Alternating groups and moduli space lifting Invariants

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Abstract. The genus of a curve discretely separates decidely different algebraic relations in two variables to focus us on the connected moduli space $\mathcal{M}_g$. Yet, modern applications also require a data variable (function) on the curve. The resulting spaces are versions, depending on our needs for this data variable, of *Hurwitz spaces*. A *Nielsen class* (§1.1) consists of $r \geq 3$ conjugacy classes $\mathbf{C}$ in the data variable monodromy G. It generalizes the genus.

Some Nielsen classes define connected spaces. To detect, however, the components of others requires further subtler invariants. We regard our Main Result (MR) as level 0 of Spin invariant information on moduli spaces.

In the MR, $G = A_n$ (the alternating group), r counts the data variable branch points and $\mathbf{C} = \mathbf{C}_{3^r}$ is r repetitions of the 3-cycle conjugacy class. This Nielsen class defines two spaces called absolute and inner: $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\rm abs}$ of degree n, genus $g = r - (n-1) > 0$ covers and $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\rm in}$ parametrizing Galois closures of such covers. The parity of a spin invariant precisely identifies the two components of each space. The inner result is the deeper.

We examine the effect of combining the MR, [**ArP05**] and $\frac{1}{2}$-canonical classes on $\mathcal{M}_g$. First: §5.2 considers an analog of a famous conjecture of Shafarevich: With H the composite group of all Galois extensions K/Q with group some alternating group, does the canonical map $G_{\mathbb{Q}} \to H$ have pro-free kernel. Second: Thm. 6.15 produces nonzero automorphic (θ-null power) functions on the reduced Hurwitz spaces $\mathcal{H}_+(A_n, \mathbf{C}_{3^r})^{\rm abs,rd}$ (resp. $\mathcal{H}_-(A_n, \mathbf{C}_{3^r})^{\rm abs,rd}$) when r is even (resp. odd), for either $g = 1$ or $n \geq 12g + 4$.

Contents

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1. Introduction and notation

1.1. Nielsen class notation. §1.1 reviews our Main Result (MR) (§1.1.1). This section's remainder reminds of Nielsen classes, Hurwitz spaces and the small lifting invariant used in the MR. We repeatedly use that components of Hurwitz spaces translate to braid (or Hurwitz monodromy) orbits on Nielsen classes.

1.1.1. *A quick review of results.* §2, §3 and §4 treat the space of projective line covers with r ($\geq n-1$) 3-cycles as branch cycles ($n \geq 4$). For $n \geq 5$, $\mathbf{C}_{3^r}$ denotes r repetitions of the conjugacy class of 3-cycles in A_n. When $n = 4$ (or 3) there are two conjugacy classes of 3-cycles, so $\mathbf{C}_{3^r}$ is ambiguous. §3.3 states precise results in that hard case, so crucial to the complete analysis. To simplify, assume here $n \geq 5$. §3.3.1 shows each allowable $\mathbf{C}_{3^r}$ replacement when $n = 4$ has a comparable result. There are results for both degree n covers and Galois covers (degree $n!/2$).

We use both cases, denoted respectively $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\mathrm{abs}}$ and $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\mathrm{in}}$. The MR says the following for either. A $\frac{1}{2}$-canonical class (spin) invariant (noted by a $\pm$ subscript) completely determines components of $\mathcal{H}(A_n, \mathbf{C}_{3^r})^*$, $* =$ in or abs.

- Thm. 1.3: For $r \geq n \geq 5, \mathcal{H}(A_n, \mathbf{C}_{3^r})^*$ has 2 components, $\mathcal{H}_{\pm}(A_n, \mathbf{C}_{3^r})^*$.
- Thm. 1.2: $\mathcal{H}(A_n, \mathbf{C}_{3^{n-1}})^*$ has exactly one (spin $(-1)^{n-1}$) component.

Cor. 5.1 shows these spaces have useful moduli properties on their own.

- Each of $\mathcal{H}_+(A_n, \mathbf{C}_{3^r})^*$ and $\mathcal{H}_-(A_n, \mathbf{C}_{3^r})^*$ has definition field $\mathbb{Q}$.
- A dense subset of $\mathcal{H}_{\pm}(A_n, \mathbf{C}_{3^r})^{\mathrm{abs}}(\bar{\mathbb{Q}})$ give $(A_n, S_n, \mathbf{C}_{3^r})$ *realizations*.

Thm. 6.15 uses that for some (r,n), $\mathcal{H}_\pm(A_n,\mathbf{C}_{3^r})^{\rm abs}$ dominates $\mathcal{M}_g$, the moduli of genus $g=r-(n-1)$ curves. That produces Hurwitz families with nonzero θ-nulls.

1.1.2. *Nielsen class preliminaries.* Let G be a subgroup of S_n. For $g\in G$, we say g contains i (or i is in the support of g) if a nontrivial disjoint cycle of g contains i. Also, for $\boldsymbol{g}\in G^r$, $\Pi(\boldsymbol{g})=g_1\cdots g_r$ and $\langle \boldsymbol{g}\rangle$ is the group the entries of $\boldsymbol{g}$ generate. Permutations from S_n act on the right of integers.

Now consider r conjugacy classes from G: $\mathbf{C}=(\mathrm{C}_1,\dots,\mathrm{C}_r)$, often one conjugacy class repeated many times. The main definitions don't depend on the order of their listing. For example, for $\boldsymbol{g}\in G^r$, $\boldsymbol{g}\in\mathbf{C}$ means entries of $\boldsymbol{g}$ are in the conjugacy classes of $\mathbf{C}$ in *some* order. This gives the *Nielsen class*, $\mathrm{Ni}(G,\mathbf{C})$ of $(G,\mathbf{C})$:

$$\{\boldsymbol{g}\in G^r \mid \Pi(\boldsymbol{g})=1,\ \boldsymbol{g}\in\mathbf{C},\ \text{and } \langle\boldsymbol{g}\rangle=G\}.$$

Denote a (ramified) cover of the sphere $\mathbb{P}^1=\mathbb{P}^1_z$ by a nonsingular connected curve X by $\varphi: X\to\mathbb{P}^1_z$. Then φ has a degree, $\deg(\varphi)=n$, and $G=G_\varphi$, a transitive subgroup —the *geometric monodromy* of φ —of S_n (symmetric group on n letters). Denote the set $\{g\in G\mid (1)g=1\}$ – the stabilizer of 1 – by $G(1)$.

A *branch point* of φ is a $z'\in\mathbb{P}^1_z$ for which the fiber $X_{z'}$ over z' has fewer than n points. So, φ has a *branch point set* $\boldsymbol{z}_\varphi=\{z_1,\dots,z_r\}\in(\mathbb{P}^1)^r\setminus\Delta_r/S_r=U_r$.

Given a labeling of points of X over z_0, the following data attaches an r-tuple $\boldsymbol{g}\in G^r$ to φ: *classical generators* $[P_1],\dots,[P_r]$ of the fundamental group $\pi_1(\mathbb{P}^1\setminus\boldsymbol{z},z_0)$ [**BaFr02**, §1.2]. Classical generators present $\pi_1(\mathbb{P}^1\setminus\boldsymbol{z},z_0)$ modulo one relation: $[P_1]\dots[P_r]=1$. These topics have down-to-earth treatments at [**Fr08b**].

Definition 1.1 (branch cycles)**.** Any cover φ then corresponds to a homomorphism $\psi_\varphi:\pi_1(\mathbb{P}^1_z\setminus\{\boldsymbol{z}_\varphi\},z_0)\to G$. The classical generators then assign $\boldsymbol{g}\in\mathrm{Ni}(G,\mathbf{C})$ by $\psi_\varphi(P_i)=g_i$, $i=1,\dots,r$: a *branch cycle description* of φ.

Another set of classical generators will produce a different $\boldsymbol{g}$ for φ, yet in the same Nielsen class. We say φ is in $\mathrm{Ni}(G,\mathbf{C})$. This with other background topics, studded with classically motivated examples, is in [**Fr08c**, Chap. 4].

In a connected family of r-branched covers, we expect $\boldsymbol{z}_\varphi$ to move with φ, while $\mathrm{Ni}(G_\varphi,\mathbf{C}_\varphi)$ is constant in φ. We say φ is in the Nielsen class. A grasp on categories of covers requires useful equivalence relations. We use *absolute*, *inner*, and their reduced versions in this paper (§A.2 and §A.3). To each equivalence of covers in a given Nielsen class, there is a corresponding equivalence on the Nielsen class.

Any of [**BaFr02**, §3.1], [**Fr95a**], [**Fr95b**, §I.F], [**MM99**, Chap. I], [**Ser92**, p. 60] or [**Vö96**, Thm. 4.32] explain Riemann's Existence Theorem (RET): how equivalence classes of φs branched over a fixed $\boldsymbol{z}\in U_r$ correspond one-one to Nielsen class representatives modulo equivalence.

1.1.3. *Families of covers.* I explain absolute and inner equivalence for the families in Thm. 1.2 and 1.3. A family of $\mathrm{Ni}(A_n,\mathbf{C}_{3^r})^{\rm abs}$ (resp. $\mathrm{Ni}(A_n,\mathbf{C}_{3^r})^{\rm in}$) covers over a space S is a degree n (resp. Galois, with group A_n) cover $\Phi:\mathcal{T}\to S\times\mathbb{P}^1_z$, where the fiber of $\mathcal{T}$ (resp. $\mathcal{T}/A_n(1)$) over $s\times\mathbb{P}^1_z$, $s\in S$, is a cover with 3-cycles as branch cycles. This determines $\Psi_\Phi: S\to\mathcal{H}(A_n,\mathbf{C}_{3^r})^{\rm abs}$ (resp. $\to\mathcal{H}(A_n,\mathbf{C}_{3^r})^{\rm in}$). Both these spaces have *fine moduli* (§A.2.3 and Rem. 6.8).

That is, Ψ_Φ determines Φ up to equivalence on families. Fine absolute (resp. inner) moduli holds because $A_n(1)$ is its own normalizer in A_n (resp. A_n has no center; §A.2). Analogous results hold for reduced equivalence (Rem. A.2.3).

In practice fine moduli means we know all families in a Nielsen class from knowing properties of one family over the corresponding space $\mathcal{H}(G,\mathbf{C})^*$. Further, properties of that family come from an action of a *mapping class group* on Nielsen classes. For inner and absolute equivalence the mapping class group is the *Hurwitz monodromy* group H_r, a braid group quotient.

With $\mathrm{Ni}(A_n,\mathbf{C})^{\mathrm{in}} = \mathrm{Ni}(A_n,\mathbf{C})/A_n$ and $\mathrm{Ni}(A_n,\mathbf{C})^{\mathrm{abs}} = \mathrm{Ni}(A_n,\mathbf{C})/S_n$:

- §2.1 reminds that H_r orbits on $\mathrm{Ni}(A_n,\mathbf{C})^{\mathrm{in}}$ (resp. $\mathrm{Ni}(A_n,\mathbf{C})^{\mathrm{abs}}$) $\Leftrightarrow$ $\mathcal{H}(A_n,\mathbf{C})^{\mathrm{in}}$ (resp. $\mathcal{H}(A_n,\mathbf{C})^{\mathrm{abs}}$) components .

Lem. 2.6 says H_r orbits on $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{in}}$ correspond one-one with orbits on $\mathrm{Ni}(G,\mathbf{C})$, but depending on $(G,\mathbf{C})$ this may not hold with "abs" replacing "in" (Ex. 1.5).

Our M(ain) R(esult) is Thms. 1.2 and 1.3: Listing absolute/inner components for $(A_n,\mathbf{C}=\mathbf{C}_{3^r})$, $n\ge 4$, $r\ge n-1$. The MR proof takes up §2, §3, §3.3 and §4.4. §5 and §6 tie to [**Fr90**], [**Mes90**], [**Ser90b**] and [**ArP05**] for a list of applications.

1.1.4. *Lifting invariants.* A Frattini cover $G'\to G$ is a group cover (surjection) where restriction to any proper subgroup of G' is not a cover. I now explain how any central Frattini extension $\psi: R\to G$ gives a lifting invariant [**Fr95b**, Part II].

A special case comes from alternating groups. Let Spin_n^+ be the (unique) non-split degree 2 cover of the connected component O_n^+ (of the identity) of the orthogonal group ([**Fr95b**, §II.C] or [**Ser90a**]). Regard S_n as a subgroup of the orthogonal group O_n. The alternating group A_n is in O_n^+, the kernel of the determinant map. Denote its pullback to Spin_n^+ by Spin_n and identify $\ker(\mathrm{Spin}_n\to A_n)$ with the multiplicative group $\{\pm1\}$. Odd order elements of S_n are in A_n. Any odd order $g\in A_n$ has a unique odd order lift, $\hat g\in \mathrm{Spin}_n$.

Let $\boldsymbol{g}\in A_n^r$, with $g_1\cdots g_r=\Pi(\boldsymbol{g})=1$. If entries of $\boldsymbol{g}$ have odd order, define the *spin* lifting invariant of $\boldsymbol{g}$ to be

$$s(\boldsymbol{g}) = s_{\mathrm{Spin}_n}(\boldsymbol{g}) = \hat g_1\cdots \hat g_r \in \{\pm1\}. \tag{1.1}$$

A degree n (absolute) cover has the same lifting invariant as its Galois closure. So, s($\mathbf{g}$) will not distinguish between absolute and inner classes.

1.2. Main Result and its corollaries. §1.2.1 states and outlines the proof of the MR. Then, the three remaining subsections discuss the applications.

1.2.1. $\mathrm{Spin}_n\to A_n$ *and 3-cycles.* Strong Coalescing Lem. 4.4 applies for $n\ge5$. It says, for $\boldsymbol{g}\in\mathrm{Ni}(A_n,\mathbf{C}_{3^r})$, there is $q\in H_r$ so $(\boldsymbol{g})q=\boldsymbol{g}'$ has $(g_2')^{-1}=g_1'$. We then induct on (r,n) to describe all components of $\mathcal{H}(A_n,\mathbf{C}_{3^r})^*$, with $*=$ abs or in.

If a curve cover $\varphi: X\to \mathbb{P}^1_z$ in $\mathrm{Ni}(A_n,\mathbf{C}_{3^r})$ corresponds to a point of a component labeled $\oplus$ (resp., $\ominus$) in the *Constellation diagram* (Fig. 1), then any branch cycle description $\boldsymbol{g}$ for φ has $s(\boldsymbol{g})=+1$ (resp. -1; §1.1.4). Lifting invariants are the same for covers representing points on the same component. Fig. 1 labels each component at (n,r) with a symbol $\oplus$ or $\ominus$ corresponding to the lifting invariant value. Thm. 1.2 generalizes [**Ser90a**, Cor. 2.3] in showing there is *exactly* one component when the curves are genus 0: row-tag $\xrightarrow{g=0}$ in Fig. 1.

Theorem 1.2. *For $r=n-1$, $n\ge5$, $\mathcal{H}(A_n,\mathbf{C}_{3^{n-1}})^{\mathrm{in}}$ has exactly one connected component. Further, $\Psi^{\mathrm{in}}_{\mathrm{abs}}:\mathcal{H}(A_n,\mathbf{C}_{3^{n-1}})^{\mathrm{in}}\to\mathcal{H}(A_n,\mathbf{C}_{3^{n-1}})^{\mathrm{abs}}$ has degree 2.*

The row with tag $\xrightarrow{g\ge1}$ illustrates this theorem.

Theorem 1.3. *For each $r\ge n\ge5$, $\mathcal{H}(A_n,\mathbf{C}_{3^r})^{\mathrm{in}}$ has exactly two connected components, $\mathcal{H}_+(A_n,\mathbf{C}_{3^r})^{\mathrm{in}}$ (symbol $\oplus$) and $\mathcal{H}_-(A_n,\mathbf{C}_{3^r})^{\mathrm{in}}$ (symbol $\ominus$). Denote their respective (connected) images in $\mathcal{H}(A_n,\mathbf{C}_{3^r})^{\mathrm{abs}}$ by $\mathcal{H}_\pm(A_n,\mathbf{C}_{3^r})^{\mathrm{abs}}$. The maps*

$$\Psi^{\mathrm{in},\pm}_{\mathrm{abs}}:\mathcal{H}_\pm(A_n,\mathbf{C}_{3^r})^{\mathrm{in}}\to\mathcal{H}_\pm(A_n,\mathbf{C}_{3^r})^{\mathrm{abs}} \text{ have degree 2.}$$

FIGURE 1. Constellation of spaces $\mathcal{H}(A_n, \mathbf{C}_{3^r})^*$

$\xrightarrow{g\ge 1}$	$\ominus\oplus$	$\ominus\oplus$	...	$\ominus\oplus$	$\ominus\oplus$	$\xleftarrow{1\le g}$
$\xrightarrow{g=0}$	$\ominus$	$\oplus$	...	$\ominus$	$\oplus$	$\xleftarrow{0=g}$
$n\ge 4$	$n=4$	$n=5$	...	n even	n odd	$4\le n$

We include a column for $n = 4$, though A_4 has two conjugacy classes of 3-cycles, with representatives $(1\,2\,3)$ and $(3\,2\,1)$. Denote the 1st by C_{+3}, the 2nd by C_{-3}. Then, $\mathbf{C}_{\pm 3^{s_1,s_2}}$ indicates s_1 (resp. s_2) repetitions of C_{+3} (resp. C_{-3}); abbreviate to $\mathrm{C}_{\pm 3^{s_1}}$ if $s_1 = s_2$. Expression (3.7) must hold for $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^{s_1,s_2}})$ to be nonempty.

Example 1.4. As in (3.8b), $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2})$ has two braid orbits, with reps.:

$$\boldsymbol{g}_{4,+} = ((1\,3\,4), (1\,4\,3), (1\,2\,3), (1\,3\,2)) \text{ and } \boldsymbol{g}_{4,-} = ((1\,2\,3), (1\,3\,4), (1\,2\,4), (1\,2\,4)).$$

Trickiest point: if $s_1 \ne s_2$, then $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^{s_1,s_2}})$ and $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^{s_2,s_1}})$ are distinct. Yet, any $\beta \in S_4 \setminus A_4$ (as in §3.3.1) conjugates between them. So

$$\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^{s_1,s_2}})^{\mathrm{abs}} = \mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^{s_2,s_1}})^{\mathrm{abs}}.$$

So, if for $n = 4$, you put any one allowed value of (s_1, s_2) for each r, then Figure 1 still is valid. Examples: For $r = 3$, $\{s_1, s_2\} = \{0, 3\}$ give one abs component; for $r = 4$, $\{s_1, s_2\} = \{2, 2\}$ and the two components (*=abs or in) are both over $\mathbb{Q}$; for $r = 5$, $\{s_1, s_2\} = \{1, 4\}$ there are two (*=abs or in) components (abs over $\mathbb{Q}$, in over $\mathbb{Q}(\sqrt{-2})$). For $r \ge 6$ there are several values of $\{s_1, s_2\}$.

Example 1.5 (in is stronger than abs)**.** Let C_d be the class of d-cycles in A_n. For $n \equiv 1 \mod 4$ and $d(n) = \frac{n+1}{2}$ consider the Nielsen class $\mathrm{Ni}(A_n, \mathbf{C}_{d(n)^4})$ of 4 reps. of $d(n)$-cycles. A special case of [**LOs08**, Thm. 5.5] gives one braid orbit on $\mathrm{Ni}(A_n, \mathbf{C}_{d(n)^4})^{\mathrm{abs}}$, suggesting there are no distinguishing properties of these spaces as n varies. Yet, [**Fr08a**, Prop. 5.15] starts their many differences with this:

(1.2a) for $n \equiv 1 \mod 8$, $\mathrm{Ni}(A_n, \mathbf{C}_{d(n)^4})^{\mathrm{in}}$ has two braid orbits, with the corresponding spaces conjugate over a quadratic extension of $\mathbb{Q}$; while
(1.2b) for $n \equiv 5 \mod 8$, $\mathrm{Ni}(A_n, \mathbf{C}_{d(n)^4})^{\mathrm{in}}$ has but one braid orbit.

1.2.2. *Corollaries on A_n realizations.* Let $\mathcal{H}'$ be an (irreducible) component of $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\mathrm{in}}$. Cor. 5.1 says $\mathcal{H}'$ (and its map to U_r) has definition field $\mathbb{Q}$.

Any $\boldsymbol{p}' \in \mathcal{H}'$ produces a field extension $\hat{L}_{\boldsymbol{p}'}/\mathbb{Q}(\boldsymbol{p}')(z)$ regular and Galois over $\mathbb{Q}(\boldsymbol{p}')$, with group A_n. Let $\mathcal{H}$ be the image of $\mathcal{H}'$ in $\mathcal{H}(G, \mathbf{C})^{\mathrm{abs}}$. Then, any $\boldsymbol{p} \in \mathcal{H}$ produces a regular degree n field extension $L_{\boldsymbol{p}}/\mathbb{Q}(\boldsymbol{p})(z)$. This is the natural extension of function fields for a cover $\varphi_{\boldsymbol{p}} : X_{\boldsymbol{p}} \to \mathbb{P}^1_z$ representing $\boldsymbol{p}$. The geometric (resp. arithmetic) Galois closure of $L_{\boldsymbol{p}}/\mathbb{Q}(\boldsymbol{p})(z)$ has group $G_{\boldsymbol{p}} = A_n$ (resp. $\hat{G}_{\boldsymbol{p}}$ between A_n and S_n). So, $\hat{G}_{\boldsymbol{p}}$ is either A_n or S_n: respectively, each $\boldsymbol{p} \in \mathcal{H}$ produces either an (A_n, A_n) or an (A_n, S_n) realization over $\mathbb{Q}(\boldsymbol{p})$.

Cor. 5.1 produces a dense set of $\boldsymbol{p} \in \mathcal{H}(\bar{\mathbb{Q}})$ with $(G_{\boldsymbol{p}}, \hat{G}_{\boldsymbol{p}})$ equal to (A_n, S_n) (resp. equal to (A_n, A_n)). It is subtler to ask if either conclusion holds restricting to $\boldsymbol{p} \in \mathcal{H}(K)$ for $[K : \mathbb{Q}] < \infty$. Combining Cor. 5.2 and [**Mes90**] shows the (A_n, A_n) conclusion if $r = n - 1$, even for $K = \mathbb{Q}$.

Let $\mathbf{C}$ be *any* odd order classes in A_n. As [**Fr08a**] shows, the algorithm behind the MR is applicable to much more than the case of 3-cycles.

Example: It makes sense to speak of – and to identify – the +-components, $\mathcal{H}_+(A_n,\mathbf{C})^*$ (as in (1.2a) with $*=\text{in}$, there may be more than one), among all components of $\mathcal{H}(A_n,\mathbf{C})^*$. [**BaFr02**, Prop. 6.8] then interprets as follows.

Lemma 1.6. *Assume $\boldsymbol{p}\in\mathcal{H}_+(A_n,\mathbf{C})^{\text{abs}}(K)$, corresponding to $\varphi_{\boldsymbol{p}}: X_{\boldsymbol{p}}\to\mathbb{P}^1_z$. With $\boldsymbol{z}_{\boldsymbol{p}}$ its branch points, assume there is $z_0\in\mathbb{P}^1_z\setminus\boldsymbol{z}$ with all points of $\varphi_{\boldsymbol{p}}^{-1}(z_0)$ in K. Then, $\boldsymbol{p}$ gives an (A_n,A_n) realization and $\boldsymbol{p}'\in\mathcal{H}(\text{Spin}_n,\mathbf{C})^{\text{in}}=\mathcal{H}_+(A_n,\mathbf{C}))^{\text{in}}$ over it gives a Spin_n regular K realization. (Rem. 6.8 for why this is nonobvious.)*

1.2.3. *$G_{\mathbb{Q}}$ and canonical fields from alternating groups.* Denote the absolute Galois group of a field K by G_K. One goal of arithmetic geometry is to present $G_{\mathbb{Q}}$ as a known group quotient $N=G(F/\mathbb{Q})$ (so F is Galois over $\mathbb{Q}$) by a known subgroup $M=G_F$. [**FV92**] produced such presentations with N a product of symmetric groups and $M=\tilde{F}_\omega$, the profree group on a countable number of generators. Fields F with known arithmetic properties enhance applications.

The archetype is *Shafarevic's conjecture*: For $F=\mathbb{Q}^{\text{cyc}}$, $N=\hat{\mathbb{Z}}^*$ (profinite invertible integers) and $M=\tilde{F}_\omega$. §5.2.1 explains the mystery of whether $\mathbb{Q}^{\text{alt}}$, the composite of all A_n extensions of $\mathbb{Q}$, $n\ge5$, should have $G_{\mathbb{Q}^{\text{alt}}}=\tilde{F}_\omega$. Our MR makes this plausible using [**FV92**, Thm. A]: For $F\subset\bar{\mathbb{Q}}$ P(seudo)A(lgebraically)C(losed), G_F is Hilbertian if and only if it is $\tilde{F}_\omega$. In particular, if $\mathbb{Q}^{\text{alt}}$ is PAC, then $G_{\mathbb{Q}^{\text{alt}}}=\tilde{F}_\omega$.

The following sequence makes a case for $\mathbb{Q}^{\text{alt}}$ being PAC. First, it is PAC if each $\mathbb{Q}$ curve X has a $\mathbb{Q}$ cover $X\to\mathbb{P}^1_z$ of degree n giving an (A_n,A_n) realization (§A.1) for some $n\ge5$ (n allowed to vary with X). Every curve of genus g appears as a geometric A_n cover with odd order branching, for many possible degrees, from [**ArP05**] (§5.2). Further, most curves give a corresponding geometric point on ∞-ly many of the spaces $\mathcal{H}(A_n,\mathbf{C}_{3^r})^{\text{in}}$. If this implies each curve over $\mathbb{Q}$ gives a corresponding $\mathbb{Q}$ point on one of these spaces, then we have the result.

That these spaces have definition field $\mathbb{Q}$ (Thm. 1.3) further encourages. Still, Prop. 5.11 shows there are many X s over $\mathbb{Q}$, of each positive even genus, with *no* odd branched $\mathbb{Q}$ cover $X\to\mathbb{P}^1_z$ of any degree, much less any (A_n,A_n) realization.

A serious issue around Hilbert's Irreducibility Theorem (HIT) arises in §5.9.

(1.3) Extending [**Mu76**, p. 36–37] to ask if $\mathcal{M}_g$ is an *Hilbertian variety*; why showing HIT for the $\frac12$-canonical covers $\mathcal{M}_{g,\pm}\to\mathcal{M}_g$ is nontrivial.

1.2.4. *Spaces supporting θ-nulls.* For each (n,r), $r\ge n$, there is a map from $\mathcal{H}_\pm(A_n,\mathbf{C}_{3^r})^{\text{abs}}$ to $\mathcal{M}_{g,\pm}$, the space of genus $g=r-(n-1)$ curves with an attached half-canonical class (§6.1.3). From Thm. 6.15 the reduced versions (§A.3) of $\mathcal{H}_+(A_n,\mathbf{C}_{3^r})^{\text{in}}$ support a canonical even θ-null (with 2-division characteristic) $\theta_{n,r}[0]$. For the absolute spaces, there is such a θ-null on $\mathcal{H}_+(A_n,\mathbf{C}_{3^r})^{\text{abs}}$ (resp. $\mathcal{H}_-(A_n,\mathbf{C}_{3^r})^{\text{abs}}$) if r is even (resp. odd). A power of this is the *Hurwitz-Torelli* analog of an automorphic function. We only, however, know it is non-zero for absolute spaces and, for given g, infinitely many explicit (n,r) (including $g=1$).

2. Coalescing and supporting lemmas

This section shows how to braid $\boldsymbol{g}\in\text{Ni}(A_n,\mathbf{C}_{3^r})$ to where its first 2 or 3 entries are in a list of precise possibities. [**Ser90a**] is a quick corollary.

2.1. Braid and Hurwitz monodromy groups. Generators $q_1, \dots, q_{r-1}$ of the *Hurwitz monodromy group*, H_r, a quotient of the braid group B_r, act as permutations on the right of $\mathrm{Ni}(G, \mathbf{C})$. For $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$:

$$(\boldsymbol{g})q_i = (g_1, \dots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+1}, \dots, g_r).$$

Generators $Q_1, \dots, Q_{r-1}$ of B_r generate it freely modulo these relations:

(2.1) $Q_i Q_j = Q_j Q_i$ for $|i-j| > 1$ and $Q_{i+1} Q_i Q_{i+1} = Q_i Q_{i+1} Q_i$.

Add to (2.1) one further relation for H_r:

(2.2) $q_1 \cdots q_{r-1} q_{r-1} \cdots q_1 = 1$.

Also, H_r is the fundamental group of projective r-space minus the discriminant locus: $\mathbb{P}^r \setminus D_r$; that is, the space of monic polynomials of degree r with no repeated roots. Another description of $\mathbb{P}^r \setminus D_r$ is as the quotient of $(\mathbb{P}^1)^r \setminus \Delta_r / S_r$.

The word $Q_1 \cdots Q_{r-1} Q_{r-1} \cdots Q_1 \stackrel{\text{def}}{=} Q^{(r-1)} \in B_r$ conjugates $\boldsymbol{g} = (g_1, \dots, g_r)$ by g_1:

$$\boldsymbol{g} \mapsto (\boldsymbol{g})Q^{(r-1)} = g_1 \boldsymbol{g} g_1^{-1} = (\dots, g_1 g_i g_1^{-1}, \dots).$$

So, to have H_r acting on $\mathrm{Ni}(G, \mathbf{C})$ requires quotienting by G: $g \in G$ has the effect

$$\boldsymbol{g} \mapsto g^{-1} \boldsymbol{g} g = (g^{-1} g_1 g, \dots, g^{-1} g_r g) \in \mathrm{Ni}(G, \mathbf{C}).$$

The resulting set $\mathrm{Ni}(G, \mathbf{C})/G = \mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$ we call *inner Nielsen classes*. Also, the element of $q_{(r-1)} \stackrel{\text{def}}{=} q_1 \cdots q_{r-1} = \mathbf{sh}$ acts as a *shift operator* on $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$:

$$(\boldsymbol{g})q_{(r-1)} = (g_2, g_3, \dots, g_r, g_1). \tag{2.3}$$

The word of (2.2) acts trivially on $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$.

Computations in the first four sections are for B_r acting on Nielsen classes. Sometimes (as in the proof of Lem. 4.1) we extend that action to a generalization of Nielsen classes. These are Nielsen *sets* $\mathrm{Ni}(G, \mathbf{C})_{g'}$, defined by $(G, \mathbf{C}, g')$ where we replace the product-one condition by $\Pi(\boldsymbol{g}) = g'$ with $g' \in G$. Only elements of G centralizing g' can act by conjugation on $\mathrm{Ni}(G, \mathbf{C})_{g'}$. Corollaries, however, then pass to H_r acting on Nielsen classes. App. A.2.2 translates between Nielsen classes and these spaces. This dictionary reduces §2-§4 to combinatorics and group theory.

There is a homomorphism $\alpha : B_r \to S_r$ by $Q_i \mapsto \alpha(Q_i) = (i\, i{+}1)$. For $\boldsymbol{h} \in G^u$, denote juxtaposition of k copies of $\boldsymbol{h}$ by $\boldsymbol{h}^{(k)}$. For example,

$$((1\,2\,3), (1\,3\,2), (1\,2\,3), (1\,3\,2)) = ((1\,2\,3), (1\,3\,2))^{(2)}.$$

Suppose $Q \in B_r$. Call Q *local* to a subset I of $\{1, \dots, r\}$ if Q is a product of braids affecting only the positions in I. Further, suppose $\boldsymbol{g} \in G^r$ and i and j are integers, with $1 \le i < j \le r$. Then, there exists $Q \in B_r$ local to the integers between i and j (inclusive) with $(\boldsymbol{g})Q = \boldsymbol{g}'$ and $g'_i = g_j$.

2.2. Coalescing. The first part of Lem. 2.1 simplifies working with alternating groups. (Regard $A_1 = \{1\}$ as the degree 1 alternating group.) §1.1.4 has the definition of the lifting invariant $s(\boldsymbol{g})$ for $\boldsymbol{g}$ having odd order entries.

2.2.1. *A starter lemma.* Lem. 2.2 uses specific braids we regard as standard.

Lemma 2.1 (3-cycle Lemma). *Let $\boldsymbol{g} \in \mathbf{C}_{3^r}$. Let $G = \langle \boldsymbol{g} \rangle$ act on $\{1, \dots, n\}$. Then, G is a product of alternating groups, one copy for each orbit of G. Up to conjugacy in S_n, here are all 3-cycle pairs with product a power of a 3-cycle.*

(2.4a) $(g, g^{-1}) \mapsto 1$.

(2.4b) $((i\,j\,k), (i\,k\,t)) \mapsto (i\,j\,t)$, *for $k \ne t$.*

(2.4c) $(g, g) \mapsto g^{-1}$.

Up to conjugacy in S_n, here are all 3-cycle pairs with product not a 3-cycle power.

(2.5a) $(g, g') \mapsto (g)(g')$ *where g and g' have no common support.*
(2.5b) $((i\,j\,k),(i\,j\,t)) \mapsto (j\,k)(i\,t)$*, for $k \neq t$.*
(2.5c) $((i\,j\,k),(i\,l\,m)) \mapsto (i\,j\,k\,l\,m)$ *with $\{i,j,k,l,m\}$ distinct integers.*

Let $(\boldsymbol{g}'_1,\dots,\boldsymbol{g}'_t) = \boldsymbol{g}$ with $\Pi(\boldsymbol{g}'_i) = 1$, $i = 1,\dots,t$. Then, $\prod_{i=1}^t s(\boldsymbol{g}'_i) = s(\boldsymbol{g})$. If g is a 3-cycle, then $s(g^{(3)}) = s(g, g^{-1}) = 1$. Finally, $s((i\,j\,k),(i\,k\,t),(i\,t\,j)) = -1$.

Proof. Assume $\boldsymbol{g}$ generates a transitive group. Then, the first statement says $\langle \boldsymbol{g} \rangle = A_n$. This is well-known from the following chain of deductions: $\langle \boldsymbol{g} \rangle$ is primitive, and a primitive group containing a 3-cycle is the alternating or symmetric group. If $\boldsymbol{g}$ isn't transitive, then each 3-cycle has support on one of the orbits. Thus, you can apply the first argument to the 3-cycles supported on each orbit of $\langle \boldsymbol{g} \rangle$.

Everything else is elementary. Example: Let $\hat{g}$ be the (unique) order 3 lift to Spin_n of the 3-cycle $g \in A_n$. Then, $s(g^{(3)}) = \hat{g}^3 = 1$ and $s(g, g^{-1}) = \hat{g}\hat{g}^{-1} = 1$. Note that $s(h\boldsymbol{g}h^{-1}) = \hat{h}s(\boldsymbol{g})\hat{h}^{-1}$ for any $h \in A_n$ and any lift of h to $\hat{h} \in \mathrm{Spin}_n$. Thus, assume $((i\,j\,k),(i\,k,t),(i\,t\,j)) = \boldsymbol{g}$ is a 3-tuple in A_5: A_5 acts on the first five integers from $\{1,\dots,n\}$. Then, $s(\boldsymbol{g})$ doesn't depend on whether we see $\boldsymbol{g}$ as elements in A_5 or in A_n, $n \geq 5$. Identify A_5 with $\mathrm{PSL}_2(\mathbb{Z}/5)$ and Spin_5 with $\mathrm{SL}_2(\mathbb{Z}/5)$. Thus, the final calculation is an explicit computation with 2×2 matrices. This appears in Part C of the proof of [**Fr95b**, Ex. 3.13] or in [**Ser90a**]. □

2.2.2. *Disappearing sequences.* The cases of (2.4) separate according to the conjugacy class in S_n of the product of the three pairs. The phrase *coalescing types* refers to this. Below we add to coalescing types (2.4b) and (2.4c) the possibility of $\boldsymbol{g}$ having as its first 3 or 4 entries these tuples (up to conjugation) having product-one:

(2.6a) $((1\,2\,3),(1\,3\,4),(1\,4\,2))$;
(2.6b) $((1\,2\,3),(1\,3\,4),(1\,2\,4),(1\,2\,4))$; and
(2.6c) $((1\,2\,3)^{(3)})$.

Then, (2.6a) and (2.6b) (resp. (2.6c)) correspond to (2.4b) (resp. (2.4c)). Only when $n = 3$ or 4 are we forced to use (2.6b) as a braiding target (see §3.3.1).

Recall the homomorphism α of §1.1. Denote the subgroup of $Q \in B_r$ with $\alpha(Q)$ permuting $\{1\,\dots,k\}$ by $B_r^{(k)}$.

For any $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$, use $g_1 \cdots g_r = 1$ to draw the following conclusion. For each $i \in \{1,\dots,n\}$ there exists $1 \leq j_1 < \cdots < j_k \leq r$ with these properties.

(2.7a) The support of g_{j_1} contains i.
(2.7b) $g_{j_1} g_{j_2} \cdots g_{j_k}$ fixes i.

Call the sequence $j_1,\dots,j_k$ a *disappearing* sequence for i. For G transitive, a braid to a conjugate of the r-tuple (say, by Lem. 2.6), replaces i by any desired integer.

Lemma 2.2 (Coalescing)**.** *For $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C}_{3^r})$, $n \geq 3, r \geq n-1$, there is a $Q \in B_r$ with $(\boldsymbol{g})Q = (g'_1,\dots,g'_r)$ and (g'_1, g'_2) is a disappearing sequence of length 2, (coalescing type* (2.4a) *or* (2.4b)*) or $(\boldsymbol{g})Q$ has coalescing type* (2.4c)*.*

Stronger still, if the first 3 terms of $\boldsymbol{h} \in \mathrm{Ni}(A_n, \mathbf{C}_{3^r})$ are a type (2.4c) *disappearing sequence, then either $(h_1, h_2, h_3) = (h_1^{(3)})$ (type* (2.6c)*), or the first two terms of $(\boldsymbol{h})Q_2^2$ are a disappearing sequence.*

Proof. Suppose we find $Q \in B_r$ with $(\boldsymbol{g})Q = \boldsymbol{g}'$ and (g_i', g_j') $(i < j)$ a pair of coalescing type from (2.4). Let $Q' = (Q_1 \cdots Q_{i-1})^{-1}(Q_2 \cdots Q_{j-1})^{-1}$. Then, $((\boldsymbol{g})Q)Q' = \boldsymbol{g}''$ has $g_1'' = g_i'$ and $g_2'' = g_j'$.

Apply an element of B_r to $\boldsymbol{g}$ to assume a given disappearing sequence for i is $1, \dots, l$. For example, to put g_{j_1} in the first position, apply $Q_1^{-1} \cdots Q_{j_1-1}^{-1}$. To simplify notation, assume $i = 1$. Such a braiding moves $g_{j_1}, \dots, g_{j_l}$. Still, it leaves them in the same order they originally appear (reading left to right).

If l is two, Lem. 2.1 lets us take (g_1', g_2') to be one of (2.4a) or (with $k = 1$) (2.4b). So, assume $l > 2$. One further assumption: For all $Q \in B_r^{(k)}$,

(2.8) l is the shortest length of a disappearing sequence for 1 in $(\boldsymbol{g})Q$.

This assumption lets us prove $l = 3$ and we may assume (g_1', g_2', g_3') satisfies (2.6c).

A disappearing sequence corresponds to integers in a chain:

$$1 \mapsto i_1 \mapsto i_2 \mapsto \cdots \mapsto i_{l-1} \mapsto 1,$$

where the 1st 3-cycle maps $1 \mapsto i_1$ and the last (lth) maps $i_{l-1} \mapsto 1$. Suppose this disappearing sequence for 1 has a 3-cycle, say g_u, not contributing to this chain. This violates (2.8) with $Q = 1$. So, too, none of $i_1, \dots, i_{k-1}$ is 1. So, the disappearing sequence has the form

$$((1\, i_1\, t_1), (i_1\, i_2\, t_2), \dots, (i_{l-2}\, i_{l-1}\, t_{l-1}), (i_{l-1}\, 1\, t_l)). \tag{2.9}$$

Now suppose in (2.9), $t_1 \ne i_2$. Apply Q_1 to (2.9) to get this replacement for the first two positions:

$$((1\, i_1\, t_1)(i_1\, i_2\, t_2)(t_1\, i_1\, 1), (1\, i_1\, t_1)) = ((1\, i_2\, \cdot), (1\, i_1\, t_1)).$$

Thus, the 1st and the 3rd through kth positions give a shorter disappearing sequence for 1. This violates (2.9) with $Q = Q_1$. Conclude $t_1 = i_2$. Similarly, applying Q_j to (2.9) forces $t_j = i_{j+1}$, $j = 1, \dots, l-1$. Note: The last of these gives $t_{l-1} = 1$. If $i_3 = 1$, then the first three positions contain the 3-cycles

$$((1\, i_1\, i_2), (i_1\, i_2\, 1), (i_2\, 1\, i_4)). \tag{2.10}$$

This is (2.4c), a disappearing sequence of length 3 for 1. From (2.8) we are done, unless $i_3 \ne 1$. In this case, apply Q_1^{-1} to (2.9). Now the first two positions are

$$((i_1\, i_2\, i_3), (i_3\, i_2\, i_1)(1\, i_1\, i_2)(i_1\, i_2\, i_3)) = ((i_1\, i_2\, i_3), (1\, i_2\, i_3)).$$

The 3-cycles in the 2nd–lth positions give a length $l - 1$ disappearing sequence for 1. This contradicts (2.8). Conclude the first paragraph by inducting on l.

Consider the hypotheses of the 2nd paragraph statement. In (2.10), if $i_4 = i_1$, then $(h_1, h_2, h_3) = (h_1^{(3)})$. Otherwise, applying Q_2^2 gives the desired conclusion: $((1\, i_1\, i_2), (i_2\, i_4\, i_1), (i_1\, i_2\, 1))Q_2 = ((1\, i_1\, i_2), (i_1\, 1\, i_4), (i_2\, i_4\, i_1))$. □

2.3. Invariance Corollary. Cor. 2.3 reproves [**Ser90a**]. For $g \in A_n$ of odd order, let $w(g)$ by the sum of $\frac{\ell^2-1}{8}$ mod 2 over the length ℓ of disjoint cycles in g.

Corollary 2.3 (Invariance). *Let $n \ge 3$. If $\varphi : X \to \mathbb{P}^1$ is in the Nielsen class $\mathrm{Ni}(A_n, \mathbf{C}_{3^{n-1}})$, then $\deg(\varphi) = n$, X has genus 0, and $s(\varphi) = (-1)^{n-1}$.*

Generally, for any genus 0 Nielsen class of odd order elements, and representing $\boldsymbol{g} = (g_1, \dots, g_r)$, $s(\boldsymbol{g})$ is constant, equal to $(-1)^{\sum_{i=1}^r w(g_i)}$.

Proof. Induct on n. Apply Lem. 2.2 to $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C}_{3^{n-1}})$. Coalesce g_1 and g_2 in any of the cases (2.4a), (2.4b) and (2.4c) to get $\boldsymbol{g}' = (g_1g_2, g_3, \dots, g_{n-1})$. In each case of (2.4), $\boldsymbol{g}'$ has fewer than $n-1$ 3-cycles as entries (still with product 1).

If $\langle \boldsymbol{g}' \rangle$ is transitive, apply RET to produce $X' \to \mathbb{P}^1$, a (connected) cover having $\boldsymbol{g}'$ as its branch cycles. R-H (in (A.2)) implies $2(n + g(X') - 1) = 2r'$ with $r' = n-2$ (case (2.4b) or (2.4c)) or $n-3$ (case (2.4a)). This is a contradiction: The genus of X' would be negative. Conclude, $\langle \boldsymbol{g}' \rangle$ has more than one orbit in each case.

In case (2.4c), $g_1 g_2 = g_1^{-1} = g_2^{-1}$: $\langle \boldsymbol{g}' \rangle$ has just one orbit. So, we can assume (2.4a) or (2.4b). The formula is clear for $n = 3$. Now do an induction.

Case (2.4a): $\boldsymbol{g}'$ has $n - 3$ branch cycles, spread on 2 or 3 orbits. First assume $\langle \boldsymbol{g}' \rangle$ has orbits of length n_1, n_2 and n_3 ($n_1+n_2+n_3 = n$). Thus, $\boldsymbol{g}'$ has $n_i - 1$ entries supported on the ith orbit, $i = 1, 2, 3$. Write $\boldsymbol{g}'$ as $(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3)$, with the 3-cycles of $\boldsymbol{g}_i$ having support on the ith orbit. According to Lem. 2.1, $s(\boldsymbol{g}') = s(\boldsymbol{g}) = \prod_{i=1}^{3} s(\boldsymbol{g}_i)$. Apply the induction assumption to conclude

$$s(\boldsymbol{g}) = (-1)^{n_1-1}(-1)^{n_2-1}(-1)^{n_3-1} = (-1)^{n-1}.$$

Now we show there cannot be just 2 orbits. The orbit of length n_i supports at least $n_i - 1$ 3-cycles. Thus, there must be at least $n_1 - 1 + n_2 - 1 = n - 2$ of these 3-cycles. There are, however, only $n - 3$ of them.

Case (2.4b): Here $\boldsymbol{g}' = (g_1 g_2, g_3, \dots, g_r)$. Let $\hat{g}_1$, $\hat{g}_2$ and $\widehat{g_1 g_2}$ be respective lifts of g_1, g_2 and $g_1 g_2$ to Spin_n. Lem. 2.1 gives $\hat{g}_1 \hat{g}_2 = -\widehat{g_1 g_2}$. Conclude: $s(\boldsymbol{g}) = -s(\boldsymbol{g}')$. In the product $g_1 g_2$, exactly one integer from the union of the support of g_1 and g_2 disappears. So, $\boldsymbol{g}'$ must have exactly two orbits of respective lengths n_1 and n_2 with $n_1 + n_2 = n$. Apply the induction assumption exactly as for (2.4a). Thus:

$$s(\boldsymbol{g}) = -s(\boldsymbol{g}') = (-1)^{n_1-1}(-1)^{n_2-1} = (-1)^{n-1}.$$

Now for the general case where $\boldsymbol{g}$ has odd order entries, but maybe not 3-cycles. Write g_i as a product of disjoint cycles, $(g_{i,1}, \dots, g_{i,k_i}) = \boldsymbol{g}_i$. Then, juxtaposed $\boldsymbol{g}_i$ s give $\boldsymbol{g}^* = (\boldsymbol{g}_1, \dots, \boldsymbol{g}_r)$ in a new Nielsen class, still of genus 0. From Lem. 2.1, $s(\boldsymbol{g}^*) = s(\boldsymbol{g})$. This reduces us to where all entries of $\boldsymbol{g}$ are cycles.

To conclude, replace each g_i (conjugate to $(1 \dots k)$) by $(h_{i,1}, \dots, h_{i,k_i}) = \boldsymbol{h}_i$, conjugate to $((1\,2\,3), (1\,4\,5), \dots, (1\,k-1\,k))$. Call the juxtaposed branch cycle $\boldsymbol{h}$. The changes are canonical and only depend on the lengths of the disjoint cycles in $\boldsymbol{g}$. Apply Lem. 2.2 to see $s(\boldsymbol{g})/s(\boldsymbol{h})$ is $\prod_{i=1}^{r} u_i$ with

$$u_i = s((1 \dots k)^{-1}, (1\,2\,3), (1\,4\,5), \dots, (1\,k-1\,k)).$$

Conclude easily from [**Ser90a**, Lem. 2] : u_i is $(-1)^{\frac{k^2-1}{8}+\frac{k-1}{2}}$ (see Rem. 2.4). □

Remark 2.4 (Clifford algebra)**.** The proof of Cor. 2.3 uses the Clifford algebra only in computing u_i from [**Ser90a**, Lem. 2]. An induction reduces this to computing directly $s((1 \dots k)^{-1}, (1\,2\,3), (1\,4\,5 \dots k))$, the lifting invariant for a polynomial map one can write down by hand. Is there a simple proof this has value $(-1)^{(k^2+(k-2)^2-2)/8+1}$ without using the Clifford algebra?

2.4. Product-one and H-M reps.

This section generalizes [**Fr95b**, §3.F].

Definition 2.5 (H-M-Nielsen class generators)**.** For $r = 2s$, let $\mathbf{C}$ be a collection of conjugacy classes from a group $G \le S_n$. We don't assume G is transitive. Suppose $\boldsymbol{g} \in \mathbf{C}$ has this form: $(g_1, g_1^{-1}, \dots, g_s, g_s^{-1})$. We say it is an H-M representative (H-M rep.) of $\langle \boldsymbol{g} \rangle$. Also, $\boldsymbol{g}$ is an *H-M rep.* of $\mathrm{Ni}(G, \mathbf{C})$ if $\langle \boldsymbol{g} \rangle = G$.

The following is from [**BF82**, Lemma 3.8].

Lemma 2.6 (Product-one)**.** *Let $\boldsymbol{g} \in \mathbf{C}$ with $\Pi(\boldsymbol{g}) = 1$. Let $i, i+1, \dots, j \bmod r$ be consecutive integers with $g_i g_{i+1} \cdots g_j = 1$ (including $i, i+1, \dots, r-1, r, 1, 2, \dots, j$). Let $\gamma \in \langle g_i, \dots, g_j \rangle$. There is $Q \in B_r$ with*

$$(\boldsymbol{g})Q = (g_1, \dots, g_{i-1}, \gamma g_i \gamma^{-1}, \gamma g_{i+1} \gamma^{-1} \dots, \gamma g_j \gamma^{-1}, g_{j+1}, \dots, g_r). \tag{2.11}$$

Easily (as in the next lemmas) find $Q \in B_r$ that takes an H-M rep. to

$$(g_1, \dots, g_u, g_u^{-1}, \dots, g_1^{-1}) = [g_1, \dots, g_u] \stackrel{\text{def}}{=} [\boldsymbol{g}]. \tag{2.12}$$

Lemma 2.7. *Take $\boldsymbol{g}$ as in* (2.12)*. For any $\pi \in S_u$, there exists $Q \in B_r$ with*

$$[g_1, \dots, g_u]Q = [g_{(1)\pi}, \dots, g_{(u)\pi}].$$

Proof. Transpositions generate S_u. It suffices to show this when $\pi = (1\,2)$ with $Q \in B_r$ local to the first four entries. Take $Q_{1,2} = Q_1^{-1} Q_3$. Then:

$$\begin{aligned}(g_1, g_2, g_2^{-1}, g_1^{-1})Q_{1,2} &= (g_2, g_2^{-1} g_1 g_2, g_2^{-1}, g_1^{-1})Q_3 \\ &= (g_2, g_2^{-1} g_1 g_2, g_2^{-1} g_1^{-1} g_2, g_2^{-1}).\end{aligned} \tag{2.13}$$

Product-one Lemma 2.6 gives $Q' \in B_r$ conjugating (2.13) by g_2 (fixing the coordinates beyond the first four). Conclude by taking $Q = Q_{1,2} Q'$. □

Lemma 2.8 (Generator)**.** *Assume the following for $\boldsymbol{g} = (\boldsymbol{g}', \boldsymbol{g}'') \in \mathbf{C}$:*

(2.14) *$\boldsymbol{g}' = [g'_1, \dots, g'_{u'}]$, and $\Pi(\boldsymbol{g}'') = 1$.*

Then, for any $h \in \langle \boldsymbol{g}' \rangle$, there is a $Q \in B_r$ with $Q(\boldsymbol{g}) = (\boldsymbol{g}', h\boldsymbol{g}'' h^{-1})$.

Suppose the following holds for each $\boldsymbol{g} = [g_1, \dots, g_u] \in \mathrm{Ni}(G, \mathbf{C})$.

(2.15) *For $\boldsymbol{g}(i) = [g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_u]$, $\langle \boldsymbol{g}(i) \rangle = G$, $i = 1, \dots, u$.*

Then, all H-M representatives of $\mathrm{Ni}(G, \mathbf{C})$ fall in one B_r orbit.

Proof. We show the statement after (2.14). Induct on the number of entries from $\boldsymbol{g}'$ to get the product h. So, it suffices to take $h = g'_j$ for some j between 1 and u'. Apply Lem. 2.7 to assume with no loss $j = u'$. Then, apply Lem. 2.8 to $([g_{u'}], \boldsymbol{g}')$. This gives the conclusion to (2.14). The conclusion following (2.15) comes from repeated application of the above to $(\boldsymbol{g}(i), g_i, g_i^{-1})$. □

Lemma 2.9 (Blocks)**.** *Suppose $\boldsymbol{g} = (\boldsymbol{g}_1, \dots, \boldsymbol{g}_u)$ with $\Pi(\boldsymbol{g}_i) = 1$, for all but possibly one $i_0 \in \{1, \dots, u\}$. For any $\pi \in S_u$, and $\tau_i \in \langle \boldsymbol{g}_i \rangle$, $i = 1, \dots, u$, there exists $Q \in B_r$ with $(\boldsymbol{g})Q = (\boldsymbol{g}_{(1)\pi}, \dots, \boldsymbol{g}_{(u)\pi})$. Also, for any i and j (excluding $j = i_0$ if i_0 exists), there exists $Q \in B_r$ with*

$$(\boldsymbol{g})Q = (\boldsymbol{g}_1, \dots, \boldsymbol{g}_{j-1}, \tau_i \boldsymbol{g}_j \tau_i^{-1}, \boldsymbol{g}_{j+1}, \dots, \boldsymbol{g}_u).$$

Proof. The case $u = 2$ suffices to show we can permute the appearance of the $\boldsymbol{g}_i$ s. For this, assume $\Pi(\boldsymbol{g}_1) = 1$, and braid every entry of $\boldsymbol{g}_1$, in order from left to right, past every entry of $\boldsymbol{g}_2$. This gives the effect of

$$(\boldsymbol{g}_1, \boldsymbol{g}_2)Q = (\alpha \boldsymbol{g}_2 \alpha^{-1}, \boldsymbol{g}_1), \alpha = \Pi(\boldsymbol{g}_1). \tag{2.16}$$

Done, since $\alpha = 1$. The last sentence reduces to cases $i = j = 1$ and $i = 1$, $j = 2$.

For the 1st, apply Lem. 2.6 to $\boldsymbol{g}_1$. For the 2nd, with g an entry of $\boldsymbol{g}_2$ ($\Pi(\boldsymbol{g}_2)$ may not be 1), braid to $(g\boldsymbol{g}_1 g^{-1}, \boldsymbol{g}_2)$ with $\boldsymbol{g}_2$ written $(\boldsymbol{h}, g, \boldsymbol{h}')$. Braid to $(\boldsymbol{h}, \boldsymbol{g}_1, g, \boldsymbol{h}')$ as above. Then, braid the sequence $\mapsto (\boldsymbol{h}, g, g^{-1}\boldsymbol{g}_1 g, \boldsymbol{h}') \mapsto (g^{-1}\boldsymbol{g}_1 g, \boldsymbol{g}_2)$. □

3. Coalescing targets

The induction goal, for given n and $n-1=r\ge 4$ (resp. $r\ge n\ge 5$), is to apply a $Q\in B_r$ to any $\boldsymbol{g}\in \mathrm{Ni}(A_n,\mathbf{C}_{3^r})$ so $(\boldsymbol{g})Q$ is an (resp. one of two) exemplar(s). §3.1 lists coalescing targets for $n\ge 5$. Yet, these require the intricate case $n=4$ (§3.2).

3.1. Coalescing Targets, $n\ge 5$. We use $[\boldsymbol{g}_{u,\bullet}]=[g_1,\dots,g_u]$ (Def. 2.5) with $g_1=(1\,2\,3),\ g_2=(1\,4\,5),\dots,g_u=(1\,2u\,2u{+}1)$ to list braid targets.

3.1.1. *Normal forms for $\mathbf{C}_{3^r}$.* The pairs of Nielsen class representatives $\boldsymbol{g}$ below have respective lifting invariants $s(\boldsymbol{g})=+1$, and $s(\boldsymbol{g})=-1$ (§1.1.4).

$$\begin{aligned}&r\ge n \text{ both odd}, ([\boldsymbol{g}_{\frac{n-3}{2},\bullet}],(1\,n{-}1\,n)^{(3)},((1\,2\,3),(3\,2\,1))^{(\frac{r-n}{2})}),\\ &([\boldsymbol{g}_{\frac{n-3}{2},\bullet}],(1\,n{-}2\,n{-}1),(1\,n{-}1\,n),(1\,n\,n{-}2),((1\,2\,3),(3\,2\,1))^{(\frac{r-n}{2})}).\end{aligned}\tag{3.1}$$

$$\begin{aligned}&\text{odd } r\ge n \text{ even}, ([\boldsymbol{g}_{\frac{n-2}{2},\bullet}],(1\,n{-}1\,n)^{(3)},((1\,2\,3),(3\,2\,1))^{(\frac{r-n-1}{2})}),\\ &([\boldsymbol{g}_{\frac{n-4}{2},\bullet}],(1\,n{-}2\,n{-}1),(1\,n{-}1\,n),(1\,n\,n{-}2),((1\,2\,3),(3\,2\,1))^{(\frac{r-n+1}{2})}).\end{aligned}\tag{3.2}$$

$$\begin{aligned}&\text{even } r\ge n \text{ odd}, ([\boldsymbol{g}_{\frac{n-1}{2},\bullet}],((1\,2\,3),(3\,2\,1))^{(\frac{r-n+1}{2})}),\\ &([\boldsymbol{g}_{\frac{n-3}{2},\bullet}],(1\,n{-}2\,n{-}1)^{(2)},(1\,n{-}2\,n),(1\,n\,n{-}1),((1\,2\,3),(3\,2\,1))^{(\frac{r-n-1}{2})}).\end{aligned}\tag{3.3}$$

$$\begin{aligned}&r\ge n \text{ both even}, ([\boldsymbol{g}_{\frac{n-2}{2},\bullet}],(1\,n{-}1\,n),(1\,n{-}1\,n)^{-1},((1\,2\,3),(3\,2\,1))^{(\frac{r-n}{2})}),\\ &([\boldsymbol{g}_{\frac{n-4}{2},\bullet}],(1\,n{-}1\,n{-}2)^{(2)},(1\,n{-}1\,n),(1\,n\,n{-}2),((1\,2\,3),(3\,2\,1))^{(\frac{r-n}{2})}).\end{aligned}\tag{3.4}$$

3.1.2. *More on $\mathbf{C}_{3^r}$ normal forms.* Each $\boldsymbol{g}$ in §3.1.1 ends with $((1\,2\,3),(3\,2\,1))^{(t)}$ for some t. Denote this end part $\boldsymbol{g}_e$, and the beginning part $\boldsymbol{g}_b$: $\boldsymbol{g}=(\boldsymbol{g}_b,\boldsymbol{g}_e)$. We chose $\boldsymbol{g}_b$ to be transitive on $\{1,\dots,n\}$, with $s(\boldsymbol{g}_b)=s(\boldsymbol{g})$. Refer to the spin lifting value by a subscript: as in $(3.4)_\pm$ indicating the two (3.4) listings.

Definition 3.1. Each $\boldsymbol{g}_b$ in §3.1.1 starts with part of an (element braid equivalent to an) H-M rep. $[\boldsymbol{g}_{\frac{n-u}{2},\bullet}]$ (Def. 2.5). The quirky part is $\boldsymbol{g}_{\mathrm{nb}}$, what is left after the H-M in $\boldsymbol{g}_b$. This is the *nub*.

For example, respective $+$ and $-$ nubs of (3.1) are

$$(1\,n{-}1\,n)^{(3)} \text{ and } (1\,n{-}2\,n{-}1),(1\,n{-}1\,n),(1\,n\,n{-}2).$$

We see the value of $s(\boldsymbol{g})$ from the nub alone.

The -1 rep. (resp. +1 rep.) of (3.2) (resp. (3.3)) also works for $r=n-1$.

Strong Coalescing Lemma 4.4 gives the tools for braiding any $\boldsymbol{g}\in\mathrm{Ni}(A_n,\mathbf{C}_r)$ to where it has the correct $\boldsymbol{g}_e$. So, in the induction of §4.4, the significant braidings are where there is no $((1\,2\,3),(3\,2\,1))^{(t)}$ tail.

For example, the $\boldsymbol{g}_b$ part of the -1 rep. of (3.3) braids to

$$([\boldsymbol{g}_{\frac{n-5}{2},\bullet}],(1\,n{-}3\,n{-}2)^{(3)},(1\,n{-}2\,n{-}1),(1\,n{-}1\,n),(1\,n\,n{-}2)).\tag{3.5}$$

Also, the $\boldsymbol{g}_b$ part of the 2nd element of (3.4) braids to

$$([\boldsymbol{g}_{\frac{n-6}{2},\bullet}],(1\,n{-}4,n{-}3)^{(3)},(1\,n{-}2\,n{-}1),(1\,n{-}1\,n),(1\,n\,n{-}2)).\tag{3.6}$$

Respectively, these are (4.6b) and (4.6a) in the proof of Lem. 4.4.

3.2. Coalescing targets for $n=3,4$. While $n=3$ is easy, $n=4$ is not.

3.2.1. *Two conjugacy classes of 3-cycles.* The Klein 4-group K is a normal subgroup of A_4. A 3-cycle in A_4 determines its conjugacy class by whether it maps to $(1\,2\,3)$ or $(1\,3\,2)$ in $A_4/K = \mathbb{Z}/3 = A_3$.

Lemma 3.2. *With $G = A_3$ or A_4, $\mathrm{Ni}(G, \mathrm{C}_{\pm 3^{s_1,s_2}})$ is nonempty if and only if*

$$s_1 - s_2 \equiv 0 \bmod 3 \ (s_1 + s_2 = r). \tag{3.7}$$

Subject to (3.7)*, (s_1, s_2) (resp. unordered pairs $\{s_1, s_2\}$) label nonempty inner (resp. absolute) Nielsen classes of 3-cycle conjugacy classes in either A_3 or A_4.*

It is convenient to select $(2\,3) = \beta$ to conjugate a 3-cycle in A_4 to the conjugacy class of its inverse. For any 3-cycle $\alpha \in A_4$, denote its conjugate $\beta\alpha\beta^{-1}$ by ${}^{\beta}\alpha$. Similarly, if $\boldsymbol{g}$ is an r-tuple of elements of A_4, its conjugate by β is ${}^{\beta}\boldsymbol{g}$.

Let $\boldsymbol{g}_i$ be an r_i-tuple of A_4 3-cycles, with $\Pi(\boldsymbol{g}_i) = 1$; r_i varies with $i = 1, \dots, t$. For $\boldsymbol{\epsilon} \in (\mathbb{Z}/2)^t$, denote $({}^{\beta^{\epsilon_1}}\boldsymbol{g}_1, \dots, {}^{\beta^{\epsilon_t}}\boldsymbol{g}_t)$ by ${}^{\boldsymbol{\epsilon}}(\boldsymbol{g}_1, \dots, \boldsymbol{g}_t)$. When no other notation suggests the division between $\boldsymbol{g}_1, \dots, \boldsymbol{g}_t$, replace the comma separators by semi-colons: ${}^{\boldsymbol{\epsilon}}(\boldsymbol{g}_1; \dots; \boldsymbol{g}_t)$ to unambiguously shows the action of $\boldsymbol{\epsilon} \in (\mathbb{Z}/2)^t$.

3.2.2. *The 3-Lemma.* We need a precise result for $G = A_3$. Assume (3.7).

Lemma 3.3 (3-Lemma)**.** *B_r applied to $((1\,2\,3)^{(s_1)}, (3\,2\,1)^{(s_2)})$ is $\mathrm{Ni}(A_3, \mathbf{C}_{\pm 3^{s_1,s_2}})$.*

If $\boldsymbol{g}^* = (\boldsymbol{g}, \boldsymbol{g}') \in \mathrm{Ni}(A_n, \mathbf{C}_{3^{r'}})$, $n \ge 5$, and $\boldsymbol{g} \in \mathrm{Ni}(A_3, \mathbf{C}_{\pm 3^{s_1,s_2}})$, $r = s_1 + s_2$, then there is $Q \in B_{r'}$ with $(\boldsymbol{g}^*)Q = (\bar{\boldsymbol{g}}, \boldsymbol{g}')$ where $\bar{\boldsymbol{g}}$ is

$$((1\,2\,3), (3\,2\,1))^{(\frac{r}{2})} \text{ if } r \text{ is even, and}$$
$$((1\,2\,3)^{(3)}, ((1\,2\,3), (3\,2\,1))^{(\frac{r-3}{2})}) \text{ if } r \text{ is odd.}$$

Proof. Since A_3 is cyclic of order 3, the first statement is obvious.

Since $n \ge 5$, apply Blocks Lem. 2.9 to conjugate $\boldsymbol{g}$ by $\gamma = (2\,3)(k\,j)$ with k and j any integers distinct from 1, 2 and 3. So, with no loss assume $s_1 \ge s_2$. Braid $\boldsymbol{g}$ to $((1\,2\,3)^{(s_1 - s_2)}, ((1\,2\,3), (3\,2\,1))^{(s_2)})$. With no loss, take $s_2 = 0$. Thus, (3.7) implies 3 divides s_1. We take s_1 even; the other case is similar. So, $\boldsymbol{g}$ is $(\boldsymbol{g}_1, \dots, \boldsymbol{g}_{s_1/3})$ with each $\boldsymbol{g}_i$ equal $(1\,2\,3)^{(3)}$. By assumption $\langle \boldsymbol{g}_i, \boldsymbol{g}' \rangle = A_n$, $n \ge 5$. Several applications of the the Blocks Lemma, using γ above, produces $Q' \in B_r$ with

$$(\boldsymbol{g}, \boldsymbol{g}')Q' = (\boldsymbol{g}_1, \gamma\boldsymbol{g}_2\gamma^{-1}, \boldsymbol{g}_3, \gamma\boldsymbol{g}_4\gamma^{-1}, \dots, \gamma\boldsymbol{g}_{s_1/3}\gamma^{-1}, \boldsymbol{g}').$$

This the desired target with r even. □

3.3. The case $n = 4$.

Most difficulties are in this induction on r for $n = 4$.

3.3.1. *A_4 targets.* Conjugating by $\beta = (2\,3)$ switches s_1 and s_2 in list (3.8).

(3.8a) $r = 3, s_1 = 3, s_2 = 0$: $\quad \boldsymbol{g}_{3,-} = ((1\,2\,3), (1\,3\,4), (1\,4\,2))$.

(3.8b) $r = 4, s_1 = s_2 = 2$: $\quad \boldsymbol{g}_{4,+} = ((1\,3\,4), (1\,4\,3), (1\,2\,3), (1\,3\,2))$, $\quad \boldsymbol{g}_{4,-} = ((1\,2\,3), (1\,3\,4), (1\,2\,4), (1\,2\,4))$.

(3.8c) $r \ge 5, s_1 = 3 + s_1', s_2 = s_2'$: $\quad \boldsymbol{g}_{r,+} = ((1\,3\,4)^{(3)}, (1\,2\,3)^{(s_1')}, (3\,2\,1)^{(s_2')})$, $\quad \boldsymbol{g}_{r,-} = (\boldsymbol{g}_{3,-}, (1\,2\,3)^{(s_1')}, (3\,2\,1)^{(s_2')})$.

Lemma 3.4 (4-Lemma)**.** *Assume* (3.7) *holds for (s_1, s_2). Then, any $\boldsymbol{g} \in \mathrm{Ni}(A_4, \mathbf{C}_{\pm \mathrm{C}_{3^{s_1,s_2}}})$ braids either to an element in* (3.8a), (3.8b) *or* (3.8c)*, or its conjugate by β. If $s_1 = s_2$, some braid achieves conjugation by β.*

We divide the proof into four subsections. The first on $r = 3$ and $r = 4$ showing how to use the **sh**-incidence matrix. The next two treat separately when (2.4a) and (2.4b) hold, inducting on r using Coalescing Lemma 2.2. The last considers the case $s_1 = s_2$ to show conjugation by β is braidable.

3.3.2. *$r = 3$ and 4, and the* **sh***-incidence matrix.* Modulo conjugation by S_4 and action of B_3 here are the strings of two or three 3-cycles with product 1.

$$((1\,2\,3),(3\,2\,1)),\ (1\,2\,3)^{(3)},\ ((1\,2\,3),(1\,3\,4),(1\,4\,2)). \tag{3.9}$$

Only the 3rd is transitive on $\{1,2,3,4\}$. This finishes the case $r = 3$.

[**BaFr02**, §2.10] has the rubric for the **sh**-incidence matrix. It works for all values of r, though for $r = 4$ it is usually possible to do it by hand. The result is information on natural j-line (curve) covers (reduced Hurwitz spaces as in §A.3) one can compare with modular curves (which are a special case). We do $\mathrm{Ni}(A_4,\mathbf{C}_{\pm 3^2})$ here. [**BaFr02**, §9] and [**Fr04**, §6, §7.2] interpret these computations.

The computation works by using these three important groups:

(3.10a) $\mathcal{Q}'' = \langle \mathbf{sh}^2, q_1q_3^{-1}\rangle$;
(3.10b) the *cusp group* $\mathrm{Cu}_4 = \langle q_2, \mathcal{Q}''\rangle/\mathcal{Q}''$; and
(3.10c) the *mapping class group* $\bar M_4 = \langle \gamma_0,\gamma_1\rangle = H_4/\mathcal{Q}''$ generated freely by $\gamma_0 = q_1q_2$, $\gamma_1 = q_1q_2q_3 = \mathbf{sh}$ (§2.1) of respective orders 3 and 2.

This induces an action of $\bar M_4$ on $\mathrm{Ni}(G,\mathbf{C})^*/\mathcal{Q}'' = \mathrm{Ni}(G,\mathbf{C})^{*,\mathrm{rd}}$ ($* =$ in or abs), *reduced* Nielsen classes. That is, $\gamma_0,\gamma_1,\gamma_\infty$ in (3.10) are names for H_4 elements on reduced classes. The orders of γ_0 and γ_1 in (3.10c) come easily from the Hurwitz relation (2.2) mod $\mathcal{Q}''$. So, too, does the relation $\gamma_0\gamma_1\gamma_\infty = 1$, with $\gamma_\infty = q_2$.

The cover $\bar{\mathcal{H}}(G,\mathbf{C})^{*,\mathrm{rd}} \to \mathbb{P}^1_j$ (as in Prop. §A.8) has as branch cycles (Def. 1.1) $(\gamma_0,\gamma_1,\gamma_\infty)$ on $\mathrm{Ni}(G,\mathbf{C})^{*,\mathrm{rd}}$. This gives us the genus of its components.

A pairing on γ_∞ orbits $(\boldsymbol{g})\mathrm{Cu}_4 = O = O_{\boldsymbol{g}}$ gives **sh**-incidence matrix entries: $(O,O') \mapsto$ the cardinality of O intersected with the shift on O':

$$\begin{pmatrix} \vdots & |O\cap(O')\gamma_1| \\ & \cdots \end{pmatrix} = \begin{pmatrix} \vdots & |O\cap(O')\gamma_0| \\ & \cdots \end{pmatrix}.$$

- Blocks $\Leftrightarrow$ components of $\mathcal{H}(G,\mathbf{C})^{*,\mathrm{rd}}$, or of $\mathcal{H}(G,\mathbf{C})^*$.
- Fixed points of γ_0 or γ_1 appear on the diagonal.

Consider $g_{1,4} = \boldsymbol{g}_{4,-} \in \mathrm{Ni}(A_4,\mathbf{C}_{\pm 3^2}) = ((1\,2\,3),(1\,3\,4),(1\,2\,4),(1\,2\,4))$. from Cor. 2.3, $s(g_{1,4}) = s((1\,2\,3),(1\,3\,4),(1\,4\,2)) = -1$.

Subdivide $\mapsto \mathrm{Ni}(A_3,\mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$ according to the sequences of conjugacy classes $\mathrm{C}_{\pm 3}$; $q_1q_3^{-1}$ and **sh** switch these rows:

[1] + - + -	[2] + + - -	[3] + - - +
[4] - + - +	[5] - - + +	[6] - + + -

Here is the notation in the charts below where $O^k_{i,j}$ appears: k is the cusp width, and i,j corresponds to a labeling of orbit representatives. The diagonal entries for $O^4_{1,1}$ and $O^4_{1,4}$ are nonzero. In detail, however, γ_1 (resp. γ_0) fixes 1 (resp. no) element of $O_{1,1}$, and neither of γ_i, $i = 0,1$, fix any element of $O_{1,4}$.

$$\begin{aligned} \text{H-M rep.} \mapsto \boldsymbol{g}_{1,1} = &\ ((1\,2\,3),(1\,3\,2),(1\,3\,4),(1\,4\,3)) \\ \boldsymbol{g}_{1,3} = &\ ((1\,2\,3),(1\,2\,4),(1\,4\,2),(1\,3\,2)) \\ \text{H-M rep.} \mapsto \boldsymbol{g}_{3,1} = &\ ((1\,2\,3),(1\,3\,2),(1\,4\,3),(1\,3\,4)) \end{aligned}$$

Proposition 3.5. *On* $\mathrm{Ni}(\mathrm{Spin}_4,\mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$ *(resp.* $\mathrm{Ni}(A_4,\mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$*)* $\bar M_4$ *has one (resp. two) orbit(s). So,* $\mathcal{H}(\mathrm{Spin}_4,\mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$ *(resp.*$\mathcal{H}(A_4,\mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$*) has one (resp. two) component(s),* $\mathcal{H}_{0,+}$ *(resp.* $\mathcal{H}_{0,+}$ *and* $\mathcal{H}_{0,-}$*).*

Ni_0^+ Orbit	$O_{1,1}^4$	$O_{1,3}^2$	$O_{3,1}^3$
$O_{1,1}^4$	1	1	2
$O_{1,3}^2$	1	0	1
$O_{3,1}^3$	2	1	0

Ni_0^- Orbit	$O_{1,4}^4$	$O_{3,4}^1$	$O_{3,5}^1$
$O_{1,4}^4$	2	1	1
$O_{3,4}^1$	1	0	0
$O_{3,5}^1$	1	0	0

Then, $\mathcal{H}(\mathrm{Spin}_4, \mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$ maps one-one to $\mathcal{H}_{0,+}$ (though changing A_4 to Spin_4 give different moduli). The compactifications of $\mathcal{H}_{0,\pm}$ both have genus 0 from Riemann-Hurwitz applied to $(\gamma_0, \gamma_1, \gamma_\infty)$ on reduced Nielsen classes (Ex. A.3).

Remark 3.6 (Complication in (3.3))**.** Prop. 3.5 says the lifting invariant separates the two B_4 orbits on $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2})$. We use this for convenient substitution, when we have all but the first 4 entries matching an item in list (3.1). For example: We can substitute $\boldsymbol{g}' = ((1\,n{-}1\,n{-}2)^{(2)}, (1\,n{-}1\,n), (1\,n\,n{-}2))$ for any 3-cycle 4-tuple $\boldsymbol{g}$, with product-one and $s(\boldsymbol{g}) = -1$, on $\{1, n{-}2, n{-}1, n\}$.

3.3.3. *Case* (2.4a). Now assume $r \ge 5$. With no loss, $\boldsymbol{g}$ braids to $(g, g^{-1}, \boldsymbol{g}')$ $\Pi(\boldsymbol{g}') = 1$. Suppose $\boldsymbol{g}'$ is intransitive on A_4. Then, $\langle \boldsymbol{g}' \rangle = A_3$. Apply Product-one Lemma 2.6 to find $Q \in B_r$, $\gamma \in A_4$ and $j \in \{2, 3\}$ with

$$\boldsymbol{g}'' = (\boldsymbol{g})Q = \gamma \boldsymbol{g} \gamma^{-1} = ((1\,j\,4), (4\,j\,1); (1\,2\,3)^{(s_1')}, (3\,2\,1)^{(s_2')}),$$

with $s_1' \equiv s_2' \mod 3$. Since $r \ge 5$, some braid of $\boldsymbol{g}''$ puts one of these at its head:

$$((1\,j\,4), (4\,j\,1), (1\,2\,3)^{(3)}), ((1\,j\,4), (4\,j\,1), (3\,2\,1)^{(3)}), ((1\,j\,4), (4\,j\,1), (1\,2\,3)^{(2)}, (3\,2\,1)^{(2)}).$$

It is easy to braid the first two to (3.8c). Example: For the 1st, if $j = 2$, conjugate by $(2\,3\,4)$ (Prod. One Lem.) and then slide $(1\,3\,4)^{(3)}$ to the front (Blocks Lem. 2.9).

Now suppose $\boldsymbol{g}'$ is transitive on A_4. Blocks Lemma 2.9 gives $Q \in B_r$ with $(\boldsymbol{g})Q = ((1\,2\,3), (3\,2\,1), \boldsymbol{g}')$. The induction assumption gives a Q' putting $\boldsymbol{g}'$ in a preferred form depending on s_1' and s_2'. If $r - 2 \ne 4$, or if $r - 2 = 4$ and $s(\boldsymbol{g}) = +1$, the Blocks Lemma allows combining the head $((1\,2\,3), (3\,2\,1))$ with the tail $((1\,2\,3)^{(s_1'')}, (3\,2\,1)^{(s_2'')})$ for a preferred form in (3.8).

That leaves only deciding how to braid $((1\,2\,3), (3\,2\,1), \boldsymbol{g}_{4,-})$ to a normal form. The Blocks Lemma braids this to $(\boldsymbol{g}_{4,-}, (1\,2\,4), (4\,2\,1))$, with $(1\,2\,4)^{(3)}$ in the 3rd through 5th entries. Apply it again to get $\boldsymbol{g}_{6,-}$ (in (3.8c)).

3.3.4. *Case* (2.4b). Without loss, assume we can't braid to case (2.4a) and $\boldsymbol{g}$ has the form $((1\,3\,4), (1\,4\,2), \boldsymbol{g}')$. Apply the induction assumption to $((1\,3\,2), \boldsymbol{g}') = \boldsymbol{g}^*$. If $\boldsymbol{g}^*$ is intransitive on A_4, then $\boldsymbol{g}^* = ((1\,2\,3)^{(s_1')}, (3\,2\,1)^{(s_2')})$. We may assume

$$\boldsymbol{g} = ((1\,3\,4), (1\,4\,2), (1\,2\,3)^{(s_1')}, (3\,2\,1)^{(s_2'-1)}).$$

Braid this to $((1\,2\,3), (1\,3\,4), (1\,4\,2), (1\,2\,3)^{(s_1'-1)}, (3\,2\,1)^{(s_2'-1)})$ $(\boldsymbol{g}_{r,-})$ by braiding $(1\,2\,3)$ past $((1\,3\,4), (1\,4\,2))$.

Now assume $\boldsymbol{g}^*$ is transitive on A_4. Recall $\alpha : B_r \to S_r$ (§2.1). Apply the induction with $r{-}1$ replacing r. Denote the ith entry of $(\boldsymbol{g})Q'$ by $(\boldsymbol{g})Q'[i]$. Let $i = (1)\alpha(Q)$. Thus, there is $Q \in B_{r-1}$ and $Q' \in B_r$ with the following properties.

(3.11a) $(\boldsymbol{g}^*)Q$ is in the list (3.8a)–(3.8c) with $r{-}1$ replacing r.

(3.11b) $(\boldsymbol{g})Q'[j] = (\boldsymbol{g}^*)Q[j]$, $j = 1, \dots, i-1$, $(\boldsymbol{g})Q'[i](\boldsymbol{g})Q'[i{+}1] = (\boldsymbol{g}^*)Q[i]$ and $(\boldsymbol{g})Q'[j{+}1] = (\boldsymbol{g}^*)Q[j]$, $j = i+1, \dots, r-1$.

Consider possibilities for reexpanding $(\boldsymbol{g}^*)Q$ to give $(\boldsymbol{g})Q'$ by putting two 3-cycles (using (3.8a)) with product $(\boldsymbol{g}^*)Q[i]$ in its place. We can dismiss all but one by showing, contrary to assumption, case (2.4b) holds. The case $r{-}1 = 3$

illustrates with $(\boldsymbol{g}^*)Q = \boldsymbol{g}_{3,-}$. With $((1\,2\,4),(1\,4\,3))$ in place of $(1\,2\,3)$, the result is $((1\,2\,4),(1\,4\,3),(1\,3\,4),(1\,4\,2))$. Thus, $(\boldsymbol{g})Q'$ is in case (2.4a). By braiding it is clear this one substitution suffices for $r-1=3$.

Now consider $r-1 \ge 4$. For $\boldsymbol{g}_{4,+}$, no matter the substitution, you are in case (2.4a). By braiding, assume substitutions in $\boldsymbol{g}_{4,-}$ are for $(1\,3\,2)$ or for $(1\,4\,2)$. For the first, this produces $((1\,3\,4),(1\,4\,2),(1\,3\,2),(1\,3\,4),(1\,4\,2))$. Apply $Q_2^{-1}Q_3^{-1}$ to get $((1\,3\,4),(1\,3\,4),(1\,2\,3),(2\,3\,4),(1\,4\,2))$. Now apply Q_3 to get

$$((1\,3\,4),(1\,3\,4),(1\,2\,4),(1\,2\,3),(1\,4\,2)).$$

With $(1\,2\,4)$ (resp. $(1\,4\,2)$) in the 3rd (resp. 5th) position, this braids to (2.4a). Similarly, substitution for $(1\,4\,2)$ gives $((1\,3\,2),(1\,3\,2),(1\,3\,4),(1\,4\,3),(1\,3\,2))$ in (2.4a).

Assume $r \ge 5$ is odd. Substitution for $(1\,3\,4)$ in $\boldsymbol{g}_{r,+}$ gives

$$((1\,3\,4)^{(2)},(1\,3\,2),(1\,2\,4),(1\,2\,3)^{(s_1')},(3\,2\,1)^{(s_2')}).$$

Since $s_1' > 0$, this is (2.4a). Substitution in $\boldsymbol{g}_{r,+}$ for $(1\,2\,3)$ gives

$$((1\,3\,4)^{(3)},(1\,2\,3)^{(s_1'-1)},(1\,2\,4),(1\,4\,3),(3\,2\,1)^{(s_2')}). \tag{3.12}$$

As $(1\,3\,4)$ and $(1\,4\,3)$ appear, this braids to (2.4a). Finally, substitute for $(3\,2\,1)$:

$$((1\,3\,4)^{(3)},(1\,2\,3)^{(s_1')},(1\,3\,4),(1\,4\,2),(3\,2\,1)^{(s_2'-1)}). \tag{3.13}$$

This is (2.4a) if $s_2' > 1$. So, let $s_2' = 1$ and $s_1' \equiv 1 \bmod 3$. Rewrite (3.13) as

$$((1\,3\,4)^{(3)},(1\,2\,3)^{(s_1'-1)},(1\,2\,3),(1\,3\,4),(1\,4\,2)). \tag{3.14}$$

Since $(2\,3\,4) \in \langle(1\,2\,3),(1\,3\,4),(1\,4\,2)\rangle$, Blocks Lem. 2.9 gives a braid that conjugates $(1\,3\,4)^{(3)}$ to $(1\,2\,3)^{(3)}$. Apply it again to braid (3.14) to

$$((1\,2\,3),(1\,3\,4),(1\,4\,2),(1\,2\,3)^{(s_1'+2)})\text{: } s_2'=0 \text{ in } \boldsymbol{g}_{r,-},\ (3.8c).$$

The $\boldsymbol{g}_{r,-}$ substitutions are easier for they imitate previous substitutions. Example: substituting in one of the beginning three entries duplicates the case $r-1=3$.

3.3.5. *Braiding β when $s_1 = s_2$.* It suffices to braid β for any braid orbit rep. If $\boldsymbol{g}$ has lifting invariant $+1$, and $s_1 = s_2$, then it braids to $\boldsymbol{g}''$ in §3.3.3 (so $s_1' = s_2'$). Example: If $j=2$, then $\beta\boldsymbol{g}''\beta^{-1} = ((1\,3\,4),(1\,4\,3);(3\,2\,1)^{(s_1')},(1\,2\,3)^{(s_1')})$. Apply the Prod. One Lem. to conjugate by $(1\,2\,3)$. Then braid the commuting pieces $(3\,2\,1)^{(s_1')}$ and $(1\,2\,3)^{(s_1')}$ past each other to return to $\boldsymbol{g}''$.

Prop. 3.5 showed, for $r=4$, any $\boldsymbol{g}$ with lifting invariant -1 braids to $\boldsymbol{g}_{4,-}$. As above, it now suffices to show we can braid β when the lifting invariant is -1 to $\boldsymbol{g}_{r,-}$, $r \ge 5$. Similar to braids above, braid $\boldsymbol{g}_{r,-}$ to

$$\boldsymbol{g}^\dagger = ((1\,3\,2)^{(2)},(1\,3\,4),(1\,4\,2);(1\,2\,3)^{(s_1'+1)},(3\,2\,1)^{(s_2'-2)}),\ \text{with } s_2' = s_1'{+}3.$$

Again, Prop. 3.5 braids the conjugate by β on the 4-entry head of $\boldsymbol{g}^\dagger$, and the paragraph above just braided it on its tail. That completes Lem. 3.4.

4. Improving the Coalescing Lemma and full induction

We improve Coalescing Lemma 2.2 to show for $n \ge 5$ we can braid to (2.4a).

4.1. Set up for Strong Coalescing. Consider how a *disappearing sequence* in Lem. 2.2 produces (2.4a), (2.4b) or (2.4c). Use l for the length of the sequence.

4.1.1. *Some tough braidings.* Coalescing types (2.4a) or (2.4b) have $l = 2$. Condition (2.8) on the shortest disappearing sequence is an induction assumption: no element of $B_r^{(l)}$ takes our choice of disappearing sequence to a smaller disappearing subsequence. From this, Lem. 2.2 concludes $l = 3$ and coalescing type (2.4c) occurs. Though $n = 3$ needs type (2.4c), we will see that case is special.

Here are hard cases with $n = 4$ and 5 ($r = 6$) for finding coalescing type (2.4a):

$$\boldsymbol{g}_0 = ((1\,2\,3)^{(3)}, (1\,4\,5)^{(3)}); \tag{4.1}$$

$$\boldsymbol{g}_1 = ((1\,2\,3)^{(3)}, (2\,1\,4)^{(3)}); \text{ or } \boldsymbol{g}_2 = ((1\,2\,3)^{(3)}, (1\,2\,4)^{(3)}). \tag{4.2}$$

We introduce an extra notation now for later use. If in addition to dropping the tail of the $(3.1)_+$ term (as in §3.1.2), we drop the head $[\boldsymbol{g}_{\frac{n-3}{2},\bullet}]$, the nub that remains is conjugate to $(1\,2\,3)^{(3)}$. All 6-tuples in (4.1) and (4.2) are juxtapositions of two of these. We refer to the resulting 6-tuples as (having type) $(3.1)_+$+$(3.1)_+$

In (4.1), each integer, $i = 1, \dots, 5$, has a disappearing sequence of length three. Also, there is no $1 \le i < j \le 6$ with (g_i, g_j) a disappearing sequence for any integer. Still, Lem. 4.1 shows we can braid (4.1) to type (2.4a).

4.1.2. *Coalescing tricks.* Lem. 4.1 conceptually finds explicit braids forced on us. Later we will be less complete; all are similar. Just as A_4 has $\mathrm{C}_{\pm 3}$, A_5 has the two 5-cycle classes $\mathrm{C}_{\pm 5}$. Similarly, as with the two B_4 orbits on $\mathrm{Ni}(A_4, \mathrm{C}_{\pm 3^2})^{\mathrm{in}}$ (§3.2.1) separated by lift invariants, there is a similar result for $(A_5, \mathbf{C}_{\pm 53^2})$.

Lemma 4.1. *There are two B_4 orbits on each of*

$$\mathrm{Ni}(A_5, \mathbf{C}_{\pm 53^2})^{\mathrm{in}}, \mathrm{Ni}(A_5, \mathbf{C}_{+5^2 3^2})^{\mathrm{in}} \textit{ and } \mathrm{Ni}(A_5, \mathbf{C}_{\pm 5^2})^{\mathrm{in}}, \tag{4.3}$$

separated by lift invariants. There is one B_4 orbit on $\mathrm{Ni}(A_5, \mathbf{C}_{3^4})^{\mathrm{in}}$.

An H-M rep. braids to $\boldsymbol{g}_0$ in (4.1) *(so, to a 6-tuple of type* (2.4a)*). Some braid takes $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ in* (4.2) *to an H-M rep. or to a juxtaposition of two conjugates to $\boldsymbol{g}_{3,-}$ (in* (3.8a)*; see Rem. 4.2).*

Proof. 4-Lemma 3.4 shows how the **sh**-incidence matrix gives two B_4 orbits on $\mathrm{Ni}(A_4, \mathrm{C}_{\pm 3^2})^{\mathrm{in}}$ (with reps. $\boldsymbol{g}_{4,\pm}$; [**BaFr02**, §2.10.1] uses $\mathrm{Ni}(A_5, \mathbf{C}_{3^4})^{\mathrm{in}}$ for similar purposes). [**BaFr02**, Prop. 5.11] showed the orbit results for (4.3).

Now consider $\boldsymbol{g}_1$. We easily braid it to

$$\boldsymbol{g}_1' = ((2\,1\,4), (1\,2\,3), (1\,2\,3), (1\,2\,3), (2\,1\,4), (2\,1\,4)).$$

Note: The first two entries, and the 4th and 5th entries have coalescing type (2.4b). Invariance Cor. 2.3 says $s(\boldsymbol{g}_1)$ is $(-1)^2 s(\boldsymbol{h}_1)$ with

$$\boldsymbol{h}_1 = ((1\,4\,3)^*, (1\,2\,3), (2\,3\,4)^*, (2\,1\,4)) \in \mathrm{Ni}(A_4, \mathrm{C}_{\pm 3^2})^{\mathrm{in}}.$$

The $*$ superscripts on $\boldsymbol{h}_1$ entries remind those positions are coalescings. Braids we now apply to $\boldsymbol{h}_1$ track the $*\,s$. Also, $(\boldsymbol{h}_1)Q_2^{-1} = ((1\,4\,3)^*, (2\,3\,4)^*, (3\,4\,1), (2\,1\,4))$.

Prop. 3.5 implies $\boldsymbol{h}_1$ braids to $\boldsymbol{g}_{4,+} = ((1\,3\,4), (1\,4\,3), (1\,2\,3), (1\,3\,2))$. Now we ask, where did the $*$s end up? If in the 1st and 2nd (resp. 3rd and 4th) positions, reexpand so $((1\,2\,3), (1\,3\,2))$ are in the 5th and 6th position; we braided $\boldsymbol{g}_1$ to have type (2.4a). If, rather, the $*$s fall one in the 1st or 2nd, the other in the 3rd or 4th, then reexpanding gives two juxtaposed (2.6a) types.

For $\boldsymbol{g}_2$, follow part of the plan for $\boldsymbol{g}_1$, braiding to

$$((1\,2\,4), (1\,2\,3), (1\,2\,3), (1\,2\,3), (1\,2\,4), (1\,2\,4)).$$

Apply $Q_1^{-1}Q_4$ to get $((1\,4\,3),(1\,2\,3))$ in the 4th and 5th entries, and $((1\,2\,3),(2\,3\,4))$ in the 1st and 2nd positions. Now use the shift to braid to where the first four positions are $((1\,2\,4),(2\,3\,4),(1\,4\,3))$. This is coalescing type (2.6a).

Now consider $\boldsymbol{g}$. Its lifting invariant is clearly one. As with $\boldsymbol{g}_1$, braid to

$$\boldsymbol{g}' = ((1\,4\,5),(1\,2\,3),(1\,2\,3),(1\,2\,3),(1\,4\,5),(1\,4\,5)).$$

The pair $((1\,4\,5),(1\,2\,3))$ has product $(1\,4\,5\,2\,3)$ and the pair $((1\,2\,3),(1\,4\,5))$ has product $(1\,2\,3\,4\,5)$. Apply the 2-orbit outcome for $\mathrm{Ni}(A_5,\mathbf{C}_{+5^23^2})$ to see that

$$\boldsymbol{h} = ((1\,4\,5\,2\,3)^*,(1\,2\,3),(1\,2\,3\,4\,5)^*,(1\,4\,5))$$

(with the *s as above) braids to $((1\,2\,3\,4\,5)^*,(5\,4\,3\,2\,1)^*,(1\,2\,3),(3\,2\,1))$, and we know where the *s must end up. So, upon their expansion we have a 6-tuple $\boldsymbol{g}'' = (\boldsymbol{g}^\dagger,(1\,2\,3),(3\,2\,1))$ of type (2.4a). Go further, still: $\boldsymbol{g}^\dagger \in \mathrm{Ni}(A_5,\mathbf{C}_{3^4})^{\mathrm{in}}$ and it has lifting invariant 1. By the first statement of the lemma, it braids to an H-M. This concludes showing $(3.1)_+ + (3.1)_+$ for $n = 5$ braids to an H-M rep. □

Remark 4.2 (Which coalescing type)**.** §4.2.2 shows $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ braid to both coalescing types (2.4a) and (2.6a), though Lem. 4.1 left this ambiguous.

Remark 4.3 (Using Braid packages)**.** [**MaSV04**] describes recent Braid package applicable to our by-hand calculation in Lem. 4.1. Our approach conceptually shows why the braidings give components separated by lifting invariants. Still, **GAP** is additional corroboration. The package documented by [**MaSV04**] does not have the **sh**-incidence matrix of [**BaFr02**] used in §3.3.2.

4.2. Strong Coalescing. Suppose $\boldsymbol{g} \in \mathrm{Ni}(A_4,\mathbf{C}_{3^r})$ with $r \ge 3$. Excluding the case where $r = 4$ and $s(\boldsymbol{g}) = -1$, 4-Lemma 3.4 gives a braid of $\boldsymbol{g}$ to $\boldsymbol{g}'$ with either (g_1',g_2') of type (2.4a) (conjugate to $((1\,2\,3),(3\,2\,1))$) or (2.6a) ((g_1',g_2',g_3') conjugate in S_4 to $((1\,2\,3),(1\,3\,4),(1\,4\,2))$.

Lem. 4.4 reduces proving Thm. 1.2 and Thm. 1.3 to cases like those of Lem. 4.1.

Lemma 4.4 (Strong Coalescing)**.** *If $n \ge 5$, $r \ge 4$, then $\boldsymbol{g} \in \mathrm{Ni}(A_n,\mathbf{C}_{3^r})$ braids to $\boldsymbol{g}'$ with (g_1',g_2') of coalescing type* (2.4a)*.*

§4.2.1 proves, for $n \ge 5$, we get either coalescing type (2.4a), or a sum of types, $T_1 + T_2$ where the T_i s are either (2.6a) or (2.6c) (see (4.5)). Then, §4.2.2 produces from this type (2.4a) and the proof of Lem. 4.4.

4.2.1. *Proof of type* (2.4a)*, or type $T_1 + T_2$ as above.* Suppose no braid of $\boldsymbol{g}$ has coalescing type either (2.4a) or (2.4b). Then Coalescing Lemma 2.2 shows each $i \in \{1,\dots,n\}$ occurs in a disappearing sequence of type $(i\,i_1\,i_2)^{(3)}$ in $(\boldsymbol{g})Q$ for every $Q \in B_r$. We show this is impossible. To simplify, take $(i\,i_1\,i_2) = (1\,2\,3)$. Then, there is $Q \in B_r$ with $(\boldsymbol{g})Q = ((1\,2\,3)^{(3)},\boldsymbol{g}^*)$. Now apply the same argument to $\boldsymbol{g}^*$, and reduce the case to one of the 6-typles $\boldsymbol{g}_0$, $\boldsymbol{g}_1$ or $\boldsymbol{g}_2$ in either (4.1) or (4.2).

Now we may assume $\boldsymbol{g}$ braids to $\boldsymbol{g}'$ with (g_1',g_2') of coalescing type (2.4b). Coalesce the first two positions of $\boldsymbol{g}'$ to $\boldsymbol{h}$ with $g_1'g_2' = h_1^*$, its remaining entries are in order from $\boldsymbol{g}'$. As in the proof of Lem. 4.1, * tracks where the coalesced entry ends up in the braiding. Suppose $\langle\boldsymbol{h}\rangle$ is transitive on a set containing at least five integers. Then, our induction assumption shows the only nontrivial case to be when the transitive set of 3-cycles includes h_1^*. With no loss, either:

(4.4a) the first two entries of $\boldsymbol{h}$ are $(h_1^*,(4\,2\,1))$ of type (2.4a); or
(4.4b) the first three entries are (h_1^*,h_2,h_3) of type (2.6a).

Expand h_1^*. For (4.4a) we can braid from $\boldsymbol{g}$ to one of type (2.6a), our desired conclusion. For (4.4b), Lem. 4.1 braids to where the 1st and 2nd (resp. 3rd and 4th) entries have type (2.4a). Now assume the orbits of $\boldsymbol{h}$ have four or three integers.

If all orbits of $\langle \boldsymbol{h} \rangle$ have cardinality 3, then restrict to one in the support of $g_1' g_2'$ (take this to be $(1\,2\,4)$ for simplicity). Exclude the previous case. Then, 3-Lemma 3.3 reduces us to $\boldsymbol{h} = ((1\,2\,4)^{(3)}, g^{(3)})$ with $g = (3\,5\,6)$. Expand $\boldsymbol{h}$ to show $\boldsymbol{g}_{4,-}$ (in (3.8b)) and $g^{(3)}$ juxtaposing. Braid $\boldsymbol{g}_{4,-}$ to get $((1\,2\,3), (1\,2\,4)^{(2)}, (3\,2\,4), g^{(3)})$. Then, coalesce $(g, (1\,2\,3))$ from the 1st and last 3-cycles. Braid the result, $(3\,5\,6\,1\,2)$, past $(1\,2\,4)^{(2)}$ to replace that by $(2\,3\,4)^2$. A braid now taking $((2\,3\,4), (3\,2\,4))$ to the 1st and 2nd position shows we can braid $\boldsymbol{g}$ to type (2.4a): our ultimate goal.

The final case reduces to where $\langle \boldsymbol{h} \rangle$ is transitive on 4 integers, and the support of (g_1', g_2') includes a 5th integer. Now apply 4-Lemma 3.4. This is similar to the case we just finished, except for one possibility: $\boldsymbol{h} = \boldsymbol{g}_{4,-}$. Cor. 2.3 shows $s(\boldsymbol{g}) = -s(\boldsymbol{h}) = +1$. In the expansion of $\boldsymbol{h}$, coalesce the last two terms from $\boldsymbol{g}_{4,-}$ to get $\boldsymbol{h}^\dagger \in \mathrm{Ni}(A_5, \mathbf{C}_{3^4})$. From Lem. 4.1 this braids to an H-M rep., so the expanded $\boldsymbol{h}^\dagger$ braids to type (2.4a) for the conclusion.

4.2.2. *Finish of producing type* (2.4a). Assume, contrary to our goal, no braid has type (2.4a). Apply §4.2.1 to braid $\boldsymbol{g}$ to have the shape $(\boldsymbol{g}_1', \boldsymbol{g}_2', \boldsymbol{g}_3')$ as follows.

(4.5a) $\boldsymbol{g}_i'$, $i = 1, 2$, is conjugate to $(1\,2\,3)^{(3)}$, $((1\,2\,3), (1\,3\,4), (1\,4\,2))$.

(4.5b) As $n \ge 5$, the Blocks Lemma allows conjugating $\boldsymbol{g}_1'$ and $\boldsymbol{g}_2'$ so each has some orbit support outside the other, though some common support.

Since $\langle (1\,2\,3), (1\,3\,4), (1\,4\,2) \rangle = A_4$, the Blocks Lemma allows anything from $\{1, 2, 3, 4\}$ to be common to all 3-cycles in $(\boldsymbol{g}_1', \boldsymbol{g}_2')$. So assume these have 1 in their support. With $\boldsymbol{g}_1'$ of type (2.4c) and $\boldsymbol{g}_2'$ of type (2.6a), with no loss, assume one of these for $(\boldsymbol{g}_1, \boldsymbol{g}_2)$:

(4.6a) $\boldsymbol{h}_1 = ((1\,2\,3)^{(3)}, (1\,4\,5), (1\,5\,6), (1\,6\,4))$ (common support 1), or

(4.6b) $\boldsymbol{h}_2 = ((1\,2\,3)^{(3)}, (1\,3\,4), (1\,4\,5), (1\,5\,3))$ (common support 1 and 3).

If both $\boldsymbol{g}_1'$ and $\boldsymbol{g}_2'$ have type (2.6a) we may assume one of the following:

(4.7a) $\boldsymbol{h}_3 = ((1\,2\,3), (1\,3\,4), (1\,4\,2), (1\,5\,6), (1\,6\,7), (1\,7\,5))$; or

(4.7b) $\boldsymbol{h}_4 = ((1\,2\,3), (1\,3\,4), (1\,4\,2), (1\,4\,5), (1\,5\,6), (1\,6\,4))$; or

(4.7c) $\boldsymbol{h}_5 = ((1\,2\,3), (1\,3\,4), (1\,4\,2), (1\,3\,4), (1\,4\,5), (1\,5\,3))$.

Both $\boldsymbol{h}_2$ and $\boldsymbol{h}_5$ have the 3-cycle subsequence $((1\,2\,3), (1\,3\,4), (1\,4\,5))$ with product $(1\,2\,5)$. Also, $\boldsymbol{h}_4$ has the subsequence $((1\,3\,4), (1\,4\,5), (1\,5\,6))$ with product $(1\,3\,6)$. Lem. 4.5 shows each braids to type (2.4a). Conclude these cases with Lem. 4.6.

Lemma 4.5. *Assume $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C}_{3^r})$ and $i < j < k$ have these properties:*

(4.8a) *$\langle g_i, g_j, g_k \rangle$ is transitive on a five integer subset from $\{1, \dots, n\}$; and*

(4.8b) *$g_i g_j g_k$ is a 3-cycle.*

Then, $\boldsymbol{g}$ braids to $\boldsymbol{g}'$ of coalescing type (2.4a).

Proof. A braid from Blocks Lem. 2.9, and a conjugation from Prod-one Lem. 2.6 allows assuming $i = 1$, $j = 2$ and $k = 3$ and the integers of transitivity in (4.8a) are $\{1, 2, 3, 4, 5\}$. Lemma 4.1 gives transitivity of B_4 on $\mathrm{Ni}(A_5, \mathbf{C}_{3^4})$. Thus, there exists $Q \in B_4$ with

$$(g_1, g_2, g_3, (g_3 g_2 g_1)^{-1})Q = ((1\,2\,3), (3\,2\,1), (1\,4\,5), (5\,4\,1)).$$

This gives $Q' \in B_r$ with $(\boldsymbol{g})Q' = ((1\,2\,3), (3\,2\,1), (1\,4\,5), g_4', \dots, g_r')$: type (2.4a). $\square$

Lemma 4.6. *The position of the 5-cycle determines a 3-typle in* $\mathrm{Ni}(A_5, \mathbf{C}_{+53^2})^{\mathrm{in}}$. *Conclude, there are braids of* $\boldsymbol{h}_1$ *and* $\boldsymbol{h}_3$ *to type* (2.4a).

PROOF. The first sentence says $\boldsymbol{g}' = ((1\,2\,3),(1\,4\,5),(5\,4\,3\,2\,1))$ has B_3 orbit $\mathrm{Ni}(A_5, \mathbf{C}_{+53^2})^{\mathrm{in}}$. Check: If $(g_1', g_2', (5\,4\,3\,2\,1)) \in \mathrm{Ni}(A_5, \mathbf{C}_{+53^2})^{\mathrm{in}}$, then conjugating by a power of $(5\,4\,3\,2\,1)$ gives $\boldsymbol{g}'$. This completes the first sentence.

The two cases $\boldsymbol{h}_1$ and $\boldsymbol{h}_3$ are similar. So we do just the harder, $\boldsymbol{h}_1$. Braid to $((1\,2\,3),(1\,2\,3),(1\,4\,5),(1\,5\,6),(1\,6\,4),(1\,2\,3))$, then coalesce the 2nd and 3rd (resp. 5th and 6th) entries to get $\boldsymbol{w} = ((1\,2\,3), w_2^*, (1\,5\,6), w_4^*)$. Cor. 2.3 shows $s(\boldsymbol{w}) = -1$. Another application of it shows $s((1\,2\,3\,4\,5),(1\,5\,6\,3\,2),(3\,6\,4)) = -1$.

According to Lem. 4.1, this implies there is a braid of $\boldsymbol{w}$ to

$$\boldsymbol{u} = ((1\,2\,3\,4\,5)^*,(1\,5\,6\,3\,2)^*,(3\,4\,6),(3\,4\,6)).$$

Reexpand $\boldsymbol{u}$ to a 6-tuple $\boldsymbol{h}_1'$. From the first sentence conclude, for some m and t, $(\boldsymbol{h}_1')Q_1^mQ_2^t$ has $((1\,2\,3),(4\,5\,6))$ (resp. $((1\,5\,6),(1\,3\,2))$) as its 1st and 2nd (resp. 3rd and 4th) entries. So, from the 1st and 4th entries, $\boldsymbol{h}_1'$ braids to type (2.4a). □

4.3. Starting induction and S_n conjugation. We prove Thms. 1.2 and 1.3 by inducting in lexicographic order on pairs (n,r), $r \ge n-1 \ge 4$. Recall: $(n,r) \ge (n',r')$ if $n > n'$ or if $n = n'$ and $r > r'$.

Strong Coalescing Lem. 4.4 braids anything in $\mathrm{Ni}(A_n, \mathbf{C}_{3^r})$ to $\boldsymbol{g} = (g_1,\dots,g_r)$ with $g_1g_2 = 1$. Rewrite $(g_3,\dots,g_r)$ as $(\boldsymbol{g}_1',\dots,\boldsymbol{g}_t')$ with this property. There are t disjoint orbits $I_1,\dots,I_t$, $t \le 3$, of $\boldsymbol{g}'$ on $\{1,\dots,n\}$. For any orbit I_j of length at least 3, $\boldsymbol{g}_j'$ consists of the 3-cycles with support in I_j. So, each $\boldsymbol{g}_j'$ generates a transitive group on I_j. It may happen I_j has length 1: $\boldsymbol{g}_j'$ would be empty, so assuring with $|I_j| = n_j$ that $\sum_j n_j = n$. Induct by assuming (on I_j) $\boldsymbol{g}_j'$ has the form (3.1), (3.2), (3.3) or (3.4). The induction relies on the nontrivial case $n = 4$.

4.3.1. (2.4a) *coalescing.* We extend $(g, g^{-1}) \mapsto 1$ in list (2.4a) of Lem. 4.4.

(4.9a) $(n,r) \mapsto (n, r-2)$, the $r-2$ elements $g_3,\dots,g_r$ remaining from coalescing (g, g^{-1}) at the beginning of $Q(\boldsymbol{g})$, are transitive.

(4.9b) $(n,r) \mapsto ((n_1,n_2), r-2)$ or $((n_1,n_2,n_3), r-2)$: $\langle g_3,\dots,g_r\rangle$ has orbits of cardinality n_1, n_2 (resp., n_1,n_2,n_3).

4.3.2. *S_n conjugation lemma.* Suppose $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C}_{3^r})$ and $\beta = (2\,3)$. If you can braid conjugation by β, then Product-one Lem. 2.6 produces S_n conjugation from braiding. For $n \ge 5$, Lem. 4.7 extends 4-Lemma 3.4.

Lemma 4.7 (S_n-Conjugation)**.** *Assume* $\boldsymbol{g} \in \mathrm{Ni}(A_n, \mathbf{C}_{3^r})$, $n \ge 5$, *has the form* $((g, g^{-1})^{(u)}, \boldsymbol{g}')$ *where* $g = (i\,j\,k)$ *and neither* j *nor* k *appear in the supports of the entries of* $\boldsymbol{g}'$. *Then, for any* $\gamma \in S_n$, *there is a braid* Q *with* $(\boldsymbol{g})Q = \gamma\boldsymbol{g}\gamma^{-1}$. *The conclusion applies to all r-tuples appearing in §3.1.1.*

PROOF. Let $\beta_0 = (j\,k)$. Since $S_n = A_n \cup \beta_0 A_n$, the Product-one Lemma gives the conclusion if we show it for β_0. By hypothesis, conjugation of $\boldsymbol{g}$ by β_0 produces $(g^{-1}, g, \boldsymbol{g}')$. Apply Q_1 to braid this back to $\boldsymbol{g}$.

For most r-tuples in §3.1.1, $\{2,3\} = \{j,k\}$ works, so long as the nub (Def. 3.1) has no support in $\{2,3\}$. These are the only exceptions.

- $((1\,2\,3),(1\,2\,3)^{-1},(1\,3\,4),(1\,4\,5),(1\,5\,3))$ from (3.1).
- $((1\,2\,3),(1\,2\,3)^{-1},(1\,3\,4)^{(2)},(1\,3\,5),(1\,5\,4))$ from (3.3).

In each case, see easily that the desired Q is the composition of Q_1 and the Blocks Lemma braid that effects conjugation by $(2\,3)(4\,5)$. □

4.4. The general induction. §3.3 treats cases $n = 3$ and 4. Now take $n \geq 5$ to complete the induction for Theorems 1.2 and 1.3. Prop. 3.5 has the initial case, $(n, r) = (5, 4)$. §4.4.1 does case (4.9a). The remaining subsections handle (4.9b).

4.4.1. *Case* (4.9a). The induction gives $Q \in B_{r-2}$ with $(g_3, \dots, g_r)Q$ a standard form for the given $r - 2$ and n. The Blocks Lemma then gives Q' with

$$(g_1, g_1^{-1}, (g_3, \dots, g_r)Q)Q' = ((g_3, \dots, g_r)Q, (1\,2\,3), (3\,2\,1)),$$

standard form for r and n.

4.4.2. *Setup for two orbit case of* (4.9b). Denote $(g_3, \dots, g_r)$ by $\boldsymbol{g}'$ and assume $\langle \boldsymbol{g}' \rangle$ has two orbits. Blocks Lem. 2.9 conjugates $\boldsymbol{g}$ by $\alpha \in A_n$ so $\alpha \boldsymbol{g}' \alpha^{-1}$ has orbits $\{1, \dots, n_1\} = O_1$ and $\{n_1 + 1, \dots, n\} = O_2$ for n_1 with $\alpha g_1 \alpha^{-1} = (1\,n_1{+}1\,n_1{+}2)$.

Lemma 4.8 (Orbits)**.** *We may assume the following.*

(4.10a) $(g_3, \dots, g_s)$ *has support in* O_1*, with one integer in common to the support of* g_1 *(*$\Pi(g_3, \dots, g_s) = 1$*).*

(4.10b) $(g_{s+1}, \dots, g_r)$ *has support in* O_2 *(*$\Pi(g_{s+1}, \dots, g_r) = 1$*) with two integers in common to the support of* g_1*.*

Proof. Let O_1 and O_2 be the orbits of $\langle \boldsymbol{g}' \rangle$. As the supports of g_i, $i \geq 3$, are in either O_1 or O_2, we may braid the former to the left of the latter. This gives $(g_3, \dots, g_s)$ and $(g_{s+1}, \dots, g_r)$ as in the lemma statement. Two integers in the support of g_1 lie in one orbit, one in the other.

Conjugate $\boldsymbol{g}$ by some $\beta' \in S_n$ to get $\boldsymbol{g}'$ satisfying the lemma's conclusion. If β' is in A_n, we are done. If not, assume some 2-cycle γ' fixing the support of g_1 switches two integers either in O_1 or in O_2. Then take the product of β' and γ' to finish the proof. We chose O_1 to have only one integer in common with the support of g_1. As O_1 contains at least two other integers, choose γ' to switch these. □

Now apply induction to $(g_3, \dots, g_s) = \bar{\boldsymbol{g}}_1$ and $(g_1, g_1^{-1}, g_{s+1}, \dots, g_r) = \bar{\boldsymbol{g}}_2$ coming from Orbits Lem. 4.8. Put each in a normal form from §3.1.1. This requires special attention to the case $|O_i| = 3$ or 4: 3-Lemma 3.3 allows braiding the endings (as in §3) appropriately. So assume $\boldsymbol{g} = (\bar{\boldsymbol{g}}_1, \bar{\boldsymbol{g}}_2)$ with $\bar{\boldsymbol{g}}_2$ in a normal form with $\{1, n_1 + 1, \dots, n\}$ replacing $\{1, 2, \dots, n{-}n_1{+}1\}$. As in §3, divide $\bar{\boldsymbol{g}}_2 = (\boldsymbol{g}_{2,b}, \boldsymbol{g}_{2,e})$ into beginning and end parts: $\bar{\boldsymbol{g}}_{2,e}$ is $((1\,n_1{+}1\,n_1{+}2), (1\,n_1{+}2\,n_1{+}1))^{(t)}$ for some t. The next lemma braids to replace $\boldsymbol{g}_{2,e}$ by $((1\,2\,3), (3\,2\,1))^{(t)}$.

Lemma 4.9 (Tail Lemma)**.** *Assume* $n \geq 5$*. Let* $\boldsymbol{h}_1$ *have 3-cycle entries transitive on* $\{1, \dots, n_1\}$ *and* $\Pi(\boldsymbol{h}_1) = 1$*. Similarly,* $\boldsymbol{h}_2$ *has 3-cycle entries transitive on* $\{1, n_1{+}1, \dots, n\}$ *and* $\Pi(\boldsymbol{h}_2) = 1$*. Then, there is a* $Q \in B_r$ *with*

$$(\boldsymbol{h}_1, \boldsymbol{h}_2, (1\,n_1{+}1\,n_1{+}2), (1\,n_1{+}2\,n_1{+}1))Q = (\boldsymbol{h}_1, \boldsymbol{h}_2, (1\,2\,3), (3\,2\,1)).$$

Proof. This is an application of the Blocks Lemma. Since $\langle \boldsymbol{h}_1, \boldsymbol{h}_2 \rangle = A_n$, this group contains γ conjugating $(1\,n_1{+}1\,n_1{+}2)$ to $(1\,2\,3)$. The Blocks Lemma says you can achieve this conjugation on the block $((1\,n_1{+}1\,n_1{+}2), (1\,n_1{+}2\,n_1{+}1))$ by an element of B_r leaving the other blocks fixed. □

Apply the Tail Lemma to assume $\boldsymbol{g}$ is $(\bar{\boldsymbol{g}}_1, \bar{\boldsymbol{g}}_2, ((1\,2\,3), (3\,2\,1))^{(t)})$ with $\bar{\boldsymbol{g}}_{1,e}$ and $\bar{\boldsymbol{g}}_{2,e}$ empty and $\bar{\boldsymbol{g}}_1$ (resp. $\bar{\boldsymbol{g}}_2$) a §3.1.1 normal form on its orbit O_1 (resp. $O_2 \cup \{1\}$). This gives the following principle, using the nub (Def. 3.1) of a normal form.

Principle 4.10. *Let $\bar{\boldsymbol{g}}_{i,\mathrm{nb}}$ be the nub of $\boldsymbol{g}_i$, $i=1,2$. If we can braid $(\bar{\boldsymbol{g}}_{1,\mathrm{nb}},\bar{\boldsymbol{g}}_{2,\mathrm{nb}})$ to one of the normal form nubs juxtaposed with an H-M rep. — call that a stable nub — then we are done.*

So, to complete the proofs of Thm. 1.2 and 1.3 requires two things.

(4.11a) Listing juxtapositions $(\bar{\boldsymbol{g}}_{1,\mathrm{nb}},\bar{\boldsymbol{g}}_{2,\mathrm{nb}})$ of normal form nubs.
(4.11b) Braiding each of the tuples in list (4.11a) to a stable nub.

4.4.3. *List 4.4.A: Repeats from list 3.1.1.* We comment on our naming conventions when both $(\bar{\boldsymbol{g}}_{1,\mathrm{nb}},\bar{\boldsymbol{g}}_{2,\mathrm{nb}})$ have the same type. If $n\ge 5$, then $(3.1)_+$ falls outside (4.11a). Still, we must consider $|O_1|=|O_2\cup\{1\}|=3$. This does fit $(3.1)_+$, though with $n=3$. So we use that to label this case in $(3.1)_+$+$(3.1)_+$ below. Also, $(3.2)_\pm$+$(3.2)_\pm$ give the same entries as $(3.1)_\pm$+$(3.1)_\pm$, so we leave them out.

With a natural renaming of integers, here are those $\bar{\boldsymbol{g}}_1$ and $\bar{\boldsymbol{g}}_2$ with both from the same place in the list §3.1.1. Case $(3.1)_-$+$(3.1)_-$ looks odd, using $n=4$ (even) for simplicity, though theoretically we only allowed n odd. Finally, in $(3.4)_\pm$+$(3.4)_\pm$, we don't use the nub (which has 4 entries), but rather a stable nub, and then we substitute the braid of Lem. 4.6 for that.

$(3.1)_+$**+**$(3.1)_+$**:** $((1\,2\,3)^{(3)},(1\,4\,5)^{(3)})$
$(3.1)_-$**+**$(3.1)_-$**:** $((1\,2\,3),(1\,3\,4),(1\,4\,2),(1\,5\,6),(1\,6\,7),(1\,7\,5))$
$(3.3)_-$**+**$(3.3)_-$**:** $((1\,2\,3)^{(3)},(1\,3\,4),(1\,4\,5),(1\,5\,4),(1\,6\,7)^{(3)},$
$(1\,7\,8),(1\,8\,9),(1\,9\,7))$
$(3.4)_+$**+**$(3.4)_+$**:** $((1\,2\,3),(1\,3\,2),(1\,3\,4),(1\,4\,3),(1\,5\,6),(1\,6\,5),(1\,6\,7),(1\,7\,6))$
$(3.4)_-$**+**$(3.4)_-$**:** $((1\,2\,3)^{(3)},(1\,4\,5),(1\,5\,6),(1\,6\,4),(1\,7\,8)^{(3)},$
$(1\,9\,10),(1\,10\,11),(1\,11\,10))$

Lemma 4.11. *Each juxtaposition in the list above braids to a stable nub.*

Proof. The Blocks Lemma braids each of $(3.3)_-$+$(3.3)_-$ and $(3.4)_\pm$+$(3.4)_\pm$ to a juxtaposing of types $(3.1)_\pm$+$(3.1)_\pm$. Lems. 4.5 and 4.6 braids these respectively to H-M reps. We are done. □

4.4.4. *List 4.4.B: Distinct pairs from §3.1.1 and concluding the proof.* Use the same principles as in §4.4.3. Example: $(3.1)_+$+$(3.1)_-$ has the shape

$$((1\,2\,3)^{(3)},(1\,4\,5),(1\,5\,6),(1\,6\,4)).$$

Lem. 4.6 braids this to a stable nub from (3.4).

The situation from $(3.1)_+$+$(3.4)_-$ is slightly different:

$$((1\,2\,3)^{(3)},(1\,4\,5)^{(3)},(1\,5\,6),(1\,6\,7),(1\,7\,5)).$$

The juxtaposition $((1\,2\,3)^{(3)},(1\,4\,5)^{(3)})$ from §4.4.3 braids to an H-M rep., so we are done. In a like manner, we find there are no serious new cases.

Finally, the Orbits Lemma has a simple variant when $\boldsymbol{g}'=(g_3,\dots,g_r)$ has three orbits. This supports the arguments following it to produce the same kind of lists.

5. Applications to $G_{\mathbb{Q}}$

Our applications of Theorems 1.2 and 1.3 – §5.1 and §5.2 – are to the Inverse Galois Problem. We use the $\frac{1}{2}$-canonical spaces $\mathcal{M}_{g,\pm}$ of §6.1.3 in §5.2. Given a field K, $\bar{K}$ indicates an algebraic closure, and G_K its absolute Galois group.

5.1. (A_n, A_n) **and** (A_n, S_n) **realizations.** § A.1 explains $(G, \hat{G})$ regular realization: (A_n, A_n) and (A_n, S_n) realizations are a special case. The *no centralizer condition* holds for the standard representation of A_n. So Prop. A.7 says K points on corresponding Hurwitz spaces correspond to finding K realizations by covers in the Nielsen class $\mathrm{Ni}(A_n, \mathbf{C}_{3^r})$. Thm. 1.2 shows the spin lifting invariant (1.1) determines the components, so [**Fr95b**, Thm. 3.16] gives part one of Cor. 5.1. The short proof for part two is from Hilbert's Irreducibility Theorem (HIT).

Corollary 5.1. *For $n \geq 5$, each component of $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\mathrm{in}}$ or $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\mathrm{abs}}$ (with its map to U_r) has definition field $\mathbb{Q}$. Further, let $\mathcal{H}^*$ be a component of $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\mathrm{abs}}$. Then, for a dense set of $\boldsymbol{p} \in \mathcal{H}^*(\bar{\mathbb{Q}})$, the corresponding cover gives an (A_n, S_n) realization over $\mathbb{Q}(\boldsymbol{p})$.*

Proof. We need only show the last sentence. Let $\mathcal{H}^{**}$ be the (unique according to Thm. 1.2) component of $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\mathrm{in}}$ lying over $\mathcal{H}^*$. So $\mathcal{H}^{**} \to \mathcal{H}^* \to U_r$ is a sequence of absolutely irreducible covers over $\mathbb{Q}$, with $\mathcal{H}^{**} \to \mathcal{H}^*$ Galois with group $\mathbb{Z}/2$. (The cover $\mathcal{H}^* \to U_r$ is far from Galois.) According to HIT (say, [**FJ86**, Chap. 10], or §5.2), there is a dense set $\boldsymbol{z} \in U_r(\mathbb{Q})$ so that for any point $\hat{\boldsymbol{p}} \in \mathcal{H}^{**}$ over $\boldsymbol{z}$, $[\mathbb{Q}(\hat{\boldsymbol{p}}) : \mathbb{Q}]$ is the degree of $\mathcal{H}^{**}/U_r$. Conclude: If $\boldsymbol{p}$ is the image of $\hat{\boldsymbol{p}}$ in $\mathcal{H}^*$, then $[\mathbb{Q}(\hat{\boldsymbol{p}}) : \mathbb{Q}(\boldsymbol{p})] = 2$, precisely what we need for an (A_n, S_n) realization. □

When $r = n-1$, [**Mes90**] shows there are an infinity of (A_n, A_n) realizations from $\mathbb{Q}$ points in $\mathcal{H}(A_n, \mathbf{C}_{3^{n-1}})$ whose images are dense in U_{n-1}. Thm. 1.2 shows there is only one component of this moduli space. Conclude the following.

Corollary 5.2. *When $r = n-1$, $\mathbb{Q}$ points giving (A_n, A_n) realizations are analytically dense in $\mathcal{H}(A_n, \mathbf{C}_{3^{n-1}})$.*

Remark 5.3 (Comments on Cors. 5.1 and 5.2)**.** There is more detail on these corollaries in [**FK97**, p. 163–167] to help a reader use the Hurwitz space interpretation. This includes the effect of special assumptions on Mestre's parametrization.

5.2. The maximal alternating extension of $\mathbb{Q}$**.** We say K is *Hilbertian* if it satisfies *HIT*: Any irreducible $f \in K[x, y]$ of degree at least 2 in y remains irreducible (over K) for ∞-ly many specializations of $x \mapsto x_0 \in K$.

Call K *projective* if G_K is projective: Surjective homomorphisms to G_K split. §5.2.1 puts a particular field $\mathbb{Q}^{\mathrm{alt}}$ in the context of [**FV92**].

5.2.1. *Hilbertian+Projective $\implies$ Pro-free Conjecture.* Pro-free groups are projective, though most projective (even finitely generated) groups are not pro-free (see Ex. 5.6). All results from [**FJ86**] are in both editions (we use the 2nd).

Let $I \leq \mathbb{N}^+$ be any infinite set of integers. Define $\mathbb{Q}^{\mathrm{alt},I}$ to be the composite of all Galois extensions $L/\mathbb{Q}$ with group A_n for some integer $n \in I$. [**FV92**, p. 476] asks if $\mathbb{Q}^{\mathrm{alt},\mathbb{N}^+} \stackrel{\mathrm{def}}{=} \mathbb{Q}^{\mathrm{alt}}$ is P(seudo)A(lgebraically)C(losed). From [**FJ86**, Thm. 10.4] this is equivalent to every absolutely irreducible curve over $\mathbb{Q}$ having a rational point in $\mathbb{Q}^{\mathrm{alt}}$. You can as well ask if any of the $\mathbb{Q}^{\mathrm{alt},I}$ are PAC.

Recall: $\tilde{F}_\omega$ is the profree group on countably many generators.

Theorem 5.4. *If $\mathbb{Q}^{\mathrm{alt},I}$ is PAC, then there is a natural short exact sequence*

$$1 \to \tilde{F}_\omega \to G_{\mathbb{Q}} \to G(\mathbb{Q}^{\mathrm{alt},I}/\mathbb{Q}) \to 1. \tag{5.1}$$

Proof. Apply [**FV92**, Thm. A]: A PAC, Hilbertian subfield of $\bar{\mathbb{Q}}$ has pro-free absolute Galois group. As $G(\mathbb{Q}^{\mathrm{alt},I}/\mathbb{Q})$ is the product over $n \in I$ of an infinite number of A_n s, it is a nontrivial finite extension of a Galois extension of $\mathbb{Q}$. It is automatic from [**FJ86**, Prop. 13.9.4] that such a field must be Hilbertian. □

[**FV92**] presents $G_{\mathbb{Q}}$ —as (5.1) would —by known groups: Products of S_n s (instead of A_n s) on the right, $\tilde{F}_\omega$ on the left. It does this by producing PAC fields $\mathbb{Q}^{\mathrm{FV}}$ that are a composite of disjoint S_n extensions.

Conjecture 5.5. [**FV92**] conjectures for $K \le \bar{\mathbb{Q}}$ that Hilbertian + Projective $\implies G_K$ is profree.

Example 5.6 (Projective, but not pro-free)**.** For G a finite group, consider its minimal projective (*universal Frattini*) cover $\tilde{\varphi} : \tilde{G} \to G$. It has pronilpotent kernel that is a direct product of pro-free pro-p groups (one for each prime p dividing $|G|$) [**FJ86**, §22.11]. Write $\tilde{G}$ as the fiber product of group covers ${}_p\tilde{G} \to G$, with pro-p kernel, $p||G|$.

Assume G is not cyclic, so neither is $\tilde{G}$. Suppose at least two primes divide $|G|$. Then, if $\tilde{G}$ is pro-free it has rank at least 2. Yet, it has a finite index subgroup that is a *product* of two nontrivial proper closed subgroups. Therefore that subgroup is not pro-free, contradicting Schreier's theorem [**FJ86**, Prop. 17.6.2].

5.2.2. *Cohomological observation on* $\mathbb{Q}^{\mathrm{alt}}$ *being projective.* For (5.1) to hold, $\mathbb{Q}^{\mathrm{alt},I}$ must be projective. [**Ser97**, Cor. p. 81] says any totally imaginary $K/\mathbb{Q}$, with each of its non-archimedian completions of infinite decomposition degree over the completion of $\mathbb{Q}$, must be projective. To see projectivity of G_K, consider each finite extension $L/\mathbb{Q}$ and any Brauer-Severi variety X over L. Inflating the corresponding Brauer group element from L to $L \cdot K$ produces (torsion) generators of the Brauer group of all finite extensions of K. From Class Field Theory, you determine $[X]$ from its $\mathbb{Q}/\mathbb{Z}$ values at the completions of L. Inflating $[X]$ from each completion $\tilde{L}$ of L to $K \cdot \tilde{L}$ kills $[X]$ (multiplication by the degree gives the inflation). So, the inflation of $[X]$ to $K \cdot L$ is trivial $\Leftrightarrow$ $[X]$ has a $K \cdot L$ point. This statement on *certain* varieties having rational points for each such L is is equivalent to projectivity. It is, however, much weaker than K is PAC.

This projectivity criterion should work to show $\mathbb{Q}^{\mathrm{alt}}$ is projective, because there are many known Galois extensions of $\mathbb{Q}$ with group an alternating group. Yet, [**FV92**] skipped showing projectivity for the fields I labeled above as $\mathbb{Q}^{\mathrm{FV}}$. Rather, it directly used [**FJ78**] to give each $\mathbb{Q}$ curve a simple-branched $\mathbb{Q}$ covering of $\mathbb{P}^1_z$.

Proposition 5.7. *This alternating group analog implies* $\mathbb{Q}^{\mathrm{alt},I}$ *is PAC:*

(5.2) *Each projective nonsingular curve* $X/\mathbb{Q}$ *appears in a cover* $\varphi : X \to \mathbb{P}^1_z$ *that gives an* (A_n, A_n) *realization over* $\mathbb{Q}$ *for some* $n \in I$.

[**Ne82**] has a well-known result: Two number fields with isomorphic absolute Galois group are conjugate. This result uses class field theory to conclude from valuation theory that abelian extensions determine the correspondence between primes. Let K be a number field, and denote the composite of all Galois extensions of K with group a Frattini cover of an alternating group extension by $\tilde{K}^{\mathrm{alt}}$.

Question 5.8. Does $G(\tilde{K}^{\mathrm{alt}}/K)$ determine K up to conjugacy.

5.2.3. *Restricting condition* (5.2) *to odd order branching.* Restrict to $g \ge 2$. [**Mu76**, p. 36] discusses that $\mathcal{M}_g$ is not *unirational* (that is, the image of a map from some projective space) for large g. Yet, a unirationity conclusion holds: "$[\mathcal{M}_g]$ has lots of rational curves:"copies of affine subsets of $\mathbb{P}^1$. These come from any algebraic surface Z and a meromorphic function $f : Z \to \mathbb{P}^1_z$ with general fiber having genus g. So, possibly $\mathcal{M}_g$ still has sufficiently many rational curves.

Question 5.9. Let $U \to W$ by any irreducible $\mathbb{Q}$ cover (finite and flat; so surjective [**Mu66**, Chap. 2 §7, Prop. 4]) with W open in $\mathcal{M}_g$. Is there a $\mathbb{Q}$ rational curve $X \subset W$ where restriction of U over X remains irreducible?

We don't know if (5.2) is true. Prop. 5.11 shows there are curves over $\mathbb{Q}$ for with no (A_n, A_n) or an (A_n, S_n) realization (over $\mathbb{Q}$), for any n, from odd order branched covers (as in §6.1). The proof would simplify if Quest. 5.9 had a yes answer; even restricting U to be one of the covers $\mathcal{M}_{g,\pm}$ (Prop. 6.4).

Lemma 5.10. *Assume $V \subset \mathcal{M}_g$ is a $\mathbb{Q}$ subvariety of dimension at least 1 satisfying the following conditions.*

- (5.3a) *There is a generically surjective $\mathbb{Q}$ morphism $W \to V$ with the function field $\mathbb{Q}(V)$ of V algebraically closed in $\mathbb{Q}(W)$.*
- (5.3b) *W is birational to an open subset of projective space $\mathbb{P}^N$ (for some N).*
- (5.3c) *Restricting $\mathcal{M}_{g,\pm}$ to V has no $\mathbb{Q}$ components of degree 1 over V.*

Conclude: A set of curves $X/\mathbb{Q}$ of genus g, corresponding to $\mathbb{Q}$ points dense in $V(\mathbb{Q})$, have no (A_n, A_n) or (A_n, S_n) realizations (over $\mathbb{Q}$) with odd order branching.

Proof. Suppose X, over $\mathbb{Q}$, of genus g, has $\varphi : X \to \mathbb{P}^1_z$, over $\mathbb{Q}$, with odd order branching. Lem. 6.2 gives a $\mathbb{Q}$ $\frac{1}{2}$-canonical class on X. To conclude we find a dense set of $v \in V(\mathbb{Q})$ so the corresponding X_v s have no $\mathbb{Q}$ $\frac{1}{2}$-canonical class.

Prop. 6.4 says the connected spaces $\mathcal{M}_{g,\pm}$ have respective degrees $2^{2g-1} \pm 2^{g-1}$ over $\mathcal{M}_g$. Denote by $V^{\pm}$ the restriction of each of these covers over V.

By condition (5.3a), over a Zariski open subset $V^* \le V$ the pullback map $\mathrm{pr}_W : V^{\pm} \times_V W \to W$ is a cover whose $\mathbb{Q}$ irreducible components W' have the same degrees over W as the corresponding components of the covers $V^{\pm} \to V$. Further, from (5.3c), none of those degrees is 1.

From (5.3b), we can apply Hilbert's Irreducibility Theorem (as in Cor. 5.1) to the collection of covers $\mathrm{pr}_W : W' \to W$, above. So, there is a dense set of $\boldsymbol{p}^* \in W(\mathbb{Q})$ with no $\mathbb{Q}$ point above them in *any* of the W' s. Conclude: The image $\boldsymbol{p} \in V(\mathbb{Q})$ of $\boldsymbol{p}^*$ gives a curve over $\mathbb{Q}$ with no $\frac{1}{2}$-canonical class over $\mathbb{Q}$. This is contrary to the above, finishing the proof of the lemma. □

We apply Lem. 5.10 to the hyperelliptic locus $\mathbf{Hyp}_g$: genus g curves with a degree 2 map to $\mathbb{P}^1$. For $g \ge 2$, a hyperelliptic curve is determined by the branch points of its canonical map to $\mathbb{P}^1_z$, up to the action of $\mathrm{PGL}_2(\mathbb{C})$ (Möbius transformations) on these unordered branch points. See §A.2.2 for the $\mathrm{PGL}_2(\mathbb{C})$ action. So, with U_r as in §1.1.2, $\mathbf{Hyp}_g$ is $U_r/\mathrm{PGL}_2(\mathbb{C})$ where $g = r/2 - 1$.

Proposition 5.11. *For each even $g \ge 2$, the space $\mathbf{Hyp}_g = V$ satisfies the hypotheses in* (5.3). *Dense in $\mathbf{Hyp}_g$ is a set of $\mathbb{Q}$ curves X where X has no presentation as a $\mathbb{Q}$ cover $\varphi : X \to \mathbb{P}^1_z$ with odd order branching. Thus, such a curve has no odd order branched cover over $\mathbb{Q}$ fulfilling* (5.2).

Proof. The components $\mathbf{Hyp}_{g,\pm}$ from restricting $\mathcal{M}_{g,\pm}$ over $\mathbf{Hyp}_g$ correspond to the orbits of the fundamental group of $\mathbf{Hyp}_g$ restricted to the action on $\frac{1}{2}$-canonical classes. Prop. 6.19 reveals this action by hand. Only when g is odd is there in this restriction a component of degree 1.

Consider the $\mathbb{Q}$ map $\mu_r : U_r \to U_r/\mathrm{PGL}_2(\mathbb{C})$, $r \ge 4$. From [**BaFr02**, Prop. 6.10], over a Zariski open subset $U' \le U_r/\mathrm{PGL}_2(\mathbb{C})$ the fibers of μ_r consist of copies of $\mathbb{P}^3$, a variety with pure transcendental function field. This gives the pullback maps $\mathbf{Hyp}_{g,\pm} \times_{U_r/\mathrm{PGL}_2(\mathbb{C})} U_r \to U_r$ the following property. Over a Zariski open subset $U^* \le U_r$, the absolutely irreducible component covers of

$$\mathbf{Hyp}_{g,\pm} \times_{U_r/\mathrm{PGL}_2(\mathbb{C})} U^* \stackrel{\mathrm{def}}{=} \mathbf{Hyp}^*_{g,\pm} \to U^*$$

have respective degrees listed in Prop. 6.19. In all cases, the covering degrees are distinct, each component is over $\mathbb{Q}$, and when the genus is even the degrees are at least r. Then the hypotheses of Lem. 5.10 apply and for a dense set of hyperelliptic curves over $\mathbb{Q}$, the proposition conclusion holds. □

6. $\frac{1}{2}$-canonical divisors and θ-functions

§6.1 explains how the irreducible components $\mathcal{H}_\pm(A_n, \mathbf{C}_{3^r})^{*,\mathrm{rd}}$, "$* =$ abs" or "in," support $\frac{1}{2}$-canonical classes, and how these then support the analytic continuation of close-to-canonical θ-functions. The difference between the two cases $\pm$: When r is even (resp. odd) the θ s for + are even (resp. odd), for - odd (resp. even) in the θ variables. §6.2.2 then discusses the even natural θ-nulls on the appropriate components. The key issue is that these be non-zero. At present we can only prove this for a given value of $g = r-(n-1) \ge 1$ for infinitely many (r, n). §6.3 computes components of $\mathcal{M}_\pm$ over the hyperelliptic locus.

6.1. Well-defined $\frac{1}{2}$-canonical classes. Let $\Phi : \mathcal{T} \to \mathcal{H} \times \mathbb{P}^1_z$ be a family of covers with odd order branching. That is, for $\boldsymbol{p} \in \mathcal{H}$, Φ restricts over the fiber $X_{\boldsymbol{p}}$ of $\mathcal{T} \to \mathcal{H}$ over $\boldsymbol{p}$ to $\varphi_{\boldsymbol{p}} : X_{\boldsymbol{p}} \to \boldsymbol{p} \times \mathbb{P}^1_z$ with odd order branching. Lem. 6.1 shows this defines a unique $\frac{1}{2}$-canonical divisor (from Φ) on $X_{\boldsymbol{p}}$. Then, Lem. 6.2 says, if Φ is a Hurwitz family, its *reduced* Hurwitz family (§A.3) supports a $\frac{1}{2}$-canonical divisor class, and so a well-defined θ divisor $\Theta_{\boldsymbol{p}}$ at each $\boldsymbol{p} \in \mathcal{H}$. Then, for a fixed $\boldsymbol{p}_0$, Prop. 6.6 gives an expression for the effect on a θ function attached to $\Theta_{\boldsymbol{p}_0}$ after it has been analytically continued around a closed path based at $\boldsymbol{p}_0$.

6.1.1. *Using differentials.* Consider $\varphi : X \to \mathbb{P}^1_z$ branched over $\boldsymbol{z} = \{z_1, \dots, z_r\}$ and having $\boldsymbol{g} = (g_1, \dots, g_r) \in G^r$ as branch cycles.

On X there is a divisor class κ that is completely canonical, being the divisor class of all global meromorphic differentials on X.

(6.1a) Any automorphism of X, in its extension to the collection of degree $2g-2$ divisor classes on X, fixes κ.

(6.1b) If X is a fiber in a family $\mathcal{X} \to \mathcal{P}$, then coordinates for $\mathcal{P}$ suffice as coordinates locally describing $\kappa_{\boldsymbol{p}}$ for $\boldsymbol{p} \in \mathcal{P}$.

Still, there is not usually a way to pick one representative *divisor* for κ explicitly. This is one difference between a general family of curves and a family of $\mathbb{P}^1_z$ covers. A member $\varphi_{\boldsymbol{p}} : X_{\boldsymbol{p}} \to \mathbb{P}^1_z$ does give such a canonical class divisor as the divisor of the differential $d\varphi_{\boldsymbol{p}}$. We accept that as part of the given data. Also, when all the branch cycles $\boldsymbol{g}$ (§1.1.2) have odd order —$\varphi_{\boldsymbol{p}}$ has *odd order* branching, this canonically produces a $\frac{1}{2}$-canonical divisor. Here is how.

In a neighborhood N_{x_0} of $x_0 \in X$, there is a one-one function $x : N_{x_0} \to \mathbb{P}^1_{x^*}$ and an integer e so the following holds. With x^*_0 the image of x_0 under x, φ composed with x^{-1} (functional inverse) looks like $x^* \mapsto (x^* - x^*_0)^e + z_0 = z$ or $1/(x^*)^e$ (corresponding to $z_0 = \infty$) in the image of N_{x_0} under x. Here e is the ramification index of x_0 over z_0: the length of a corresponding disjoint cycle in g_i if $z_0 = z_i$. Thus, dz has order $e-1$ (resp. $e+1$) at x_0 between the two cases $z_0 \in \mathbb{C}$ (resp. $z_0 = \infty$). So, each e being odd, implies $e \pm 1$ is even.

Lemma 6.1. *If $\varphi = \varphi_{\boldsymbol{p}}$ has odd order branching, then the divisor $(d\varphi)$ of the meromorphic differential $d\varphi$ is $2D_\varphi$ with D_φ a well-defined divisor on X. If φ has definition field a perfect field K, then D_φ does, too, and so does its divisor class.*

The divisor D_φ in Lem. 6.1 is a well-defined $\frac{1}{2}$-canonical divisor. Any divisor *class* ι on X with $2 \cdot \iota = \kappa$ is called a $\frac{1}{2}$-canonical (divisor) class. There are 2^{2g} of these, differing pairwise by some 2-division point on the Jacobian $\mathrm{Pic}^{(0)}(X)$ (identified with divisor classes of degree 0 by Abel's Theorem, §6.1.4).

Denote the set of $\frac{1}{2}$-canonical classes by $S_{\kappa/2}(X)$. They canonically sit in $\mathrm{Pic}^{(g-1)}(X)$, the degree $g-1$ divisor classes. [**Ser90b**] quotes [**A71**] and [**Mu71**] for basics on $S_{\kappa/2}(X)$. Closest to our start is [**Fay73**], for that works with moduli spaces of curves as do we. Still, we switch to [**Sh98**] of necessity, for the production of automorphic functions, for that works with global moduli as do we, though our spaces are reduced Hurwitz spaces, not (say) Siegel upper half-spaces.

There are two types of $y \in S_{\kappa/2}(X)$. Assume the divisor D represents y. Let $L(D)$ be the linear system of meromorphic functions f on X satisfying $(f)+D \geq 0$. Call y *even* (resp. *odd*) if $\dim_{\mathbb{C}} L(D) = \dim(y)$ is even (resp. odd).

6.1.2. *Odd order branching and reduced equivalence.* Continue with the family Φ of §6.1.1. Assume some Nielsen class $\mathrm{Ni}(G, \mathbf{C})$ with odd order conjugacy classes defines Φ. Then Lem. 6.1 smoothly assigns a well defined $\frac{1}{2}$-canonical divisor on the Riemann surfaces attached to points of $\mathcal{H}(G, \mathbf{C})^{\mathrm{abs}}$ (or $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$).

Such representing divisors, however, disappear if we use *reduced* equivalence of covers (§A.3; equivalencing by a $\mathrm{PGL}_2(\mathbb{C})$ action). Still, even the reduced Hurwitz spaces $\mathcal{H}(G, \mathbf{C})^{\mathrm{abs,rd}}$ (and $\mathcal{H}(G, \mathbf{C})^{\mathrm{in,rd}}$) carry well defined $\frac{1}{2}$-canonical classes.

Lemma 6.2. *Assume $\mathcal{H}$ parametrizes a family of covers in a reduced Nielsen class* $\mathrm{Ni}(G, \mathbf{C})^{*,\mathrm{rd}}$ *(as above) with odd order classes. This canonically defines* $\boldsymbol{p} \in \mathcal{H} \mapsto$ *a* $\frac{1}{2}$-canonical *class on the associated curve $X_{\boldsymbol{p}}$.*

Proof. Lem. 6.1 produces a unique $\frac{1}{2}$-canonical divisor D_φ on $\varphi : X \to \mathbb{P}^1_z$ representing the Nielsen class before it is reduced. We have only to show the divisor attached to $\alpha \circ \varphi$ for $\alpha \in \mathrm{PGL}_2(\mathbb{C})$ is linearly equivalent to D_φ.

Replacing φ by $\alpha \circ \varphi$ produces $d(\alpha \circ \varphi)$ as the differential. With no loss assume an element in $\mathrm{SL}_2(\mathbb{C})$ represents α. If $\alpha(z) = \frac{az+b}{cz+d}$, then

$$d(\alpha \circ \varphi) = d\varphi/(c\varphi + d)^2.$$

Thus, this has the same divisor as does $d\varphi$ with the subtraction of two times the divisor of the function $c\varphi + d$. Therefore, the $\frac{1}{2}$-canonical class is well-defined. □

Example 6.3 (non-odd order branching and $\frac{1}{2}$-canonical classes)**.** For $g > 1$ a $\frac{1}{2}$-canonical class, as in Lem. 6.2, on the family of a Nielsen class might be rare, unless the branch cycles have odd order. Still, Prop. 6.19 shows for $\mathrm{Ni}(\mathbb{Z}/2, \mathbf{C}_{2^{2s}})$ with $g = \frac{2s-2}{2}$ odd, and $s \geq 2$, there is a globally defined $\frac{1}{2}$-canonical class on the Hurwitz spaces. If, however, $s \geq 4$, this class gives a degenerate θ (Rem. 6.20).

6.1.3. $\frac{1}{2}$-canonical *classes in families of covers.* Consider any smooth family $\Psi : \mathcal{T} \to \mathcal{H}$ of genus $g \geq 1$ curves. There is a natural map $\Psi_{\mathcal{H},\mathcal{M}_g} : \mathcal{H} \to \mathcal{M}_g$, by $\boldsymbol{p} \mapsto [\mathcal{T}_{\boldsymbol{p}}]$. More generally, we can define $\Psi_{\mathcal{H},\mathcal{M}_g}$ if $\mathcal{H}$ is just a *stack* of compact Riemann surfaces in the Grothendieck topology of finite covers (see Rem. 6.8).

Also, for any integer k, there is a canonical family $\Psi_k : \mathcal{P}^{(k)} \to \mathcal{H}$ with the fiber $\mathcal{P}^{(k)}_{\boldsymbol{p}}$ over $\boldsymbol{p}$ the variety $\text{Pic}^{(k)}(\mathcal{T}_{\boldsymbol{p}})$ of degree k divisor classes on $\mathcal{T}_{\boldsymbol{p}}$. For $\mathcal{H} = \mathcal{M}_g$ (the moduli space of projective non-singular curves of genus g as in §1.1.1), this defines a cover $\mathcal{M}_{g,\pm} \to \mathcal{M}_g$, with the fiber of $\mathcal{M}_{g,\pm}$ over $m \in \mathcal{M}_g$ consisting of the 2^{2g} points $S_{\kappa/2}(X_m) \subset \mathcal{P}^{(g-1)}$ (§6.1.1).

This has disjoint irreducible components $\mathcal{M}_{g,+}$ and $\mathcal{M}_{g,-}$. Prop. 6.4 distinguishes, on the fiber product $\mathcal{H} \times_{\mathcal{M}_g} \mathcal{M}_{g,\pm} \stackrel{\text{def}}{=} \mathcal{H}_{\pm}$, the points on the two components. We introduce the divisors Θ on $\text{Pic}^{(0)}(\mathcal{T}_{\boldsymbol{p}})$. These pull back to its universal covering space, $\widetilde{\text{Pic}^{(0)}}(\mathcal{T}_{\boldsymbol{p}})$, where Riemann's θ functions live.

Proposition 6.4. *Each $\boldsymbol{p}^+ \in \mathcal{H}_+$ ($\boldsymbol{p}^- \in \mathcal{H}_-$) lying over $\boldsymbol{p} \in \mathcal{H}$ corresponds (uniquely) to the divisor $\Theta_{\boldsymbol{p}^+}$ (resp. $\Theta_{\boldsymbol{p}^-}$) of zeros of one of the $2^{2g-1} + 2^{g-1}$ (resp. $2^{2g-1} - 2^{g-1}$) even (resp. odd) θ functions (as defined by Riemann, up to an exponential factor) on $\mathcal{T}_{\boldsymbol{p}}$. Given an even (resp. odd) $\theta_{\boldsymbol{p}_0}$ at $\boldsymbol{p}_0 \in \mathcal{H}$, there are normalizations that give a unique analytic continuation of it to even (resp. odd) θ s along any path in $\mathcal{H}$ based at $\boldsymbol{p}_0$.*

The next two subsections do the proof. §6.1.4 describes the 2^{2g} θ s (even and odd) attached to a Riemann surface. Our aim to get θ s to depend on just the coordinates describing those families. That isn't possible, as Riemann knew, though for families of say, Thm. 1.3, the coordinates for the θ s are especially good.

§6.1.4 introduces coordinates from the infinite degree Torelli space cover of $\mathcal{M}_g$. That cover has the $2g \times 2g$ symplectic group over $\mathbb{Z}$, $\text{Sp}_{2g}(\mathbb{Z})$, as its monodromy group. Then, §6.1.5 uses the finite cover $\mathcal{M}_{g,\pm} \to \mathcal{M}_g$. I tie this not-easily found classical result to the telegraphic discussion in [**Fay73**].

6.1.4. *Torelli space coordinates.* I now explain Torelli space $\mathcal{T}_g$, an unramified covering of $\mathcal{M}_g$. For $m \in \mathcal{M}_g$, the fiber $\mathcal{T}_{g,m}$ over m consists of all possible canonical (first) homology bases for X_m. Typical notation has such a basis as $\boldsymbol{\ell} = (\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ with the matrix of cup-product intersections looking like $J_{2g} = \left(\begin{smallmatrix} 0_g & I_g \\ -I_g & 0_g \end{smallmatrix}\right)$ using the $g \times g$ zero, 0_g, and identity, 1_g, matrices. With $\mathbb{M}_g(\mathbb{Z})$ ordinary $g \times g$ matrices in $\mathbb{Z}$, that leaves possible choices (see the precise notation of §B.1) of basis as a homogenous space for

$$\text{Sp}_{2g}(\mathbb{Z}) = \left\{U = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \mid A, B, C, D \in \mathbb{M}_g(\mathbb{Z}), U J_{2g} U^{\text{tr}} = J_{2g}\right\}. \tag{6.2}$$

Denote the $\mathbb{Z}$ module that $\boldsymbol{\ell}$ generates by $\langle \boldsymbol{\ell} \rangle$.

Giving $\boldsymbol{\ell}$ fixes an identification of $\text{Pic}^{(0)}(X)$ with $\mathbb{C}^g/\langle \Pi(\boldsymbol{\ell}) \rangle$ with $\Pi(\boldsymbol{\ell})$ a lattice in $\mathbb{C}^g$, thereby starting Riemann's generalization of Abel's Theorem. I explain $\Pi(\boldsymbol{\ell})$.

Choose a basis $\boldsymbol{\omega} = (\omega_1, \dots, \omega_g)$ of holomorphic differentials on X_m, using one of the normalizations also typical in the literature. For example, [**Fay73**, p. 3] takes the integral of the g-tuple of differentials along the paths $\alpha_1, \dots, \alpha_g$ to be the $g \times g$ matrix $2\pi i I_g$, while others, respectively, replace $\alpha_1, \dots, \alpha_g$ and $2\pi i I_g$ with $\beta_1, \dots, \beta_g$ and I_g. Riemann showed such a choice determines $\boldsymbol{\omega}$ and the resulting $g \times 2g$ matrix has columns listing the transpose of $(\boldsymbol{\omega})$ integrated in order along

$\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g$. Those columns form the matrix $\Pi(\boldsymbol{\ell}) = \big(2\pi i I_g|\tau_{\boldsymbol{\ell}}\big)$, with $\tau_{\boldsymbol{\ell}}$ symmetric. Then, (Riemann showed) $\langle\Pi(\boldsymbol{\ell})\rangle$ is the lattice those columns span.

Now we require (noncanonical) choices: Choose a set of g points $(x'_1,\dots,x'_g)$ on X, so the complex vector

$$(6.3)\qquad (\sum_{i=1}^{g}\int_{x'_i}^{x_i}\omega_1,\dots,\sum_{i=1}^{g}\int_{x'_i}^{x_i}\omega_g) = \sum_{i=1}^{g}(\int_{x'_i}^{x_i}\omega_1,\dots,\int_{x'_i}^{x_i}\omega_g) = \boldsymbol{w}\in\mathbb{C}^g$$

represents a degree 0 divisor class $[D=\sum_{i=1}^{g}(x_i-x'_i)]$ in $\mathbb{C}^g/\langle\Pi(\boldsymbol{\ell})\rangle$. Modding out by $\langle\Pi(\boldsymbol{\ell})\rangle$ assures an integral independent of the path choices from x'_i to x_i.

Some discussions choose $x'_1,\dots,x'_g$ to be the same point repeated g times; still not canonical. (As $\boldsymbol{z}$ appears in this paper from coordinates on $\mathbb{P}^1_z$, we use $\boldsymbol{w}$ rather than the traditional $\boldsymbol{z}$.) With tr meaning transpose, Riemann's θ function,

$$(6.4)\qquad \theta(\boldsymbol{\ell},\boldsymbol{w}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^g} e^{\boldsymbol{n}\tau_{\boldsymbol{\ell}}\boldsymbol{n}^{\rm tr}+\boldsymbol{n}\boldsymbol{w}^{\rm tr}}\ \text{[\textbf{Fay73}, p. 1] on } \mathcal{T}_{g,m}\times\mathbb{C}^g,$$

is invariant under $\boldsymbol{w}\mapsto-\boldsymbol{w}$ (it is *even*), and under translating $\boldsymbol{w}$ by its $\boldsymbol{\alpha}$ periods.

Significantly, (6.4) depends on $\boldsymbol{\ell}$; even its divisor of zeros – denote this $\Theta_{\boldsymbol{\ell}}$ – depends on $\boldsymbol{\ell}$ mod $2\boldsymbol{\ell}$. Giving $m\in\mathcal{M}_g$ does not canonically give $\boldsymbol{\ell}$, so I comment on that dependence now: How can we compare info on X_m varying in a *given* family (a Hurwitz space, or $\mathcal{M}_g$) with variation of $\theta(\boldsymbol{\ell},\boldsymbol{w})$. The quotient of X_m^k by the symmetric group S_k, permutating its coordinates, gives the degree k divisors.

First, we compare the zero set $\Theta_{\boldsymbol{\ell}}$ on $\boldsymbol{\ell}\times\mathbb{C}^g$ with the positive, degree $g-1$, divisor classes on X_m: $W_{g-1}\stackrel{\rm def}{=}W_{g-1,m}$. Fundamental Fact: W_{g-1} is birational to X_m^{g-1}/S_{g-1}, degree $g{-}1$ divisors. If δ is a $\frac{1}{2}$-canonical divisor class on X_m, then the translate $W_{g-1,m}-\delta$ by δ is a $g{-}1$-dimensional set in $\mathrm{Pic}^{(0)}(X_m)$.

Theorem 6.5. *Pullback of $W_{g-1,m}-\delta$ from $\mathrm{Pic}^{(0)}(X_m)$ to its universal covering space $\boldsymbol{\ell}\times\mathbb{C}^g$ gives a divisor of the form $\Theta_{\boldsymbol{\ell}}+\mu$ with $\mu=\mu_m$ representing a point of order 2 on $\mathbb{C}^g/\langle\Pi(\boldsymbol{\ell})\rangle$ (2-division point).*

As μ runs over 2-division representatives, translates, $\theta(\boldsymbol{\ell},\boldsymbol{w}+\mu)$, of $\theta(\boldsymbol{\ell},\boldsymbol{w})$ run over a collection of 2^{2g} functions, each either even or odd. Each has zero divisor of form $W_{g-1,m}-\delta$, for some half-canonical class representative δ, and $\theta(\boldsymbol{\ell},\boldsymbol{w}+\mu)$ is zero at $\boldsymbol{w}=\mathbf{0}$ if and only if the class of δ contains a positive divisor. Both $\boldsymbol{\ell}_0$ and $\theta(\boldsymbol{\ell}_0,\boldsymbol{w}+\mu_0)$ (uniquely) analytically continue along any path P in $\mathcal{T}_g$ based at $\boldsymbol{\ell}_0$.

Comments on the proof. [**Fay73**, p. 7, Thm. 1.1] states the characterization of $W_{g-1,m}-\delta$ as being some translate of the θ divisor. It comes from Riemann's precise solution of the Jacobi Inversion Problem. The minimal expression of that says the map (6.3) from X_m^g to $\mathbb{C}^g/\langle\Pi(\boldsymbol{\ell})\rangle$ is onto. The characterization of the θ divisor is that these are the (degree 0) divisor classes of form $[D-\delta]$ where this map fails to be one-one, having as fiber a copy of projective space of dimension one less than that of the linear system of D. He quotes [**Le64**] or [**Ma61**] for a proof.

From Riemann-Roch: $W_{g-1,m}-\delta$ is closed under multiplication by -1; and it determines the θ function with it as divisor up to a holomorphic exponential (in $\boldsymbol{w}$), which we can take to be even in $\boldsymbol{w}$. So, $\theta(\boldsymbol{\ell},\boldsymbol{w}+\mu)$, $2\mu\equiv\mathbf{0}\mod\langle\boldsymbol{\ell}\rangle$, is either:

$$(6.5)\qquad \begin{aligned}\text{even: }\theta(\boldsymbol{\ell},-\boldsymbol{w}+\mu)\ &=\theta(\boldsymbol{\ell},\boldsymbol{w}+\mu);\ \text{or}\\ \text{odd: }\theta(\boldsymbol{\ell},-\boldsymbol{w}+\mu)\ &=-\theta(\boldsymbol{\ell},\boldsymbol{w}+\mu).\end{aligned}$$

Suppose a path $P:[0,1]\to\mathcal{T}_g$ starts at $\boldsymbol{\ell}_0$. Points on $\mathbb{C}^g$ representing 2-division points on $\mathbb{C}^g/\langle\Pi(\boldsymbol{\ell})\rangle$ form a discrete set. So, you can uniquely assign

$t \in [0,1] \mapsto \mu_t \in \mathbb{C}^g$ representing a 2-division point on $\mathbb{C}^g/\langle \Pi(P(t)) \stackrel{\text{def}}{=} \Pi(\boldsymbol{\ell}_t)\rangle$ to be continuous in t. Then,

$$\theta(\boldsymbol{\ell}_t, \boldsymbol{w} + \mu_t) \text{ on } \boldsymbol{\ell}_t \times \mathbb{C}^g \tag{6.6}$$

analytically continues $\theta(\boldsymbol{\ell}_0, \boldsymbol{w} + \mu_0)$ along P. □

6.1.5. *Even and odd θ s and the spaces $\mathcal{M}_{g,\pm}$.* A function $\theta(\boldsymbol{\ell}_0, \boldsymbol{w} + \mu_0)$ in Thm. 6.5 is called a θ with (a 2-division) characteristic. Even if P is a closed path, we don't expect $\boldsymbol{\ell}$ or μ at the beginning and end of P to be the same.

Now return to the family $\mathcal{H}$ in §6.1.3, $\boldsymbol{p}_0 \in \mathcal{H}$ and let $\delta_0 \in \mathcal{M}_{g,\pm}$ be a $\frac{1}{2}$-canonical class on $X_{\boldsymbol{p}_0}$. Any $\boldsymbol{\ell}_0$ in Torelli space over $\boldsymbol{p}_0$ determines the unique theta $\theta(\boldsymbol{\ell}_0, \boldsymbol{w} + \mu_0)$ whose divisor $\Theta_{\boldsymbol{p}_0}$ is the pullback of $W_{g-1,\boldsymbol{p}_0} - \delta_0$. Let δ_t be the value at t of the unique lift of the path P in $\mathcal{M}_{g,\pm}$ starting at δ_0. Then, the divisor of expression (6.6) is the pullback of $W_{g-1,P(t)} - \delta_t$.

Proposition 6.6. *If $\theta(\boldsymbol{\ell}_0, \boldsymbol{w} + \mu_0)$ is odd (resp. even), then so is $\theta(\boldsymbol{\ell}_t, \boldsymbol{w} + \mu_t)$ for all $t \in [0,1]$. Suppose $\mathcal{H}$ is a family of covers with odd order branching, δ_0 is the $\frac{1}{2}$-canonical class defined by Lem. 6.2 at $\boldsymbol{p}_0 \in \mathcal{H}$, and the divisor of $\theta(\boldsymbol{\ell}_0, \boldsymbol{w} + \mu_0)$ is the pullback of $W_{g-1,\boldsymbol{p}_0} - \delta_0$. Then, if $P : [0,1] \to \mathcal{H}$ is a closed path, the divisors of $\theta(\boldsymbol{\ell}_0, \boldsymbol{w} + \mu_0)$ and $\theta(\boldsymbol{\ell}_1, \boldsymbol{w} + \mu_1)$ are the same.*

PROOF. Apply Thm. 6.5 to the path $t \mapsto (\Psi_{\mathcal{H},\mathcal{M}_g} \circ P)(t)$ to get the analytic continuation. We know for each t, $\theta(\boldsymbol{\ell}_t, \boldsymbol{w} + \mu_t)$ is one of $\pm\theta(\boldsymbol{\ell}_t, -\boldsymbol{w} + \mu_t)$. Their ratio is continuous in all variables, avoiding $(t, \boldsymbol{w})$ that make the denominator 0. So, the value is either $+1$ or -1 giving the first conclusion. The 2nd conclusion follows from Lem. 6.2, saying $P(t)$ determines δ_t, and $P(0) = P(1)$. □

Now consider in Prop. 6.4 the respective degrees of $\mathcal{M}_{g,\pm}$ over $\mathcal{M}_g$. Prop. 6.6 says analytic continuation of $\frac{1}{2}$-canonical classes from one point on the connected space $\mathcal{M}_g$ to another moves even (resp. odd) classes to even (resp. odd) classes. So, for each g, use connectedness of $\mathcal{M}_g$ to analytically continue to where we can count these classes. Conclude this count using hyperelliptic curves in Cor. 6.20.

Next, consider for $\mathcal{H} = \mathcal{M}_g$ why the monodromy action is transitive on even (resp. odd) $\frac{1}{2}$-canonical classes. That means, for paths P running over lifts to $\mathcal{T}_g$ of closed paths in $\mathcal{H}$, the action is transitive on both even and odd θ s.

Assume along P based at $\boldsymbol{\ell}_0 \in \mathcal{T}_{g,m}$, the endpoint is $(\boldsymbol{\ell}_0)\Psi_P \stackrel{\text{def}}{=} \boldsymbol{\ell}_1 \in \mathcal{T}_{g,m}$, and μ_0 continues to μ_1. A formula explains how θ functions transform with an application of ψ_P to $\Pi(\boldsymbol{\ell}_0) = (2\pi i I_g, \tau_{\boldsymbol{\ell}_0})$ ([**Fay73**, p. 7], from [**Ig72**, p. 84]). Write

$$\tau_{\boldsymbol{\ell}_1} \stackrel{\text{def}}{=} 2\pi i(A_P\tau_{\boldsymbol{\ell}_0} + 2\pi i B_P)(C_P\tau_{\boldsymbol{\ell}_0} + 2\pi i D_P)^{-1}.$$

Denote $C_P\tau_{\boldsymbol{\ell}} + 2\pi i D_P$ – its entries are functions in the entries of $\tau_{\boldsymbol{\ell}}$ – by $M_{\boldsymbol{\ell}}$. If M^* is any matrix with entries that are functions of the entries of $\tau_{\boldsymbol{\ell}}$, then $\nabla_\Pi(M^*)$ is the matrix whose (i,j) entry is the partial with respect to the variable in the (i,j) position of $\tau_{\boldsymbol{\ell}}$. Then, $\Pi(\boldsymbol{\ell}_1) = \left(\begin{smallmatrix} D & C \\ B & A \end{smallmatrix}\right)\Pi(\boldsymbol{\ell}_0)$ and $2\pi i\boldsymbol{w} = \tilde{\boldsymbol{w}}M$. [**Fay73**, p. 8] notation is compatible with §B.1 by writing $\Pi(\boldsymbol{\ell})$ as $\left(\begin{smallmatrix}\boldsymbol{\alpha} \\ \boldsymbol{\beta}\end{smallmatrix}\right)$ and having $\Psi(P)$ from the left as $U_P = \left(\begin{smallmatrix} A_P & B_P \\ C_P & D_P \end{smallmatrix}\right)$ in (6.2). The result is $\left(\begin{smallmatrix}\tilde{\boldsymbol{\alpha}} \\ \tilde{\boldsymbol{\beta}}\end{smallmatrix}\right)$, with the following two provisos. First: Normalize by multiplying on the right by the inverse of the matrix

$\tilde{\boldsymbol{\alpha}}$, and then multiply by $2\pi i$. Second: Fay changes U_P to $\left(\begin{smallmatrix} D_P & C_P \\ B_P & A_P \end{smallmatrix}\right)$ (this is also in $\mathrm{Sp}_{2g}(\mathbb{Z})$): He thought the result in more standard notation?

For a $g \times g$ matrix N, use bracket notation $\{N\}$ for the vector of diagonal elements. The transformation formula:

$$\text{with}\quad \mu_1^{\mathrm{tr}} = \left(\begin{smallmatrix} D & -C \\ -B & A \end{smallmatrix}\right)\mu^{\mathrm{tr}} + \tfrac{1}{2}\begin{bmatrix} \{CD^{\mathrm{tr}}\} \\ \{AB^{\mathrm{tr}}\} \end{bmatrix},\ \text{then } \theta(\boldsymbol{\ell}_1, \tilde{\boldsymbol{w}} + \mu_1) = \tag{6.7}$$
$$K_{\mu_0,P} \det(M)^{\frac{1}{2}}_{|\tau_{\boldsymbol{\ell}_0}} \exp(\tfrac{1}{2}(\boldsymbol{w}\nabla_\Pi \ln(\det(M))_{|\tau_{\boldsymbol{\ell}_0}} \boldsymbol{w}^{\mathrm{tr}})\theta(\boldsymbol{\ell}_0, \boldsymbol{w} + \mu_0).$$

Transitivity on even and odd θs is very old; a corollary of (6.7) by applying elements of $\mathrm{Sp}_{2g}(\mathbb{Z})$. That all even θs are nonzero at the origin on a general surface is explained at [**Fay73**, p. 7] by alluding to [**Fay73**, Cor. 3.2]. Hershel Farkas – in the late 60's when we were Stony Brook colleagues – attributed this argument to [**Po1895**]. Fay expands along these lines: Deforming from period matrices of "products of elliptic curves" to discern objects on a general Riemann surface.

Denote the function of $(t, \boldsymbol{w})$ in Prop. 6.6 corresponding to a path P by $\theta_P(\boldsymbol{w})$. As in §6.1.3, for $\Psi : \mathcal{T} \to \mathcal{H}$ a family of surfaces, form $\mathcal{H}_\pm$.

Definition 6.7. We say Ψ supports an even (resp. odd) $\frac{1}{2}$-canonical class if a component of $\mathcal{H}_+$ (resp. $\mathcal{H}_-$) maps one-one from the fiber product to $\mathcal{H}$. We say an even class has a nontrivial θ-null if there is a path P on this component (based at $(\boldsymbol{p}_0, \delta_0)$ on the component), and some corresponding $\boldsymbol{\ell}_0$, so that for some P, $\theta_P(\boldsymbol{w})$ in Prop. 6.6, $\theta_P(\mathbf{0}) \not\equiv 0$ (as a function of t).

If Ψ supports a $\frac{1}{2}$-canonical class, but the component is in $\mathcal{H}_-$, then $\theta_P(\boldsymbol{w})$ is odd in $\boldsymbol{w}$ (Thm. 6.15). Conclude: $\theta_P(\mathbf{0})$ is an identically 0 function of t. Still, we can ask if such a component defines a *nondegenerate* odd θ. That is, there is a path P so that for most values of t, the gradient of $\theta_P(\boldsymbol{w})$ in $\boldsymbol{w}$ doesn't vanish at $\boldsymbol{w} = \mathbf{0}$.

Thm. 6.15 shows the $\oplus$ families of Figure 1 (Thm. 1.3) have a nontrivial θ-null for many values of (n, g). For $g = 1$ (and $\mathcal{H} = \mathcal{M}_1$) there is a unique nondegenerate odd θ, but for $g \geq 2$, there is more than one (as in Prop. 6.19). See §B.2.2.

Remark 6.8 (Stacks of compact Riemann surfaces)**.** Significantly, $\mathcal{M}_g$ has no total family $\mathcal{X} \to \mathcal{M}_g$ representing its points. *Stacks* arise in such situations to produce the *algebraic* map $\Psi_{\mathcal{H},\mathcal{M}_g} : \mathcal{H} \to \mathcal{M}_g$ (before Prop. 6.6). [**Fr77**, §4] shows, for an ordinary (say, absolute or inner) Hurwitz space (§A.2.2) $\mathcal{H} = \mathcal{H}(G, \mathbf{C})$, the space has (a finite number of explicit) Zariski open subsets $\{U_i\}_{i=1}^s$, each having an étale cover $\Phi_i : W_i \to U_i$ with this property. There is a total family $\mathcal{T}_i \to W_i \times \mathbb{P}^1_z$ so that for $w_i \in W_i$, the fiber $\mathcal{T}_{i,w_i} \to w_i \times \mathbb{P}^1_z$ over w_i represents the equivalence class of covers of the image of w_i in $\mathcal{H}$. That is, the stack exists in the étale topology. This produces a *stacky* definition $\Psi_{\mathcal{H},\mathcal{M}_g}$. There is always a unique global total family if the Hurwitz space has fine moduli: as in §A.2.3, self-normalizing in the absolute case; G has no center in the inner case. Example: In Lem. 1.6, Spin_n has a nontrivial center, so $\boldsymbol{p} \in \mathcal{H}_+(\mathrm{Spin}_n, \mathbf{C})^{\mathrm{in}}(K)$ may not guarantee a K cover.

[**We98**] uses stack language; behind it is our construction. Pull reduced Hurwitz spaces (§A.3) back to ordinary Hurwitz spaces; define the map from this.

Remark 6.9 (Warning!)**.** Don't confuse 2-division points with $\frac{1}{2}$-canonical classes. Any $\frac{1}{2}$-canonical class translates 2-division points into $\frac{1}{2}$-canonical classes: The former is a homogeneous space for the latter. Still, the monodromy action on 2-division points over $\mathcal{M}_g$ has a length 1 orbit (the origin) and another of length $2^{2g}-1$; different from the two orbit lengths on $\frac{1}{2}$-canonical classes, in Prop. 6.4.

6.2. θ-nulls and roots of 1. [**Sh98**, Thm. 27.7] (in greater generality, quoting [**KP1892**], but [**Sh98**, §28] has a detailed proof) says $K_{\mu_0,P}$ in (6.7) is a root of 1 that depends on which branch of $\det(M)^{\frac{1}{2}}$ we have chosen.

6.2.1. *Hurwitz-Torelli automorphic functions and roots of 1.* Consider a braid orbit O on a reduced Nielsen class $\mathrm{Ni}(G,\mathbf{C})^{*,\mathrm{rd}}$, $*=$ abs or in, and denote its corresponding reduced Hurwitz space component by $\mathcal{H}_O$. Regard $\boldsymbol{p}_0\in\mathcal{H}_O$ as a base point for analytic continuation. Suppose f is a meromorphic function around $\boldsymbol{p}_0$ and it continues along *any* path on $\mathcal{H}_O$ based at $\boldsymbol{p}_0$. Denote the subgroup, of H_r that fixes a particular element, $\boldsymbol{g}_O\in O$, by $H_{r,\boldsymbol{g}_O}$. Then, the monodromy action on f explicitly interprets as an action of $H_{r,\boldsymbol{g}_O}$: $q\in H_{r,\boldsymbol{g}_O}$ takes f to f_q.

Choose a classical generators (§1.1.2) of $\pi_1(U_{\boldsymbol{z}},z_0)$ to determine how H_r acts in §2. If in this identification $\boldsymbol{g}_0$ is a branch cycle description of $\varphi_0: X_0\to\mathbb{P}^1_z$, then by restriction, $H_{r,\boldsymbol{g}_O}$ acts on the fundamental group of $X_0\setminus\{\varphi^{-1}(\boldsymbol{z})\}$.

Definition 6.10. Refer to f as a π_1-H(urwitz)-T(orelli) (resp. H_1-H-T) function if the $H_{r,\boldsymbol{g}_O}$ action determined by it factors through $\pi_1(X_{\boldsymbol{p}_0})$ (resp. $H_1(X_{\boldsymbol{p}_0,\mathbb{Z}})$). For an H_1-H-T function, in the notation of (6.7), $q\in H_{r,\boldsymbol{g}_O}$ acts through $\left(\begin{smallmatrix} D_q & C_q\\ B_q & A_q\end{smallmatrix}\right)$ with an associated $g\times g$ matrix $M_q(\boldsymbol{\ell})=C_q\tau_{\boldsymbol{\ell}}+2\pi i D_q$. With P a path based at $\boldsymbol{p}_0$, for $\boldsymbol{p}=P(t)$ denote the analytic continuation of $\boldsymbol{\ell}_0$ over $P(t)$ by $\boldsymbol{\ell}=\tilde P(t)$. An H_1-H-T function f is *(H-T) automorphic*, of weight m, if:

for each $q\in H_{r,\boldsymbol{g}_O}$, and path P based at $\boldsymbol{p}_0$, $f_q(P(t))=M_q(\tilde P(t))^m f(P(t))$.

The automorphic definition matches the form of [**Sh98**, §25.1]. We now give examples of H_1-H-T functions. Start with $\boldsymbol{p}_0\in\mathcal{H}(G,\mathbf{C})^{*,\mathrm{rd}}$, and let $\delta_{\boldsymbol{p}_0}$ be any $\frac{1}{2}$-canonical class on $X_{\boldsymbol{p}_0}$ defining the $\Theta_{\boldsymbol{p}_0}$ as in Thm. 6.5.

Assume $\boldsymbol{\ell}_0$ and μ_0 in Prop. 6.5, over $\boldsymbol{p}_0$, give an even θ with zero divisor $\Theta_{\boldsymbol{p}_0}$. For $P:[0,1]\to\mathcal{H}(G,\mathbf{C})^{*,\mathrm{rd}}$ a path based at $\boldsymbol{p}_0$, denote analytic continuation of the θ (resp. $\delta_{\boldsymbol{p}_0}$) along P by $\theta_P(\boldsymbol{\ell}_0,\boldsymbol{w}+\mu_0)$ (resp. δ_P). It's value at $t\in[0,1]$ is then $\theta(\boldsymbol{\ell}_t,\boldsymbol{w}+\mu_t)$ (resp. $\delta_{P(t)}$) compatible with the notation of §6.1.5. Denote the connected component of $\mathcal{H}_O\times_{\mathcal{M}_g}\mathcal{M}_{g,+}$ containing $(\boldsymbol{p}_0,\delta_{\boldsymbol{p}_0})$ by $\mathcal{H}_{O,\delta_{\boldsymbol{p}_0}}$.

Definition 6.11. The result, $\theta_P(\boldsymbol{\ell}_0,\mu_0)$, of setting $\boldsymbol{w}=\mathbf{0}$ and letting P vary over all paths P based at $\boldsymbol{p}_0$ is a *θ-null* on $\mathcal{H}(G,\mathbf{C})^{*,\mathrm{rd}}$.

Let P' be a closed path on $\mathcal{H}(G,\mathbf{C})^{*,\mathrm{rd}}$ representing $q\in H_{r,\boldsymbol{g}_O}$ with $\boldsymbol{\ell}_1$ and μ_1 the analytic continuations of $\boldsymbol{\ell}_0$ and μ_0 to the end of P'. With P running over all paths based at $\boldsymbol{p}_0$, (6.7) compares $\theta_{P'\cdot P}(\boldsymbol{\ell}_0,\mu_0)=\theta_P(\boldsymbol{\ell}_1,\mu_1)$ and $\theta_P(\boldsymbol{\ell}_0,\mu_0)$. Analytic continuations only depend on the homotopy class of a path with the homotopy keeping the endpoints fixed. So, we can replace the fixed closed path P' by q.

Proposition 6.12. *The theta-null $\theta_P(\boldsymbol{\ell}_0,\mu_0)$ is an H_1-H-T function. It is identically zero if and only if $\delta_{\boldsymbol{p}}$ contains a positive divisor for each $(\boldsymbol{p},\delta_{\boldsymbol{p}})\in\mathcal{H}_{O,\delta_{\boldsymbol{p}_0}}$.*

For $\mathrm{Ni}(G,\mathbf{C})$ *a Nielsen class of odd-order branching, and $q=[P']$ as above,*

$$\theta_{q\cdot P}(\boldsymbol{\ell}_0,\mu_0)=\theta_P(\boldsymbol{\ell}_1,\mu_1)=K_{\mu_0,q}\det(M_q(\tilde P))^{\frac{1}{2}}\theta_P(\boldsymbol{\ell}_0,\mu_0).$$

For some minimal positive integer m, $(\theta_P(\boldsymbol{\ell}_0,\mu_0))^m$ is automorphic under $H_{r,\boldsymbol{g}_O}$.

PROOF. Most everything follows from the definition of the θ-null and (6.7), except these two. The criterion for nonzeroness at the origin of a θ is from Thm. 6.5, and we apply Prop. 6.6 to a Nielsen class of odd order branching. The minimal integer m in the last paragraph is one for which $K^m_{\mu_0,q} = 1$ for q running over any finite set of generators of $H_{r,\boldsymbol{g}_0}$. See Rem. 6.13. $\square$

When $g = 1$, and $H_{r,\boldsymbol{g}_O}$ identifies with $\mathrm{PSL}_2(\mathbb{Z})$, [**FaK01**, p. 101] shows explicitly $K_{\mu_0,q} \in \langle\sqrt{-i}\rangle$: an 8th root of 1, where the serious calculation is q representing $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. [**FaK01**, p. 176] repeats the classical definition of automorphic form in "dimension 1," though as with everything in this book the functions are supported on congruence subgroups. In particular, they explicitly compute that root of 1 in [**FaK01**, Prop. 2.1] for a θ-null with other rational characteristics – notating these as χ – where the denominator in them is an integer k. Such functions are supported on the upper-half plane quotient of the congruence subgroup $\Gamma(k)$. The precise result is [**FaK01**, Prop. 2.1] where the $K_{\chi,q}$ are $8k$-th roots of 1.

Remark 6.13 (Effective monodromy)**.** Since H_r is a presented group, and $H_{r,\boldsymbol{g}_0}$ is a stabilizer subgroup, an algorithm for computing generators of $H_{r,\boldsymbol{g}_O}$ comes from the Schreier algorithm for generators of a subgroup of finite index in a finitely generated free group [**FJ86**, p. 351]. Suppose $X_{\boldsymbol{p}_0} \to \mathbb{P}^1_z$ has branch cycles $\boldsymbol{g}_0$, and $\hat{X}_{\boldsymbol{p}_0} \to \mathbb{P}^1_z$ is its Galois closure. Let $X^{\mathrm{un}}_{\boldsymbol{p}_0}$ be the maximal unramified cover of $X_{\boldsymbol{p}_0}$ lying between $\hat{X}_{\boldsymbol{p}_0}$ and $X_{\boldsymbol{p}_0}$. [**Fr89**, Ex. 3.4] gives examples where $X^{\mathrm{un}}_{\boldsymbol{p}_0} \ne X_{\boldsymbol{p}_0}$. Yet, typically they are equal, as for the Nielsen classes $\mathrm{Ni}(A_n, \mathbf{C}_{3^r})^{\mathrm{abs}}$. Given explicit $q \in H_{r,\boldsymbol{g}_O}$, [**Fr89**, §3.2] displays $\pi_1(X^{\mathrm{un}}_{\boldsymbol{p}_0})$ (and so H_1) *in terms of the branch cycles* $\boldsymbol{g}_O$. That is, the presentation supports an explicit action of q.

Remark 6.14 (Modular Towers)**.** Each space $\mathcal{H}_+(A_n, \mathbf{C}_{3^r})^{\mathrm{abs}}$ in Thm. 1.3 has above it an infinite collection of Hurwitz spaces for which Prop. 6.12 produces an even θ-null. The case $n = 3$ is included, where for each prime p (excluding $p = 3$) and for each nonnegative prime power p^ℓ, the system has a Hurwitz space attached to a centerless group $(\mathbb{Z}/p^\ell)^2 \times^s \mathbb{Z}/3$, and the four conjugacy classes $\mathbf{C}_{\pm 3^2}$ (Ex. 1.4). [**Fr06**, §6] discusses the modular curve-like properties of this system. Similarly, starting with $n = 5$ and $r = 4$ in Thm. 1.2, though less obvious what are the ingredients for a system. We can compare the case of Prop. 3.5 with [**FaK01**]: Like a modular curve it is a family of genus 1 covers. Indeed, all these spaces are 1-dimensional quotients of the upper half-plane, but they aren't modular curves. [**Fr08a**, Main Thm.] puts these families in a bigger context, and extends the modular curve-like properties. As in the Farkas-Kra discussion above, if you go "up" in these systems, higher levels support higher characteristic θ-nulls.

6.2.2. *θ-nulls on $\mathcal{H}_\pm(A_n, \mathbf{C}_{3^r})^{*,\mathrm{rd}}$.* Consider $\mathcal{H}$, a reduced Hurwitz space component of odd order branched covers. Prop. 6.6 canonically gives on it an analytic continuation of an even (resp. odd) θ, if the $\frac{1}{2}$-canonical classes on the component are even (resp. odd; Def. 6.7). [**Ser90b**] determines which from the Nielsen class and the component lifting invariant. Prop. 6.12 gives the transformation formula for the corresponding θ-null. It can only be nonzero if the θ is even. Assume $n \ge 4$.

Theorem 6.15. *Assume $g = r-n+1$, the genus of the degree n covers parametrized by $\mathcal{H}_\pm(A_n, \mathbf{C}_{3^r})^{\mathrm{abs,rd}}$ (Thm. 1.3) is at least 1. For r even, the θ is even (resp. odd) when $\mathcal{H} = \mathcal{H}_+(A_n, \mathbf{C}_{3^r})^{\mathrm{abs,rd}}$ (resp. $\mathcal{H}_-(A_n, \mathbf{C}_{3^r})^{\mathrm{abs,rd}}$). For r odd, the results are* *switched. For the inner case, the result is independent of the parity of r: the θ is even (resp. odd) when $\mathcal{H} = \mathcal{H}_+(A_n, \mathbf{C}_{3^r})^{\text{in,rd}}$ (resp. $\mathcal{H}_-(A_n, \mathbf{C}_{3^r})^{\text{in,rd}}$).*

For $g = 1$ or for $n \geq 12 \cdot g + 4$, the natural map $\Psi_{\mathcal{H}_\pm, \mathcal{M}_g} : \mathcal{H}_\pm(A_n, \mathbf{C}_{3^r})^{\text{abs,rd}} \to \mathcal{M}_g$ restricted to each component is dominant. If also, r is even (resp. odd), then $\mathcal{H}_+(A_n, \mathbf{C}_{3^r})^{\text{abs,rd}}$ (resp. $\mathcal{H}_-(A_n, \mathbf{C}_{3^r})^{\text{abs,rd}}$) supports a nonzero θ-null.

Proof. In [**Ser90b**, Thm. 2] we take the special case $X = \mathbb{P}^1_z$, so [**Ser90b**, exp. (17)] applies. Serre notes this reverts to [**Ser90a**, exp. (10)] (which references an early version of Cor. 2.3): For $\boldsymbol{g} = (g_1, \dots, g_r)$ in the braid orbit of a Nielsen class $\text{Ni}(G, \mathbf{C})$, we get even for the $\frac{1}{2}$-canonical class exactly when the product of the lift invariant (1.1) and $(-1)^{\sum_{i=1}^r w(g_i)}$ is 1: Serre's formula written multiplicatively. This only depends on the Nielsen class and the lifting invariant. In the 3-cycle absolute case each $w(g_i)$ is 1. For, however, the inner case, each $w(g_i)$ is $n!/6$, which is even (for $n \geq 4$).

We review ingredients from [**ArP05**, Thm. 1]. Let X be a compact Riemann surface of genus g. Consider $n \geq 12g + 4$ an integer, $\boldsymbol{k}$ a $\frac{1}{2}$-canonical class on X and x_1, x_2, x_3 any distinct points on X. Assume also:

(6.8a) There exists a meromorphic $\frac{1}{2}$-canonical differential (expression (B.1)) μ whose divisor of poles is $\leq D_{X,\boldsymbol{n}} \stackrel{\text{def}}{=} n_1x_1 + n_2x_2 + n_3x_3$; and
(6.8b) the square $\mu \otimes \mu$ of μ is the differential df of a meromorphic f on X.

From the Riemann-Roch Theorem (see §B.2.2), (6.8a) follows if a $\frac{1}{2}$-canonical class has sufficient polar degree to guarantee a global section. Denote the linear system of sections of $\boldsymbol{k}$ with polar divisor $\leq$ a divisor D by $H^0(X, O(\boldsymbol{k}, D))$.

The argument for (6.8b) crucially gives a differential satisfying Square Hypothesis (6.11). Consider the $\mathbb{C}$ bilinear pairing

$$\Gamma : H^0(X, O(\boldsymbol{k}, D)) \times H^0(X, O(\boldsymbol{k}, D)) \to H^1(X \setminus D, \mathbb{C})$$

by $(s_1, s_2) \mapsto s_1 \otimes s_2$, a differential with pole divisor supported in D. Let $\Delta_{\boldsymbol{k}}$ be restriction of Γ to the diagonal. Then, $\Delta_{\boldsymbol{k}}^{-1}(0)$ is sections whose square is exact:

(6.9) df whose divisor satisfies the square hypothesis (6.11), so $f : X \to \mathbb{P}^1_z$ has odd ramification.

If $\dim(H^0(X, O(\boldsymbol{k}, D))) > \dim(H^1(X \setminus D, \mathbb{C}))$, by putting Γ in standard form see that $\Delta_{\boldsymbol{k}}^{-1}(0) \setminus \{0\}$ is nonempty. We can change $\boldsymbol{n}$ to have this happen because D has just three support points, so the target dimension doesn't change with $\boldsymbol{n}$. We want to know in increasing order for which (n, r) the following hold:

(6.10a) Whether (and which) $\Psi_{\mathcal{H}_+, \mathcal{M}_g}$ and/or $\Psi_{\mathcal{H}_-, \mathcal{M}_g}$ is dominant; and
(6.10b) if some cover $\varphi_{\boldsymbol{p}} : X_{\boldsymbol{p}} \to \mathbb{P}^1_z$ in such a component produces a θ whose value at $\mathbf{0}$ is nonzero.

All even θ s on a general curve of genus g are nonzero at $\mathbf{0}$ (above Def. 6.7). Suppose $\mathcal{H}$ is a component which supports an even θ-null. Conclude: (6.10a) holding for $\Psi_{\mathcal{H}, \mathcal{M}_g}$ implies (6.10b). When $g = 1$, (6.10a) holds [**FKK01**].

Since, however, the divisor of poles has just three points of support, it doesn't tell us which X s are in the image of points in $\mathcal{H}_\pm(A_n, \mathbf{C}_{3^r})$. It is easy that a dense set of $m \in \mathcal{M}_g$ represent X_m with an odd order branched cover not in any Nielsen classes $\text{Ni}(A_n, \mathbf{C}_{3^r})$. Yet, X_m may *also* be a cover in such a Nielsen class.

It is also easy to show that if $m \in \mathcal{M}_g$ has complex coordinates that over $\mathbb{Q}$ generate the function field of $\mathcal{M}_g$ (m a generic point), then any odd order branched cover $f_m : X_m \to \mathbb{P}^1_z$ (degree n) is primitive [**Fr77**, p. 26].

It is much harder that if $g \geq 3$ (and m generic) then the monodromy group of f_m must be A_u for u given in Rem. 6.17. To show (6.10a) requires knowing when we can find an alternate f_m^* to f_m, so $f_m^* : X_m \to \mathbb{P}^1_z$ has 3-cycle branching.

The answer is always. Start by writing each of the branch cycles $\boldsymbol{g} = (g_1, \dots, g_r)$ for f_m as a product of 3-cycles, to give a (possibly) larger value r^* and branch cycles $\boldsymbol{g}^* \in \mathrm{Ni}(A_n, \mathbf{C}_{3^{r^*}})$ with $r^* - n + 1 = g$. Then, apply Riemann's existence theorem to produce a cover that must also represent a generic surface of genus g, and so by specialization represents X_m as a 3-cycle cover of $\mathbb{P}^1_z$ ([**Fr77**, §4] or [**ArP05**, §5]).

Notice, however, this argument "deforms" between different Nielsen classes. To be certain both components of $\mathcal{H}_\pm$ (for the appropriate (n, r)) contain a "generic" Riemann surface, we need to know this deformation preserves information about the evenness and oddness of the $\frac{1}{2}$-canonical classes.

Both [**Ser90b**, following Thm. 2] and [**ArP05**, §5] have arguments that handle this: The former uses topology to characterize $\frac{1}{2}$-canonical parity. The latter would adjust to use that (in our 3-cycle case) the even $\frac{1}{2}$-canonical classes correspond to an unramified spin cover of the Galois closure of $\varphi : X \to \mathbb{P}^1_z$ (Cor. 2.3). □

Question 6.16. For those absolute and inner Hurwitz space components carrying an even θ-null in Thm. 6.15 (or as in Rem. 6.14), generalize the theorem to find those for which the θ-null is nonzero.

Remark 6.17 (Generic alternating group monodromy)**.** Let $f_m : X_m \to \mathbb{P}^1_z$ present the generic compact Riemann surface of genus g as an odd branched cover following the proof of Thm. 6.15. If $g \geq 3$, then the monodromy group is a copy of $A_{\deg(f_m)}$ according to [**GN95**], [**GM98**] and [**GS07**]. The case $g = 0$ is a source of considerable applications. Then, excluding a finite number of significant special cases, either the monodromy group is $A_{\deg(f_m)}$, or it is A_l with $\deg(f_m) = \frac{l(l-1)}{2}$ (A_l acting on unordered pairs of integers from $\{1, \dots, l\}$). [**Fr04**, p. 76] lists all cases, not just odd order branching.

6.3. The Hyperelliptic Locus. Suppose the affine part of a hyperelliptic curve X is $\{(z, w) \mid w^2 = h(z)\}$. §6.3.2 lists differentials ω satisfying square hypothesis (6.11). §6.3.3 uses these to list representative divisors for all $\frac{1}{2}$-canonical classes, and Prop. 6.19 computes the monodromy orbit lengths.

6.3.1. *Half-canonical divisors.* Suppose ω is a meromorphic differential on a Riemann surface X, written locally as $f_\alpha(z_\alpha)dz_\alpha$, as in §B.2 (or in detail in [**Fr08c**, Chap. 2, §2.4]), on simply connected domains U_α. On U_α its divisor is the divisor $(f_\alpha(z_\alpha))$ of the function. Assume also the *square hypothesis*:

(6.11) $(f_\alpha(z_\alpha))$ has the form $2D_\alpha$ for U_α running over a subchart covering X. Then, there is a branch $h_\alpha(z_\alpha)$ of square root (of $f_\alpha(z_\alpha)$) on U_α [**Fr08c**, Chap. 2, §6.1]. Of course, there are two of these; our notation means we have chosen one. Call the symbol collection $\{\tau_\alpha = h_\alpha(z_\alpha)\sqrt{dz_\alpha}\}_{\alpha \in I}$, a *half-canonical 1-chain* on U_α. In bundle language, this is a 1-chain with values in the square-root of a canonical bundle. Still, it is more than that, for the squares of these form a global differential on X. So, we refer to $\{h_\alpha(z_\alpha)\}_{\alpha \in I}$ by $\boldsymbol{h}$ and call it a square-root of ω.

Lemma 6.18 (Half-canonical divisor)**.** *The 1-chain $\{(h_\alpha(z_\alpha))\}_{\alpha \in I}$ from a square root of ω give a well-defined divisor: a half-canonical divisor on X.*

PROOF. Let $D = (\omega)$ be the divisor of ω. Since, $h_\alpha^2 = f_\alpha$, the support multiplicities of D are all even integers. So, a square-root of ω defines $D_{1/2} = (\omega)/2$, a divisor uniquely given by the zeros and poles of the h_α s. □

6.3.2. *Square-hypothesis for hyperelliptic curves.* App. B.2 tells precisely what it means – expression (B.1) – for there to be a $\frac{1}{2}$-canonical differential representing a $\frac{1}{2}$-canonical divisor class. This subsection presents the much easier case of hyperelliptic curves. With no loss, assume an odd degree polynomial h with distinct zeros $z_1,\dots,z_{r-1}$ (use $z_r=\infty$). Denote the point on $X \overset{\text{def}}{=} X_{\boldsymbol{z}}$ over z_i by x_i, with x_∞ lying over $z=\infty$. As in [**Mu76**, p. 7], form

$$\omega_i = \frac{(z-z_i)^{\frac{1}{2}}}{(\prod_{j\ne i} z-z_j)^{\frac{1}{2}}}\, dz,\ i=1,\dots,r-1.$$

Since $w=\sqrt{h(z)}$, the factor in front of the dz (a meromorphic differential on X) in ω_i is just $\frac{z-z_i}{w}$, a meromorphic function on X.

Express ω_i in a local parameter $t_{z'}$ – a fractional power of $(z-z')$ – over each $z'\in\mathbb{P}^1_z$. Take $t_{z'}$ to be $z-z'$ for $z'\in U_{\boldsymbol{z}}=\mathbb{P}^1_z\setminus\{\boldsymbol{z}\}$, $(z-z')^{\frac{1}{2}}$ for $z'\in\{z_1,\dots,z_{r-1}\}$ and $1/z^{\frac{1}{2}}$ for $z'=\infty$. The multiplicity of ω_i at $t_{z'}=t=0$ is 0 for $z'\in U_{\boldsymbol{z}}$; the multiplicity, 2, of $t^2/h(t)\,d(t^2)$ (including t in the denominator) for $z'=z_i$; 0 for $z'=z_j$, with $j\ne i$; and the multiplicity, $r-6$, of $((1/t^2-z_i)/h(1/t^2))^{\frac{1}{2}}\,d(1/t^2)$ at $t=0$ at $z'=\infty$. Conclude: The total divisor of ω_i is

$$2x_i+(r-6)x_\infty = 2\cdot D_i; x_i + (\frac{r-6}{2})x_\infty \text{ is a } \frac{1}{2}\text{-canonical divisor.} \tag{6.12}$$

The following divisors are linearly equivalent to 0:

(6.13a) $2(x_i-x_j)$ (divisor of $\frac{z-z_i}{z-z_j}$); and
(6.13b) $\sum_{i=1}^{r-1}(x_i-x_\infty)$ (divisor of $\sqrt{h(z)}$).

6.3.3. $\mathcal{M}_{g,\pm}$ *over* $\mathbf{Hyp}_g$. Use $\boldsymbol{z}$ as the basepoint of U_r. Start with ω_1 and multiply it by powers of the $(z-z_i)$ s to get differentials satisfying square hypothesis (6.11). Divide by 2 to get representatives of all $\frac{1}{2}$-canonical classes on $X_{\boldsymbol{z}}$ from

$$\mathcal{D} \overset{\text{def}}{=} \{D_1 + \sum_{1\le i\le r-1} \epsilon_i(x_i-x_\infty)\}_{\epsilon_i\in\{0,1,-1\}}.$$

The expression *orbit* refers to the action of $\pi_1(U_r,\boldsymbol{z})$ (or of $\pi_1(U_r/\mathrm{PGL}_2(\mathbb{C}),\boldsymbol{z}_0)$; §1.1.2) on a divisor class. For convenience, we sometimes write $x_\infty = x_r$.

The divisor mx_i, m odd (resp.even), is equivalent to $x_i+(m-1)x_\infty$ (resp.mx_∞). The collection $\mathcal{D}$ modulo linear equivalence, represent all $\frac{1}{2}$-canonical divisors on $X_{\boldsymbol{z}}$. From (6.13a), you can replace x_i-x_j by x_j-x_i: each $D\in\mathcal{D}$ is equivalent to

$$\mathcal{D}_s \overset{\text{def}}{=} \{\sum_{u=1}^{s} x_{i_u} + mx_r\}_{1\le i_1<\cdots<i_s<r} \text{ with } s+m=g-1. \tag{6.14}$$

Add (6.13b) to any D in (6.14)to conclude it is equivalent to one in $\cup_{s=0}^{\frac{r-2}{2}}\mathcal{D}_s$.

Proposition 6.19. *The set* $\cup_{s=0}^{\frac{r-2}{2}}\mathcal{D}_s$ *consists of inequivalent divisors. With* $r = 2+4\ell$ *(resp.* 4ℓ*), here are the* $\pi_1(U_r,\boldsymbol{z})$ *orbits. For* $s=0,2,\dots,g-2$ *(resp.* $s=1,3,\dots,g-2$*),* $\mathcal{D}_s\cup\mathcal{D}_{s+1}$ *forms 1 orbit of of length* $\binom{r-1}{s}+\binom{r-1}{s+1}$*, leaving* *one (resp. two) orbit(s) $\mathcal{D}_g$ (resp. $\mathcal{D}_0$ and $\mathcal{D}_g$) of length(s) $\binom{r-1}{\frac{r-2}{2}}$ (resp. $\binom{r-1}{\frac{r-2}{2}}$ and $\binom{r-1}{0}$). There are $1+\ell$ orbits on the $\sum_{s=0}^{\frac{r-2}{2}}\binom{r-1}{s}=2^{2g}$ distinct classes in $\mathcal{D}$.*

Proof. Suppose $\cup_{s=0}^{\frac{r-2}{2}}\mathcal{D}_s$ contains equivalent divisors. Then, (6.13a) implies, for some $m\le r-1$, mx_∞ is equivalent to $\sum_{u=1}^{s'}x_{i_u}$, contrary to h is the lowest degree polynomial defining X as a hyperelliptic curve.

We start with the cases $r=6$ ($g=2$) and $r=8$ ($g=3$). For $r=6$:

(6.15a) $\mathcal{D}_0\cup\mathcal{D}_1=\{x_{i_1}, 1\le i\le 6\}$): $6=\binom{6}{1}=\binom{5}{0}+\binom{5}{1}$ elements, and

(6.15b) $\mathcal{D}_2=\{x_{i_1}+x_{i_2}-x_\infty,\ 2\le i<j\le 5\}$: $10=\binom{5}{2}$ elements.

For all r, it is easy to compute the monodromy action. Each element of $\pi_1(U_r,\boldsymbol{z}_0)$ is represented by a permutation of entries of $\boldsymbol{z}$ (§2.1), inducing the same action on $x_1,\dots,x_r$. Then, $\mathcal{D}_0\cup\mathcal{D}_1$ (resp. $\mathcal{D}_2$), an orbit of length 6 (resp. 10), consists of the odd (resp. even) $\frac{1}{2}$-canonical classes.

When $s=0$ and m is even (say, when $r=8$), then $\mathcal{D}_0$ contains mx_∞ whose orbit has length 1. Here are the rest of the orbits for $r=8$:

$$\text{(6.16)}\qquad \begin{array}{rl}\mathcal{D}_1\cup\mathcal{D}_2: & \{x_i+x_\infty, 1\le i\le 7\}\cup\{x_i+x_j, 1\le i<j\le 7\}.\\ \mathcal{D}_3: & \{x_i+x_j+x_k-x_\infty, 1\le i<j<k\le 7\}.\end{array}$$

This case has three orbits, with respective representatives (and orbit lengths) $2x_\infty$ (1), x_1+x_2 $(28=\binom{8}{2}=\binom{7}{1}+\binom{7}{2})$, $x_1+x_2+x_3-x_\infty$ $(35=\binom{7}{3})$.

From this format, an easy induction counts the orbits as in the last paragraph of the Proposition's statement. The quantity $\sum_{s=0}^{\frac{r-2}{2}}\binom{r-1}{s}$ is

$$\text{(6.17)}\qquad \sum_{s=0}^{r-1}\binom{r-1}{s}/2=(1+1)^{r-1}/2=2^{2g},$$

giving the count of the classes represented in $\mathcal{D}$. □

In Prop. 6.19 notation, refer to $\mathcal{D}_s\cup\mathcal{D}_{s+1}$, as appropriate, by $\mathcal{D}'_s$. A $\frac{1}{2}$-canonical class is, respectively, even or odd if the dimension of its linear system (§6.1.1) is even or odd. *Nondegenerate* if this dimension is, respectively, 0 or 1.

Corollary 6.20 (Even and odd θ s). *The set $\mathcal{D}_g$ (resp. $\mathcal{D}'_{g-2}$) consists of nondegenerate even (resp. odd) $\frac{1}{2}$-canonical classes. Similarly, for even (resp. odd) g, the divisors in $\mathcal{D}'_{g-2k}$, $k=1,\dots,\frac{g}{2}$ (resp. $k=1,\dots,\frac{g-1}{2}$) are odd (resp. even) if k is odd (resp. even). When g is odd, $\mathcal{D}_0$ has the same parity as $\frac{g+1}{2}$.*

The total number of even (resp. odd) classes is $2^{2g-1}+2^{g-1}$ (resp. $2^{2g-1}-2^{g-1}$).

Proof. There is one pole of order 1 for a divisor in $\mathcal{D}_g$, so there is no divisor of a function that adds to one of these to give a positive divisor. All divisors in $\mathcal{D}_{g-1}$ have support consisting of points of multiplicity 1, so only constant functions can make them positive divisors.

Given r, let a_r and b_r be the respective count of even and odd $\frac{1}{2}$-canonical classes. From Prop. 6.19, $a_r+b_r=2^{2g}$. If we show $a_r-b_r=2^g$. Then, solving for a_r and b_r gives the 2nd paragraph statement. The two cases are similar, so we do just the tougher, $r=4\ell$. For each odd s use $\binom{r-1}{s}+\binom{r-1}{s+1}=\binom{r}{s+1}$. Write a_r-b_r as

$$\text{(6.18)}\qquad \binom{r-1}{g}-\binom{r}{g-1}+\binom{r}{g-3}+\cdots+(-1)^{\ell-1}\binom{r}{2}+(-1)^{\ell}\binom{r-1}{0}.$$

Use $\binom{r-1}{g}=\binom{r-1}{g+1}$ and $\binom{r-1}{0}=\binom{r}{0}$ to write (6.18) – akin to (6.17) – as the real part of $(-1)^\ell(1+i)^r/2=(\sqrt{2}e^{i\pi/4})^{4\ell}/2$. The result is $2^{2\ell-1}=2^g$, and we are done. □

Remark 6.21 (Nondegenerate even and odd θ s)**.** Riemann showed every Riemann surface of genus g has nongenerate odd θ s (see §6.1.5). Apply Rem. 6.20 to label these θ s explicitly using the corresponding $\frac{1}{2}$-canonical class. Example: For odd g, the component of $\mathcal{M}_{g,\pm}$ over $\mathbf{Hyp}_g$ of degree 1 is degenerate for $g\ge 3$.

Appendix A. Interpreting points on Hurwitz spaces

I review how Hurwitz spaces interpret Inverse Galois aspects. [**Fr08b**] has examples. For z transcendental over a field K, a field extension $L/K(z)$ is *regular* if $L\cap\hat{K}=K$. As in §1.1.2, we abbreviate Riemann's Existence Theorem by RET. As in the rest of the paper we denote the degree of $L/K(z)$ by n.

A.1. RIGP and AIGP.

The Regular Inverse Galois Problem over a field K ($\supset\mathbb{Q}$ for simplicity) asks when does a group G ($\le S_n$) appear as the group of a Galois extension $\hat{L}/K(z)$ with $L\cap\bar{K}=K$. We use the acronym RIGP for this.

More often we seek regular extensions $L/K(z)$ with Galois closure $\hat{L}/K(z)$ where, with $\hat{K}$ the constant field of $\hat{L}$, $G(\hat{L}/\hat{K}(z))$ is G. Then, G is the *geometric* monodromy group (of $L/K(z)$), a normal subgroup of the *arithmetic* monodromy $\hat{G}=G(\hat{L}/K(z))$ (also, $\le S_n$). This is a $(G,\hat{G})$ *realization*. Finding such a realization for some $\hat{G}$ is the A(bsolute)IGP (we often care which $\hat{G}$ s are achieved).

Attached to a cover of projective curves $\varphi: X\to\mathbb{P}^1_z$ over K is a collection of branch points $\boldsymbol{z}=\{z_1,\dots,z_r\}$ invariant under G_K. Each z_i corresponds to a conjugacy class C_i in the geometric monodromy group. The collection $\mathrm{C}_1,\dots,\mathrm{C}_r$ is a deformation invariant of the cover. We say all equivalence classes of covers with the datum $(G,\mathbf{C})$ are in the Nielsen class $\mathrm{Ni}(G,\mathbf{C})$, using the notation of §1.1 including for the braid and Hurwitz monodromy groups B_r and H_r.

This section reviews equivalences on covers, and the corresponding spaces associated to $(G,\mathbf{C})$ whose points interpret a solution to the RIGP or AIGP ([**BaFr02**, §2], much from [**Fr77**] or [**FV91**]). That material identifies H_r with the fundamental group of projective r-space, $\mathbb{P}^r$ minus the discriminant locus D_r: $U_r=\mathbb{P}^r\setminus D_r$. The natural map $(\mathbb{P}^1)^r\to\mathbb{P}^r$, modulo the action of S_r, takes the fat diagonal Δ_r to D_r. This interprets U_r as the space of r distinct unordered points in $\mathbb{P}^1$.

A.2. Absolute equivalence.

With $\mathbf{C}$ a collection of conjugacy classes of G, denote the automorphisms of G, from conjugations in S_n, that permute entries in $\mathbf{C}$ by $\mathrm{Aut}_{\mathbf{C}}(G)$. Conjugation by G induces inner automorphisms of G. Denote the subgroup of $\mathrm{Aut}_{\mathbf{C}}(G)$ induced by inner automorphisms by $\mathrm{Inn}(G)$. With G' any group between $\mathrm{Aut}_{\mathbf{C}}(G)$ and $\mathrm{Inn}(G)$ (allow both extremes), denote the quotient of the conjugation action of G' on $\mathrm{Ni}(G,\mathbf{C})$ by $\mathrm{Ni}(G,\mathbf{C})/G'$. When $G'=G$, use $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{in}}$ for $\mathrm{Ni}(G,\mathbf{C})/G'$.

Definition A.1. For $H\le G$, call H self-normalizing in G if $N_G(H)=H$. This is equivalent to no element of $S_{n(H)}\setminus H$ centralizes H [**Fr77**, Lem. 2.1]. Primitivity of $T_H: G\to S_{n(H)}$ is sufficient but not necessary (Ex. A.4).

A.2.1. *Absolute Nielsen classes.* For $H \le G$, of index $n(H) = (G : H)$, denote elements in $S_{n(H)}$ that normalize G and permute the conjugacy classes $\mathbf{C}$ by $S_{n(H)}(G, \mathbf{C})$. Denote the quotient $\mathrm{Ni}(G, \mathbf{C})/S_{n(H)}(G, \mathbf{C})$ by $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}(H)}$.

Lemma A.2 (Equivalence Lemma)**.** *The action of B_r on $\mathrm{Ni}(G, \mathbf{C})/G$ induces an action of B_r (or H_r) on $\mathrm{Ni}(G, \mathbf{C})/G'$. Suppose $H \le G$ is self-normalizing in G and H contains no nontrivial normal subgroup of G. Let $T_H : G \to S_{n(H)}$ be the faithful representation on the $n(H)$ cosets of H. Then, T_H extends to G' if and only if G' maps the conjugates of H in G among themselves. This case induces an H_r equivariant map from $\mathrm{Ni}(G, \mathbf{C})/G'$ to $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}(H)}$.*

Proof. Two transitive permutation representations of G are (permutation) equivalent if and only they have the same point stabilizers. Let $H_1, \dots, H_t$ (with $H_1 = H$) be the conjugates of H in G. An element of G acts by conjugation on $H_1, \dots, H_t$. Since H is self-normalizing, the stabilizer of H in this action is just H. Let $T : G \to S_{n(H)}$ be the corresponding permutation representation. As T and T_H have the same point stabilizers, these representations are permutation equivalent. To extend T_H to G', it suffices to extend T to G'. This is now automatic. Everything else in the lemma follows immediately from the definitions. □

As above, G' is any group between $\mathrm{Aut}_{\mathbf{C}}(G)$ and G. From fundamental group theory, as in [**Fr77**] or [**FV91**], H_r acting on $\mathrm{Ni}(G, \mathbf{C})/G'$ produces an unramified cover $\Phi^{G'} : \mathcal{H}(G, \mathbf{C})^{G'} \to U_r$. So, $\mathcal{H}(G, \mathbf{C})^{G'}$ is a manifold. Connected components of $\mathcal{H}(G, \mathbf{C})^{G'}$ correspond one-one with B_r (or H_r) orbits on $\mathrm{Ni}(G, \mathbf{C})/G'$.

There are two natural equivalences of covers.

- (A.1a) $\varphi_i : X_i \to \mathbb{P}^1$, $i = 1, 2$, are equivalent if some continuous $\alpha : X_1 \to X_2$ satisfies $\varphi_2 \circ \alpha = \varphi_1$.
- (A.1b) As in (A.1a), except there is $\beta \in \mathrm{PGL}_2(\mathbb{C})$ (§5.2) with $\varphi_2 \circ \alpha = \beta \circ \varphi_1$.

Call equivalence (A.1a) (resp. (A.1b)) *absolute* (resp. *reduced* absolute) equivalence. Any (degree n) cover produces a (degree n) permutation representation of the geometric monodromy group of the cover.

A.2.2. *Hurwitz spaces.* RET says elements of $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}(H)}$ correspond one-one to covers of $\mathbb{P}^1$ with unordered branch points $\boldsymbol{z}$ (modulo absolute equivalence) in the Nielsen class $\mathrm{Ni}(G, \mathbf{C})$. The complete collection of these equivalence classes of covers as $\boldsymbol{z}$ varies in U_r forms a covering space, $\mathcal{H}(G, \mathbf{C})^{\mathrm{abs}(H)}$ of U_r. We drop the H in $\mathrm{abs}(H)$ if we know it from the context.

There is a natural compactification $\bar{\mathcal{H}}(G, \mathbf{C})^{\mathrm{abs}(H)}$ of $\mathcal{H}(G, \mathbf{C})^{\mathrm{abs}(H)}$ over $\mathbb{P}^r$. If $\mathcal{H}'$ is any component of $\mathcal{H}(G, \mathbf{C})^{\mathrm{abs}(H)}$, let $\bar{\mathcal{H}}'$ be the normalization of $\mathbb{P}^r$ in the function field of $\mathcal{H}'$, and take the disjoint union of these $\bar{\mathcal{H}}'$, a projective variety by Grauert-Remmert ([**GrR57**], as used in [**Fr77**, §4]).

[**Ser92**, p. 56] succinctly goes through the literature to show an analytic cover of a Zariski subspace of a projective variety extends to a general cover of projective varieties. It notes that [**GrR57**], extending the original cover to a cover of complete analytic spaces, is delicate. By contrast, [**FV91**, p. 788] notes: When G has a self-normalizing representation, we see the affine structure on $\mathcal{H}(G, \mathbf{C})^{\mathrm{abs}(H)}$ from the 1-dimensional RET at generic branch points and normalization in a function field. So, this gives the quasiprojective structure for all absolute and inner spaces directly.

This also works for reduced Hurwitz spaces, equivalence (A.1b). [**Vö96**, §10.2] overlooked this. When $r > 4$ there are other compactifications, related to admissible covers [**We98**], but this one is useful.

The index of $g \in S_n$ is $n - m$ where m is the number of orbits of g. Covers in an absolute Nielsen class $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}(H)}$ have a genus $g_{G,\mathbf{C},H} = g_{\boldsymbol{g}}$ defined by

$$\text{(A.2)} \qquad \text{Riemann-Hurwitz: } 2(n + g_{\boldsymbol{g}} - 1) = \sum_{i=1}^{r} \mathrm{ind}(g_i).$$

This is the genus of a cover of $\mathbb{P}^1_z$ with branch cycles given by $\boldsymbol{g}$. As above, equivalence (A.1a) presents a space $\mathcal{H}$ of covers with r branch points as a cover of U_r. Equivalence (A.1b) gives a different target, J_r: the bi-quotient of $(\mathbb{P}^1)^r \setminus \Delta_r$ by $\mathrm{PGL}_2(\mathbb{C})$ (linear fractional transformations) and S_r. Here, $\mathrm{PGL}_2(\mathbb{C})$ acts diagonally on $(\mathbb{P}^1)^r$ and S_r acts by permuting these coordinates. These actions commute. Example: J_4 is the traditional j-line minus the cusp at ∞.

A.2.3. *Interpreting self-normalizing Def. A.1.* Self-normalizing is equivalent to $\varphi_{\boldsymbol{p}} : X_{\boldsymbol{p}} \to \mathbb{P}^1$ – representing $\boldsymbol{p} \in \mathcal{H}(G, \mathbf{C})^{\mathrm{abs}(H)}$ – has no automorphisms commuting with $\varphi_{\boldsymbol{p}}$. *Fine moduli* for $\mathcal{H}(G, \mathbf{C})^{\mathrm{abs}(H)}$ means there is a *unique* family of representing covers (as in §1.1.3) and self-normalizing is equivalent [**Fr77**, Lem. 2.1].

Let $\Psi^{\mathrm{abs}(H)} : \mathcal{T}^{\mathrm{abs}(H)} \to \mathcal{H}(G, \mathbf{C})^{\mathrm{abs}(H)} \times \mathbb{P}^1$ be the corresponding family: For $\boldsymbol{p} \in \mathcal{H}(G, \mathbf{C})^{\mathrm{abs}(H)}$, restricting $\mathcal{T}^{\mathrm{abs}(H)}$ to the fiber of $\mathcal{T}^{\mathrm{abs}(H)}$ over $\boldsymbol{p} \times \mathbb{P}^1$ is a cover representing $\boldsymbol{p}$. [**FV91**, §4] shows $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$ has a unique representing family if G has no center. When there is a self-normalizing H, §A.2.4 constructs $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$ directly using geometric Galois closure.

Example A.3. Use $(\gamma_0, \gamma_1, \gamma_\infty)$ from the **sh**-incidence calculation in Prop. 3.5. Denote their restrictions to lifting invariant +1 (resp. -1) orbit by $(\gamma_0^+, \gamma_1^+, \gamma_\infty^+)$ (resp. $(\gamma_0^-, \gamma_1^-, \gamma_\infty^-)$). We read indices of the $+$ (resp. $-$) elements from the Ni_0^+ (resp. Ni_0^-) matrix block: Cusp widths over ∞ add to the degree $4+2+3=9$ (resp. $4+1+1=6$) to give $\mathrm{ind}(\gamma_\infty^+) = 6$ (resp. $\mathrm{ind}(\gamma_\infty^+ = 3)$; since γ_1^+ (resp. γ_1^-) has 1 (resp. no) fixed point and $\gamma_0^\pm$ have no fixed points, $\mathrm{ind}(\gamma_1^+) = 4$ (resp. $\mathrm{ind}(\gamma_1^-) = 3$) and $\mathrm{ind}(\gamma_0^+) = 6$ (resp. $\mathrm{ind}(\gamma_0^+) = 4$). The genus of $\bar{\mathcal{H}}_{0,\pm}$ is $g_\pm = 0$:

$$2(9 + g_+ - 1) = 6 + 4 + 6 = 16 \text{ and } 2(6 + g_- - 1) = 3 + 3 + 4 = 10.$$

Example A.4. Consider the dihedral group, $D_{p^{k+1}}$, of order $2p^{k+1}$, with p an odd prime and $k > 0$. The standard permutation representation of $D_{p^{k+1}}$ on $\mathbb{Z}/p^{k+1}$ (an involution generates H) is imprimitive though H is self-normalizing.

A.2.4. *Inner Hurwitz spaces.* [**BaFr02**, §3.1.3] gives a Galois closure process (more direct than in [**FV91**]) for inner Hurwitz spaces $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$ from the absolute spaces. To form the Galois closure of a separable, finite flat morphism $\Phi : \mathcal{T} \to W$ of normal varieties over a field K, form the fiber product of Φ, $n = \deg(\Phi)$ times:

$$\{(t_1, \dots, t_n) \in \mathcal{T}^n \mid \Phi(t_i) = \Phi(t_j)\}.$$

The Galois closure (of Φ over K) identifies with the normalization of a non-diagonal absolutely irreducible component $\hat{\mathcal{T}}$ of this algebraic set. Prop. A.5 shows how to go from $\mathcal{H}^{\mathrm{abs}}$ to $\mathcal{H}^{\mathrm{in}}$ giving the self-centralizing fine moduli condition.

The space we seek is an unramified cover $\mathcal{H}^{\mathrm{in}} \to \mathcal{H}^{\mathrm{abs}}$: Its points $\boldsymbol{p}^{\mathrm{in}} \in \mathcal{H}^{\mathrm{in}}$ over $\boldsymbol{p}^{\mathrm{abs}} \in \mathcal{H}^{\mathrm{abs}}$ representing an absolute Nielsen class, represent the class of pairs

$$(\hat{X} \to \mathbb{P}^1, h : G \to \mathrm{Aut}(\hat{X}/\mathbb{P}^1))$$

in the inner Nielsen class $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{in}}$. Then, $\hat{X} \to \mathbb{P}^1$ is a geometrically Galois cover with group G having branch points $\boldsymbol{z}$; and h is an isomorphism between G and the automorphism group of the cover. Mapping between inner and absolute spaces takes $\boldsymbol{p}^{\mathrm{in}}$ to $\boldsymbol{p}^{\mathrm{abs}} = \Phi^{\mathrm{in}}_{\mathrm{abs}(H)}(\boldsymbol{p}^{\mathrm{in}})$ and $\boldsymbol{z} = \Phi^{\mathrm{abs}} \circ \Phi^{\mathrm{in}}_{\mathrm{abs}(H)}(\boldsymbol{p}^{\mathrm{in}})$.

Since $\hat{X}$ is a subset of X^n, identify G as the subgroup of S_n mapping $\hat{X}$ into itself. Given self-normalizing, Prop. A.5 goes from a family of absolute covers over any parameter space to a family of Galois closures of these covers.

Proposition A.5. *If $H \le G$ is self-normalizing (§A.2.3) a K component of the Galois closure of a total family $\Phi^{\mathrm{abs}} : \mathcal{T}^{\mathrm{abs}} \to \mathcal{H}^{\mathrm{abs}} \times \mathbb{P}^1_z$ gives $\Phi^{\mathrm{in}} : \mathcal{T}^{\mathrm{in}} \to \mathcal{H}^{\mathrm{in}} \times \mathbb{P}^1_z$, with $\mathcal{H}^{\mathrm{in}}$ the normalization of $\mathcal{H}^{\mathrm{abs}}$ in $\mathcal{T}^{\mathrm{in}}$. Suppose $L \supset K$. Then, an L point $\boldsymbol{p}^{\mathrm{in}} \in \mathcal{H}(G,\mathbf{C})^{\mathrm{in}}$ corresponds to an L component of the Galois closure of $\varphi_{\boldsymbol{p}^{\mathrm{abs}}}$.*

We have a sequence of covers

$$\mathcal{H}(G,\mathbf{C})^{\mathrm{in}} \xrightarrow{\Phi^{\mathrm{in}}_{\mathrm{abs}(H)}} \mathcal{H}(G,\mathbf{C})^{\mathrm{abs}(H)} \xrightarrow{\Phi^{\mathrm{abs}}} U_r$$

from inner to absolute Hurwitz space. For $\mathcal{H}$ a component of $\mathcal{H}(G,\mathbf{C})^{\mathrm{in}}$ and $\mathcal{H}'$ its image by $\Phi^{\mathrm{in}}_{\mathrm{abs}(H)}$, $\mathcal{H} \to \mathcal{H}'$ is Galois and unramified. Its group is the subgroup of $N_{S_{n(H)}}(G,\mathbf{C})/G$ that stabilizes the braid orbit in $\mathrm{Ni}(G,\mathbf{C})$ associated to $\mathcal{H}'$.

COMMENTS. Let $\Phi : \mathcal{T} \to \mathcal{H} \times \mathbb{P}^1_z$ be any r branch point, degree n, family of $\mathbb{P}^1_z$ covers. Form the n-fold fiber product of Φ. Normalize the result, and take a (non-diagonal) connected component $\hat{\mathcal{T}}$. This induces $\Phi^{(n)} : \hat{\mathcal{T}} \to \mathcal{H} \times \mathbb{P}^1_z$. Then, take $\hat{\mathcal{H}}$ to be the normalization of $\mathcal{H}$ in the function field of $\hat{\mathcal{T}}$.

We can (geometrically) compare the Galois closure of the sphere covers at any fiber $\Phi_{\boldsymbol{p}} : \mathcal{T}_{\boldsymbol{p}} \to \boldsymbol{p} \times \mathbb{P}^1_z$, for $\boldsymbol{p} \in \mathcal{H}$, to the components of the fiber $\hat{\mathcal{T}}_{\boldsymbol{p}} \to \boldsymbol{p} \times \mathbb{P}^1_z$. We expect several geometric copies (u, say, locally constant in $\boldsymbol{p}$) of the Galois closure of $\mathcal{T}_{\boldsymbol{p}} \to \boldsymbol{p} \times \mathbb{P}^1_z$ for each $\boldsymbol{p}$. Then, u is the degree of $\hat{\mathcal{H}}$ over $\mathcal{H}$. Local constancy of u results from deforming classical generators of the r-punctured sphere, as in [**Fr77**, §4]. This works in positive characteristic only for tamely ramified covers. □

Example A.6 (Ex. 1.5 cont.)**.** Even for $G = A_n$, the value of $u = \deg \Phi^{\mathrm{in}}_{\mathrm{abs}(H)}$ in Prop. A.5 is nonobvious. It is 1 for $K = \bar{\mathbb{Q}}$ in Ex. 1.5, $n \equiv 1 \mod 8$, while it is 2 for $K = \mathbb{Q}$. It is always 2 in Ex. 1.5, $n \equiv 5 \mod 8$, or in Thms. 1.2 or 1.3.

Here is the RIGP-AIGP interpretation, a variant on [**FV91**, Main Thm.].

Proposition A.7. *A regular extension $L/K(z)$ over K in the Nielsen class $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{abs}(H)}$ corresponds to $\boldsymbol{p} \in \mathcal{H}(G,\mathbf{C})^{\mathrm{abs}(H)}(K)$. If H is self-normalizing, then conversely, any $\boldsymbol{p} \in \mathcal{H}(G,\mathbf{C})^{\mathrm{abs}(H)}(K)$ corresponds to a regular extension over K in the Nielsen class. This gives a $(G,\hat{G})$ (K) realization (as in §A.1) with $\hat{G}/G = G(K(\hat{\mathbf{p}})/K)$ for $\hat{\mathbf{p}} \in \mathcal{H}(G,\mathbf{C})^{\mathrm{in}}$ over $\boldsymbol{p}$.*

Regular Galois $\hat{L}/K(z)$ in $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{in}}$ corresponds to $\boldsymbol{p} \in \mathcal{H}(G,\mathbf{C})^{\mathrm{in}}(K)$: A solution of the RIGP over K. Conversely, for H centerless, any $\hat{\mathbf{p}} \in \mathcal{H}(G,\mathbf{C})^{\mathrm{in}}(K)$ gives a regular Galois extension in the Nielsen class.

A.3. Reduced versions of moduli spaces. Points on nonreduced absolute moduli spaces correspond to sphere covers in a given Nielsen class. Suppose G is centerless and $H \le G$ is self-normalizing. We see the Inverse Galois problem structure from the relation between $\mathcal{H}(G,\mathbf{C})^{\mathrm{abs}(H)}$ and $\mathcal{H}(G,\mathbf{C})^{\mathrm{in}}$ (§5.1). Even with self-normalizing, conveniently interpreting points for the reduced moduli spaces $\mathcal{H}(G,\mathbf{C})^{\mathrm{abs}(H)}/\mathrm{PGL}_2(\mathbb{C})$ and $\mathcal{H}(G,\mathbf{C})^{\mathrm{in}}/\mathrm{PGL}_2(\mathbb{C})$ depends on the situation.

To see this, consider the case tied to modular curves: $G = D_p$ (p an odd prime) with $\mathbf{C} = \mathbf{C}_{2^4}$, four repetitions of the involution conjugacy class ([**Fr90**] and [**DF94**, §5.1–§5.2]) Then, $\mathcal{H}(D_p, \mathbf{C}_{3^4})^{\rm in}/\mathrm{PGL}_2(\mathbb{C})$ identifies with the classical $Y_1(p)$ (modular curve without cusps). Points of $Y_1(p)$ correspond to equivalence classes of pairs $(E, \boldsymbol{e})$ with E an elliptic curve and $\boldsymbol{e}$ an order p torsion point on E.

If multiplication by -1 is the only automorphism of E, then $(E, -\boldsymbol{e})$ is also in this class. Also, if $(E, \boldsymbol{e}')$ is here, then $\boldsymbol{e}' = \pm\boldsymbol{e}$. Let O be the $\mathrm{PGL}_2(\mathbb{C})$ orbit in U_r that maps to the j-line value of E. Choose any $\boldsymbol{x} \in O$. To show $Y_1(p)$ is $\mathcal{H}(D_p, \mathbf{C}_{3^4})^{\rm in}/\mathrm{PGL}_2(\mathbb{C})$ requires recovering $(\hat{X} \to \mathbb{P}^1, h : D_p \to \mathrm{Aut}(\hat{X}/\mathbb{P}^1))$ up to conjugation by D_p from the triple $(E, \boldsymbol{e}, \boldsymbol{x})$ up to equivalence. This works; $\boldsymbol{x}$ determines a degree 2 map from $E/\langle \boldsymbol{e}\rangle \to \mathbb{P}^1$. The sequence $E \to E/\langle \boldsymbol{e}\rangle \to \mathbb{P}^1$ is $E \to \mathbb{P}^1$, a (geometric) Galois cover with group D_p. The collection of points $\pm\boldsymbol{e}$ determines an isomorphism of this group with D_p up to conjugacy by D_p.

Finally, the reduced absolute space in this case interprets naturally as $Y_0(p)$ in a diagram coming from the map $\mathcal{H}(G, \mathbf{C})^{\rm in} \to \mathcal{H}(G, \mathbf{C})^{\mathrm{abs}(H)}$. That is, map the equivalence class of $(E, \boldsymbol{e})$ to the equivalence class $(E, E/\langle \boldsymbol{e}\rangle)$.

Now we quote the literature for the algebraic structure on the reduced spaces.

Proposition A.8 (Reduction Proposition)**.** *With $G \le G' \le \mathrm{Aut}_{\mathbf{C}}(G)$ as above, the quotients $U_r^{\rm rd} = J_r$ (§A.2.2), and $\mathcal{H}(G, \mathbf{C})^{G'}/\mathrm{PGL}_2(\mathbb{C}) \stackrel{\rm def}{=} \mathcal{H}(G, \mathbf{C})^{G',\rm rd}$ are affine algebraic varieties with an induced finite map $\Psi^{\rm rd} : \mathcal{H}(G, \mathbf{C})^{G',\rm rd} \to J_r$.*

Proof. The discriminant locus in $\mathbb{P}^r$ is a hypersurface. Therefore, its complement, U_r, is affine. Since $\mathcal{H}(G, \mathbf{C})^{G'}$ doesn't ramify over U_r, it is normal. Further, the Grauert-Remmert version of RET applies [**Har77**, p. 442]. There is a unique normal projective variety $\overline{\mathcal{H}(G, \mathbf{C})^{G'}}$ with a finite covering $\bar{\Psi} : \overline{\mathcal{H}(G, \mathbf{C})^{G'}} \to \mathbb{P}^r$ whose restriction over U_r gives the natural covering map $\Psi : \mathcal{H}(G, \mathbf{C})^{G'} \to U_r$. Since Ψ is a finite cover, it is an affine cover and $\mathcal{H}(G, \mathbf{C})^{G'}$ is an affine variety.

Apply [**MuFo82**, Thm. 1.1, p. 27] to the affine scheme $\mathcal{H}(G, \mathbf{C})^{G'}$ and the reductive group $\mathrm{PGL}_2(\mathbb{C})$. If the action of $\mathrm{PGL}_2(\mathbb{C})$ is closed, $\mathcal{H}(G, \mathbf{C})^{G',\rm rd}$ with the natural map $\Gamma : \mathcal{H}(G, \mathbf{C})^{G'} \to \mathcal{H}(G, \mathbf{C})^{G',\rm rd}$ is a *universal geometric quotient* and affine. We see the $\mathrm{PGL}_2(\mathbb{C})$ orbit of $\boldsymbol{p} \in \mathcal{H}(G, \mathbf{C})^{G'}(\mathbb{C}) \leftrightarrow \varphi$ (with branch set $\boldsymbol{z}$) is closed by considering any limit $\alpha_n \circ \varphi$ with $\alpha_n \in \mathrm{PGL}_2(\mathbb{C})$. This comes to showing any limit of $\boldsymbol{z}$ under $\{\alpha_n\}_{n=0}^{\infty}$ is in U_r; or the analog for $(z_1, \dots, z_r) \in U^r \stackrel{\rm def}{=} (\mathbb{P}^1)^r \setminus \Delta_r$ (§1.1.2) replacing $\boldsymbol{z}$. What α_n does to $(z_1, \dots, z_r)$ determines it, so this determines the limit of the α_n in $\mathrm{PGL}_n(\mathbb{C})$. When $r = 4$, $\mathcal{H}(G, \mathbf{C})^{G',\rm rd}$ is a curve; its completion ramifies over $\bar{J}_4 = \mathbb{P}^1_j$ with connected components one-one to those of $\mathcal{H}(G, \mathbf{C})^{G'}$. For $r \ge 5$, $\mathcal{H}(G, \mathbf{C})^{G',\rm rd}$ may have singularities. □

For $r = 4$, recall $\mathcal{Q}''$ of (3.10a) and the elements γ_0 and γ_1 of (3.10c).

Remark A.9 (Fine reduced moduli)**.** §A.2.3 discussed fine moduli for inner (no center in G) and absolute (H self-normalizing) Hurwitz spaces. These respective conditions must hold for reduced fine moduli. Yet, you need more [**BaFr02**, Prop. 4.7]. For $r \ge 5$, it is that the reduced space $\mathcal{H}^*/\mathrm{PGL}_2(\mathbb{C})$ ($* =$ in or abs) is nonsingular. The analog for $r = 4$ is that γ_0 and γ_1 on *reduced Nielsen classes*, $\mathrm{Ni}(G, \mathbf{C})^*/\mathcal{Q}''$, have no fixed points. For $r = 4$ there is also one more condition: $\mathcal{Q}''$ has *only maximal length* (that would be 4; action through a Klein 4-group) orbits on $\mathrm{Ni}(G, \mathbf{C})^*$. [**BaFr02**, Ex. 8.5] notes that neither condition holds for $\mathrm{Ni}(A_5, \mathbf{C}_{3^4})^{\rm in}$

(in Thm. 1.2). The two components $\mathcal{H}_\pm$ attached to $\mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2})^{\mathrm{in,rd}}$ both fail the $\mathcal{Q}''$ condition (its orbit lengths on both are 2), while neither γ_0 or γ_1 have fixed point on $\bar{\mathcal{H}}_-$, but γ_1 does on $\bar{\mathcal{H}}_+$ [**Fr06**, §6.6.3].

Appendix B. Producing $\frac{1}{2}$-canonical differentials

Let X be a compact Riemann surface. Riemann used *certain* theta functions on X to give a constructive approach to all its functions and differential forms. We collects observations on the key ingredient, $\frac{1}{2}$-canonical differentials.

B.1. Θ data. Suppose X appears in a smooth family $\Psi : \mathcal{X} \to \mathcal{P}$ of Riemann surfaces as the fiber $X_{\boldsymbol{p}}$ over $\boldsymbol{p} \in \mathcal{P}$. Let $(\boldsymbol{\alpha} \stackrel{\text{def}}{=} (\alpha_1, \dots, \alpha_g), \boldsymbol{\beta} \stackrel{\text{def}}{=} (\beta_1, \dots, \beta_g)$ be a *canonical homology basis* (of $H_1(X, \mathbb{Z})$) for X: The cup-product image of (α_i, β_j) (resp. (α_i, α_j) and (β_i, β_j)) in $H_2(X, \mathbb{Z}) \equiv \mathbb{Z}$ is $\delta_{i,j}$ (resp. 0) for all $1 \le i, j \le g$.

If so, we can represent the cup product as a skew-symmetric $2g \times 2g$ matrix E which extends to an $\mathbb{R}$-bilinear form on $H_1(X, \mathbb{R})$. We can write

$$\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} E(\boldsymbol{\alpha}^{\mathrm{tr}} | \boldsymbol{\beta}^{\mathrm{tr}}) = J_{2g} \text{ (on the left the } i\text{th row is } \alpha_i,\ 1 \le i \le g, \text{ etc.).}$$

Then, an element $U = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{Z})$ as in §6.1.4, acting on the left, transforms $\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}$ so as to preserve the pairing. [**Sh98**, §3.1] calls $f(\boldsymbol{w})$ a θ function if $f(\boldsymbol{w} + \gamma) = f(\boldsymbol{w}) e^{l_\gamma(\boldsymbol{w}) + c_\gamma}$ for $\gamma \in H_1(X, \mathbb{Z})$, $l_\gamma(\boldsymbol{w})$ linear in $\boldsymbol{w}$ and $c_\gamma \in \mathbb{C}$.

This treatment expresses that any θ comes from a slight generalization of E. Conversely, there is a θ on a complex torus if and only if there is an associated E. Riemann's θ is that attached to this particular E.

Denote positive divisors of degree $g-1$ on X by W_{g-1}. Note that W_{g-1} is independent of $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, but the $\frac{1}{2}$-canonical class and $\Theta_{X_{\boldsymbol{p}}}$ is not.

[**BaFr02**, App. B] explains that for his generalization of Abel's Theorem, Riemann wanted θ_X *odd* and nondegenerate: θ has nonzero gradient (see §B.2.2) at the origin of $\widetilde{\mathrm{Pic}}^{(0)} = \mathbf{C}^g$, as in §6.1.3. This gave his generalization of Abel's Theorem. So, instead of the even $\theta(\boldsymbol{\ell}, \boldsymbol{w})$ in (6.4), he needed a different θ with 2-division characteristic. We want θ_X even, but also nondegenerate: not zero at the origin.

In Thm. 6.15, we needed a point on the Hurwitz space so that *no* θ with 2-division characteristic was zero at the origin. Riemann showed some θs with 2-division characteristic will work at each Riemann surface. Still, for a general family of covers $\mathcal{H}$ with odd order branching (as in Prop. 6.6) it may be that none of those match the $\frac{1}{2}$-canonical class θ-null defined by Lem. 6.2. Here is the exact criterion.

Proposition B.1. *The corresponding θ-null is nontrivial exactly when there is $\boldsymbol{p} \in \mathcal{H}$ so the $\frac{1}{2}$-canonical class $\delta_{\boldsymbol{p}}$ attached to $\boldsymbol{p}$ contains no positive divisor.*

B.2. $\frac{1}{2}$-canonical differentials from coordinate charts. Suppose X is an n-dimensional complex manifold. Let $\{U_\alpha, \varphi_\alpha\}_{\alpha \in I}$ be the coordinate chart, with $\{\psi_{\beta,\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1}\}_{\alpha,\beta \in I}$ the corresponding transition functions. Each $\psi_{\beta,\alpha}$ is one-one and analytic on an open subset of $\mathbb{C}^n$ whose coordinates we label $z_{\alpha,1}, \dots, z_{\alpha,n}$.

B.2.1. *Reminder on cocycles.* Denote the $n \times n$ *complex Jacobian matrix* for $\psi_{\beta,\alpha}$ by $J(\psi_{\beta,\alpha})$. Call the matrices $\{J(\psi_{\beta,\alpha})\}_{\alpha,\beta \in I}$ the (transformation) *cocycle* attached to meromorphic differentials.

Similarly $\{J(\psi_{\beta,\alpha})^{-1}\}_{\alpha,\beta\in I}$ is the cocycle attached to meromorphic tangent vectors. Recall the notation for $n\times n$ matrices, $\mathbb{M}_n(R)$ with entries in an integral domain R and for the *invertible* matrices $\mathrm{GL}_n(R)$ with entries in R under multiplication. Cramer's rule says for each $A\in\mathbb{M}_n(R)$ there is an adjoint matrix A^* so that AA^* is the scalar matrix $\det(A)I_n$ given by the determinant of A. This shows the invertibility of $A\in\mathbb{M}_n(R)$ is equivalent to $\det(A)$ being a *unit* (in the multiplicatively invertible elements R^*) of R. Denote the $n\times n$ identity matrix (resp. zero matrix) in $\mathrm{GL}_n(R)$ by I_n (resp. $\mathbf{0}_n$). If $U\subset X$ is an open set, denote the *holomorphic* functions on U by $\mathrm{Hol}(U)$.

Definition B.2 (1-cycocle)**.** Suppose $g_{\beta,\alpha}\in\mathrm{GL}_n(\mathrm{Hol}(U_\alpha\cap U_\beta))$, $\alpha,\beta\in I$. Assume $g_{\gamma,\beta}g_{\beta,\alpha}=g_{\gamma,\alpha}$ for all $\alpha,\beta,\gamma\in I$ on $U_\alpha\cap U_\beta\cap U_\gamma$ (if nonempty). Then, $\{g_{\beta,\alpha}\}_{\alpha,\beta\in I}$ is a multiplicative *1-cocycle with values in* $\mathcal{GL}_{n,X}$. Similarly, suppose $g_{\beta,\alpha}\in\mathbb{M}_n(\mathcal{H}(U_\alpha\cap U_\beta))$, $\alpha,\beta\in I$. Suppose $g_{\gamma,\beta}+g_{\beta,\alpha}=g_{\gamma,\alpha}$ for all $\alpha,\beta,\gamma\in I$ on $U_\alpha\cap U_\beta\cap U_\gamma$. Then, $\{g_{\beta,\alpha}\}_{\alpha,\beta\in I}$ is an additive *1-cocycle with values in* $\mathbb{M}_{n,X}$.

When there are k-cocycles, there are also (k-1)-chains and their associated k-boundaries. We write the definition for GL_n, recognizing there are analogous versions for all other types of cocycles.

Definition B.3 (1-boundary)**.** Consider $u_\alpha\in\mathrm{GL}_n(\mathrm{Hol}(U_\alpha))$, $\alpha\in I$. If

$$g_{\beta,\alpha}=u_\beta(u_\alpha)^{-1}\text{ in }U_\alpha\cap U_\beta$$

(if nonempty) for all $\alpha,\beta\in I$), then $\{g_{\beta,\alpha}\}_{\alpha,\beta\in I}$ is a 1-cocycle, called a *1-boundary* with values in $\mathcal{GL}_{n,X}$. Call the set $\{u_\alpha\}_{\alpha\in I}$ a *0-chain* with values in $\mathcal{GL}_{n,X}$.

B.2.2. *Square hypothesis versus sections of* $\frac12$-canonical *cocycles.* Let ω – represented by $\{f_\alpha(z_\alpha)dz_\alpha\}_{\alpha\in I}$ – be a differential on a compact Riemann surface X satisfying Square Hypothesis (6.11). Example: One produced by the differential of a function with odd order branching, as in Lem. 6.2. If each U_α is simply connected, then $f_\alpha(z_\alpha)$ has two meromorphic square-roots $\pm h_\alpha(z_\alpha)$ on $\varphi_\alpha(U_\alpha)$.

Use transition function notation $\psi_{\beta,\alpha}$ from §B.2.1 to consider existence of a well-defined $\frac12$-canonical *differential* whose divisor (on X) is $\boldsymbol{k}=(\omega)/2$. For that, we must choose signs on the h_α s so as, on $\varphi_\alpha(U_\alpha\cap U_\beta)$, to assert equality:

$$\tau_\alpha(z_\alpha)=h_\alpha(z_\alpha)\sqrt{dz_\alpha}=\tau_\beta(\psi_{\beta,\alpha}(z_\alpha))=h_\beta(\psi_{\beta,\alpha}(z_\alpha))\sqrt{d\psi_{\beta,\alpha}(z_\alpha)}. \tag{B.1}$$

If so, call such a collection $\{h_\alpha\}_{\alpha\in I}$ a (meromorphic) section of (the *bundle* of) $\boldsymbol{k}$.

Proposition B.4. *Assume $U_\alpha\cap U_\beta$, $(\alpha,\beta)\in I\times I$ is simply connected and you have chosen $\sqrt{J(\psi_{\beta,\alpha})}=k_{\beta,\alpha}$ on $U_\alpha\cap U_\beta$. If you can sign the h_α s to give equality in* (B.1) *for all (α,β), then $\{k_{\beta,\alpha}\}_{(\alpha,\beta)\in I\times I}=\boldsymbol{k}$ is a (*$\frac12$-canonical*) cocycle.*

Suppose $\boldsymbol{k}$ and $\boldsymbol{k}'$ are two $\frac12$-canonical *cocycles that differ by a coboundary. Then, one has a section if and only if the other does. So, there are 2^{2g} such* $\frac12$-canonical *cocycles modulo coboundaries corresponding to h_α sign changes.*

Proof. Assume there is a choice of signs that gives equality in (B.1). Then, on a triple α,β,γ with $U_\alpha\cap U_\beta\cap U_\gamma\neq\emptyset$, dropping the extra evaluation notation, check the compatibility of the equations $h_\gamma=h_\alpha k_{\gamma,\alpha}$, $h_\gamma=h_\beta k_{\gamma,\beta}$ and $h_\beta=h_\alpha k_{\beta,\alpha}$. Substitute the 3rd in the 2nd, then equate to the first to see $\boldsymbol{k}$ is a co-cycle.

If $\boldsymbol{k}$ and $\boldsymbol{k}'$ differ by a coboundary from $\boldsymbol{m}=\{m_\alpha(\in\{\pm1\})\}_{\alpha\in I}$, then we can multiply the section $\{h_\alpha\}_{\alpha\in I}$ by $\boldsymbol{m}$ to get $\{h_\alpha\cdot m_\alpha\}_{\alpha\in I}$ to get a section of $\boldsymbol{k}'$. More generally, all choices of $\boldsymbol{m}$ as above, modulo coboundaries, that give allowable sign changes in the h_α s correspond to homomorphisms of the fundamental group of X into $\{\pm 1\}$. There are 2^{2g} such homomorphisms. □

The square of a $\frac{1}{2}$-canonical differential ((B.1) holds) gives a differential satisfying the Square Hypothesis. Yet, the converse may not hold. Consider those $\frac{1}{2}$-canonical classes on a compact surface X for which there is an h_δ cocycle section with no poles. [**Fay73**, Thm. 4.21, due to Mumford] says the collection of their squares generates the space of holomorphic differentials. [**Fay73**, p. 16] says this produces a nondegenerate (after Def. 6.7) odd $\frac{1}{2}$-canonical class δ. Since Riemann used this result, it is disconcerting the "proof" quotes only modern papers.

Given such a δ, let μ be the 2-division point giving $\theta(\boldsymbol{\ell}, \boldsymbol{w} + \mu)$ with $W_{g-1,X} - \delta$ (compatible with §6.1.4 notation) as zero divisor. This produces the *prime form*,

$$\frac{\theta(\psi(x - x') + \mu)}{h_\delta(x) h_\delta(x')} \text{ with } x, x' \in X, \text{ and } \psi \text{ an embedding of } X \text{ in } \mathrm{Pic}^{(0)}(X).$$

From the prime form, [**Fay73**, Chap. II] constructs all the important objects on X, including functions as expressing Riemann's generalization of Abel's Theorem. Unattributed, but calling it classical, [**Fay73**, p. 17] directly presents the square of the prime form for a surface X presented as a $\mathbb{P}^1_z$ cover.

Problem B.5. Work out ingredients of the prime form along Hurwitz spaces of odd order branching, in Thms. 1.2 and 1.3 and the many more spaces in Rem. 6.14.

Thm. 6.15 uses Riemann-Roch to produce sections of any $\frac{1}{2}$-canonical cocycle with degree bounds on the polar divisor D of the meromorphic section. Further, the existence of $\frac{1}{2}$-canonical sections is what makes the argument using the map $H^0(X, O(\boldsymbol{k}, D)) \to H^1(X \setminus D, \mathbb{C})$ work. Still, there may be a difference between even and odd $\boldsymbol{k}$ s in the fiber dimensions of this "quadratic" map.

Problem B.6. With $\varphi : X \to \mathbb{P}^1_z$ having odd order branching, characterize when some section of a $\frac{1}{2}$-canonical bundle (§6.3.1) has divisor half of $(d\varphi)$.

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