_{n_2} a_{n_3} \bar{a}_{n_4} \right|^2,$$

where  $\tilde{C}$  is of size approximately  $N_5$ . We define the set

$$S_{(n_5, n_1, \zeta)} = \left\{ \begin{array}{l} n = n_5 - n_2 + n_3 - n_4 + n_1, \\ (n, n_2, n_3, n_4) : n_1, n_3, n_5 \neq n_2, n_4, n \in \tilde{C} \\ \zeta = |n_5|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 \end{array} \right\}$$

and note that that  $|S_{(n_5, n_1, \zeta)}| \lesssim N_4^3 N_3^3 N_2^2$ . Note also that  $\Delta\zeta \lesssim N_5^2$  hence we can continue for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$  with

$$\begin{aligned} (8.11) &\lesssim \delta^{-2\mu} \|\gamma\|_{\ell_\zeta^2}^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 N_5^2 N_5^{-3} \sup_{\zeta} \sum_{n_1, n_5} \left| \sum_{S_{(n_5, n_1, \zeta)}} \chi_{\tilde{C}}(n) k_n \bar{a}_{n_2} a_{n_3} \bar{a}_{n_4} \right|^2 \\ &\lesssim \delta^{-2\mu} \|\gamma\|_{\ell_\zeta^2}^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 N_5^{-1} \sup_{\zeta} \sum_{n_1, n_5} \sum_{S_{(n_5, n_1, \zeta)}} |S_{(n_5, n_1, \zeta)}| |a_{n_2}|^2 |a_{n_3}|^2 |a_{n_4}|^2 |\chi_{\tilde{C}}(n) k_n|^2 \\ &\lesssim \delta^{-2\mu} \|\gamma\|_{\ell_\zeta^2}^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 N_5^{-1} N_4^3 N_3^3 N_2^2 \\ &\quad \sum_{n, n_2, n_3, n_4} |a_{n_2}|^2 |a_{n_3}|^2 |a_{n_4}|^2 |\chi_{\tilde{C}}(n) k_n|^2 |S_{(n, n_2, n_3, n_4, \zeta)}| \\ &\lesssim \delta^{-2\mu} \|\chi_C a_{n_1}\|_{\ell^2}^2 N_4^3 N_3^3 N_2^2 \prod_{i=2}^4 \|a_{n_i}\|_{\ell^2}^2 \|k_n\|_{\ell^2}^2 \|\gamma\|_{\ell_\zeta^2}^2 \end{aligned}$$

where

$$S_{(n, n_2, n_3, n_4, \zeta)} = \left\{ (n, n_5) : \begin{array}{l} n = n_5 - n_2 + n_3 - n_4 + n_1, n_5 \in C \\ \zeta = |n_5|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 \end{array} \right\}$$

and in the last inequality we used that  $|S_{(n, n_2, n_3, n_4, \zeta)}| \leq N_5$ . After renormalizing and taking square root we obtain the bound of  $N_5^{-3s+\alpha+3}$  which entails  $s > 1 + \frac{\alpha}{3}$ .

**Case c):** This is like Case b), but now we do not need to cut the support of the window  $N_1$  by  $N_5$ .

**Case d):** In this case we proceed as in subsection 8.5.1, the only difference being in the treatment of the terms in (8.7). More precisely here we show how to estimate the random term in (8.9). We have for  $v_0^\omega$  in (5.1)

$$(8.12) \quad N_0^s N_3^{-1+\alpha} \|P_{N_0} P_{N_3} v_0^\omega\|_{L_t^\infty H_x^{1-\alpha}} \prod_{j=1}^4 N_j^{\frac{1}{4}-s} \|P_{N_j} u_j(x, t)\|_{U_\Delta^4 H^s}.$$

where we notice that  $N_3 \sim N_0$  otherwise the contribution would be null. This is enough to obtain the desired bound.

**Case e):** This is like Case d), but now we do not need to cut the support of the window  $N_1$  by  $N_2$ .

**8.5.3. The DDDRR Case.** To estimate the expression in (8.5) we will assume without any loss of generality that  $u_4, u_5$  are random and  $N_4 \geq N_5$ . We can also assume that  $N_1 \geq N_2 \geq N_3$ . We have two different scenarios: Case 1:  $u_4 u_5$  or Case 2:  $\bar{u}_4 u_5$ , the other cases being obtained by complex conjugation since we do not care about bars on deterministic functions. The only difference between Case 1 and 2 is that in Case 2 we automatically have that  $n_4 \neq n_5$  which still allows us to use Proposition 3.5, and hence the

same argument as in Case 1 applies. We discuss Case 1 within the context of the following cases ( Case 2 being analogous after appropriately rewriting the corresponding constraints):

- **Case a):**
  - Case i):  $N_4 \sim N_5 \geq N_0, N_1$ .
  - Case ii):  $N_4 \sim N_1 \geq N_0$ .
- **Case b):**  $N_4 \sim N_0$  and
  - Case i):  $N_5 \geq N_1$ .
  - Case ii):  $N_4 \geq N_1 \geq N_5 \geq N_2$ .
  - Case iii):  $N_4 \geq N_1$  and  $N_2 \geq N_5 \geq N_3$ .
  - Case iv):  $N_4 \geq N_1$  and  $N_3 \geq N_5$ .
- **Case c):**  $N_1 \sim N_0$  and
  - Case i):  $N_1 \geq N_4, N_5 \geq N_2$ .
  - Case ii):  $N_1 \geq N_4 \geq N_2 \geq N_5 \geq N_3$ .
  - Case iii):  $N_1 \geq N_4 \geq N_2 \geq N_3 \geq N_5$ .
  - Case iv):  $N_2 \geq N_4, N_5 \geq N_3$ .
  - Case v):  $N_2 \geq N_4 \geq N_3 \geq N_5$ .
  - Case vi):  $N_3 \geq N_4$ .
- **Case d):**  $N_1 \sim N_2 \geq N_0, N_4$

Below we always treat Case 1 and without any loss of generality we may assume  $\tilde{u}_1 = u_1, \tilde{u}_j = \bar{u}_j, j = 2, 3$ .

• **Case a), i):** In this case,  $N_4 \sim N_5 \geq N_0, N_1$ . By Cauchy-Schwarz, the transfer principle and Plancherel we are reduced to estimating

$$(8.13) \quad \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z}^3} \left| \sum_{n_4, n_5} \left[ \sum_{S_{(n_4, n_5, n, m)}} a_{n_1} \bar{a}_{n_2} \bar{a}_{n_3} \frac{1}{|n_4|^{\frac{3}{2}}} \frac{1}{|n_5|^{\frac{3}{2}}} \right] g_{n_4}(\omega) g_{n_5}(\omega) \right|^2,$$

where

$$S_{(n_4, n_5, n, m)} = \left\{ (n_1, n_2, n_3) : \begin{array}{l} n = n_4 + n_5 + n_1 - n_2 - n_3, \\ n_2, n_3 \neq n_1, n_4, n_5, \\ m = |n_4|^2 + |n_5|^2 + |n_1|^2 - |n_2|^2 - |n_3|^2 \end{array} \right\}$$

with  $|S_{(n_4, n_5, n, m)}| \lesssim N_3^3 N_2^2$  and the variation of  $m$ ,  $\Delta m \sim N_4^2$ . We then have, for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$  that

$$(8.13) \lesssim \delta^{-2\mu r} N_4^2 N_4^{-3} N_5^{-3} \sup_m \sum_{n_4, n_5, n} \left| \sum_{S_{(n_4, n_5, n, m)}} a_{n_1} \bar{a}_{n_2} \bar{a}_{n_3} \right|^2$$

$$\lesssim \delta^{-2\mu r} N_4^{-1} N_5^{-3} \sup_m \sum_{n_4, n_5, n} \sum_{S_{(n_4, n_5, n, m)}} |S_{(n_4, n_5, n, m)}| |a_{n_1}|^2 |a_{n_2}|^2 |a_{n_3}|^2$$

$$\lesssim \delta^{-2\mu r} N_4^{-1} N_5^{-3} N_2^2 N_3^3 \sup_m \sum_{n_1, n_2, n_3} |a_{n_1}|^2 |a_{n_2}|^2 |a_{n_3}|^2 |S_{(n_1, n_2, n_3, m)}|$$

$$\lesssim \delta^{-2\mu r} N_4^{-1} N_5^{-3} N_2^2 N_3^3 N_4 N_5^3 \sum_{n_1, n_2, n_3} |a_{n_1}|^2 |a_{n_2}|^2 |a_{n_3}|^2$$

$$(8.14) \lesssim \delta^{-2\mu r} N_2^2 N_3^3 \prod_{i=1}^3 \|a_{n_i}\|_{\ell^2}^2$$

where we used that

$$S_{(n_1, n_2, n_3, m)} = \left\{ \begin{array}{l} n = n_4 + n_5 + n_1 - n_2 - n_3, \\ (n, n_4, n_5) : n_2, n_3 \neq n_1, n_4, n_5, \\ m = |n_4|^2 + |n_5|^2 + |n_1|^2 - |n_2|^2 - |n_3|^2 \end{array} \right\},$$

and  $|S_{(n_1, n_2, n_3, m)}| \lesssim N_5^3 N_4$ . Taking square root and normalizing we then obtain the bound  $N_4^{s-2+2\alpha}$  which requires  $s < 2 - 2\alpha$ .

• **Case a), ii)** In this case  $N_4 \sim N_1 \geq N_0$ , we repeat the argument in Case a), i), but in this case after taking square root and normalizing we obtain the bound  $N_4^{\frac{3}{2}-2s+\alpha}$ .

• **Case b), i)** In this case,  $N_4 \sim N_0$  and  $N_5 \geq N_1$ . From (8.5), we first decompose the support of  $\chi_{N_0} \hat{h}$  with cubes  $C$  of sidelength  $N_5$  and then apply Cauchy-Schwarz, the transfer principle and Plancherel. We are thus reduced to estimating an expression just as in (8.13) but where now the variation in  $m$ ,  $\Delta m \sim N_4 N_5$  and thus we obtain instead of (8.14) the estimate  $\delta^{-2\mu r} N_4^{-1} N_5 N_2^2 N_3^3 \prod_{i=1}^3 \|a_{n_i}\|_{\ell^2}^2$ . Taking square root and normalizing we now obtain the bound  $N_4^{s-\frac{3}{2}+\alpha}$  provided  $\alpha < \frac{1}{2}$ , which in turn entails  $1 \leq s < \frac{3}{2} - \alpha$ .

• **Case b), ii)** In this case we have  $N_4 \sim N_0$  and  $N_1 \geq N_5$ . The proof follows that of Case b), i) except that now we first decompose the support of  $\chi_{N_0} \hat{h}$  -and hence the  $N_4$  Fourier window- with cubes  $C$  of sidelength  $N_1$ . We then have that  $\Delta m \sim N_4 N_1$  and we now obtain instead of (8.14) the estimate  $\delta^{-2\mu r} N_4^{-1} N_1 N_2^2 N_3^3 \prod_{i=1}^3 \|a_{n_i}\|_{\ell^2}^2$ . Taking square root and normalizing we now obtain as before the bound  $N_4^{s-\frac{3}{2}+\alpha}$ .

• **Cases b, iii) iv)** are analogous to case **Case b, ii)**.

• **Case c, i)** In this case we have that  $N_0 \sim N_1 \geq N_4$ ,  $N_5 \geq N_2$ . We will proceed by duality and a change of variables  $\zeta = m - |n_1|^2$  as in the proof of Proposition 7.3, (7.15) and also as in (8.11) in subsection 8.5.2 above. We also cut the window  $N_1$  by cubes  $C$  of sidelength  $N_4$ . We have to bound

$$(8.15) \quad \|\chi_C a_{n_1}\|_{\ell^2}^2 \|\gamma\|_{\ell_\zeta^2}^2 \sum_{\zeta \in \mathbb{Z}, n_1 \in \mathbb{Z}^3} \left| \sum_{\substack{n=n_1+n_4+n_5-n_2-n_3, \\ n_2, n_3 \neq n_1, n_4, n_5 \\ \zeta=|n_4|^2+|n_5|^2-|n_2|^2-|n_3|^2}} \frac{g_{n_4}(\omega)}{|n_4|^{\frac{3}{2}}} \frac{g_{n_5}(\omega)}{|n_5|^{\frac{3}{2}}} \chi_{\tilde{C}}(n) k_n \bar{a}_{n_2} \bar{a}_{n_3} \right|^2,$$

where  $\tilde{C}$  is of size approximately  $N_4$ . Let us now define

$$(8.16) \quad \sigma_{n_1, n_2} = \sum_{\substack{n=n_1+n_4+n_5-n_2-n_3, \\ n_2, n_3 \neq n_1, n_4, n_5 \\ \zeta=|n_4|^2+|n_5|^2-|n_2|^2-|n_3|^2}} \chi_{\tilde{C}}(n) k_n \bar{a}_{n_3} \frac{g_{n_4}(\omega)}{|n_4|^{\frac{3}{2}}} \frac{g_{n_5}(\omega)}{|n_5|^{\frac{3}{2}}},$$

and note that then  $\Delta \zeta \sim N_4^2$ . Then

$$(8.15) \lesssim \|\chi_C a_{n_1}\|_{\ell^2}^2 \|\gamma\|_{\ell_\zeta^2}^2 N_4^2 \sup_{\zeta} \sum_{n_1 \in C} |\sigma_{n_1, n_2} \bar{a}_{n_2}|^2$$

$$(8.17) \leq N_4^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|\gamma\|_{\ell_\zeta^2}^2 \|a_{n_2}\|_{\ell^2}^2 \sup_{\zeta} \|\mathcal{G} \mathcal{G}^*\|.$$



As in Section 7, we write

$$(8.18) \quad \|\mathcal{G}\mathcal{G}^*\| \lesssim \max_{n_1} \sum_{n_2, n_2 \neq n_1} |\sigma_{n_1, n_2}|^2 + \left( \sum_{n_1 \neq n'_1} \left| \sum_{n_2} \sigma_{n_1, n_2} \bar{\sigma}_{n'_1, n_2} \right|^2 \right)^{\frac{1}{2}} =: M_1 + M_2,$$

and estimate each term separately. For  $M_1$  we proceed as follows:

$$(8.19) \quad \begin{aligned} M_1 &= \sup_{n_1} \sum_{n_2, n_2 \neq n_1} \left| \sum_{n_4, n_5} \left[ \sum_{S_{(n_1, n_2, n_4, n_5, \zeta)}} \chi_{\tilde{C}}(n) k_n \bar{a}_{n_3} \frac{1}{|n_4|^{\frac{3}{2}}} \frac{1}{|n_5|^{\frac{3}{2}}} \right] g_{n_4}(\omega) g_{n_5}(\omega) \right|^2, \\ &\lesssim \delta^{-2\mu r} \sup_{n_1} \sum_{n_2 \neq n_1, n_4, n_5} N_4^{-3} N_5^{-3} \left| \sum_{S_{(n_1, n_2, n_4, n_5, \zeta)}} \chi_{\tilde{C}}(n) k_n \bar{a}_{n_3} \right|^2 \\ &\lesssim \delta^{-2\mu r} \sup_{n_1} \sum_{n_2, n_4, n_5} N_4^{-3} N_5^{-3} |S_{(n_1, n_2, n_4, n_5, \zeta)}| \sum_{S_{(n_1, n_2, n_4, n_5, \zeta)}} |\chi_{\tilde{C}}(n) k_n|^2 |\bar{a}_{n_3}|^2, \end{aligned}$$

for  $\omega$  out side a set of measure  $e^{-\frac{1}{\delta^r}}$ , where

$$S_{(n_1, n_2, n_4, n_5, \zeta)} := \left\{ \begin{array}{l} n = n_1 + n_4 + n_5 - n_2 - n_3, \\ (n, n_3) : n_2, n_3 \neq n_1, n_4, n_5, n \in \tilde{C} \\ \zeta = |n_4|^2 + |n_5|^2 - |n_2|^2 - |n_3|^2 \end{array} \right\}$$

and  $|S_{(n_1, n_2, n_4, n_5, \zeta)}| \lesssim N_3^2$ . Hence for

$$S_{(n, n_3, \zeta)} := \left\{ \begin{array}{l} n = n_1 + n_4 + n_5 - n_2 - n_3, \\ (n_2, n_4, n_5) : n_2, n_3 \neq n_1, n_4, n_5, n \in \tilde{C} \\ \zeta = |n_4|^2 + |n_5|^2 - |n_2|^2 - |n_3|^2 \end{array} \right\}$$

we have that

$$(8.19) \quad \begin{aligned} &\lesssim \delta^{-2\mu r} N_4^{-3} N_5^{-3} N_3^2 \sum_{n, n_3} |\chi_{\tilde{C}}(n) k_n|^2 |\bar{a}_{n_3}|^2 |S_{(n, n_3, \zeta)}| \\ &\lesssim \delta^{-2\mu r} N_4^{-3} N_5^{-3} N_3^2 N_2^3 N_5^3 N_4 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \\ &\lesssim \delta^{-2\mu r} N_4^{-2} N_2^3 N_3^2 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2. \end{aligned}$$

Hence the contribution of  $M_1$  to (8.17) is

$$\delta^{-2\mu r} N_4^2 N_4^{-2} N_2^3 N_3^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|a_{n_2}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|\gamma\|_{\ell^2}^2 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2.$$

After taking square root and normalizing we then obtain a bound of  $N_4^{-1+\alpha} N_5^{-s+\frac{1}{2}+\alpha}$  which suffices provided  $s > \frac{1}{2} + \alpha$ .

To estimate  $M_2$  we first write

$$(8.20) \quad \begin{aligned} M_2^2 &= \sum_{n_1 \neq n'_1} \left| \sum_{n_2} \sigma_{n_1, n_2} \bar{\sigma}_{n'_1, n_2} \right|^2 \\ &\sim \sum_{n_1 \neq n'_1} \left| \sum_{S_{(n_1, n'_1, \zeta)}} \chi_{\tilde{C}}(n) k_n \chi_{\tilde{C}}(n') k_{n'} \bar{a}_{n_3} a_{n'_3} \frac{g_{n_4}(\omega)}{|n_4|^{\frac{3}{2}}} \frac{g_{n_5}(\omega)}{|n_5|^{\frac{3}{2}}} \frac{\bar{g}_{n'_4}(\omega)}{|n'_4|^{\frac{3}{2}}} \frac{\bar{g}_{n'_5}(\omega)}{|n'_5|^{\frac{3}{2}}} \right|^2 \end{aligned}$$

where

(8.21)

$$S_{(n_1, n'_1, \zeta)} = \left\{ \begin{array}{l} n = n_1 + n_4 + n_5 - n_2 - n_3, \\ n' = n'_1 + n'_4 + n'_5 - n_2 - n'_3, \\ (n, n_2, n_3, n'_3, n_4, n'_4, n_5, n'_5) : n_2, n_3 \neq n_1, n_4, n_5; \quad n'_2, n'_3 \neq n'_1, n'_4, n'_5; \quad n, n' \in \tilde{C} \\ \zeta = |n_4|^2 + |n_5|^2 - |n_2|^2 - |n_3|^2, \\ \zeta = |n'_4|^2 + |n'_5|^2 - |n_2|^2 - |n'_3|^2 \end{array} \right\}.$$

To streamline the exposition let

$$\mathcal{C} := \left\{ \begin{array}{l} n = n_1 + n_4 + n_5 - n_2 - n_3, \quad n' = n'_1 + n'_4 + n'_5 - n_2 - n'_3; \\ \zeta = |n_4|^2 + |n_5|^2 - |n_2|^2 - |n_3|^2, \quad \zeta = |n'_4|^2 + |n'_5|^2 - |n_2|^2 - |n'_3|^2; \\ n_2, n_3 \neq n_1, n_4, n_5; \quad n'_2, n'_3 \neq n'_1, n'_4, n'_5; \quad n, n' \in \tilde{C}. \end{array} \right.$$

We need to organize the estimates according to whether some frequencies are the same or not, in all we have seven cases:

- **Case  $\beta_1$ :**  $n_4, n_5 \neq n'_4, n'_5$ .
- **Case  $\beta_2$ :**  $n_4 = n'_4; n_5 \neq n'_5$ .
- **Case  $\beta_3$ :**  $n_4 \neq n'_4; n_5 = n'_5$ .
- **Case  $\beta_4$ :**  $n_4 \neq n'_5; n_5 = n'_4$ .
- **Case  $\beta_5$ :**  $n_4 = n'_5; n_5 \neq n'_4$ .
- **Case  $\beta_6$ :**  $n_4 = n'_5; n_5 = n'_4$ .
- **Case  $\beta_7$ :**  $n_4 = n'_4; n_5 = n'_5$ .

**Case  $\beta_1$ :** To estimate the contribution of  $M_2$ , we first define the set

$$S_{(n_1, n'_1, n_4, n'_4, n_5, n'_5, \zeta)} := \{(n, n', n_2, n_3, n'_3) \text{ satisfying } \mathcal{C}\},$$

with  $|S_{(n_1, n'_1, n_4, n'_4, n_5, n'_5, \zeta)}| \lesssim N_3^6 N_2^2$ . Next, for  $\omega$  out side a set of measure  $e^{-\frac{1}{\delta^r}}$ , we estimate  $M_2^2$  as follows:

$$\begin{aligned} (8.20) &\lesssim \delta^{-4\mu r} \sum_{n_1 \neq n'_1} N_4^{-6} N_5^{-6} \sum_{n_4 \neq n'_4, n_5 \neq n'_5} \left[ \sum_{S_{(n_1, n'_1, n_4, n'_4, n_5, n'_5, \zeta)}} \chi_C(n) k_n \chi_C(n') k_{n'} \bar{a}_{n_3} a_{n'_3} \right]^2 \\ &\lesssim \delta^{-4\mu r} \sum_{n_1 \neq n'_1} N_4^{-6} N_5^{-6} N_3^6 N_2^2 \sum_{n_4, n'_4, n_5, n'_5} \sum_{S_{(n_1, n'_1, n_4, n'_4, n_5, n'_5, \zeta)}} |\chi_{\tilde{C}}(n) k_n|^2 |\chi_{\tilde{C}}(n') k_{n'}|^2 |\bar{a}_{n_3}|^2 |a_{n'_3}|^2 \\ &\lesssim \delta^{-4\mu r} N_4^{-6} N_5^{-6} N_3^6 N_2^2 \sum_{n, n', n_3, n'_3} |S_{(n, n', n_3, n'_3, \zeta)}| |\chi_{\tilde{C}}(n) k_n|^2 |\chi_{\tilde{C}}(n') k_{n'}|^2 |\bar{a}_{n_3}|^2 |a_{n'_3}|^2 \\ &\lesssim \delta^{-4\mu r} N_4^{-6} N_5^{-6} N_3^6 N_2^2 N_2^2 N_5^6 N_4^6 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|\chi_{\tilde{C}}(n') k_{n'}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n'_3}\|_{\ell^2}^2 \\ &\lesssim \delta^{-4\mu r} N_4^{-4} N_3^6 N_5^5 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|\chi_{\tilde{C}}(n') k_{n'}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n'_3}\|_{\ell^2}^2, \end{aligned}$$

where we have used that

$$S_{(n, n', n_3, n'_3, \zeta)} := \{(n_1, n'_1, n_2, n_4, n'_4, n_5, n'_5) \text{ satisfying } \mathcal{C}\}$$

has cardinality less than or equal to  $N_2^3 N_5^6 N_4^2$ .

All in all we then have that the contribution of  $\Delta\zeta M_2$  is bounded by  $N_3^3 N_2^{\frac{5}{2}}$ . Taking square root and normalizing we finally obtain the bound  $N_4^{-1+\alpha} N_5^{\frac{7}{4}-2s+\alpha}$  in this case which suffices provided  $s > \frac{7}{8} + \frac{\alpha}{2}$ .

**Case  $\beta_2$ :** Now we have that  $n_4 = n'_4$  while  $n_5 \neq n'_5$  rendering (8.20) equal to

$$(8.22) \quad \sum_{n_1 \neq n'_1} \left| \sum_{S_{(n_1, n'_1, \zeta)}} \chi_{\tilde{C}}(n) k_n \chi_{\tilde{C}}(n') k_{n'} \bar{a}_{n_3} a_{n'_3} \frac{|g_{n_4}(\omega)|^2}{|n_4|^3} \frac{g_{n_5}(\omega)}{|n_5|^{\frac{3}{2}}} \frac{\bar{g}_{n'_5}(\omega)}{|n'_5|^{\frac{3}{2}}} \right|^2.$$

We proceed in a similar fashion as we did in (7.17)-(7.23) and define

$$(8.23) \quad \mathcal{Q}_1 := \sum_{n_1 \neq n'_1} \left| \sum_{S_{(n_1, n'_1, \zeta)}} k_n^{\tilde{C}} k_{n'}^{\tilde{C}} \bar{a}_{n_3} a_{n'_3} \frac{(|g_{n_4}(\omega)|^2 - 1)}{|n_4|^3} \frac{g_{n_5}(\omega)}{|n_5|^{\frac{3}{2}}} \frac{\bar{g}_{n'_5}(\omega)}{|n'_5|^{\frac{3}{2}}} \right|^2$$

$$(8.24) \quad \mathcal{Q}_2 := \sum_{n_1 \neq n'_1} \left| \sum_{S_{(n_1, n'_1, \zeta)}} k_n^{\tilde{C}} k_{n'}^{\tilde{C}} \bar{a}_{n_3} a_{n'_3} \frac{1}{|n_4|^3} \frac{g_{n_5}(\omega)}{|n_5|^{\frac{3}{2}}} \frac{\bar{g}_{n'_5}(\omega)}{|n'_5|^{\frac{3}{2}}} \right|^2$$

where we have denoted by  $k_n^{\tilde{C}} := \chi_{\tilde{C}}(n) k_n$  and similarly for  $k_{n'}^{\tilde{C}}$ .

To estimate  $\mathcal{Q}_2$  define the set

$$S_{(n_1, n'_1, n_5, n'_5, \zeta)} := \{(n, n', n_2, n_4, n_3, n'_3) \text{ satisfying } \mathcal{C}\},$$

with  $|S_{(n_1, n'_1, n_5, n'_5, \zeta)}| \lesssim N_3^6 N_2^3 N_4$ . Then for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$

$$(8.24) \lesssim \delta^{-4\mu r} \sum_{n_1 \neq n'_1} N_4^{-6} N_5^{-6} \sum_{n_5 \neq n'_5} \left[ \sum_{S_{(n_1, n'_1, n_5, n'_5, \zeta)}} k_n^{\tilde{C}} k_{n'}^{\tilde{C}} \bar{a}_{n_3} a_{n'_3} \right]^2 \\ \lesssim \delta^{-4\mu r} \sum_{n_1 \neq n'_1} N_4^{-6} N_5^{-6} N_3^6 N_2^3 N_4 \sum_{n_5, n'_5} \sum_{S_{(n_1, n'_1, n_5, n'_5, \zeta)}} |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |\bar{a}_{n_3}|^2 |a_{n'_3}|^2 \\ \lesssim \delta^{-4\mu r} N_4^{-6} N_5^{-6} N_3^6 N_2^3 N_4 \sum_{n, n', n_3, n'_3} |S_{(n, n', n_3, n'_3, \zeta)}| |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |\bar{a}_{n_3}|^2 |a_{n'_3}|^2 \\ \lesssim \delta^{-4\mu r} N_4^{-6} N_5^{-6} N_3^6 N_2^3 N_4 N_2^3 N_5^6 N_4 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|\chi_{\tilde{C}}(n') k_{n'}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n'_3}\|_{\ell^2}^2 \\ \lesssim \delta^{-4\mu r} N_4^{-4} N_3^6 N_2^6 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|\chi_{\tilde{C}}(n') k_{n'}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n'_3}\|_{\ell^2}^2,$$

where we have used that

$$S_{(n, n', n_3, n'_3, \zeta)} := \{(n_1, n'_1, n_2, n_4, n_5, n'_5) \text{ satisfying } \mathcal{C}\}$$

has cardinality less than or equal to  $N_2^3 N_5^6 N_4$ .

The bound for  $\mathcal{Q}_1$  is smaller, just as we saw it was the case in the proof of Proposition 7.3, (7.17)-(7.23) in Section 7. We omit the details.

Thus the contribution of  $\Delta\zeta M_2$ , is bounded by  $N_3^3 N_2^3$  which after taking the square root and normalizing gives a bound of  $N_4^{-1+\alpha} N_5^{2-2s+\alpha}$  which suffices provided  $s > 1 + \frac{\alpha}{2}$ .

**Case  $\beta_3$ :** Now we have that  $n_4 \neq n'_4$  while  $n_5 = n'_5$  rendering (8.20) equal to

$$(8.25) \quad \sum_{n_1 \neq n'_1} \left| \sum_{S_{(n_1, n'_1, \zeta)}} k_n^{\tilde{C}} k_{n'}^{\tilde{C}} \bar{a}_{n_3} a_{n'_3} \frac{|g_{n_5}(\omega)|^2}{|n_5|^3} \frac{g_{n_4}(\omega)}{|n_4|^{\frac{3}{2}}} \frac{\bar{g}_{n'_4}(\omega)}{|n'_4|^{\frac{3}{2}}} \right|^2.$$

We proceed as above, defining analogous  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  terms bounding (8.25) in this case.

To estimate  $\mathcal{Q}_2$  we define the set,

$$S_{(n_1, n'_1, n_4, n'_4, \zeta)} := \{(n, n', n_2, n_5, n_3, n'_3) \text{ satisfying } \mathcal{C}\},$$

with  $|S_{(n_1, n'_1, n_4, n'_4, \zeta)}| \lesssim N_3^6 N_2^3 \min(N_5^2, N_4) \leq N_3^6 N_2^3 N_5^2$ . Then for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$

$$(8.26) \quad \begin{aligned} \mathcal{Q}_2 &\lesssim \delta^{-4\mu r} \sum_{n_1 \neq n'_1} N_4^{-6} N_5^{-6} \sum_{n_4 \neq n'_4} \left[ \sum_{S_{(n_1, n'_1, n_4, n'_4, \zeta)}} k_n^{\tilde{C}} k_{n'}^{\tilde{C}} \bar{a}_{n_3} a_{n'_3} \right]^2 \\ &\lesssim \delta^{-4\mu r} \sum_{n_1 \neq n'_1} N_4^{-6} N_5^{-6} N_3^6 N_2^3 N_5^2 \sum_{n_4, n'_4} \sum_{S_{(n_1, n'_1, n_4, n'_4, \zeta)}} |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |\bar{a}_{n_3}|^2 |a_{n'_3}|^2 \\ &\lesssim \delta^{-4\mu r} N_4^{-6} N_5^{-6} N_3^6 N_2^3 N_5^2 \sum_{n, n', n_3, n'_3} |S_{(n, n', n_3, n'_3, \zeta)}| |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |\bar{a}_{n_3}|^2 |a_{n'_3}|^2 \\ &\lesssim \delta^{-4\mu r} N_4^{-6} N_5^{-6} N_3^6 N_2^3 N_5^2 N_2^3 N_5^3 N_4^2 \|k_n^{\tilde{C}}\|_{\ell^2}^2 \|k_{n'}^{\tilde{C}}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n'_3}\|_{\ell^2}^2 \\ &\lesssim \delta^{-4\mu r} N_4^{-4} N_5^{-1} N_3^6 N_2^6 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|\chi_{\tilde{C}}(n') k_{n'}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n'_3}\|_{\ell^2}^2, \end{aligned}$$

where we have now used that,

$$S_{(n, n', n_3, n'_3, \zeta)} := \{(n_1, n'_1, n_2, n_4, n'_4, n_5) \text{ satisfying } \mathcal{C}\}$$

has cardinality less than or equal to  $N_2^3 N_5^3 N_4^2$ . Note this is a better bound than that obtained in Case  $\beta_2$ .

Since just as before, the bound for  $\mathcal{Q}_1$  is smaller, we have that the contribution of  $\Delta\zeta M_2$ , is bounded by  $N_3^3 N_2^3$ . After taking the square root and normalizing the latter gives the same bound as in Case  $\beta_2$ .

**Case  $\beta_4$ :** In this case we have that  $n_4 \neq n'_5$  while  $n_5 = n'_4$  rendering (8.20) equal to

$$(8.27) \quad \sum_{n_1 \neq n'_1} \left| \sum_{S_{(n_1, n'_1, \zeta)}} k_n^{\tilde{C}} k_{n'}^{\tilde{C}} \bar{a}_{n_3} a_{n'_3} \frac{|g_{n_5}(\omega)|^2}{|n_5|^3} \frac{g_{n_4}(\omega)}{|n_4|^{\frac{3}{2}}} \frac{\bar{g}_{n'_5}(\omega)}{|n'_5|^{\frac{3}{2}}} \right|^2.$$

Once again, we proceed by defining the corresponding  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  terms bounding (8.27) and note the estimate for  $\mathcal{Q}_1$  is better than that for  $\mathcal{Q}_2$ . In the latter case, we proceed as in (8.26) in case  $\beta_3$  but now we have

$$S_{(n_1, n'_1, n_4, n'_5, \zeta)} := \{(n, n', n_2, n_5, n_3, n'_3) \text{ satisfying } \mathcal{C}\},$$

which has cardinality,  $|S_{(n_1, n'_1, n_4, n'_5, \zeta)}| \lesssim N_3^6 N_2^3 N_4$ . Furthermore, since  $n_5 = n'_4$ , we have from the definition of  $\mathcal{C}$  that  $\Delta\zeta \lesssim N_5^2$ .

Then for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ ,

$$\begin{aligned}
(8.28) \quad \mathcal{Q}_2 &\lesssim \delta^{-4\mu r} \sum_{n_1 \neq n'_1} N_4^{-3} N_5^{-9} \sum_{n_4 \neq n'_5} \left[ \sum_{S_{(n_1, n'_1, n_4, n'_5, \zeta)}} k_n^{\tilde{C}} k_{n'}^{\tilde{C}} \bar{a}_{n_3} a_{n'_3} \right]^2 \\
&\lesssim \delta^{-4\mu r} \sum_{n_1 \neq n'_1} N_4^{-3} N_5^{-9} N_3^6 N_2^3 N_4 \sum_{n_4, n'_5} \sum_{S_{(n_1, n'_1, n_4, n'_5, \zeta)}} |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |\bar{a}_{n_3}|^2 |a_{n'_3}|^2 \\
&\lesssim N_4^{-3} N_5^{-9} N_3^6 N_2^3 N_4 \sum_{n, n', n_3, n'_3} |S_{(n, n', n_3, n'_3, \zeta)}| |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |\bar{a}_{n_3}|^2 |a_{n'_3}|^2 \\
&\lesssim \delta^{-4\mu r} N_4^{-3} N_5^{-9} N_3^6 N_2^3 N_4 N_2^3 N_5^5 N_4 \|k_n^{\tilde{C}}\|_{\ell^2}^2 \|k_{n'}^{\tilde{C}}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n'_3}\|_{\ell^2}^2 \\
&\lesssim \delta^{-4\mu r} N_4^{-1} N_5^{-4} N_3^6 N_2^6 \|\chi_{\tilde{C}}(n)k_n\|_{\ell^2}^2 \|\chi_{\tilde{C}}(n')k_{n'}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n'_3}\|_{\ell^2}^2,
\end{aligned}$$

where now

$$S_{(n, n', n_3, n'_3, \zeta)} := \{(n_1, n'_1, n_2, n_4, n'_5, n_5) \text{ satisfying } \mathcal{C}\}$$

has cardinality less than or equal to  $N_2^3 N_5^3 N_4$ .

Thus

$$\Delta \zeta M_2 \lesssim \delta^{-4\mu r} N_5^2 N_4^{-\frac{1}{2}} N_5^{-2} N_3^3 N_2^3 \|\chi_{\tilde{C}}(n)k_n\|_{\ell^2} \|\chi_{\tilde{C}}(n')k_{n'}\|_{\ell^2} \|a_{n_3}\|_{\ell^2} \|a_{n'_3}\|_{\ell^2},$$

whence after taking square root and normalizing we obtain in this case a bound of  $N_4^{-\frac{5}{4}+\alpha} N_5^{-2s+\alpha}$  which suffices provided  $s > 1 + \frac{\alpha}{2}$ .

**Case  $\beta_5$ :** In this case we have that  $n_4 = n'_5$  while  $n_5 \neq n'_4$  rendering (8.20) equal to

$$(8.29) \quad \sum_{n_1 \neq n'_1} \left| \sum_{S_{(n_1, n'_1, \zeta)}} k_n^{\tilde{C}} k_{n'}^{\tilde{C}} \bar{a}_{n_3} a_{n'_3} \frac{|g_{n_4}(\omega)|^2}{|n_4|^3} \frac{g_{n'_4}(\omega)}{|n'_4|^{\frac{3}{2}}} \frac{\bar{g}_{n_5}(\omega)}{|n_5|^{\frac{3}{2}}} \right|^2.$$

Once again, we proceed by defining the corresponding  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  terms bounding (8.29).

We treat the estimate  $\mathcal{Q}_2$  as in (8.26) in case  $\beta_3$  but with the set

$$S_{(n_1, n'_1, n'_4, n_5, \zeta)} := \{(n, n', n_2, n_4, n_3, n'_3) \text{ satisfying } \mathcal{C}\},$$

instead with cardinality,  $|S_{(n_1, n'_1, n'_4, n_5, \zeta)}| \lesssim N_3^6 N_2^3 N_4$ .

Then for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ ,

$$\begin{aligned}
(8.30) \quad \mathcal{Q}_2 &\lesssim \delta^{-4\mu r} \sum_{n_1 \neq n'_1} N_4^{-9} N_5^{-3} \sum_{n'_4 \neq n_5} \left[ \sum_{S_{(n_1, n'_1, n'_4, n_5, \zeta)}} k_n^{\tilde{C}} k_{n'}^{\tilde{C}} \bar{a}_{n_3} a_{n'_3} \right]^2 \\
&\lesssim \delta^{-4\mu r} \sum_{n_1 \neq n'_1} N_4^{-9} N_5^{-3} N_3^6 N_2^3 N_4 \sum_{n'_4, n_5} \sum_{S_{(n_1, n'_1, n'_4, n_5, \zeta)}} |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |\bar{a}_{n_3}|^2 |a_{n'_3}|^2 \\
&\lesssim N_4^{-9} N_5^{-3} N_3^6 N_2^3 N_4 \sum_{n, n', n_3, n'_3} |S_{(n, n', n_3, n'_3, \zeta)}| |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |\bar{a}_{n_3}|^2 |a_{n'_3}|^2 \\
&\lesssim \delta^{-4\mu r} N_4^{-9} N_5^{-3} N_3^6 N_2^3 N_4 N_2^3 N_5^5 N_4 \|k_n^{\tilde{C}}\|_{\ell^2}^2 \|k_{n'}^{\tilde{C}}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n'_3}\|_{\ell^2}^2 \\
&\lesssim \delta^{-4\mu r} N_4^{-6} N_3^6 N_2^6 \|\chi_{\tilde{C}}(n)k_n\|_{\ell^2}^2 \|\chi_{\tilde{C}}(n')k_{n'}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n'_3}\|_{\ell^2}^2,
\end{aligned}$$

where now

$$S_{(n,n',n_3,n'_3,\zeta)} := \{(n_1, n'_1, n_2, n_4, n'_5, n_5) \text{ satisfying } \mathcal{C}\}$$

has cardinality less than or equal to  $N_2^3 N_5^3 N_4^2$ .

Thus the contribution of  $\Delta\zeta M_2$ , is bounded by  $N_4^{-1} N_5^{-2} N_3^3 N_2^3$ . After taking the square root and normalizing we obtain in this case a bound of  $N_4^{-\frac{3}{2}+\alpha} N_5^{2-2s+\alpha}$  which suffices provided  $s > 1 + \frac{\alpha}{2}$ .

**Case  $\beta_6$ :** In this case we have that  $n_4 = n'_5$  and that  $n_5 = n'_4$  and (8.20) now has enough decay to use Lemma 3.7. We define

$$S_{(n_1,n'_1,\zeta)} := \{(n, n', n_2, n_3, n'_3, n_4, n'_4) \text{ satisfying } \mathcal{C}\}$$

with cardinality  $|S_{(n_1,n'_1,\zeta)}| \lesssim N_3^6 N_2^3 N_4^4$  and proceed as follows for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ :

$$\begin{aligned} (8.31) \quad & \sum_{n_1 \neq n'_1} \left| \sum_{S_{(n_1,n'_1,\zeta)}} k_n^{\tilde{C}} k_{n'}^{\tilde{C}} \bar{a}_{n_3} a_{n'_3} \frac{|g_{n_4}(\omega)|^2}{|n_4|^3} \frac{|g_{n'_4}(\omega)|^2}{|n'_4|^3} \right|^2 \\ & \lesssim N_4^{-12+\varepsilon} \sum_{n_1 \neq n'_1} |S_{(n_1,n'_1,\zeta)}| \sum_{S_{(n_1,n'_1,\zeta)}} |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |\bar{a}_{n_3}|^2 |a_{n'_3}|^2, \\ & \lesssim N_4^{-12+\varepsilon} N_3^6 N_2^3 N_4^4 N_2^3 N_4^4 \sum_{n,n',n_3,n'_3} |S_{(n,n',n_3,n'_3,\zeta)}| |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |\bar{a}_{n_3}|^2 |a_{n'_3}|^2 \\ (8.32) \quad & \lesssim N_4^{-4+\varepsilon} N_3^6 N_2^6 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|\chi_{\tilde{C}}(n') k_{n'}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n'_3}\|_{\ell^2}^2, \end{aligned}$$

where now to obtain (8.32) we have used that

$$S_{(n,n',n_3,n'_3,\zeta)} := \{(n_1, n'_1, n_2, n_4, n'_4) \text{ satisfying } \mathcal{C}\}$$

has cardinality less than or equal to  $N_2^3 N_4^4$ .

Thus the contribution of  $\Delta\zeta M_2$ , is bounded by  $N_4^\varepsilon N_3^3 N_2^3$ . After taking the square root and normalizing we obtain in this case a bound of  $N_4^{-1+\alpha+\varepsilon} N_5^{2-2s+\alpha}$  which suffices provided  $s > 1 + \frac{\alpha}{2}$ .

**Case  $\beta_7$ :** In this case we have that  $n_4 = n'_4$  and that  $n_5 = n'_5$  and once again (8.20) has enough decay to use Lemma 3.7. Define

$$S_{(n_1,n'_1,\zeta)} := \{(n, n', n_2, n_3, n'_3, n_4, n_5) \text{ satisfying } \mathcal{C}\}$$

with cardinality  $|S_{(n_1,n'_1,\zeta)}| \lesssim N_3^6 N_2^3 N_5^3 N_4$  and proceed as follows for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ :

$$\begin{aligned}
(8.33) \quad & \sum_{n_1 \neq n'_1} \left| \sum_{S_{(n_1, n'_1, \zeta)}} k_n^{\tilde{C}} k_{n'}^{\tilde{C}} \bar{a}_{n_3} a_{n'_3} \frac{|g_{n_4}(\omega)|^2}{|n_4|^3} \frac{|g_{n_5}(\omega)|^2}{|n_5|^3} \right|^2 \\
& \lesssim N_4^{-6+\varepsilon} N_5^{-6} \sum_{n_1 \neq n'_1} |S_{(n_1, n'_1, \zeta)}| \sum_{S_{(n_1, n'_1, \zeta)}} |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |\bar{a}_{n_3}|^2 |a_{n'_3}|^2, \\
& \lesssim N_4^{-6+\varepsilon} N_5^{-6} N_3^6 N_2^3 N_5^3 N_4 \sum_{n, n', n_3, n'_3} |S_{(n, n', n_3, n'_3, \zeta)}| |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |a_{n_3}|^2 |a_{n'_3}|^2 \\
(8.34) \quad & \lesssim N_4^{-4+\varepsilon} N_3^6 N_2^6 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|\chi_{\tilde{C}}(n') k_{n'}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n'_3}\|_{\ell^2}^2,
\end{aligned}$$

where now to obtain (8.34) we have used that

$$S_{(n, n', n_3, n'_3, \zeta)} := \{(n_1, n'_1, n_2, n_4, n_5) \text{ satisfying } \mathcal{C} \text{ for fixed } (n, n', n_3, n'_3, \zeta)\}$$

has cardinality less than or equal to  $N_2^3 N_5^3 N_4$ .

Thus the contribution of  $\Delta\zeta M_2$ , is bounded by  $N_4^\varepsilon N_3^3 N_2^3$ . After taking the square root and normalizing we obtain in this case a bound of  $N_4^{-1+\alpha+\varepsilon} N_5^{2-2s+\alpha}$  which suffices provided  $s > 1 + \frac{\alpha}{2}$ .

• **Case c, ii)** In this case we have that  $N_0 \sim N_1 \geq N_4 \geq N_2 \geq N_5 \geq N_3$ . As in **Case c) i)** after duality, changing variables  $\zeta := m - |n_1|^2$  and cutting the window  $N_1$  by cubes  $C$  of sidelength  $N_4$  we have to estimate expression (8.15). Since  $\Delta\zeta \sim N_4^2$  we once again bound (8.15) by

$$\|\chi_C a_{n_1}\|_{\ell^2}^2 \|\gamma\|_{\ell^2_\zeta}^2 N_4^2 \sup_{\zeta} \sum_{n_1 \in C} |\sigma_{n_1, n_2} \bar{a}_{n_2}|^2 \leq N_4^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|\gamma\|_{\ell^2_\zeta}^2 \|a_{n_2}\|_{\ell^2}^2 \sup_{\zeta} \|\mathcal{G}\mathcal{G}^*\|,$$

where  $\sigma_{n_1, n_2}$  is defined as in (8.16) and  $\mathcal{G}$  denotes, as usual, the matrix of entries  $\sigma_{n_1, n_2}$ . Just as in **Case c) i)** we are then reduced to estimating  $M_1$  and  $M_2$  as defined (8.18).

To estimate  $M_1$  we proceed just as in (8.19) to obtain for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$  the same bound

$$(8.35) \quad M_1 \lesssim \delta^{-2\mu r} N_4^{-2} N_2^3 N_3^2 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2.$$

Hence  $\Delta\zeta M_1$  is bounded once again by

$$\delta^{-2\mu r} N_2^3 N_3^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|a_{n_2}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|\gamma\|_{\ell^2_\zeta}^2 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2.$$

which after taking square root and normalizing gives in this case the bound  $N_4^{-s+\frac{1}{2}+\alpha}$ , which suffices provided  $s > \frac{1}{2} + \alpha$ .

The estimate for  $M_2$  proceeds as in **Case c) i)** by analyzing cases  $\beta_1 - \beta_7$  as stated there, yielding the same bounds for  $\Delta\zeta M_2$ . We do not repeat the arguments but rather indicate the bound we obtain in each case after taking square root and normalizing since now  $N_2 \geq N_5 \geq N_3$ , so we need to trade the slower decay of the random term  $\tilde{u}_5$  for the better regularity of the deterministic function  $\tilde{u}_2$ .

**Case  $\beta_1$ :** In this case we have that the contribution of  $\Delta\zeta M_2$  is bounded by  $N_3^3 N_2^{\frac{5}{2}}$ . Taking square root and normalizing we now obtain the bound  $N_4^{-1+\alpha} N_2^{\frac{5}{2}-s} N_5^{\frac{1}{2}-s+\alpha}$ , which suffices provided  $s > \frac{1}{2} + \alpha$  and  $\alpha < 1$ .

**Case  $\beta_2$  and  $\beta_3$ :** In these cases we have that the contribution of  $\Delta\zeta M_2$  is bounded by  $N_3^3 N_2^3$ . Taking square root and normalizing we thus obtain the bound  $N_4^{\frac{1}{2}-s+\alpha} N_5^{\frac{1}{2}-s+\alpha}$ , which suffices provided  $s > \frac{1}{2} + \alpha$ .

**Case  $\beta_4$ :** In this case we have that the contribution of  $\Delta\zeta M_2$  is bounded by  $N_4^{-\frac{1}{2}} N_2^3 N_3^3$ . Taking square root and normalizing gives the bound  $N_4^{\frac{1}{4}-s+\alpha} N_5^{\frac{1}{2}-s+\alpha}$ , which suffices provided  $s > \frac{1}{2} + \alpha$ .

**Case  $\beta_5$ :** In this case we have that the contribution of  $\Delta\zeta M_2$  is bounded by  $N_4^{-1} N_5^{-2} N_2^3 N_3^3$  which is smaller than the bound in **Case  $\beta_4$** .

**Case  $\beta_6$  and **Case  $\beta_7$ :** In this case we have that the contribution of  $\Delta\zeta M_2$  is bounded by  $N_4^\varepsilon N_2^3 N_3^3$ . After taking square root and normalizing we get the bound  $N_4^{\frac{1}{2}-s+\alpha+\varepsilon} N_5^{\frac{1}{2}-s+\alpha}$ , which once again suffices provided  $s > \frac{1}{2} + \alpha$ .**

• **Case c, iii)** In this case we have that  $N_0 \sim N_1 \geq N_4 \geq N_2 \geq N_3 \geq N_5$ . Since  $\Delta\zeta M_1$  is bounded by

$$\delta^{-2\mu r} N_2^3 N_3^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|a_{n_2}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|\gamma\|_{\ell^2}^2 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2,$$

we have, after taking square root and normalizing the bound  $N_4^{-s+\frac{1}{2}+\alpha}$  just as before. The latter suffices provided  $s > \frac{1}{2} + \alpha$ . For  $M_2$ , following the scheme presented above for **Case c, ii)** we now have:

**Case  $\beta_1$ :** Since the contribution of  $\Delta\zeta M_2$  is bounded by  $N_3^3 N_2^{\frac{5}{2}}$ , we obtain, after taking square root and normalizing, the bound  $N_4^{\frac{7}{4}-2s+\alpha}$  which suffices provided  $s > \frac{7}{8} + \frac{\alpha}{2}$ .

**Case  $\beta_2$  and  $\beta_3$ :** In these cases we have that the contribution of  $\Delta\zeta M_2$  is bounded by  $N_3^3 N_2^3$ . Taking square root and normalizing we thus obtain the bound  $N_4^{2-2s+\alpha}$ , which suffices provided  $s > 1 + \frac{\alpha}{2}$ .

**Case  $\beta_4$ :** In this case we have that the contribution of  $\Delta\zeta M_2$  is bounded by  $N_4^{-\frac{1}{2}} N_2^3 N_3^3$  which is smaller than the bound in **Cases  $\beta_2, \beta_3$** .

**Case  $\beta_5$ :** In this case we have that the contribution of  $\Delta\zeta M_2$  is bounded by  $N_4^{-1} N_5^{-2} N_2^3 N_3^3$  which is smaller than the bound in **Case  $\beta_4$** .

**Case  $\beta_6$  and **Case  $\beta_7$ :** In these cases we have that the contribution of  $\Delta\zeta M_2$  is bounded by  $N_4^\varepsilon N_2^3 N_3^3$ . After taking square root and normalizing we get the bound  $N_4^{2-2s+\alpha+\varepsilon}$ , which once again suffices provided  $s > 1 + \frac{\alpha}{2}$ .**

• **Case c), iv), v), vi)**  $N_1 \sim N_0$  and  $N_2 \geq N_4$  and **Case d):**  $N_1 \sim N_2 \geq N_0, N_4$

In this case we proceed as in Subsection 8.5.1. Assume  $N_0 \sim N_1 \geq N_2 \geq N_4$ , Case d) having similar or better bounds. The estimates of the trilinear expressions will give after normalization

$$N_0^s N_1^{-s} N_2^{-s+1} N_3^{1-s} N_4^\alpha N_5^\alpha$$

and we assume that  $s > 1 + \alpha$ .

One also needs to estimate the terms in (8.7). Here we show how to estimate the term involving the random function at frequency  $N_4$  in (8.9). We first observe that in order for



this term not to be zero it must be that  $N_4 \sim N_0$ . Then for  $v_0^\omega$  in (5.1), after normalization we have the bound

$$\begin{aligned} & N_0^s N_1^{-s} N_2^{-s} N_3^{-s} N_4^{-1+\alpha} N_5^{-1+\alpha} \\ \times & \|P_{N_0} P_{N_4} v_0^\omega\|_{L_t^\infty H_x^{1-\alpha}} N_5^{\frac{1}{4}} \|D^{1-\alpha}(P_{N_0} P_{N_5} v_0^\omega)\|_{L_t^4 L_x^4} \prod_{j=1,2,3} N_j^{\frac{1}{4}} \|P_{N_j} u_j(x, t)\|_{U_\Delta^4 H^s}. \end{aligned}$$

The latter together with the Strichartz estimate (4.15) are enough to obtain the desired bound since for  $\alpha < \frac{3}{4}$ , we have:

$$N_0^s N_0^{-s+\frac{1}{4}-1+\alpha} \sim N_0^{-\frac{3}{4}+\alpha} < 1.$$

**8.5.4. The DDRRR Case.** To estimate the expression in (8.5) we first observe that in terms of bars we need to estimate only the following cases: Case 1:  $u_1, u_3, u_5$  are random, that is none of the random functions are conjugated, or Case 2: only one of these functions is conjugated, the other cases are obtained by conjugating the whole expression in (8.5). We will remark later on how the estimates change depending on these two Cases.

We now assume that the first three functions are random and the last two are deterministic. We also assume that  $N_1 \geq N_2 \geq N_3$  and  $N_4 \geq N_5$ . We then have the following subcases:

- **Case a)**:  $N_4 = \max(N_1, N_4)$ 
  - Case i)  $N_2 \leq N_5 \leq N_4$
  - Case ii)  $N_2 \leq N_5 \leq N_1 \leq N_4$
  - Case iii)  $N_3 \leq N_5 \leq N_2 \leq N_1 \leq N_4$
  - Case iv)  $N_5 \leq N_3 \leq N_2 \leq N_1 \leq N_4$
- **Case b)**:  $N_1 = \max(N_1, N_4)$  and  $N_2 \geq N_4$ 
  - Case i)  $N_3 \geq N_4$
  - Case ii)  $N_5 \leq N_3 \leq N_4 \leq N_2$
  - Case iii)  $N_3 \leq N_5 \leq N_4 \leq N_2$
- **Case c)**:  $N_1 = \max(N_1, N_4)$  and  $N_4 \geq N_2$ 
  - Case i)  $N_2 \leq N_5 \leq N_4 \leq N_1$
  - Case ii)  $N_3 \leq N_5 \leq N_2 \leq N_4 \leq N_1$
  - Case iii)  $N_5 \leq N_3 \leq N_2 \leq N_4 \leq N_1$

**Case a), i)**: In this case we proceed as in Subsection 8.5.1. Assume for simplicity that  $N_0 \sim N_4$ , the other cases are smoother. The estimates of the trilinear expressions will give after normalization

$$N_0^s N_4^{-s} N_5^{-s+1} N_3^\alpha N_2^\alpha N_1^\alpha$$

and we assume that  $s > 1 + 3\alpha$ . One also needs to estimate the terms in (8.7). Here we show how to estimate the factor involving the random term at frequency  $N_1$  in (8.9). We have for  $v_0^\omega$  in (5.1)

$$(8.36) \quad N_0^s N_1^{-1+\alpha} N_2^{-1+\alpha} N_3^{-1+\alpha} \|P_{N_0} P_{N_3} v_0^\omega\|_{L_t^\infty H_x^{1-\alpha}} \prod_{j=4,5} N_j^{\frac{1}{4}-s} \|P_{N_j} u_j(x, t)\|_{U_\Delta^4 H^s}.$$

where we notice that  $N_1 \sim N_0$  otherwise the contribution would be null. This is enough to obtain the desired bound since

$$N_0^s N_0^{-1+\alpha} N_4^{\frac{1}{4}-s} \sim N_4^{-\frac{3}{4}+\alpha}.$$

Also note that this case is not effected by conjugation hence it is the same both in Case 1 and Case 2.

**Case a), ii).** We also assume that  $N_4 \sim N_0$ , this is the least favorable situation. We proceed by duality and a change of variables  $\zeta = m \pm |n_4|^2$  as in the proof of Proposition 7.3 in particular see (7.15). We have to bound

$$(8.37) \quad \|\gamma\|_{\ell_\zeta^2}^2 \|a_{n_4}\|_{\ell^2}^2 \sum_{(\zeta, n_4) \in \mathbb{Z} \times \mathbb{Z}^3} \left| \sum_{\substack{n = \pm n_1 \pm n_2 \pm n_3 \pm n_4 \pm n_5, \\ n_i, n_j, n_k \neq n_r, n_p \\ \zeta = \pm |n_1|^2 \pm |n_2|^2 \pm |n_3|^2 \pm |n_5|^2}} \frac{\tilde{g}_{n_1}(\omega)}{|n_1|^{\frac{3}{2}}} \frac{\tilde{g}_{n_2}(\omega)}{|n_2|^{\frac{3}{2}}} \frac{\tilde{g}_{n_3}(\omega)}{|n_3|^{\frac{3}{2}}} k_n \tilde{a}_{n_5} \right|^2.$$

We now consider two cases:

- **Case  $A_0$ :**  $n_1, n_2, n_3$  are all different from each other.
- **Case  $A_1$ :** At least two of the frequencies  $n_1, n_2, n_3$  are equal.

**Case  $A_0$ :** We define the set

$$S_{(\zeta, n_4, n_1, n_2, n_3)} = \left\{ (n, n_5) : \begin{array}{l} n = \pm n_1 \pm n_2 \pm n_3 \pm n_4 \pm n_5, \\ n_i, n_j, n_k \neq n_r, n_p, \\ \zeta = \pm |n_1|^2 \pm |n_2|^2 \pm |n_3|^2 \pm |n_5|^2 \end{array} \right\}$$

with  $|S_{(\zeta, n_4, n_1, n_2, n_3)}| \lesssim N_5^2$  and we write

$$(8.37) \lesssim \|\gamma\|_{\ell_\zeta^2}^2 \|a_{n_4}\|_{\ell^2}^2 \sum_{(\zeta, n_4) \in \mathbb{Z} \times \mathbb{Z}^3} N_1^{-3} N_2^{-3} N_3^{-3} \left| \sum_{n_1, n_2, n_3} \tilde{g}_{n_1}(\omega) \tilde{g}_{n_2}(\omega) \tilde{g}_{n_3}(\omega) \sum_{S_{(\zeta, n_4, n_1, n_2, n_3)}} k_n \tilde{a}_{n_5} \right|^2$$

By using Lemma 3.4 we can continue, for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ , with

$$\begin{aligned} &\lesssim \delta^{-2\mu r} \|\gamma\|_{\ell_\zeta^2}^2 \|a_{n_4}\|_{\ell^2}^2 \\ &\quad \sum_{(\zeta, n_4) \in \mathbb{Z} \times \mathbb{Z}^3} N_1^{-3} N_2^{-3} N_3^{-3} \sum_{n_1, n_2, n_3} \sum_{S_{(\zeta, n_4, n_1, n_2, n_3)}} |S_{(\zeta, n_4, n_1, n_2, n_3)}| |k_n|^2 |a_{n_5}|^2 \\ &\lesssim \delta^{-2\mu r} \|\gamma\|_{\ell_\zeta^2}^2 \|a_{n_4}\|_{\ell^2}^2 N_1^{-3} N_2^{-3} N_3^{-3} N_5^2 \sum_{n, n_5} |k_n|^2 |a_{n_5}|^2 |S_{(n_5, n)}| \end{aligned}$$

where

$$S_{(n, n_5)} = \left\{ (\zeta, n_4, n_1, n_2, n_3) : \begin{array}{l} n = \pm n_1 \pm n_2 \pm n_3 \pm n_4 \pm n_5, \\ n_i, n_j, n_k \neq n_r, n_p, \\ \zeta = \pm |n_1|^2 \pm |n_2|^2 \pm |n_3|^2 \pm |n_5|^2 \end{array} \right\}$$

and  $|S_{(n, n_5)}| \lesssim N_1^3 N_2^3 N_3^3$ , where we used that  $\Delta \zeta \lesssim N_1^2$ . Hence we can continue with

$$\lesssim \delta^{-2\mu r} \|\gamma\|_{\ell_\zeta^2}^2 \|a_{n_4}\|_{\ell^2}^2 N_5^2 \|k_n\|_{\ell^2}^2 \|a_{n_5}\|_{\ell^2}^2$$

and after taking square root and normalizing we obtain the bound

$$N_5^{1-s} N_1^{-1+\alpha} N_2^{-1+\alpha} N_3^{-1+\alpha}.$$

We note that this case is the same both in Case 1 and Case 2.

**Case A<sub>1</sub>:** We first assume that only two frequencies are equal. The important remark is that we have removed the frequencies that would give rise to  $|g_n(\omega)|^2$  so in (8.37) we would see either  $(\tilde{g}_{n_1})^2(\omega)\tilde{g}_{n_3}(\omega)$  or  $\tilde{g}_{n_1}(\omega)(\tilde{g}_{n_2})^2(\omega)$ . In both cases we can still use Lemma 3.4 and proceed as above to obtain in fact better estimates since the cardinality of the sets involved are smaller due to the collapse of the frequencies that are equal.

If all three frequencies are equal, and this can happen only in Case 2, then we have that  $N_1 \sim N_2 \sim N_3$  and

$$(8.37) \lesssim \|\gamma\|_{\ell_\zeta^2}^2 \|a_{n_4}\|_{\ell^2}^2 \sum_{(\zeta, n_4) \in \mathbb{Z} \times \mathbb{Z}^3} N_1^{-12} \sum_{n_3} |g_{n_3}(\omega)|^3 \left| \sum_{S(\zeta, n_4, n_3)} k_n \tilde{a}_{n_5} \right|^2$$

where

$$S_{(\zeta, n_4, n_3)} = \left\{ (n, n_5) : \begin{array}{l} n = \pm 3n_3 \pm n_4 \pm n_5, \\ \zeta = \pm 3|n_3|^2 \pm |n_5|^2 \end{array} \right\}.$$

Then by using Lemma 3.7 we can continue, for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ , with

$$\lesssim \|\gamma\|_{\ell_\zeta^2}^2 \|a_{n_4}\|_{\ell^2}^2 \sum_{(\zeta, n_4) \in \mathbb{Z} \times \mathbb{Z}^3} N_1^{-12+\varepsilon} \sum_{S(\zeta, n_4)} |k_n|^2 |a_{n_5}|^2 |S_{(\zeta, n_4)}|,$$

where

$$S_{(\zeta, n_4)} = \left\{ (n, n_3, n_5) : \begin{array}{l} n = \pm 3n_3 \pm n_4 \pm n_5, \\ \zeta = \pm 3|n_3|^2 \pm |n_5|^2 \end{array} \right\}.$$

with  $|S_{(\zeta, n_4)}| \lesssim N_5^2 N_3$ , and we continue with

$$\lesssim \|\gamma\|_{\ell_\zeta^2}^2 \|a_{n_4}\|_{\ell^2}^2 N_1^{-12+\varepsilon} N_5^2 N_3 \sum_{n, n_5} |k_n|^2 |a_{n_5}|^2 |S_{(n, n_5)}|,$$

where

$$S_{(n, n_5)} = \left\{ (\zeta, n, n_3, n_4) : \begin{array}{l} n = \pm 3n_3 \pm n_4 \pm n_5, \\ \zeta = \pm 3|n_3|^2 \pm |n_5|^2 \end{array} \right\}.$$

with  $|S_{(n, n_5)}| \lesssim N_3^3$ . We obtain the bound  $N_1^{-6+\varepsilon}$  which clearly suffices without any further restriction when we take square root and normalize.

We now observe that **Cases a) iii)** and **iv)** can be analyzed just like **Cases a) i)** since  $N_4$  and  $N_1$  are still the top frequencies and the order of the rest is not relevant.

**Case b), i)** We assume first that  $N_1 \sim N_0$ . We cut the frequency windows  $N_0$  and  $N_1$  by cubes  $C$  of sidelength  $N_2$ . After using Cauchy-Schwarz we need to estimate

$$(8.38) \sum_{m \in \mathbb{Z}, n \in C} \left| \sum_{n_1, n_2, n_3; n_1 \in C} \tilde{g}_{n_1}(\omega) \tilde{g}_{n_2}(\omega) \tilde{g}_{n_3}(\omega) \left[ \sum_{S(m, n, n_1, n_2, n_3)} \frac{1}{|n_1|^{\frac{3}{2}}} \frac{1}{|n_2|^{\frac{3}{2}}} \frac{1}{|n_3|^{\frac{3}{2}}} \tilde{a}_{n_5} \tilde{a}_{n_4} \right] \right|^2$$

where

$$S_{(m, n, n_1, n_2, n_3)} = \left\{ (n_4, n_5) : \begin{array}{l} n = \pm n_1 \pm n_2 \pm n_3 \pm n_4 \pm n_5, \\ n_i, n_j, n_k \neq n_r, n_p, m = \pm |n_1|^2 \pm |n_2|^2 \pm |n_3|^2 \pm |n_5|^2 \pm |n_4|^2 \end{array} \right\},$$

with  $|S_{(m, n, n_1, n_2, n_3)}| \lesssim N_5^2$ . We now consider two cases:

- **Case A<sub>0</sub>:**  $n_1, n_2, n_3$  are all different from each other.

- **Case  $A_1$ :** At least two of the frequencies  $n_1, n_2, n_3$  are equal.

**Case  $A_0$ :** We use Lemma 3.4 and, for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ , we have

$$\begin{aligned}
(8.38) &\lesssim \delta^{-2\mu r} \sum_{m \in \mathbb{Z}, n \in C} \sum_{n_1, n_2, n_3; n_1 \in C} \left[ \sum_{S(m, n, n_1, n_2, n_3)} \frac{1}{|n_1|^{\frac{3}{2}}} \frac{1}{|n_2|^{\frac{3}{2}}} \frac{1}{|n_3|^{\frac{3}{2}}} \tilde{a}_{n_5} \tilde{a}_{n_4} \right]^2 \\
&\lesssim \delta^{-2\mu r} N_1^{-3} N_2^{-3} N_3^{-3} \sum_{m \in \mathbb{Z}, n \in C} \sum_{S(m, n)} \#S(m, n, n_1, n_2, n_3) |a_{n_5}|^2 |a_{n_4}|^2 \\
&\lesssim \delta^{-2\mu r} N_1^{-3} N_2^{-3} N_3^{-3} N_5^2 \sum_{n_4, n_5} |a_{n_5}|^2 |a_{n_4}|^2 |S_{(n_4, n_5)}|
\end{aligned}$$

where

$$S_{(n_4, n_5)} = \left\{ (m, n, n_1, n_2, n_3) : \begin{array}{l} n = \pm n_1 \pm n_2 \pm n_3 \pm n_4 \pm n_5, \\ m = \pm |n_1|^2 \pm |n_2|^2 \pm |n_3|^2 \pm |n_5|^2 \pm |n_4|^2 \end{array} \right\},$$

with  $|S_{(n_4, n_5)}| \lesssim N_1 N_2 N_1 N_2^3 N_3^3$ , which finally gives

$$(8.38) \lesssim \delta^{-2\mu r} N_1^{-1} N_2 \|a_{n_5}\|_{\ell^2}^2 \|a_{n_4}\|_{\ell^2}^2.$$

By taking square root and normalizing we require that

$$N_0^s N_1^{-\frac{3}{2} + \alpha} N_2^{-\frac{1}{2} + \alpha} N_3^{-1 + \alpha} N_4^{-s} N_5^{-s} \lesssim N_1^{-\beta}$$

and this follows from assuming  $s < \frac{3}{2} - \alpha$ .

**Case  $A_1$ :** We proceed just like in the same case for **Case a), ii)**. Here we only work out the details for the case when all frequencies are equal, again this can happen only in Case 2. We have that  $N_1 \sim N_2 \sim N_3$  and

$$(8.38) \lesssim \sum_{(m, n) \in \mathbb{Z} \times \mathbb{Z}^3} N_1^{-12} \left| \sum_{n_3} |g_{n_3}(\omega)|^3 \sum_{S(m, n, n_3)} \tilde{a}_{n_4} \tilde{a}_{n_5} \right|^2$$

where

$$S_{(m, n, n_3)} = \left\{ (n_4, n_5) : \begin{array}{l} n = \pm 3n_3 \pm n_4 \pm n_5, \\ m = \pm 3|n_3|^2 \pm |n_4|^2 \pm |n_5|^2 \end{array} \right\}.$$

Then by using Lemma 3.7, for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ , we can continue with

$$\lesssim \sum_{(m, n) \in \mathbb{Z} \times \mathbb{Z}^3} N_1^{-12 + \varepsilon} \sum_{S(m, n)} |a_{n_4}|^2 |a_{n_5}|^2 |S_{(m, n)}|,$$

where

$$S_{(m, n)} = \left\{ (n_3, n_4, n_5) : \begin{array}{l} n = \pm 3n_3 \pm n_4 \pm n_5, \\ m = \pm 3|n_3|^2 \pm |n_4|^2 \pm |n_5|^2 \end{array} \right\}.$$

with  $|S_{(m, n)}| \lesssim N_5^2 N_3$ , and we continue with

$$\lesssim N_1^{-12 + \varepsilon} N_5^2 N_3 \sum_{n_4, n_5} |a_{n_4}|^2 |a_{n_5}|^2 |S_{(n_4, n_5)}|,$$

where

$$S_{(n_4, n_5)} = \left\{ (m, n, n_3) : \begin{array}{l} n = \pm 3n_3 \pm n_4 \pm n_5, \\ m = \pm 3|n_3|^2 \pm |n_4|^2 \pm |n_5|^2 \end{array} \right\}.$$

with  $|S_{(n_4, n_5)}| \lesssim N_1^2 N_3$ . We obtain the bound  $N_1^{-6+\varepsilon}$  which clearly suffices without any further restriction when we take square root and normalize.

Now assume that  $N_1 \sim N_2$ . Here we do not need to cut with cubes  $C$ , but the argument and the estimates are similar as the ones we just analyzed.

**Case b), ii), iii):** These cases are estimated just like the case we just analyzed since the two highest frequencies are still  $N_1$  and  $N_2$  and the order of the others is not relevant.

**Case c), i):** Assume first  $N_0 \sim N_1$ . This case is handled like **Case b) i)** above. Here we cut with cubes  $C$  of sidelength  $N_4$ . This gives in particular that  $\Delta m \lesssim N_1 N_4$ .

**Case  $A_0$ :** Just like in **Case b) i)** we have, for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ , that

$$(8.38) \lesssim \delta^{-2\mu r} N_1^{-3} N_2^{-3} N_3^{-3} N_5^2 \sum_{n_4, n_5} |a_{n_5}|^2 |a_{n_4}|^2 |S_{(n_4, n_5)}|$$

where now  $|S_{(n_4, n_5)}| \lesssim N_1 N_4 N_1 N_2^3 N_3^3$ , since  $\Delta m \lesssim N_1 N_4$ . This finally gives

$$(8.38) \lesssim \delta^{-2\mu r} N_1^{-1} N_4 \|a_{n_5}\|_{\ell^2}^2 \|a_{n_4}\|_{\ell^2}^2.$$

By taking square root and normalizing we require that

$$N_0^s N_1^{-\frac{3}{2}+\alpha} N_2^{-1+\alpha} N_3^{-1+\alpha} N_4^{-s+\frac{1}{2}} N_5^{-s} \lesssim N_1^{-\beta}$$

and this follows from assuming again  $s < \frac{3}{2} - \alpha$ .

**Case  $A_1$ :** This is like the same case for **Case b) i)**.

**Case c), i):** Now assume  $N_4 \sim N_1$ . Here we do not need to cut and the same estimates as the ones we just presented hold.

**Case c), ii), iii):** These cases are estimated just like the case we just analyzed since the two highest frequencies are still  $N_1$  and  $N_4$  and the order of the others is not relevant.

8.5.5. *The DRRRR Case.* To estimate the expression in (8.5) we assume without any loss of generality that  $u_5$  is the deterministic function and it is not conjugated. By Cauchy-Schwarz and Proposition 4.8 we are reduced to estimate

$$(8.39) \sum_{m \in \mathbb{Z}, n \in C} \left| \sum_{\substack{n_1, n_2, n_3, n_4 \\ n_1, n_3 \neq n_2, n_4}} \left[ \sum_{\substack{n_5 = -n_1 + n_2 - n_3 + n_4 - n, \\ n_5 \neq n_2, n_4 \\ m = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_5|^2}} a_{n_5} \frac{g_{n_1}(\omega) \bar{g}_{n_2}(\omega) g_{n_3}(\omega) \bar{g}_{n_4}(\omega)}{|n_1|^{\frac{3}{2}} |n_2|^{\frac{3}{2}} |n_3|^{\frac{3}{2}} |n_4|^{\frac{3}{2}}} \right] \right|^2,$$

where we have assumed that  $\widehat{u_5}(n_5, t) = e^{it|n_5|^2} a_{n_5}$  and  $C$  is a cube of sidelength to be determined later.

Since we have removed the frequencies  $n_1, n_3 = n_2$  or  $n_1, n_3 = n_4$ , which would give rise to terms of the form  $|g_i(\omega)|^2$ , we can invoke Lemma 3.4 and proceed by further considering the following subcases. For  $i, j \in \{1, 2, 3, 4\}$ ,

- **Case a):** There exists  $j$  such that  $N_0 \sim N_j$ ,  $N_5 \lesssim N_j$ .
- **Case b):** There exist  $j \neq i$  such that  $N_i \sim N_j$  and  $N_5, N_0 \lesssim N_i$ .
- **Case c):**  $N_0 \sim N_5$  and  $N_j \lesssim N_5$ .
- **Case d):** There exist  $j \neq i$  such that  $N_5 \sim N_j$  and  $N_0, N_i \lesssim N_j$ .

**Case a):** Assume  $N_k$ ,  $k \in \{1, 2, 3, 4, 5\}$ ,  $k \neq j$  is the second largest frequency. Then let  $C$  be of sidelength  $N_k$  and let

$$S_{(n,m,n_1,n_2,n_3,n_4)} = \left\{ \begin{array}{l} n_5 = -n_1 + n_2 - n_3 + n_4 - n, \\ n_5 : n_5 \neq n_2, n_4, n_j \in C \\ m = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_5|^2 \end{array} \right\}.$$

By Lemma 3.4, for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ , we have

$$\begin{aligned} (8.39) &\lesssim \delta^{-2\mu r} \sum_{m \in \mathbb{Z}, n \in C} N_1^{-3} N_2^{-3} N_3^{-3} N_4^{-3} \sum_{n_1, n_2, n_3, n_4} \left| \sum_{S_{(n,m,n_1,n_2,n_3,n_4)}} a_{n_5} \right|^2 \\ &\lesssim \delta^{-2\mu r} \sum_{m \in \mathbb{Z}, n \in C} N_1^{-3} N_2^{-3} N_3^{-3} N_4^{-3} \sum_{S_{(n,m)}} |a_{n_5}|^2 \end{aligned}$$

where

$$S_{(n,m)} = \left\{ \begin{array}{l} n = n_1 - n_2 + n_3 - n_4 + n_5, \\ (n_1, n_2, n_3, n_4, n_5) : n_j \in C \\ m = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_5|^2 \end{array} \right\}.$$

We now define the set

$$S_{(n_5)} = \left\{ \begin{array}{l} n = n_1 - n_2 + n_3 - n_4 + n_5, \\ (m, n, n_1, n_2, n_3, n_4) : n_j \in C \\ m = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_5|^2 \end{array} \right\}$$

where  $|S_{(n_5)}| \lesssim N_j^2 N_k^4 N_p^3 N_q^3$ . Then we continue with

$$(8.39) \lesssim \delta^{-2\mu r} N_1^{-3} N_2^{-3} N_3^{-3} N_4^{-3} \sum_{n_5} |a_{n_5}|^2 |S_{(n_5)}| \lesssim \delta^{-2\mu r} N_j^{-1} N_k \|a_{n_5}\|_{\ell^2}^2,$$

By taking square root and normalizing we obtain the bound  $N_j^{s+\alpha-\frac{3}{2}} N_k^{-\frac{1}{2}+\alpha}$  which entails  $s + \alpha < \frac{3}{2}$  and  $\alpha < \frac{1}{2}$ .

**Case b):** We go back to (8.39) and we let  $C$  be of its natural sidelength  $N_0$ . We then repeat the argument above with the role of  $N_k$  played by  $N_j$  and we count the set

$$S_{(n_5)} = \left\{ \begin{array}{l} n = n_1 - n_2 + n_3 - n_4 + n_5, \\ (m, n, n_1, n_2, n_3, n_4) : \\ m = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_5|^2 \end{array} \right\}$$

where  $|S_{(n_5)}| \lesssim N_j^3 N_i^3 N_p^3 N_q^3$ . By taking square root and normalizing we obtain the bound  $N_j^{s+2\alpha-2}$  which entails  $s + 2\alpha < 2$ .

**Case c):** We proceed as in **Case b)** of Subsection 8.5.2. More precisely by duality and a change of variables  $\zeta = m - |n_5|^2$  as in the proof of Proposition 7.3 in particular see (7.15). Here we let  $C$  be of its natural sidelength  $N_0$ . Let also  $N_k$ , where  $N_k$ ,  $k \in \{1, \dots, 4\}$  be the

second largest frequency. We have to bound  
(8.40)

$$\|\gamma\|_{\ell^2_\zeta}^2 \|\chi_C a_{n_5}\|_{\ell^2}^2 \sum_{(\zeta, n_5) \in \mathbb{Z} \times \mathbb{Z}^3} \left| \sum_{\substack{n=n_5-n_2+n_3-n_4+n_1, \\ n_1, n_3, n_5 \neq n_2, n_4 \\ \zeta=|n_1|^2-|n_2|^2+|n_3|^2-|n_4|^2}} \chi_C(n) k_n \frac{g_{n_1}(\omega) \bar{g}_{n_2}(\omega) g_{n_3}(\omega) \bar{g}_{n_4}(\omega)}{|n_1|^{\frac{3}{2}} |n_2|^{\frac{3}{2}} |n_3|^{\frac{3}{2}} |n_4|^{\frac{3}{2}}} \right|^2.$$

We proceed again as above where now we have to replace  $S_{(n_5)}$  by

$$S_{(n)} = \left\{ (\zeta, n_1, n_2, n_3, n_4, n_5) : \begin{array}{l} n = n_1 - n_2 + n_3 - n_4 + n_5, \\ \zeta = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 \end{array} \right\}$$

where  $|S_{(n)}| \lesssim N_k^3 N_i^3 N_p^3 N_q^3$ , where we used that  $\Delta\zeta \lesssim N_k^2$ . By taking square root and normalizing we obtain the bound  $N_k^{\alpha-1}$ .

**Case d):** This case is analogous to **Case c)**.

8.5.6. *The all random  $\overline{RRRR}$  Case.* Since we have removed the frequencies  $n_1, n_3 = n_2$  or  $n_1, n_3 = n_4$ , which would give rise to terms of the form  $|g_i(\omega)|^2$ , we can invoke Lemma 3.4 and proceed to estimate the expression in (8.5) by further considering the following two subcases, **Case a):**  $N_0 \sim N_i$  for some  $i = 1, \dots, 5$  and **Case b):**  $N_i \sim N_j$  for  $i, j \neq 0$ .

**Case a):** Let  $N_j$  be the second largest frequency size after  $N_i$ . We cut the window  $N_0$  with cubes  $C$  of sidelength  $N_j$ . By Cauchy-Schwarz and Plancherel we estimate

$$(8.41) \quad \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}^3} \left| \sum_{\substack{n=n_1-n_2+n_3-n_4+n_5, \\ n_1, n_3, n_5 \neq n_2, n_4 \\ m=|n_1|^2-|n_2|^2+|n_3|^2-|n_4|^2+|n_5|^2}} \frac{g_{n_1}(\omega) \bar{g}_{n_2}(\omega) g_{n_3}(\omega) \bar{g}_{n_4}(\omega) g_{n_5}(\omega)}{|n_1|^{\frac{3}{2}} |n_2|^{\frac{3}{2}} |n_3|^{\frac{3}{2}} |n_4|^{\frac{3}{2}} |n_5|^{\frac{3}{2}}} \right|^2$$

By Lemma 3.4 we have that, for  $\omega$  outside a set of measure  $e^{-\frac{1}{\delta^r}}$ ,

$$(8.41) \lesssim \delta^{-2\mu r} \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}^3} \sum_{\substack{n=n_1-n_2+n_3-n_4+n_5, \\ n_1, n_3, n_5 \neq n_2, n_4, n_i \in C \\ m=|n_1|^2-|n_2|^2+|n_3|^2-|n_4|^2+|n_5|^2}} \frac{1}{|n_1|^3} \frac{1}{|n_2|^3} \frac{1}{|n_3|^3} \frac{1}{|n_4|^3} \frac{1}{|n_5|^3} \\ \lesssim \delta^{-2\mu r} |S| \prod_{k=1}^5 N_k^{-3} \lesssim N_i^{-1} N_j$$

where

$$S = \left\{ (m, n, n_1, \dots, n_5) : \begin{array}{l} n = n_1 - n_2 + n_3 - n_4 + n_5, \quad n_i \in C \\ m = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_5|^2 \end{array} \right\}$$

and  $|S| \lesssim N_i^2 N_j^4 \prod_{k \neq i, j, 0} N_k^3$ . Taking square root and normalizing we obtain the bound

$$N_i^{s+\alpha-\frac{3}{2}} N_j^{-\frac{1}{2}+\alpha} \prod_{k \neq i, j, 0} N_k^{-1+\alpha}$$

which suffices provided  $s + \alpha < \frac{3}{2}$  and  $\alpha < \frac{1}{2}$ .

**Case b):** This is like **Case a)**, but now we do not need to cut the support of the window  $N_0$  by  $N_j$ .

8.5.7. *The  $U_{\Delta}^4 L^2$  Estimates.* Assume that  $N_i$  are dyadic numbers and without loss of generality that  $N_1 \geq N_2 \geq \dots \geq N_5$ . We start by rewriting

$$\begin{aligned} & \int_0^{2\pi} \int_{\mathbb{T}^3} D^s(\mathcal{N}(P_{N_i}(w + v_0^\omega))) \overline{P_{N_0} h} dx dt = \int_0^{2\pi} \int_{\mathbb{T}^3} D^s(\mathcal{N}(P_{N_i} w)) \overline{P_{N_0} h} dx dt \\ & + \int_0^{2\pi} \int_{\mathbb{T}^3} D^s(\mathcal{N}(P_{N_i} v_0^\omega)) \overline{P_{N_0} h} dx dt \\ & + \int_0^{2\pi} \int_{\mathbb{T}^3} D^s(\mathcal{N}(P_{N_i} w, P_{N_i} v_0^\omega)) \overline{P_{N_0} h} dx dt, \\ & = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3, \end{aligned}$$

where in the term  $\mathcal{T}_3$  we include all the nonlinear expressions with both random and deterministic terms. Our goal is to obtain an estimate for the first and last term with the  $U_{\Delta}^4 L^2$  norms of  $w$  in the right hand side possibly paying the prize of  $N_2^\gamma$ , with  $\gamma > 0$ . Then using the interpolation Proposition 4.9, we combine this estimate with the ones involving the norms  $U_{\Delta}^2 L^2$  in previous sections and the embeddings (4.5) and (4.7) to finally conclude the proof of Proposition 5.2.

Clearly we do not need to estimate  $\mathcal{T}_2$  that involves purely random terms. For the other two we have

$$\mathcal{T}_1 + \mathcal{T}_3 \lesssim [\|\mathcal{N}(P_{N_i} w)\|_{L_t^4 L_x^2} + \|\mathcal{N}(P_{N_i} w, P_{N_i} v_0^\omega)\|_{L_t^4 L_x^2}] \|P_{N_0} h\|_{L_t^\infty L_x^2},$$

and from (2.9), we a certain abuse of notation,

$$\begin{aligned} & \|\mathcal{N}(P_{N_i} w)\|_{L_t^4 L_x^2} + \|\mathcal{N}(P_{N_i} w, P_{N_i} v_0^\omega)\|_{L_t^4 L_x^2} \\ & \lesssim \sum_{i=1}^9 \|\mathcal{F}^{-1} J_i(w)\|_{L_t^4 L_x^2} + \sum_{j=1}^9 \|\mathcal{F}^{-1} J_j(P_{N_i} w, P_{N_i} v_0^\omega)\|_{L_t^4 L_x^2} \\ & = \sum_{i=1}^9 (\mathcal{S}_1^i + \mathcal{S}_2^i) \end{aligned}$$

where  $J_i(w, v_0^\omega)$  indicates that the functions involved could be both  $w$  and  $v_0^\omega$ . To estimate  $\mathcal{S}_1^i$  and  $\mathcal{S}_2^i$  we use the transfer principle in Proposition 4.8 and we assume that  $\hat{w}(t, n) = e^{it|n|^2} b_n(t)$ . Below we write  $a_{n_j}^j$  to indicate  $b_n$  or the Fourier coefficients of  $v_0^\omega$  or their conjugates. Now define

$$\Phi_i(n, t) := \left| \sum_{\substack{n = \sum_{i=1}^5 \pm n_i, n_i \sim N_i, \\ W_i(n_1, n_2, n_3, n_4, n_5)}} a_{n_1}^1 e^{\pm it|n_1|^2} a_{n_2}^2 e^{\pm it|n_2|^2} a_{n_3}^3 e^{\pm it|n_3|^2} a_{n_4}^4 e^{\pm it|n_4|^2} a_{n_5}^5 e^{\pm it|n_5|^2} \right|^2,$$

where  $W_i(n_1, n_2, n_3, n_4, n_5)$  indicates the constraints among the five frequencies in  $J_i$ . Then for  $i = 1, \dots, 9$  and  $k = 1, 2$

$$(\mathcal{S}_k^i)^2 \lesssim \sup_{t \in [0, 2\pi]} \sum_n \Phi_i(n, t) \lesssim \sup_{t \in [0, 2\pi]} \sum_n \left| \sum_{S(n)} |a_{n_1}^1| |a_{n_2}^2| |a_{n_3}^3| |a_{n_4}^4| |a_{n_5}^5| \right|^2,$$



where

$$\mathcal{S}_{(n)} = \{(n_1, n_2, n_3, n_4, n_5) : \sum_{j=1}^5 \pm n_j, n_j \sim N_j\}.$$

Assume now that  $N_1$ , the highest frequency, is such that  $N_1 \sim N_0$ , which is in fact the least favorable situation. Then  $|\mathcal{S}_{(n)}| \lesssim N_2^3 N_3^3 N_4^3 N_5^3$  and by Cauchy-Schwarz

$$(8.42) \quad \left| \sum_{\mathcal{S}_{(n)}} |a_{n_1}^1| |a_{n_2}^2| |a_{n_3}^3| |a_{n_4}^4| |a_{n_5}^5| \right|^2 \lesssim N_2^3 N_3^3 N_4^3 N_5^3 \|a_{n_1}^1\|_{\ell^2}^2 \prod_{j=2}^5 \|a_{n_j}^j\|_{\ell^2}^2.$$

We then have

$$(8.43) \quad \mathcal{S}_1^i \lesssim N_2^{6-4s} \|P_{N_1} w\|_{U_{\Delta}^4 H^s} \prod_{j=2}^5 \|P_{N_j} w\|_{U_{\Delta}^4 H^s}.$$

We observe that a similar estimate holds for  $\mathcal{S}_2^i$  when the function associated to frequency  $N_1$  is also deterministic. In fact in this case we have

$$(8.44) \quad \mathcal{S}_2^i \lesssim N_2^{2+4\alpha} \|P_{N_1} w\|_{U_{\Delta}^4 H^s} \prod_{j \notin J, j \neq 1} \|P_{N_j} w\|_{U_{\Delta}^4 H^s}.$$

Finally if the function associated to frequency  $N_1$  is random, then we have

$$(8.45) \quad \mathcal{S}_2^i \lesssim N_1^{s-1+\alpha} N_2^{2+4\alpha} \prod_{j \notin J} \|P_{N_j} w\|_{U_{\Delta}^4 H^s}.$$

We conclude by using the interpolation Proposition 4.9. Note here that in both (8.43) and (8.44) the interpolation at most introduces a factor of  $N_2^\varepsilon$  which can be easily absorbed by the negative power of  $N_2$  in the estimates involving norms  $U_{\Delta}^2 L^2$ , see previous subsections. On the other hand (8.45) and interpolation introduce a factor  $N_1^\varepsilon$ . But this too can be absorbed thanks to the presence of a negative power of  $N_1$  in the estimates involving norms  $U_{\Delta}^2 L^2$  in those cases in which the highest frequency is associated to a random function.

This concludes the proof of Proposition 5.2.

## 9. PROOF OF PROPOSITION 5.3

*Proof of Proposition 5.3.* We first show an improved version of Proposition 5.2, that is we show that if  $r > 0$  is small enough then there exists  $\theta > 0$  such that for  $\omega \in \Omega_\delta$  we have: if  $N_1 \gg N_0$  or  $P_{N_1} w = P_{N_1} v_0^\omega$

$$(9.1) \quad \left| \int_0^{2\pi} \int_{\mathbb{T}^3} D^s (\psi_\delta(t) \mathcal{N}(P_{N_i}(w + v_0^\omega))) \overline{P_{N_0} h} dx dt \right| \lesssim \delta^\theta N_1^{-\varepsilon} \|P_{N_0} h\|_{Y^{-s}} \left( 1 + \prod_{i \notin J} \|\psi_\delta P_{N_i} w\|_{X^s} \right),$$

and if  $N_1 \sim N_0$  and  $P_{N_1} w \neq P_{N_1} v_0^\omega$

$$(9.2) \quad \left| \int_0^{2\pi} \int_{\mathbb{T}^3} D^s (\psi_\delta(t) \mathcal{N}(P_{N_i}(w + v_0^\omega))) \overline{P_{N_0} h} dx dt \right| \lesssim \delta^\theta N_2^{-\varepsilon} \|P_{N_0} h\|_{Y^{-s}} \|\psi_\delta P_{N_1} w\|_{X^s} \left( 1 + \prod_{i \notin J, i \neq 1} \|\psi_\delta P_{N_i} w\|_{X^s} \right),$$

for some small  $\varepsilon > 0$ .

To prove (9.1) and (9.2) we first observe that in the proof of Proposition 5.2, in particular the estimates involving the terms  $J_2, \dots, J_7$  in (5.3), we always have the factor  $\|P_{N_0} h\|_{L_t^2 L_x^2}$  in the right hand side. We can then replace this by

$$(9.3) \quad \|\psi_\delta(t) P_{N_0} h\|_{L_t^2 L_x^2} \lesssim \delta^{\frac{1}{2}} \|\psi_\delta(t) P_{N_0} h\|_{L_t^\infty L_x^2} \lesssim \delta^{\frac{1}{2}} \|\psi_\delta(t) P_{N_0} h\|_{Y^0},$$

where we used (4.7) and obtain the proof of Proposition 5.3 for the nonlinear terms involving  $J_2, \dots, J_7$ .

To estimate the term involving  $J_1$  we go back to Subsections 8.5.1-8.5.6. We recall that except when the top frequencies, say  $N_1$  and  $N_2$ , are associated to two deterministic functions, also in this case we have  $\|P_{N_0} h\|_{L_t^2 L_x^2}$  in the right hand side and (9.3) can be used again.

We are then reduced to estimating the term involving  $J_1$  where the top frequencies  $N_1$  and  $N_2$  are associated to two deterministic functions. So we consider

$$(9.4) \quad \left| \int_0^{2\pi} \int_{\mathbb{T}^3} \mathcal{F}_{-1} J_1(\psi_\delta(t) P_{N_i}(u_i) \psi_\delta(t) \overline{P_{N_0} h}) dx dt \right|$$

where without loss of generality  $N_1 \geq N_2 \geq N_3 \geq N_4 \geq N_5$  and  $u_1$  and  $u_2$  are deterministic functions while  $u_{N_i}$ ,  $i = 3, 4, 5$  represents either  $w$  or  $v_0^\omega$ . We consider two cases:

- **Case 1:**  $\delta^{-\sigma} > N_2$
- **Case 2:**  $\delta^{-\sigma} \leq N_2$

where  $\sigma > 0$  will be determined later.

**Case 1:** We observe that the estimate of (9.4) can be reduced to analyzing an expression such as

$$(9.5) \quad \left| \int_0^\delta \int_{\mathbb{T}^3} \tilde{u}_{N_1} \tilde{u}_{N_2} \tilde{u}_{N_3} \tilde{u}_{N_4} \tilde{u}_{N_5} \tilde{h}_{N_0} dx dt \right|$$

where  $u_{N_i}$  are as above. In fact to obtain the full product as in (9.5) one needs to put back some frequencies, and hence some terms, see for example (8.7) in Subsection 8.5.1. But these terms are similar to those involved in  $J_2, \dots, J_7$  and again the gain on  $\delta$  is guaranteed by (9.3).

We then go back to (9.5) and we further assume that  $N_1 \sim N_0$ , which is the least favorable situation. We cut the frequency window  $N_0$ , and hence  $N_1$ , with respect to cubes  $C$  of sidelength  $N_2$  and we obtain the bound

$$\begin{aligned} & \left| \int_0^\delta \int_{\mathbb{T}^3} \tilde{u}_{N_1} \tilde{u}_{N_2} \tilde{u}_{N_3} \tilde{u}_{N_4} \tilde{u}_{N_5} \tilde{h}_{N_0} dx dt \right|^2 \lesssim \delta \sum_C \|P_C \tilde{u}_{N_1}\|_{L_t^{12} L_x^{12}}^2 \|P_C \tilde{h}_{N_0}\|_{L_t^{12} L_x^{12}}^2 \\ & \times \|\tilde{u}_{N_2}\|_{L_t^{12} L_x^{12}}^2 \|\tilde{u}_{N_3}\|_{L_t^{12} L_x^{12}}^2 \|\tilde{u}_{N_4}\|_{L_t^{12} L_x^{12}}^2 \|\tilde{u}_{N_5}\|_{L_t^{12} L_x^{12}}^2 \end{aligned}$$

and from (4.15), (4.16) and (4.17) we can continue with

$$(9.6) \quad \lesssim \delta N_2^{m(\alpha, s)} \sum_C \|P_C u_{N_1}\|_{U_\Delta^{12} L^2}^2 \|P_C h_{N_0}\|_{U_\Delta^{12} L^2}^2 \prod_{i \notin J, i \neq 1} \|u_{N_i}\|_{U_\Delta^{12} L^2}^2,$$

where  $J \subset \{2, 3, 4, 5\}$  is the set of indices corresponding to random linear solutions. Then normalizing, interpolating through Proposition 4.9, and using the embedding (4.7) combined with (4.6), we have

$$\begin{aligned} & \left| \int_0^\delta \int_{\mathbb{T}^3} \tilde{u}_{N_1} \tilde{u}_{N_2} \tilde{u}_{N_3} \tilde{u}_{N_4} \tilde{u}_{N_5} \tilde{h}_{N_0} dx dt \right| \\ & \lesssim \delta^{\frac{1}{2}} N_2^{m(\alpha, s)} \|P_{N_0} h\|_{Y^{-s}} \|\psi_\delta P_{N_1} w\|_{X^s} \left( 1 + \prod_{i \notin J, i \neq 1} \|\psi_\delta P_{N_i} w\|_{X^s} \right) \\ & \lesssim \delta^{\frac{1}{3}} \|P_{N_0} h\|_{Y^{-s}} \|\psi_\delta P_{N_1} w\|_{X^s} N_2^{-\varepsilon} \left( 1 + \prod_{i \notin J, i \neq 1} \|\psi_\delta P_{N_i} w\|_{X^s} \right), \end{aligned}$$

if we take  $\sigma < \frac{1}{100m(\alpha, s)}$ .

**Case 2:** Here we go back to (5.4) and (5.5). We recall that  $P_{N_1} u_1$  is deterministic and again we assume that  $N_1 \sim N_0$ , the other cases can be treated similarly. Then we use (5.4) and we have

$$\begin{aligned} & \left| \int_0^{2\pi} \int_{\mathbb{T}^3} \mathcal{F}_{-1} J_1(\psi_\delta(t) P_{N_i}(u_i) \psi_\delta(t) \overline{P_{N_0} h}) dx dt \right| \\ & \lesssim \delta^\gamma \delta^{-\gamma - \mu r} N_2^{-\rho(\alpha, s)} \|P_{N_0} h\|_{Y^{-s}} \|\psi_\delta P_{N_1} w\|_{X^s} \prod_{i \notin J, i \neq 1} \|\psi_\delta P_{N_i} w\|_{X^s} \\ & \lesssim \delta^\gamma N_2^{-\varepsilon} \|P_{N_0} h\|_{Y^{-s}} \|\psi_\delta P_{N_1} w\|_{X^s} \left( 1 + \prod_{i \notin J, i \neq 1} \|\psi_\delta P_{N_i} w\|_{X^s} \right), \end{aligned}$$

provided  $\sigma \geq \frac{\gamma + \mu r}{\rho(\alpha, s)}$  which is satisfied for  $\gamma, r$  small enough.

To finish the proof we now need to sum the dyadic blocks just as in [23]. In (9.1) we have enough decay in the highest frequency  $N_1$  that we can use Cauchy-Schwarz in all the smaller frequencies terms and just pay with a  $N_1^{-\frac{\varepsilon}{2}}$ . In (9.2) instead we use Cauchy-Schwarz for the lower frequencies  $N_5, \dots, N_3$  and pay with a  $N_2^{-\frac{\varepsilon}{2}}$  that can be absorbed and use Cauchy-Schwarz on  $N_0 \sim N_1$ . □

## REFERENCES

- [1] Bourgain, J., *Periodic nonlinear Schrödinger equation and invariant measures*, Commun. Math. Phys. **166** pp. 1-26, [1994].
- [2] Bourgain, J., *Nonlinear Schrödinger equations*, IAS/ Park City Math. Series **5** pp. 3-157, [1999].
- [3] Bourgain, J., *Global solutions of nonlinear Schrödinger equations*, American Mathematical Society, Colloquium Publications **46** [1999].
- [4] Bourgain, J., *Invariant measures for the 2D-defocusing nonlinear Schrödinger equation*. Comm. Math. Phys. 176, no. 2, 421- 445, [1996].
- [5] Bourgain, J., *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger Equations*, Geom. Funct. Anal. **3**, no. 2, 107-156, [1993].
- [6] Bourgain, J. and Bulut, A., *Gibbs measure evolution in radial nonlinear wave and Schrödinger equations on the ball*. C. R. Math. Acad. Sci. Paris 350, no. 11-12, 571575, [2012].
- [7] Bourgain, J. and Bulut, A., *Almost sure global well posedness for the radial nonlinear Schrödinger equation on the unit ball II: the 3D case*. Preprint, arXiv:1302.5409 .

- [8] Bourgain, J. and Bulut, A., *Invariant Gibbs measure evolution for the radial nonlinear wave equation on the 3D ball*. Preprint, arXiv:1304.1477.
- [9] Brydges, D. and Slade, G., *Statistical mechanics of the 2-dimensional focusing nonlinear Schrödinger equation*. Comm. Math. Phys., 182, 485-504, [1996].
- [10] Burq, N., *Random data Cauchy Theory for Dispersive Partial Differential Equations* Proceedings of the International Congress of Mathematicians, Hyderabad, India, 1862-1883, [2010].
- [11] Burq, N., Tzvetkov, N., *Random data Cauchy theory for supercritical wave equations. I. Local theory*. Invent. Math. 173, no. 3, 449475, [2008].
- [12] Burq, N., Tzvetkov, N., *Random data Cauchy theory for supercritical wave equations. II. A global theory*. Invent. Math. 173, 477496 [2008].
- [13] Burq, N., Tzvetkov, N., *Probabilistic well posedness for the cubic wave equation*. JEMS, to appear [2011].
- [14] Burq, N., Thomann, L. and Tzvetkov, N., *Global infinite energy solutions for the cubic wave equation*. Preprint, arXiv:1210.2086.
- [15] Burq, N., Thomann, L. and Tzvetkov, N., *Long time dynamics for the one dimensional non linear Schrödinger equation*. Preprint, arXiv:1002.4054, [2010].
- [16] Colliander, J.; Keel, M.; Staffilani, G.; Takaoka, H.; Tao, T., *Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in  $\mathbb{R}^3$* . Ann. of Math. (2) 167, no. 3, 767865, [2008].
- [17] Colliander, J. and Oh, T., *Almost sure well-posedness of the cubic nonlinear Schrödinger equation below  $L^2(\mathbb{T})$* . Duke Math. J. 161, no. 3, 367414, [2012].
- [18] Deng, Y., *Two Dimensional NLS Equation With Random Radial Data*. Preprint, arXiv:1008.2657, preprint [2010].
- [19] Deng, Y., *Invariance of the Gibbs measure for the Benjamin-Ono equation*. Preprint, arXiv:1210.1542, [2012].
- [20] de Suzzoni, A.-S., *Invariant measure for the cubic wave equation on the unit ball of  $R^3$* . Dynamics of PDE, vol 8, n 2, pp 127-148, [2011].
- [21] de Suzzoni, A.-S., *Consequences of the choice of a particular basis of  $L^2(S^3)$  for the cubic wave equation on the sphere and the Euclidian space*, arXiv:1207.2384, preprint, [2012].
- [22] Hadac, M., Herr, S. and Koch, H. *Well-posedness and scattering for the KP-II equation in a critical space*, Ann. Inst. H. Poincaré, **3** pp. 917-941 [2009].
- [23] Herr, S., Tataru, D. and Tzvetkov, N. *Global well-posedness of the energy-critical nonlinear Schrödinger equation with small initial data in  $H^1(\mathbb{T}^3)$* . , Duke Math. J., **159** pp. 329-349 [2011].
- [24] Ionescu, A. D. and Pausader, B., *The energy-critical defocusing NLS on  $\mathbb{T}^3$* . Duke Math. J. 161 no. 8, 1581 - 1612, [2012].
- [25] Nahmod, A. R., Pavlovic, N. and Staffilani, G., *Almost sure existence of global weak solutions for super-critical Navier-Stokes equations*, arXiv:1204.5444, preprint [2012].
- [26] Nahmod, A. R., Oh, T., Rey-Bellet, L. and Staffilani, G., *Invariant weighted Wiener measures and almost sure global well-posedness for the periodic derivative NLS*. J. Eur. Math. Soc. (JEMS) 14, no. 4, 1275 - 1330, [2012].
- [27] Nahmod, A. R., Rey-Bellet, L., Sheffield, S. and Staffilani, G., *Absolute continuity of Brownian bridges under certain gauge transformations*. Math. Res. Lett. 18, no. 5, 875 - 887 [2011].
- [28] Oh, T., *Invariant Gibbs measures and a.s. global well-posedness for coupled KdV systems*. Differential Integral Equations 22, 637 - 668, [2009].
- [29] Oh, T., *Invariance of the Gibbs measure for the Schrödinger-Benjamin-Ono system*. SIAM J. Math. Anal. 41, no. 6, 2207 - 2225, [2010].
- [30] Salmhofer, M. *Renormalization. An Introduction*. Texts and Monographs in Physics. Springer-Verlag, Berlin, [1999].
- [31] *On solutions for periodic generalized KdV equations*. Internat. Math. Res. Notices, no. 18, 899 - 917, [1997].
- [32] Thomann, L. *Random data Cauchy problem for supercritical equations*. Ann. I. H. Poincaré-AN 26 2385 - 2402 [2009].
- [33] Thomann, L. and Tzvetkov, N., *Gibbs measure for the periodic derivative nonlinear Schrödinger equation*. Nonlinearity 23), no. 11, 2771 - 2791, [2010].
- [34] Tzvetkov, N. *Invariant measures for the defocusing nonlinear Schrödinger equation*. Ann. Inst. Fourier (Grenoble) 58, no. 7, 2543 - 2604, [2008].

- [35] Tzvetkov, N. *Construction of a Gibbs measure associated to the periodic Benjamin-Ono equation*. Probab. Theory Related Fields 146 no. 3-4, 481514, [2010].

\* DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MASSACHUSETTS AT AMHERST, 710 N. PLEASANT STREET, AMHERST MA 01003  
*E-mail address:* `nahmod@math.umass.edu`

† DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, 77 MASSACHUSETTS AVENUE, CAMBRIDGE, MA 02139  
*E-mail address:* `gigliola@math.mit.edu`