



Alpha Power Odd Generalized Exponential Family of Distributions: Model, Properties and Applications

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Highlights

- This paper focuses on a new distribution family.
- Three new distributions are defined from the new proposed distribution family.
- Applications based on real data sets are the importance and flexibility of the proposed models.

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Abstract

In this paper, we exhibit a general family of distributions called alpha power odd generalized exponential family of distributions. The new family is very flexible with increasing, decreasing, J, reversed-J, bathtub shapes. Statistical properties of the family such as quantile, expansion of density function, moments, incomplete moments, mean deviation, Bonferroni and Lorenz curves are proposed. The method of maximum likelihood to estimate the model parameters is used. Three applications based on real data sets are the importance and flexibility of the three proposed models.

Keywords

Alpha power transformation,
Odd generalized exponential family,
Moments
Maximum likelihood estimation

1. INTRODUCTION

The generation of new distributions beginning with a baseline distribution with parameters to existing distributions for building groups to illustrate flexibility is one of the favored research areas in the probability distribution submitted. In several applied fields, such as economics, environmental sciences, architecture, biological research, etc., many methods of applying a parameter to distributions have been proposed and used to model results. Some well-known families are the Marshall-Olkin-G by Marshall and Olkin [1], the beta-G by Eugene et al. [2], odd log-logistic-G by Gleaton and Lynch [3,4] the transmuted-G by Shaw and Buckley [5], the gamma-G by Zografos and Balakrishnan [6], the Kumaraswamy-G by Cordeiro and de Castro [7], the logistic-G by Torabi and Montazeri [8], exponentiated generalized-G by Cordeiro et al. [9], the McDonald-G by Alexander et al. [10], T-X family by Alzaatreh et al. [11], the Weibull-G by Bourguignon et al. [12], the generalized Lindley family by Cakmakyapan and Ozel [13], the odd generalized exponential (OGE) family by Tahir et al. [14], the beta odd log-logistic generalized by Cordeiro et al. [15], the generalized transmuted-G by Nofal et al. [16], alpha power transformed family of

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distributions introduced by Mahdavi and Kundu [17], alpha logarithmic transformation family of distributions introduced by Karakaya et al. [18] etc.

This article's purpose is to extract a new flexible distribution family, namely the distribution family of alpha power odd generalized exponential (AP_{OGE}). The feature of the hazard rate will increase, decrease, J, reverse J, shaped bathtub and upside-down bathtub. In this case, the odd generalized exponential (OGE) family is called the baseline distribution G for the transformed alpha power family and some of the new family's general properties are obtained.

The rest of the paper is organized as follows: In section 2, we present new alpha power odd-generalized exponential family (AP_{OGE}) of distributions, and some special cases. Useful expansions for density and cumulative density functions of this family are derived in section 3. Statistical properties of AP_{OGE} included quantile, ordinary, generating functions, conditional moments, mean deviation and Bonferroni and Lorenz curves are provided in section 4. Maximum likelihood estimation is performed in section 5. A simulation study is conducted to indicate performance for new family in section 6. Then, applications are presented in section 7. The conclusion is given in section 8.

2. NEW FAMILY

In this section we consider a cumulative distribution function (cdf) $G(x, \varphi)$ with the probability density function (pdf) $g(x, \varphi)$ and survival function $\bar{G}(x, \varphi)$ of a baseline model with parameter vector φ . Mahdavi and Kundu [17] proposed alpha power transformed family of distributions with the cdf and the pdf given by

$$F_{AP}(x; \alpha, \varphi) = \begin{cases} \frac{\alpha^{G(x; \varphi)} - 1}{\alpha - 1}, & \alpha > 0, \alpha \neq 1, x > 0 \\ G(x; \varphi), & \alpha = 1, x > 0 \end{cases} \quad (1)$$

and

$$f_{AP}(x; \alpha, \varphi) = \begin{cases} \frac{\log(\alpha) \alpha^{G(x; \varphi)} g(x; \varphi)}{\alpha - 1}, & \alpha > 0, \alpha \neq 1, x > 0 \\ g(x), & \alpha = 1, x > 0 \end{cases}, \quad (2)$$

respectively. Here, φ is the vector of parameter with baseline cdf $G(x; \varphi)$. Tahir et al. [14] presented the OGE family of distributions with the cdf and

$$H_{OGE}(x; \lambda, \theta, \varphi) = \left(1 - e^{-\lambda \left(\frac{G(x; \varphi)}{\bar{G}(x; \varphi)} \right)} \right)^\theta, \quad x > 0 \quad (3)$$

and the pdf

$$h_{OGE}(x; \lambda, \theta, \varphi) = \frac{\lambda \theta g(x; \varphi)}{\bar{G}^2(x; \varphi)} e^{-\lambda \left(\frac{G(x; \varphi)}{\bar{G}(x; \varphi)} \right)} \left[1 - e^{-\lambda \left(\frac{G(x; \varphi)}{\bar{G}(x; \varphi)} \right)} \right]^{\theta-1}, \quad x > 0, \quad (4)$$

respectively. Here, $\lambda > 0$ and $\theta > 0$ are two additional parameters and $g(x; \varphi)$, $G(x; \varphi)$ are the pdf and the cdf of any baseline distribution depending on a vector of parameter φ . Motivated from the alpha power transformation and OGE family, we construct a new generalized family by inserting (3) into (1). The cdf and pdf for new class AP_{OGE} family of distributions are

$$F_{AP_{OGE}}(x; \alpha, \lambda, \theta, \varphi) = \begin{cases} \frac{\alpha \left(1 - e^{-\lambda \left(\frac{G(x;\varphi)}{\bar{G}(x;\varphi)}\right)}\right)^{\theta} - 1}{\alpha - 1}, & \alpha > 0, \alpha \neq 1, x > 0 \\ \left(1 - e^{-\lambda \left(\frac{G(x;\varphi)}{\bar{G}(x;\varphi)}\right)}\right)^{\theta}, & \alpha = 1, x > 0 \end{cases} \quad (5)$$

and

$$f_{AP_{OGE}}(x; \alpha, \lambda, \theta, \varphi) = \begin{cases} \frac{\log(\alpha) \lambda \theta g(x; \varphi) e^{-\lambda \left(\frac{G(x;\varphi)}{\bar{G}(x;\varphi)}\right)} \left[1 - e^{-\lambda \left(\frac{G(x;\varphi)}{\bar{G}(x;\varphi)}\right)}\right]^{\theta-1} \alpha \left(1 - e^{-\lambda \left(\frac{G(x;\varphi)}{\bar{G}(x;\varphi)}\right)}\right)^{\theta}}{\bar{G}^2(x; \varphi) (\alpha - 1)}, & \alpha > 0, \alpha \neq 0, x > 0 \\ \frac{\lambda \theta g(x; \varphi) e^{-\lambda \left(\frac{G(x;\varphi)}{\bar{G}(x;\varphi)}\right)} \left[1 - e^{-\lambda \left(\frac{G(x;\varphi)}{\bar{G}(x;\varphi)}\right)}\right]^{\theta-1}}{\bar{G}^2(x; \varphi)}, & \alpha = 1, x > 0 \end{cases} \quad (6)$$

A random variable has the density function (6) is denoted by $X \sim AP_{OGE}(\eta)$ and $\eta = (\alpha, \lambda, \theta, \varphi)$. The hazard rate function (hrf) is given by

$$h(x; \alpha, \lambda, \theta, \varphi) = \begin{cases} \frac{\log(\alpha) \lambda \theta g(x; \varphi) e^{-\lambda \left(\frac{G(x;\varphi)}{\bar{G}(x;\varphi)}\right)} \left[1 - e^{-\lambda \left(\frac{G(x;\varphi)}{\bar{G}(x;\varphi)}\right)}\right]^{\theta-1} \alpha \left(1 - e^{-\lambda \left(\frac{G(x;\varphi)}{\bar{G}(x;\varphi)}\right)}\right)^{\theta}}{\bar{G}^2(x; \varphi) (\alpha - 1)}, & x > 0. \\ \frac{\left(1 - e^{-\lambda \left(\frac{G(x;\varphi)}{\bar{G}(x;\varphi)}\right)}\right)^{\theta}}{1 - \frac{\alpha \left(1 - e^{-\lambda \left(\frac{G(x;\varphi)}{\bar{G}(x;\varphi)}\right)}\right)^{\theta} - 1}{\alpha - 1}} \end{cases}$$

2.1. Some Special Cases of the AP_{OGE}-G Family

In this subsection, we introduce special models for AP_{OGE} family of distributions, "namely alpha power odd generalized exponential Lindley distribution, alpha power odd generalized exponential Rayleigh distribution and alpha power odd generalized exponential uniform distribution.

2.1.1. Alpha power odd generalized exponential Lindley AP_{OGE}-L distribution

Consider the $G(x) = 1 - (1 + \frac{ax}{1+a})e^{-ax}$ and $g(x) = \frac{a^2}{1+a} (1+x)e^{-ax}$ of the Lindley distribution, then the cdf and pdf of the AP_{OGE}-L model (for $x > 0$) are

$$F_{AP_{OGE-L}}(x; \alpha, \lambda, \theta, a) = \begin{cases} \frac{\alpha \left(1 - e^{-\lambda \left(1 + \frac{ax}{1+a}\right)^{-1} e^{-ax-1}}\right)^{\theta} - 1}{\alpha - 1}, & \alpha > 0, \alpha \neq 1, x > 0 \\ \left(1 - e^{-\lambda \left(1 + \frac{ax}{1+a}\right)^{-1} e^{-ax-1}}\right)^{\theta}, & \alpha = 1, x > 0 \end{cases}$$

and

$$f_{AP_{OGE}-L}(x; \alpha, \lambda, \theta, a) = \begin{cases} \frac{\log(\alpha) \frac{a^2}{1+a} (1+x)e^{-ax} e^{-\lambda \left((1+\frac{ax}{1+a})^{-1} e^{-ax} - 1 \right)} \left[1 - e^{-\lambda \left((1+\frac{ax}{1+a})^{-1} e^{-ax} - 1 \right)} \right]^{\theta-1}}{\left(1 + \frac{ax}{1+a} \right)^2 e^{-2ax} (\alpha - 1)} \alpha \left(1 - e^{-\lambda \left((1+\frac{ax}{1+a})^{-1} e^{-ax} - 1 \right)} \right)^\theta, & \alpha > 0, \alpha \neq 1, x > 0 \\ \frac{\frac{a^2}{1+a} (1+x)e^{-ax} e^{-\lambda \left((1+\frac{ax}{1+a})^{-1} e^{-ax} - 1 \right)}}{\left(1 + \frac{ax}{1+a} \right)^2 e^{-2ax}} \left[1 - e^{-\lambda \left((1+\frac{ax}{1+a})^{-1} e^{-ax} - 1 \right)} \right]^{\theta-1}, & \alpha = 1, x > 0 \end{cases}$$

Figure 1 presents some hazard shapes of $AP_{OGE} - L$ distribution for arbitrary parameters. As seen in Figure 1, the hrf should be increasing, decreasing and bathtub-shaped.

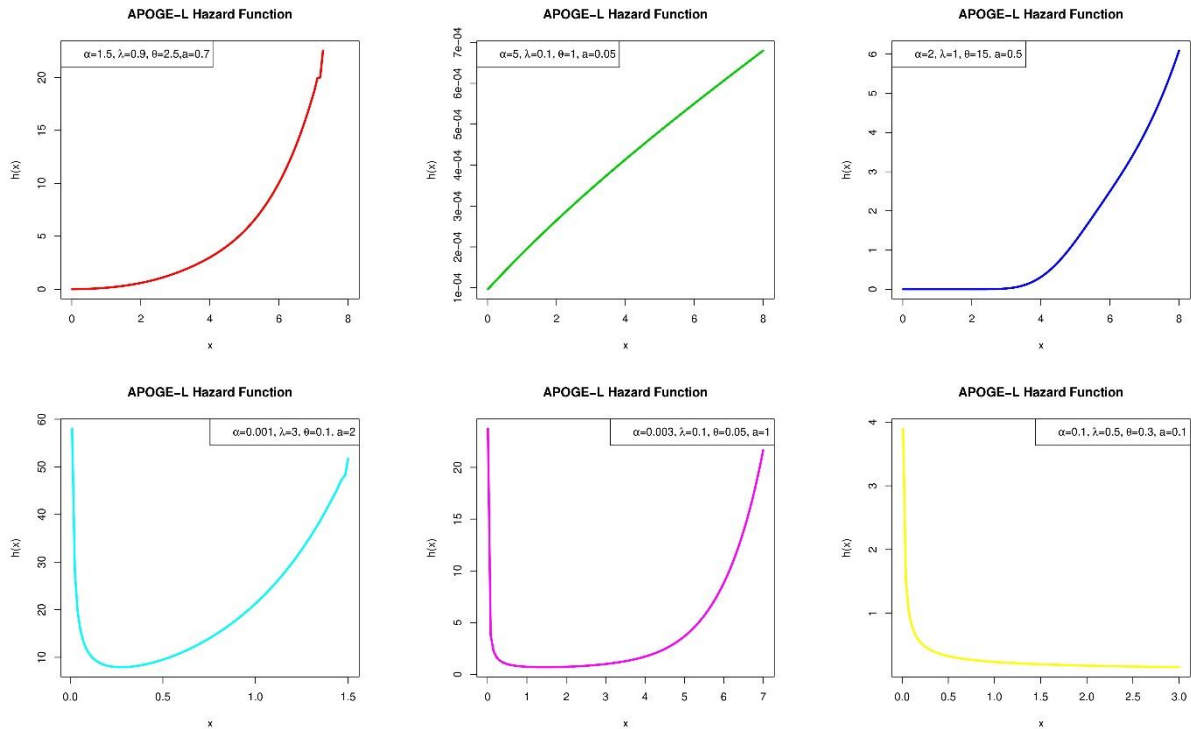


Figure 1. Some hazard shapes of $AP_{OGE} - L$ distribution for arbitrary parameters

2.1.2. Alpha power odd generalized exponential Rayleigh ($AP_{OGE} - R$) distribution

Consider the cdf $G(x) = 1 - e^{-\frac{\mu}{2}x^2}$ and the pdf $g(x) = \mu x e^{-\frac{\mu}{2}x^2}$ of the Rayleigh distribution, then the cdf and pdf of the $AP_{OGE} - R$ model (for $x > 0$) are

$$F_{AP_{OGE}-R}(x; \alpha, \lambda, \theta, \mu) = \begin{cases} \frac{\alpha \left(1 - e^{-\lambda \left(e^{-\frac{\mu}{2}x^2} - 1\right)}\right)^\theta - 1}{\alpha - 1}, & \alpha > 0, \alpha \neq 1, x > 0 \\ \left(1 - e^{-\lambda \left(e^{-\frac{\mu}{2}x^2} - 1\right)}\right)^\theta, & \alpha = 1, x > 0 \end{cases}$$

and

$$f_{AP_{OGE}-R}(x; \alpha, \lambda, \theta, \mu) = \begin{cases} \frac{\log(\alpha) \mu x e^{-\frac{\mu}{2}x^2} e^{-\lambda \left(e^{-\frac{\mu}{2}x^2} - 1\right)} \left[1 - e^{-\lambda \left(e^{-\frac{\mu}{2}x^2} - 1\right)}\right]^{\theta-1}}{e^{-\mu x^2} (\alpha - 1)} \alpha \left(1 - e^{-\lambda \left(e^{-\frac{\mu}{2}x^2} - 1\right)}\right)^\theta, & \alpha > 0, \alpha \neq 1, x > 0 \\ \frac{\mu x e^{-\frac{\mu}{2}x^2} e^{-\lambda \left(e^{-\frac{\mu}{2}x^2} - 1\right)} \left[1 - e^{-\lambda \left(e^{-\frac{\mu}{2}x^2} - 1\right)}\right]^{\theta-1}}{e^{-\mu x^2}}, & \alpha = 1, x > 0 \end{cases}$$

We present the hrf of the $AP_{OGE} - R$ distribution for arbitrary parameters in Figure 2. The plots show that the hrf of $AP_{OGE} - R$ distribution should be increasing, decreasing, and bathtub shaped.

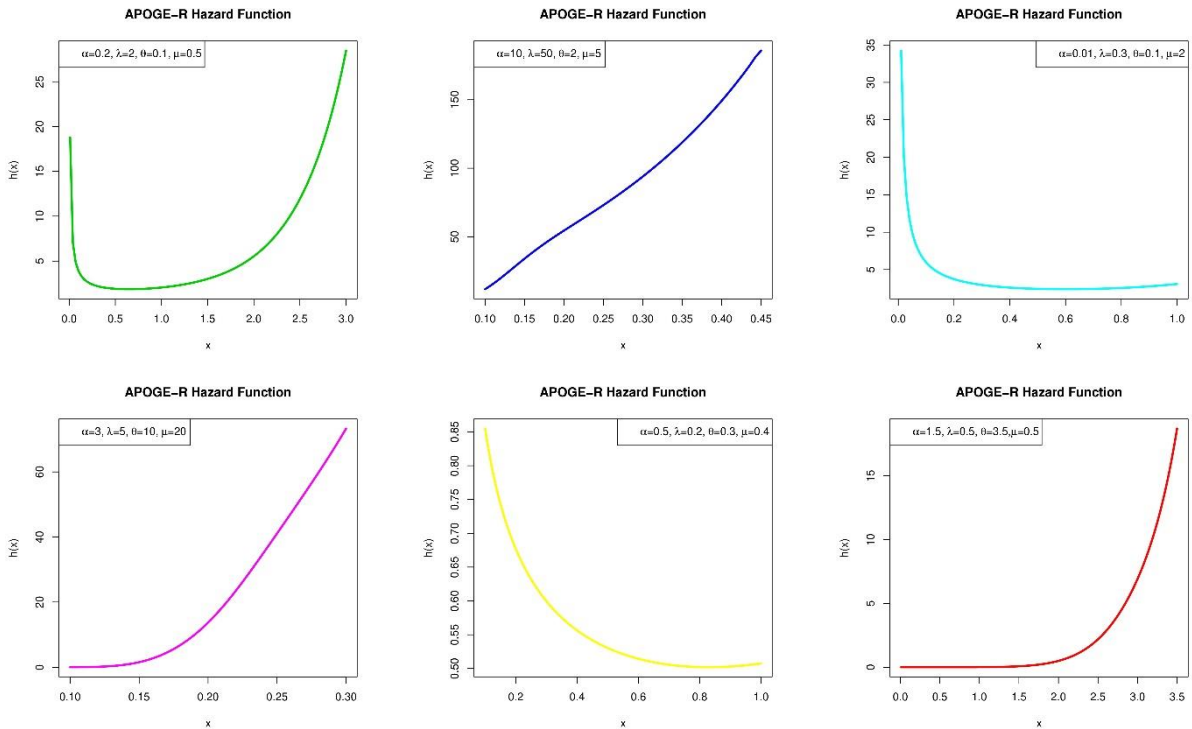


Figure 2. Some hazard shapes of $AP_{OGE} - R$ distribution for arbitrary parameters

2.1.3. Alpha power odd generalized exponential uniform ($AP_{OGE} - U$) distribution

Consider the uniform distribution on the interval $(0, \beta)$ with $G(x) = \frac{x}{\beta}$ and $g(x) = \frac{1}{\beta}$. The cdf and pdf for $AP_{OGE} - U$ distribution are, respectively, given by

$$F_{AP_{OGE}-U}(x; \alpha, \lambda, \theta, \beta) = \begin{cases} \frac{\alpha \left(1 - e^{-\lambda\left(\frac{x}{\beta}-1\right)}\right)^\theta - 1}{\alpha - 1}, & \alpha > 0, \alpha \neq 0, x > 0 \\ \left(1 - e^{-\lambda\left(\frac{x}{\beta}-1\right)}\right)^\theta, & \alpha = 1, x > 0 \end{cases}$$

$$f_{AP_{OGE}-U}(x; \alpha, \lambda, \theta, \beta) = \begin{cases} \frac{\log(\alpha) \frac{\lambda\theta}{\beta} e^{-\lambda\left(\frac{x}{\beta}-1\right)} \left[1 - e^{-\lambda\left(\frac{x}{\beta}-1\right)}\right]^{\theta-1} \alpha \left(1 - e^{-\lambda\left(\frac{x}{\beta}-1\right)}\right)^\theta}{\left(1 - \frac{x}{\beta}\right)^2 (\alpha - 1)}, & \alpha > 0, \alpha \neq 1, x > 0 \\ \frac{\frac{\lambda\theta}{\beta} e^{-\lambda\left(\frac{x}{\beta}-1\right)} \left[1 - e^{-\lambda\left(\frac{x}{\beta}-1\right)}\right]^{\theta-1}}{\left(1 - \frac{x}{\beta}\right)^2}, & \alpha = 1, x > 0 \end{cases}$$

The hrf for $AP_{OGE} - U$ distribution for several values of the parameters are given in Figure 3. The plots indicate that the hrf should be increasing, decreasing, J, reversed-J, bathtub shaped.

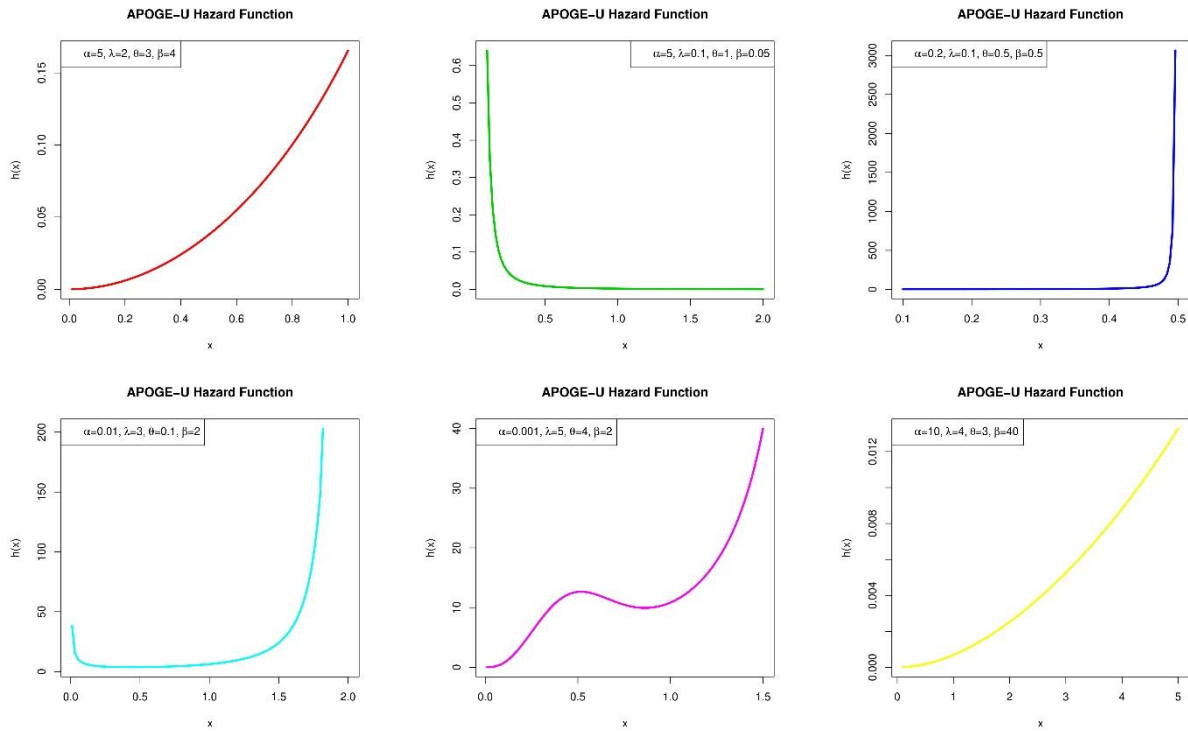


Figure 3. Some hazard shapes of $AP_{OGE} - U$ distribution for arbitrary parameters

3. USEFUL EXPANSIONS

We give a representation of the AP_{OGE} family using the series representation

$$\alpha^Z = \sum_{k=0}^{\infty} \frac{(\log \alpha)^k}{k!} Z^k \tag{7}$$

and the binomial series expansion given by

$$(1 - z)^{b-1} = \sum_{k=0}^{\infty} (-1)^j \binom{b-1}{j} z^j = \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(b)}{\Gamma(b-j)j!} z^j. \tag{8}$$

Here, $b > 0$ is the positive real non-integer and $|z| < 1$. Density function for AP_{OGE} family is expressed as

$$f_{AP_{OGE}}(x; \eta) = \frac{\lambda \theta g(x; \varphi)}{G^2(x; \varphi)} \sum_{i,j=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i!(\alpha-1)} (-1)^j \binom{\theta(i+1)-1}{j} e^{-\lambda(j+1)\left(\frac{G(x; \varphi)}{G(x; \varphi)}\right)}. \tag{9}$$

Expanding the term $e^{-\lambda(j+1)\left(\frac{G(x; \varphi)}{G(x; \varphi)}\right)}$ in power series, we can write

$$e^{-\lambda(j+1)\left(\frac{G(x; \varphi)}{G(x; \varphi)}\right)} = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k (j+1)^k}{k!} \frac{G(x; \varphi)^k}{G(x; \varphi)^k}. \tag{10}$$

Inserting (10) in (9), the AP_{OGE} density can be written as

$$f_{AP_{OGE}}(x; \eta) = \sum_{i,j,k=0}^{\infty} \frac{\lambda \theta (\log \alpha)^{i+1} (-1)^{j+k} \lambda^k (j+1)^k}{(\alpha-1) i! k!} \binom{\theta(i+1)-1}{j} \times g(x; \varphi) \frac{G(x; \varphi)^k}{G(x; \varphi)^{k+2}}. \tag{11}$$

Using the general binomial expansion to $(1 - G(x; \varphi))^{-(k+2)}$, we have

$$(1 - G(x; \varphi))^{-(k+2)} = \sum_{d=0}^{\infty} \frac{\Gamma(k+d+2)}{\Gamma(k+2)d!} G(x; \varphi)^d. \quad (12)$$

Applying the expansion in (12) to (11), AP_{OGE} family is expressed as an infinite linear combination of the pdf for the exponentiated- G (exp- G) family where

$$f_{AP_{OGE}}(x; \eta) = \sum_{k,d=0}^{\infty} \omega_{k,d} (k + d + 1) g(x; \varphi) G(x; \varphi)^{k+d} = \sum_{k,d=0}^{\infty} \omega_{k,d} \pi_{(k+d+1)}(x; \varphi). \quad (13)$$

Here, we define

$$\omega_{k,d} = \sum_{i,j=0}^{\infty} \frac{\lambda \theta (\log \alpha)^{i+1} (-1)^{j+k} \lambda^k (j+1)^k}{i! k! (k+d+1)(\alpha-1)} \binom{\theta(i+1)-1}{j} \frac{\Gamma(k+d+2)}{\Gamma(k+2)d!}$$

and $\pi_{\nu}(x) = \nu g(x)G(x)^{\nu-1}$ denotes the exp- G family with the parameter $\nu > 0$. Equation (13) shows that density of X is expressed as a linear mixture of exp- G densities with power parameter $(k + d + 1)$ [9]. Therefore, several statistical properties are obtained from exp- G distribution. The cdf for AP_{OGE} family is expressed as a mixture of the cdfs for exp- G . Then, we have

$$F_{AP_{OGE}}(x; \eta) = \sum_{k,d=0}^{\infty} \omega_{k,d} \Pi_{(k+d+1)}.$$

4. PROPERTIES

4.1. Quantile Function

The AP_{OGE} quantile function $x = Q(u)$ is obtained by inverting (5) as

$$F^{-1}(u) = Q_G(u) = G^{-1} \left[\frac{\frac{-1}{\lambda} \log \left[1 - \left(\frac{\log(1+u(\alpha-1))}{\log \alpha} \right)^{\frac{1}{\theta}} \right]}{1 + \left\{ \frac{-1}{\lambda} \log \left[1 - \left(\frac{\log(1+u(\alpha-1))}{\log \alpha} \right)^{\frac{1}{\theta}} \right] \right\}} \right]. \quad (14)$$

We generate X by taking u as a uniform random variable in $(0,1)$. One of the earliest skewness measures to be suggested is the Bowley skewness [19] defined by

$$SK = \frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2Q(\frac{1}{2})}{Q(\frac{3}{4}) - Q(\frac{1}{4})}.$$

On the other hand, the Moors kurtosis [20] based on cotiles is given by

$$KU = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})},$$

where $Q(\cdot)$ is the quantile function. SK and KU are less sensitive to outliers. Positive kurtosis indicates heavy tails and peakness in comparison to the normal distribution for symmetric unimodal distributions, whereas negative kurtosis indicates light tails and flatness. For the normal distribution, $SK = KU = 0$.

4.2. Moments and Moment Generating Functions

4.2.1. Moments

Let $Z_{(k+d+1)}$ be a random variable with pdf of Exp-G family which has power parameter $k + d + 1$. r_{th} moment for AP_{OGE} family of distributions can be obtained from (13)

$$\mu_r' = E(X^r) = \sum_{k,d=0}^{\infty} \omega_{k,d} E(Z_{(k+d+1)}^r) \quad (15)$$

where $Z_{(k+d+1)}$ denotes exp-G family by power parameter $k + d + 1$. Another formula of r_{th} moment follows from (13) as

$$\mu_r' = E(X^r) = \sum_{k,d=0}^{\infty} \omega_{k,d} E(Z_{(k+d+1)}^r)$$

where

$$E(Z_v^r) = \int_0^{\infty} x^r g(x; \varphi) G(x; \varphi)^{v-1} dx, \quad v > 0$$

is calculated via baseline quantile function $Q_G(u) = G^{-1}(u)$ as

$$E(Z_v^r) = \int_0^1 u^{v-1} Q_G(u)^r du.$$

Now, we introduce two formulae for the moment generating function (mgf). The first formula can be calculated from Equation (13) as follows:

$$M_X(t) = E(e^{tX}) = \sum_{k,d=0}^{\infty} \omega_{k,d} M_{k+d+1}(t), \quad (16)$$

where $M_{k+d+1}(t)$ is the mgf of $Z_{(k+d+1)}$. Consequently, we can be easily determined $M_X(t)$ from generating function for exp-G. The second formula of $M_X(t)$ follows from (13) as

$$M_X(t) = E(e^{tX}) = \sum_{k,d=0}^{\infty} \omega_{k,d} M_{k+d+1}(t)$$

where $M_{\kappa}(t)$ is the mgf of random variable Z_{κ} given by

$$M_{\kappa}(t) = \int_0^{\infty} e^{tx} g(x; \varphi) G(x; \varphi)^{\kappa-1} dx = \int_0^1 u^{\kappa-1} e^{tQ_G(u)} du, \quad \kappa > 0$$

which is calculated from baseline quantile function $Q_G(u) = G^{-1}(u)$.

4.2.2. Conditional Moments

The s_{th} incomplete moments of X defined by $\vartheta_s(t)$ for any real $s > 0$ is expressed from (13) as

$$\vartheta_s(t) = \int_{-\infty}^t x^s f(x) dx = \sum_{k,d=0}^{\infty} \omega_{k,d} \int_{-\infty}^t x^s \vartheta_{s,(k+d+1)}(t) dx \quad (17)$$

where

$$\vartheta_{s,\kappa}(t) = \int_0^{G(t)} u^{\kappa-1} Q_G(u)^s du$$

and $\vartheta_{s,\mathcal{X}}(t)$ can be evaluated numerically.

4.2.3. Mean Deviation

The first incomplete moment is important and it is useful to obtain mean deviation about the mean and about the median, where the mean deviations give important information about characteristic of population and also have been applied of income fields and property in economics. If X has the AP_{OGE} family of distributions. The mean deviations about mean $\mu = E(X)$ and mean deviations about median M are defined by

$$\delta_1(x) = E|X - \mu'_1| = 2\mu'_1 F(\mu'_1) - 2\vartheta_1(\mu'_1) \quad (18)$$

and

$$\delta_2(x) = E|X - M| = \mu'_1 - 2\vartheta_1(M) \quad (19)$$

respectively, where $\mu'_1 = E(X)$, $M = \text{median}(X) = Q(\frac{1}{2})$ and $\vartheta_1(t)$ is the first complete moment given by (17) with $s = 1$.

4.3. Bonferroni and Lorenz Curves

For a positive random variable X and a given probability p , Bonferroni and Lorenz curves are given by

$$B(p) = \frac{1}{p\mu'_1} \vartheta_1(q) \quad (20)$$

respectively, where $\mu'_1 = E(X)$, and $q = Q(p)$ is the quantile function of X at p .

$$L(p) = \frac{1}{\mu'_1} \vartheta_1(q) \quad (21)$$

5. MAXIMUM LIKELIHOOD ESTIMATION

In this section, we obtain the maximum likelihood estimates (MLEs) for parameters of AP_{OGE} family. Let x_1, \dots, x_n be a random sample of size n from the AP_{OGE} family given by (6). Let $\eta = (\alpha, \lambda, \theta, \varphi)^T$ be $p \times 1$ vector of parameters. The log-likelihood function of AP_{OGE} family is given by

$$\begin{aligned} L_n = & n \log(\log(\alpha)) - n \log(\alpha - 1) + n \log(\lambda) + n \log(\theta) + \sum_{i=1}^n \log g(x_i; \varphi) - \lambda \sum_{i=1}^n w_i + (\theta \\ & - 1) \sum_{i=1}^n \log(1 - e^{-\lambda w_i}) + \sum_{i=1}^n (1 - e^{-\lambda w_i})^\theta \log(\alpha) - 2 \sum_{i=1}^n \log \bar{G}(x_i; \varphi) \end{aligned} \quad (22)$$

where $w_i = \frac{G(x_i; \varphi)}{\bar{G}(x_i; \varphi)}$. The log-likelihood of Equation (22) is maximized by using SAS or R-language or by solving the nonlinear likelihood equations obtained by differentiating (22). The components of the score function $U_n(\eta) = \frac{\partial L_n}{\partial \eta} = \left[\frac{\partial L_n}{\partial \alpha}, \frac{\partial L_n}{\partial \lambda}, \frac{\partial L_n}{\partial \theta}, \frac{\partial L_n}{\partial \varphi} \right]^T$ are given by

$$U_n(\alpha) = \frac{n}{\alpha \log(\alpha)} - \frac{n}{\alpha - 1} + \frac{1}{\alpha} \sum_{i=1}^n (1 - e^{-\lambda w_i})^\theta, \quad (23)$$

$$U_n(\lambda) = \frac{n}{\lambda} - \sum_{i=1}^n w_i + (\theta - 1) \sum_{i=1}^n \frac{w_i e^{-\lambda w_i}}{(1 - e^{-\lambda w_i})} + \theta \log(\alpha) \sum_{i=1}^n w_i e^{-\lambda w_i} (1 - e^{-\lambda w_i})^{\theta-1}, \quad (24)$$

and

$$\frac{\partial L_n}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^n \log(1 - e^{-\lambda w_i}) + \log(\alpha) \sum_{i=1}^n \log(1 - e^{-\lambda w_i})(1 - e^{-\lambda w_i})^\theta \quad (25)$$

and

$$\begin{aligned} \frac{\partial L_n}{\partial \varphi_k} = & \sum_{i=1}^n \frac{g'(x_i; \varphi)}{g(x_i; \varphi)} - \lambda \sum_{i=1}^n \frac{\partial w_i}{\partial \varphi_k} + (\theta - 1) \sum_{i=1}^n \frac{\lambda e^{-\lambda w_i}}{(1 - e^{-\lambda w_i})} \frac{\partial w_i}{\partial \varphi_k} + \log(\alpha) \theta \sum_{i=1}^n \lambda e^{-\lambda w_i} (1 \\ & - e^{-\lambda w_i})^{\theta-1} \frac{\partial w_i}{\partial \varphi_k} - 2 \sum_{i=1}^n \frac{\overline{G}'_{\varphi_k}(x; \varphi)}{\overline{G}(x; \varphi)}, \end{aligned}$$

where $g'(x_i; \varphi) = \frac{\partial g(x_i; \varphi)}{\partial \varphi_k}$ and $\overline{G}'_{\varphi_k}(x; \varphi) = \frac{\partial \overline{G}(x; \varphi)}{\partial \varphi_k}$. The MLEs of η , say $\hat{\eta}$, is obtained by solving nonlinear system $U_n(\eta) = 0$.

6. SIMULATION

In this section, we perform a Monte Carlo simulation study for verifying finite sample behavior of MLEs of the parameters of the related parameters. We use a package programme in statistical software R, named Adequacy Model. In addition, we have shown that parameters can be obtained with a simple package program although pdfs seems complex. Simulation results are obtained from 1000 Monte Carlo replications. We generated random numbers using quantile functions of $AP_{OGE} - U$, $AP_{OGE} - L$ and $AP_{OGE} - R$ distributions with arbitrary parameters. Tables 1 and 2 show simulation results of MSEs (mean square errors) and their biases for estimating different combinations of real model parameters. We obtain estimations for three different the sample sizes $n = 30, 100, 500$ and 1000 . The results indicate that the MSEs and biases are very small and decrease while the sample sizes increase as expected.

Table 1. Simulation results of MSEs for different combinations of real model parameters

Distribution	Real Parameters	n			
		30	100	500	1000
$AP_{OGE} - L$	$\alpha = 0.2$	0.025189	0.037011	0.003698	0.000281
	$\lambda = 0.3$	0.022367	0.022140	0.008773	0.000015
	$\theta = 0.4$	0.039744	0.017640	0.000889	0.000920
	$a = 0.5$	0.016787	0.004739	0.001992	0.000968
	$\alpha = 0.1$	0.033053	0.006444	0.000064	0.000027
	$\lambda = 0.5$	0.009195	0.002429	0.000016	0.000009
	$\theta = 0.2$	0.002030	0.000421	0.000009	0.000005
	$a = 0.3$	0.011813	0.010506	0.000951	0.000095
	$\alpha = 0.5$	0.019496	0.004753	0.001107	0.005587
	$\lambda = 0.2$	0.034107	0.013732	0.002941	0.002407
	$\theta = 0.7$	0.050224	0.016333	0.002492	0.001332
	$a = 0.3$	0.011180	0.005164	0.000518	0.000339
$AP_{OGE} - R$	$\alpha = 2.5$	0.109218	0.052775	0.026602	0.005181
	$\lambda = 0.3$	0.192581	0.045081	0.008524	0.003723
	$\theta = 0.4$	0.025768	0.005646	0.001097	0.000500
	$\mu = 0.5$	0.065921	0.015989	0.003234	0.001519
	$\alpha = 0.1$	0.049193	0.006204	0.000960	0.000417
	$\lambda = 0.5$	0.056515	0.018375	0.001002	0.000404
	$\theta = 0.2$	0.001637	0.000519	0.000111	0.000057
	$\mu = 0.3$	0.160303	0.015602	0.001009	0.000414
	$\alpha = 0.5$	0.021387	0.007730	0.001556	0.003073
	$\lambda = 0.2$	0.034777	0.013587	0.002960	0.001648
	$\theta = 0.7$	0.062698	0.017554	0.003248	0.001453
	$\mu = 0.3$	0.017494	0.004527	0.000879	0.000432
$AP_{OGE} - U$	$\alpha = 2.5$	0.035370	0.037013	0.000344	0.000064
	$\lambda = 0.3$	0.036367	0.007290	0.000122	0.000061
	$\theta = 0.4$	0.018456	0.003997	0.000068	0.000033
	$\beta = 0.5$	0.000783	0.000147	0.000002	0.000001
	$\alpha = 0.1$	0.019447	0.007586	0.000933	0.000040
	$\lambda = 0.5$	0.106557	0.063994	0.008135	0.000244
	$\theta = 0.2$	0.001520	0.000521	0.000104	0.000005
	$\beta = 0.3$	0.010271	0.001916	0.000179	0.000006
	$\alpha = 0.5$	0.003850	0.000088	0.000002	0.000000
	$\lambda = 0.2$	0.008224	0.000258	0.000042	0.000016
	$\theta = 0.7$	0.024025	0.000781	0.000115	0.000039
	$\beta = 0.3$	0.000118	0.000028	0.000004	0.000000

Table 2. Simulation results of biases for different combinations of real model parameters

Distribution	True Parameters	n			
		30	100	500	1000
$AP_{OGE} - L$	$\alpha = 2.5$	-0.091028	-0.027278	-0.008180	0.004060
	$\lambda = 0.3$	-0.019731	-0.032178	-0.025291	0.009620
	$\theta = 0.4$	0.065187	0.014504	-0.009957	0.002382
	$a = 0.5$	0.001539	0.017192	0.014969	0.003009
	$\alpha = 0.1$	0.055664	0.020568	-0.001749	-0.002810
	$\lambda = 0.5$	-0.018288	-0.004247	-0.001111	-0.001094
	$\theta = 0.2$	0.001083	-0.000422	0.002909	0.002409
	$a = 0.3$	0.051784	0.017365	0.000949	-0.000858
	$\alpha = 0.5$	-0.04704	-0.005287	-0.003992	-0.003300
	$\lambda = 0.2$	0.01138	-0.020660	0.002399	0.003115
	$\theta = 0.7$	0.03450	-0.024322	0.000571	0.001102
	$a = 0.3$	0.04363	0.030873	0.002327	0.001243
$AP_{OGE} - R$	$\alpha = 2.5$	-0.105635	-0.055816	-0.011895	-0.002856
	$\lambda = 0.3$	0.104285	0.033896	0.009330	0.003022
	$\theta = 0.4$	0.034219	0.009924	0.002346	0.000965
	$\mu = 0.5$	0.086713	0.026021	0.004650	0.003505
	$\alpha = 0.1$	0.074180	0.018094	0.000780	-0.001945
	$\lambda = 0.5$	-0.103943	-0.042121	-0.000883	0.000652
	$\theta = 0.2$	0.003854	0.002590	0.002624	0.002552
	$\mu = 0.3$	0.244807	0.059231	0.005339	0.001720
	$\alpha = 0.5$	-0.049061	-0.024423	-0.004273	-0.001199
	$\lambda = 0.2$	0.013277	0.014192	-0.000280	-0.005280
	$\theta = 0.7$	0.048102	0.018736	0.000917	-0.005601
	$\mu = 0.3$	0.053661	0.014032	0.004566	0.003241
$AP_{OGE} - U$	$\alpha = 2.5$	-0.034725	-0.016541	-0.004471	-0.001156
	$\lambda = 0.3$	0.014303	-0.001337	-0.004069	-0.000262
	$\theta = 0.4$	0.021489	0.005209	-0.000309	0.000581
	$\beta = 0.5$	-0.006823	-0.002354	-0.000856	-0.000306
	$\alpha = 0.1$	0.010003	-0.003987	-0.003571	-0.003363
	$\lambda = 0.5$	-0.244889	-0.110825	-0.023084	-0.008070
	$\theta = 0.2$	0.003217	0.004214	0.002755	0.002565
	$\beta = 0.3$	-0.080985	-0.028905	-0.005878	-0.002396
	$\alpha = 0.5$	-0.020170	-0.0073651	-0.000746	-2.198e-04
	$\lambda = 0.2$	-0.014190	0.0003659	-0.001195	-8.768e-05
	$\theta = 0.7$	0.007113	0.0089225	-0.000556	-1.322e-04
	$\beta = 0.3$	-0.004796	-0.0012513	-0.000267	-1.673e-04

7. APPLICATIONS

We demonstrate the utility of three real data sets for the $AP_{OGE} - G$ family, notably $AP_{OGE} - U$, $AP_{OGE} - R$ and $AP_{OGE} - L$. The first data collection comprises 69 carbon fibers' tensile strength tests. They were measured at gauge lengths of 20 mm under strain as shown in Table 3. The data set has already fitted other probability distributions by the authors [21-23]. We fit the data set also using $AP_{OGE} - U$ distribution. The results of the $AP_{OGE} - U$, Generalized Lindley (GL), Generalized Gamma (GG), Prakaamy (P), Sujatha (S), Ram Awadh (Ra), Pranav (PR), Ishita (I), Lindley (L), Exponential (E), weighted quasi Akash (WQA), three parameter weighted Lindley (TWL), Quasi Akash(QA) and Akash (A) distributions are given in Table 3. Table 3 contains MLEs and values for distribution statistics from AIC (Akaike information criterion), CAIC (consistent AIC) and BIC (Bayesian information criterion). As seen in Table 4, $AP_{OGE} - U$ has the best fit to the data.

Table 3. Carbon Fibers Data

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, .272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, .511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, .697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, .067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585

Table 4. MLEs and the values of AIC, CAIC, and BIC statistics

Distribution	MLEs	AIC	CAIC	BIC
$AP_{OGE} - U(\alpha, \lambda, \theta, \beta)$	10.495897, 5.636075, 13.438384, 6.070391	105.443	108.799	104.799
GL $(\alpha, \lambda, \theta)$	22.7198, 4.7710, 9.3907	107.959	110.475	107.959
GG (α, θ, β)	3.586, 12.6483, 0.3044	106,581	109.097	106.097
P (α)	2.0098	190.77	191.609	193.00
S (λ)	0.936119	223.60	224.439	225.52
Ra (β)	1.84921	209.41	210.249	209.60
Pr (α)	1.22514	219.12	219.959	221.03
I (α)	0.931565	225.14	225.979	227.05
L (θ)	0.659000	240.38	241.219	242.61
E (λ)	0.407941	263.74	264.579	265.97
WQA (α, λ, β)	9.43570, 7.2356 ,21.4177	107.877	104.716	107.393
TWL (λ, θ, β)	9.35345, 22.765 ,9.74956	107.966	104.805	107.482
QAD (λ, θ)	0.71313, 59.325	268.496	270.174	268.173
A (θ)	1.35544	165.727	166.566	165.565

The second data collection reflects the relaxation periods of 20 analgesic patients in Table 5. Fayomi [24] suggested odd Frechet inverse Weibull (OFIW) distribution to better fit for the data set with AIC=44.253. On the other hand, APOGE-L distribution is more preferable than others with the smaller AIC value. The results for OFIW, new modified Weibull (NMW), generalized Sujatha (GS), S, Aradhana (Ar), L, E and Weibull (W) distributions are given in Table 6.

Table 5. Relief Times Data

1.1, 1.4,1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2

Table 6. MLEs and the values of AIC, CAIC, and BIC statistics

Distribution	MLEs	AIC	CAIC	BIC
$AP_{OGE} - L(\alpha, \lambda, \theta, a)$	0.067918, 18.251489, 8.215363, 0.198803	41.278	43.945	45.261
OFIW (a, b, α, β)	0.208, 25.815, 13.215	44.253	45.753	42.156
NMW $(\mu, \alpha, \lambda, \theta, a)$	0.1215, 2.7837, 0.000008227, 0.0003, 2.7871	51.173	55.459	47.678
GS (α, λ)	1.5712, 222.235	49.96	50.67	51.96
S (λ)	11,367	59.49	59.71	60.49

$Ar(\theta)$	11,232	58.37	58.59	59.36
$L(\theta)$	0.8161	62.49	62.71	63.49
$E(\lambda)$	0.5263	67.67	67.89	68.67
$W(\alpha, \theta)$	2.7870, 2.1300	45.173	47.164	45.879

The third data set covers the glass fibre strengths of 1.5 cm given in Table 7. Firstly, knowledge was collected by employees at the UK National Physical Laboratory. The data set is also very popular and it has been used many studies. Abdelfattah et al. [25] suggested a new distribution namely, Weibull generalized exponential distribution and fitted the data by the distribution. They obtained the best fit with $AIC=35.656$. We used $AP_{OGE} - R$ and suggest the distribution to fit with smaller AIC value given in Table 7.

Table 6. Glass Fibres Data

0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2, 0.74, 1.04, 1.27, 1.39, 1.49, .53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69, .76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, .48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89
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Table 7. MLEs and the values of AIC, CAIC, and BIC statistics

Distribution	MLEs	AIC	CAIC	BIC
$AP_{OGE} - R(\alpha, \lambda, \theta, \mu)$	8.485560, 0.873138, 1.907908, 0.951858	34.651	35.340	43.223
$E(\lambda)$	0.664	179.660	179.726	181.803
$GE(\alpha, \lambda)$	2.6105, 31.3032	66.767	66.967	71.053
$BE(\alpha, \lambda, \theta)$	17.7786, 22.7222, 0.3898	54.254	54.661	60.683
$BGE(\alpha, \lambda, \theta, a)$	0.4125, 93.4655, 0.92271, 22.6124	39.199	39.889	47.772
$WGE(\lambda, \theta, a)$	56.881, 4.893, 0.222	35.656	36.063	42.085

Finally, we obtain graphs of density and cumulative density functions and histograms to see how better the distributions fitting the data sets in Figures 4 and 5. These figures show that the AP_{OGE} family is can be used to model symmetric, left and right skewed data sets.

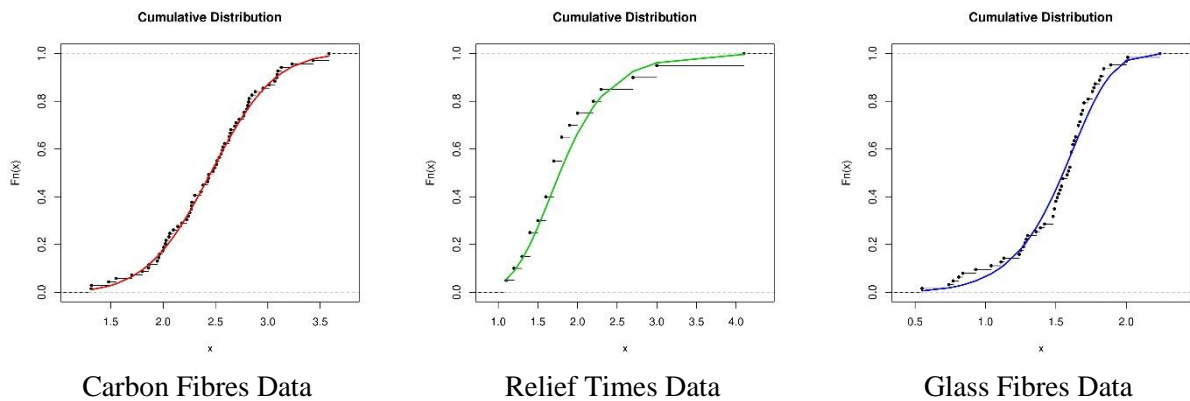


Figure 4. Graphs of empirical cdf and cdf for $AP_{OGE} - U, AP_{OGE} - R, AP_{OGE} - L$

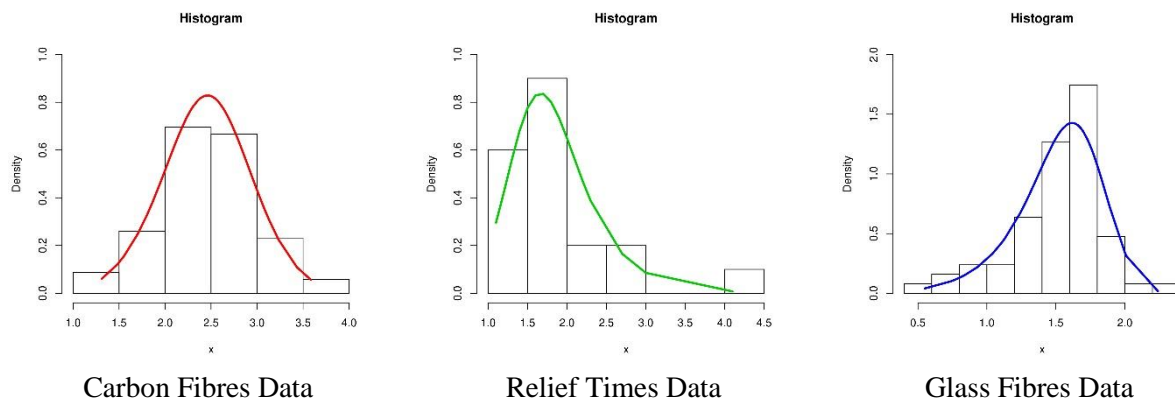


Figure 5. Histograms and fitted lines for $AP_{OGE} - U$, $AP_{OGE} - R$, $AP_{OGE} - L$

8. CONCLUSION

We propose a new family of distributions in this paper, called the alpha power odd generalized exponential distribution family. Several new family properties are discussed, including moments, moment generating function, quantile function, and special cases. The parameters are calculated using the maximum likelihood method. Then, the results of the Monte Carlo simulation show that the MLE outputs are very satisfactory. Applications show that, of all versions, the AP_{OGE} family offers the best fit. The AP_{OGE} family provides a closer fit to the histogram than the other models from plots for suited densities and histograms. The new family model can therefore be used in the study of data very efficiently. We hope that the proposed model can be efficiently used to fit real data as a competitive model.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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