Alternating Multiple Mixed Values

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Abstract. In this paper, we define and study a variant of multiple zeta values (MZVs) of level four, called alternating multiple mixed values or alternating multiple M-values (AMMVs), which forms a subspace of the space of colored MZVs of level four as $\mathbb{Q}[i]$-vector spaces. This variant includes the alternating version of Hoffman's multiple t-values, Kaneko-Tsumura's multiple T-values, and the multiple S-values studied by the authors previously as special cases. We exhibit nice properties similar to the ordinary MZVs such as the duality, integral shuffle and series stuffle relations. After setting up the algebraic framework we derive the regularized double shuffle relations of the AMMVs. We also investigate several alternating multiple T- and S-values by establishing some explicit relations of integrals involving arctangent function. In the end, we discuss the explicit evaluations of a kind of AMMVs at depth three and compute the dimensions of a few interesting subspaces of AMMVs for weight less than 7.

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1 Introduction

We begin with some basic notation. Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of positive integers, the set of real numbers and complex numbers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A finite sequence $\boldsymbol{s} = \boldsymbol{s}_r := (s_1, \dots, s_r)$ of positive integers is called a *composition*. We put

$$|\boldsymbol{s}| := s_1 + \cdots + s_r, \quad \mathrm{dep}(\boldsymbol{s}) := r,$$

and call them the *weight* and the *depth* of $\boldsymbol{s}$, respectively. If $s_1 > 1$, $\boldsymbol{s}$ is called *admissible*.

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1.1 Multiple mixed values and alternating version

In [31], we define the *multiple mixed values* (or *multiple M-values*, MMVs for short) for an admissible composition $\boldsymbol{s} = (s_1, \dots, s_r)$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_r) \in \{\pm 1\}^r$ by

$$\begin{aligned} M^{\boldsymbol{\varepsilon}}(s_1, \dots, s_r) &:= \sum_{m_1 > \cdots > m_r > 0} \frac{(1+\varepsilon_1(-1)^{m_1}) \cdots (1+\varepsilon_r(-1)^{m_r})}{m_1^{s_1} \cdots m_r^{s_r}} \in \mathbb{R} \\ &= \int_0^1 w_0^{s_1-1} w_{\varepsilon_1\varepsilon_2} w_0^{s_2-1} w_{\varepsilon_2\varepsilon_3} \cdots w_0^{s_{r-1}-1} w_{\varepsilon_{r-1}\varepsilon_r} w_0^{s_r-1} w_{\varepsilon_r}, \end{aligned} \tag{1.1}$$

where

$$w_0 := \frac{dt}{t}, \quad w_{-1} := \frac{2dt}{1-t^2}, \quad w_1 := \frac{2tdt}{1-t^2}.$$

The theory of iterated integrals was developed first by K.T. Chen in the 1960's. It has played important roles in the study of algebraic topology and algebraic geometry since then. Its simplest form is

$$\int_0^1 f_1(t)dt f_2(t)dt \cdots f_p(t)dt := \int_{1>t_1>\cdots>t_p>0} f_1(t_1) f_2(t_2) \cdots f_p(t_p) dt_1 dt_2 \cdots dt_p.$$

One can extend these to iterated integrals over any piecewise smooth path on the complex plane via pull-backs. We refer the interested reader to Chen's original work [8–10] for more details. In particular, we have

- MMVs satisfy the series stuffle relations and integral shuffle relations;
- All $\varepsilon_j = 1$ in MMVs $\Longrightarrow$ MZVs$\times \dfrac{1}{2^{s_1+s_2+\cdots+s_r-r}}$;
- All $\varepsilon_j = -1$ in MMVs $\Longrightarrow$ MtVs$\times 2^r$;
- All $\varepsilon_j = (-1)^{r+1-j}$ in MMVs $\Longrightarrow$ MTVs;
- All $\varepsilon_j = (-1)^{r-j}$ in MMVs $\Longrightarrow$ MSVs.

For $\boldsymbol{s} := (s_1, \dots, s_r) \in \mathbb{N}^r$ and $s_1 > 1$, the *multiple zeta values* (MZVs) are defined by (cf. [14, 34])

$$\zeta(\boldsymbol{s}) \equiv \zeta(s_1, \dots, s_r) := \sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}} \in \mathbb{R}.$$

For $\boldsymbol{s} := (s_1, \dots, s_r) \in \mathbb{N}^r$ and $s_1 > 1$, Hoffman's *multiple t-values* (MtVs) are defined by (see [16])

$$t(\boldsymbol{s}) \equiv t(s_1, \dots, s_r) := \sum_{\substack{n_1 > \cdots > n_r > 0 \\ n_i \text{ odd}}} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}$$

$$= \sum_{n_1>\cdots>n_r>0} \frac{1}{(2n_1-1)^{s_1}\cdots(2n_r-1)^{s_r}} \in \mathbb{R}.$$

For $\boldsymbol{s} := (s_1,\ldots,s_r) \in \mathbb{N}^r$ and $s_1 > 1$, the Kaneko-Tsumura *multiple T-values* (MTVs) are defined by (see [18, 19])

$$\begin{aligned} T(\boldsymbol{s}) \equiv T(s_1,\ldots,s_r) &:= 2^r \sum_{\substack{n_1>\cdots>n_r>0 \\ n_i \equiv r-i+1 \pmod 2}} \frac{1}{n_1^{s_1}\cdots n_r^{s_r}} \\ &= 2^r \sum_{n_1>\cdots>n_r>0} \frac{1}{(2n_1-r)^{s_1}\cdots(2n_r-1)^{s_r}} \in \mathbb{R}. \end{aligned}$$

For $\boldsymbol{s} := (s_1,\ldots,s_r) \in \mathbb{N}^r$ and $s_1 > 1$, we defined in [31] the *multiple S-values* (MSVs) by

$$\begin{aligned} S(\boldsymbol{s}) \equiv S(s_1,\ldots,s_r) &:= 2^r \sum_{\substack{n_1>\cdots>n_r>0 \\ n_i \equiv r-i \pmod 2}} \frac{1}{n_1^{s_1}\cdots n_r^{s_r}} \\ &= 2^r \sum_{n_1>\cdots>n_r>0} \frac{1}{(2n_1-r+1)^{s_1}\cdots(2n_r)^{s_r}} \in \mathbb{R}. \end{aligned}$$

As a normalized version of the multiple t-values, we call

$$\begin{aligned} \widetilde{t}(s_1,s_2,\ldots,s_r) &:= \sum_{n_1>n_2>\cdots>n_r\geq 1} \frac{1}{(n_1-1/2)^{s_1}(n_2-1/2)^{s_2}\cdots(n_r-1/2)^{s_r}} \\ &= 2^{s_1+s_2+\cdots+s_r} t(s_1,s_2,\ldots,s_r) \end{aligned}$$

the *multiple $\widetilde{t}$-values.* According to the definitions, $\widetilde{t}(s) = 2^s t(s) = \zeta_H(s;1/2) = (2^s - 1)\zeta(s)$ for integer $s \geq 2$, where $\zeta_H(s;a)$ is the *Hurwitz zeta function* and $\zeta(s)$ is the *Riemann zeta function*.

The systematic study of MZVs began in the early 1990s with the works of Hoffman [14] and Zagier [34]. Due to their surprising and sometimes mysterious appearance in the study of many branches of mathematics and theoretical physics, these special values have attracted a lot of attention and interest in the past three decades (for example, see the book by the third author [38]). For Hoffman's MtVs and Kaneko-Tsumura's MTVs, a number of mathematicians also studied their various formulas similar to MZVs by applying various methods, some recent results can be found in [4, 6, 7, 20, 22–24, 27, 30, 37] and references therein.

In general, let $\boldsymbol{k} = (k_1,\ldots,k_r) \in \mathbb{N}^r$ and $\boldsymbol{z} = (z_1,\ldots,z_r)$, where $z_1,\ldots,z_r$ are Nth roots of unity. We can define the *colored MZVs* (CMZVs) of level N by

$$\mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{z}) := \sum_{n_1>\cdots>n_r>0} \frac{z_1^{n_1}\ldots z_r^{n_r}}{n_1^{k_1}\ldots n_r^{k_r}}, \tag{1.2}$$

which converges if $(k_1, z_1) \neq (1,1)$ (see [33] and [38, Ch. 15]), in which case we call $(\boldsymbol{k};\boldsymbol{z})$ *admissible.* The level two colored MZVs are often called *Euler sums* or *alternating*

MZVs. In this case, namely, when $(z_1,\dots,z_r)\in\{\pm1\}^r$ and $(k_1,z_1)\neq(1,1)$, we set $\zeta(\boldsymbol{k};\boldsymbol{z})=\mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{z})$. Further, we put a bar on top of k_j if $z_j=-1$. For example,

$$\zeta(\bar{2},3,\bar{1},4)=\zeta(2,3,1,4;-1,1,-1,1).$$

More generally, for any composition $(k_1,\dots,k_r)\in\mathbb{N}^r$, the *classical multiple polylogarithm function* (MPL) with r-variables is defined by

$$\mathrm{Li}_{k_1,\dots,k_r}(x_1,\dots,x_r):=\sum_{n_1>n_2>\cdots>n_r>0}\frac{x_1^{n_1}\cdots x_r^{n_r}}{n_1^{k_1}\cdots n_r^{k_r}} \tag{1.3}$$

which converges if $|x_1\cdots x_j|<1$ for all $j=1,\dots,r$. It can be analytically continued to a multi-valued meromorphic function on $\mathbb{C}^r$ (see [35]). In particular, if $x_1=x, x_2=\cdots=x_r=1$, then $\mathrm{Li}_{k_1,\dots,k_r}(x,\{1\}_{r-1})$ is the *single-variable multiple polylogarithm function*, also called *generalized polylogarithm* in by some authors. Here the $\{l\}_m$ denotes the sequence obtained by repeating l exactly m times.

We now consider the MMVs in some more details. Denote by ev (resp. od) the set of even (resp. odd) numbers. By abuse of notation it is sometimes more transparent to write $\varepsilon=\mathrm{ev}$ if $\varepsilon=1$ and $\varepsilon=\mathrm{od}$ if $\varepsilon=-1$. To save space, we may put a check on top of s_j if and only if $\varepsilon_j=\mathrm{od}$. For example,

$$M(\check{s}_1,s_2)=M^{\mathrm{od},\mathrm{ev}}(s_1,s_2)=\sum_{\substack{m_1>m_2>0\\ m_1\in\mathrm{od},m_2\in\mathrm{ev}}}\frac{4}{m_1^{s_1}m_2^{s_2}}.$$

To extend the ideas in [32], we can also define the *alternating multiple mixed values* (AMMVs for short) for any composition $\boldsymbol{s}=(s_1,\dots,s_r)$, $\boldsymbol{\varepsilon}=(\varepsilon_1,\dots,\varepsilon_r)\in\{\pm1\}^r$, and $\boldsymbol{\sigma}=(\sigma_1,\dots,\sigma_r)\in\{\pm1\}^r$ with $(s_1,\sigma_1)\neq(1,1)$ by

$$M_{\boldsymbol{\sigma}}^{\boldsymbol{\varepsilon}}(\boldsymbol{s}):=\sum_{m_1>\cdots>m_r>0}\frac{(1+\varepsilon_1(-1)^{m_1})\sigma_1^{(2m_1+1-\varepsilon_1)/4}\cdots(1+\varepsilon_r(-1)^{m_r})\sigma_r^{(2m_r+1-\varepsilon_r)/4}}{m_1^{s_1}\cdots m_r^{s_r}}. \tag{1.4}$$

As usual, we call $s_1+\cdots+s_r$ and r the *weight* and *depth*, respectively. We may use the convention for Euler sums by putting a bar on top of s_j if and only if $\sigma_j=-1$. For example,

$$\begin{aligned}M(\check{\bar{a}},\bar{b},\check{c},d)&=M_{-1,-1,1,1}^{\mathrm{od},\mathrm{ev},\mathrm{od},\mathrm{ev}}(a,b,c,d)\\ &=\sum_{\substack{m_1>m_2>m_3>m_4>0\\ m_1,m_3\in\mathrm{od},\ m_2,m_4\in\mathrm{ev}}}\frac{16(-1)^{(m_1+1)/2}(-1)^{m_2/2}}{m_1^a m_2^b m_3^c m_4^d}.\end{aligned}$$

It is not too hard to see that the AMMVs can be is intimately related to level four CMZVs.

Set

$$w_0:=\frac{dt}{t},\quad w_{+1}^{-1}:=\frac{2dt}{1-t^2},\quad w_{-1}^{-1}:=\frac{-2dt}{1+t^2},\quad w_{+1}^{+1}:=\frac{2tdt}{1-t^2},\quad w_{-1}^{+1}:=\frac{-2tdt}{1+t^2}$$

and

$$w_\sigma^{\varepsilon_1,\varepsilon_2} := \max\{\sigma, \operatorname{sgn}(1+\varepsilon_2-\varepsilon_1)\} w_\sigma^{\varepsilon_1\varepsilon_2}.$$

Namely, $w_\sigma^{\varepsilon_1,\varepsilon_2} = w_\sigma^{\varepsilon_1\varepsilon_2}$ unless $\sigma=\varepsilon_2=-\varepsilon_1=-1$ when $w_\sigma^{\varepsilon_1,\varepsilon_2} = -w_\sigma^{\varepsilon_1\varepsilon_2}$. It is straightforward to deduce that AMMVs can be expressed by the following iterated integrals

$$M_{\boldsymbol{\sigma}}^{\boldsymbol{\varepsilon}}(\boldsymbol{s}) = \int_0^1 w_0^{s_1-1} w_{\sigma_1}^{\varepsilon_1,\varepsilon_2} w_0^{s_2-1} w_{\sigma_1\sigma_2}^{\varepsilon_2,\varepsilon_3} \cdots w_0^{s_r-1} w_{\sigma_1\sigma_2\cdots\sigma_r}^{\varepsilon_r}. \tag{1.5}$$

For example,

$$M(\check{3},\bar{2}) = M_{1,-1}^{\mathrm{od},\mathrm{ev}}(3,2) = \sum_{n_1>n_2>0} \frac{4(-1)^{n_2}}{(2n_1-1)^3(2n_2)^2} = \int_0^1 w_0^2 w_{+1}^{-1} w_0 w_{-1}^{+1}$$

and

$$\begin{aligned} M(\bar{2},3,\check{\bar{4}}) &= \sum_{n_1>n_2>n_3>0} \frac{8(-1)^{n_1+n_3-1}}{(2n_1-2)^2(2n_2-2)^3(2n_3-1)^4} \\ &= \int_0^1 w_0 w_{-1}^{+1} w_0^2 (-w_{-1}^{-1}) w_0^3 w_{+1}^{-1}. \end{aligned}$$

For $\boldsymbol{\sigma},\boldsymbol{\varepsilon} \in \{\pm 1\}^r$ define

$$\operatorname{sgn}(\boldsymbol{\sigma},\boldsymbol{\varepsilon}) := (-1)^{\sharp\{i<r \mid \sigma_i=\varepsilon_i=\varepsilon_{i+1}\varepsilon_{i+2}\cdots\varepsilon_r=-1\}}.$$

Then for all $\boldsymbol{s}=(s_1,\dots,s_r)\in\mathbb{N}^r$ with $(s_1,\sigma_1)\neq(1,1)$, we have

$$\int_0^1 w_0^{s_1-1} w_{\sigma_1}^{\varepsilon_1} \cdots w_0^{s_r-1} w_{\sigma_r}^{\varepsilon_r} = \operatorname{sgn}(\boldsymbol{\sigma},\boldsymbol{\varepsilon}) M_{\sigma_1,\sigma_2\sigma_1,\dots,\sigma_r\sigma_{r-1}}^{\varepsilon_1\cdots\varepsilon_r,\varepsilon_2\cdots\varepsilon_r,\dots,\varepsilon_{r-1}\varepsilon_r,\varepsilon_r}(\boldsymbol{s}). \tag{1.6}$$

For example, if $\boldsymbol{\varepsilon}=(1,-1,-1,1)$ and $\boldsymbol{\sigma}=(-1,-1,-1,1)$ then $\operatorname{sgn}(\boldsymbol{\sigma},\boldsymbol{\varepsilon})=-1$ and we can easily verify that

$$-\int_0^1 w_0^{s_1-1} w_{-1}^{+1} w_0^{s_2-1} w_{-1}^{-1} w_0^{s_3-1} w_{-1}^{-1} w_0^{s_4-1} w_{+1}^{+1} = M_{-1,1,1,-1}^{1,1,-1,1}(s_1,s_2,s_3,s_4).$$

By Chen's theory of iterated integrals (see e.g., [38, Ch 4]) we know that AMMVs must satisfy the integral shuffle relations. For example,

$$\begin{aligned} &M(\check{\bar{1}})M(\check{\bar{2}}) = M_{-1}^{\mathrm{od}}(1) M_{-1}^{\mathrm{od}}(2) \\ &= \int_0^1 w_{-1}^{-1} \int_0^1 w_0 w_{-1}^{-1} = \int_0^1 w_{-1}^{-1} w_0 w_{-1}^{-1} + 2\int_0^1 w_0 w_{-1}^{-1} w_{-1}^{-1} = -M(\bar{1},\check{2}) - 2M(\bar{2},\check{1}) \end{aligned}$$

and

$$M(\bar{3})M(\check{2}) = M_{-1}^{\mathrm{ev}}(3) M_1^{\mathrm{od}}(2) = \int_0^1 w_0^2 w_{-1}^{+1} \int_0^1 w_0 w_{+1}^{-1}$$

$$
\begin{aligned}
=&\int_0^1 w_0^2 w_{-1}^{+1} w_0 w_{+1}^{-1} + \int_0^1 w_0 w_{+1}^{-1} w_0^2 w_{-1}^{+1} + 2\int_0^1 w_0^2 w_{+1}^{-1} w_0 w_{-1}^{+1} + 3\int_0^1 w_0^3 w_{+1}^{-1} w_{-1}^{+1} + 3\int_0^1 w_0^3 w_{-1}^{+1} w_{+1}^{-1} \\
=&M(\check{\bar{3}},\check{\bar{2}}) + M(\check{2},\bar{3}) + 2M(\check{3},\bar{2}) + 3M(\check{4},\bar{1}) + 3M(\check{\bar{4}},\check{\bar{1}}).
\end{aligned}
$$

It is apparent that AMMVs also satisfy the series stuffle relations. For examples,

$$
\begin{aligned}
M(\bar{2},3,\check{\bar{4}})M(\check{\bar{2}}) =&M(\bar{2},3,\check{\bar{4}},\check{\bar{2}}) + M(\bar{2},3,\check{\bar{2}},\check{\bar{4}}) \\
&+ M(\bar{2},\check{\bar{2}},3,\check{\bar{4}}) + M(\check{\bar{2}},\bar{2},3,\check{\bar{4}}) + 2M(\bar{2},3,\check{6})
\end{aligned}
$$

and

$$
\begin{aligned}
M(\bar{1},\check{\bar{2}})M(3,\check{\bar{2}}) =&M(\bar{1},\check{\bar{2}},3,\check{\bar{2}}) + M(3,\check{\bar{2}},\bar{1},\check{\bar{2}}) + 2M(\bar{1},3,\check{\bar{2}},\check{\bar{2}}) + 2M(3,\bar{1},\check{\bar{2}},\check{\bar{2}}) \\
&+ 2M(\bar{1},3,\check{4}) + 2M(3,\bar{1},\check{4}) + 4M(\bar{4},\check{\bar{2}},\check{\bar{2}}) + 4M(\bar{4},\check{4}).
\end{aligned}
$$

Note that stuffing can happen only when the two composition components correspond to two equal ε's. For each incidence of such stuffing, the two signs (i.e., σ's) corresponding to the two composition components are multiplied and an extra factor of 2 is produced.

1.2 Alternating multiple T/S-harmonic sums

For positive integers m and n such that $n \ge m$, we define the following subsets of $\mathbb{N}^m$:

$$
D_{n,m} := \begin{cases} \Big\{ \boldsymbol{n} \in \mathbb{N}^m \mid n \ge n_1 > n_2 \ge \cdots \ge n_{m-2} > n_{m-1} \ge n_m > 0 \Big\}, & \text{if } 2 \nmid m; \\ \Big\{ \boldsymbol{n} \in \mathbb{N}^m \mid n > n_1 \ge n_2 > \cdots \ge n_{m-2} > n_{m-1} \ge n_m > 0 \Big\}, & \text{if } 2 \mid m, \end{cases}
$$

$$
E_{n,m} := \begin{cases} \Big\{ \boldsymbol{n} \in \mathbb{N}^m \mid n > n_1 \ge n_2 > \cdots > n_{m-2} \ge n_{m-1} > n_m > 0 \Big\}, & \text{if } 2 \nmid m; \\ \Big\{ \boldsymbol{n} \in \mathbb{N}^m \mid n \ge n_1 > n_2 \ge \cdots > n_{m-2} \ge n_{m-1} > n_m > 0 \Big\}, & \text{if } 2 \mid m. \end{cases}
$$

Here we have put $\boldsymbol{n} = (n_1, \dots, n_m)$.

Definition 1.1. For positive integer m and $\boldsymbol{\sigma}_r := (\sigma_1, \sigma_2, \dots, \sigma_r) \in \{\pm 1\}^r$, define

$$
T_n^{\boldsymbol{\sigma}_{2m-1}}(\boldsymbol{k}_{2m-1}) := \sum_{\boldsymbol{n} \in D_{n,2m-1}} \frac{2^{2m-1} \sigma_1^{n_1} \sigma_2^{n_2} \cdots \sigma_{2m-1}^{n_{2m-1}}}{\left(\prod_{j=1}^{m-1} (2n_{2j-1}-1)^{k_{2j-1}} (2n_{2j})^{k_{2j}}\right) (2n_{2m-1}-1)^{k_{2m-1}}}, \tag{1.7}
$$

$$
T_n^{\boldsymbol{\sigma}_{2m}}(\boldsymbol{k}_{2m}) := \sum_{\boldsymbol{n} \in D_{n,2m}} \frac{2^{2m} \sigma_1^{n_1} \sigma_2^{n_2} \cdots \sigma_{2m}^{n_{2m}}}{\prod_{j=1}^{m} (2n_{2j-1})^{k_{2j-1}} (2n_{2j}-1)^{k_{2j}}}, \tag{1.8}
$$

$$
S_n^{\boldsymbol{\sigma}_{2m-1}}(\boldsymbol{k}_{2m-1}) := \sum_{\boldsymbol{n} \in E_{n,2m-1}} \frac{2^{2m-1} \sigma_1^{n_1} \sigma_2^{n_2} \cdots \sigma_{2m-1}^{n_{2m-1}}}{\left(\prod_{j=1}^{m-1} (2n_{2j-1})^{k_{2j-1}} (2n_{2j}-1)^{k_{2j}}\right) (2n_{2m-1})^{k_{2m-1}}}, \tag{1.9}
$$

$$
S_n^{\boldsymbol{\sigma}_{2m}}(\boldsymbol{k}_{2m}) := \sum_{\boldsymbol{n} \in E_{n,2m}} \frac{2^{2m} \sigma_1^{n_1} \sigma_2^{n_2} \cdots \sigma_{2m}^{n_{2m}}}{\prod_{j=1}^{m} (2n_{2j-1}-1)^{k_{2j-1}} (2n_{2j})^{k_{2j}}}, \tag{1.10}
$$

where $T_n^{\boldsymbol{\sigma}_{2m-1}}(\boldsymbol{k}_{2m-1}) := 0$ if $n < m$, and $T_n^{\boldsymbol{\sigma}_{2m}}(\boldsymbol{k}_{2m}) = S_n^{\boldsymbol{\sigma}_{2m-1}}(\boldsymbol{k}_{2m-1}) = S_n^{\boldsymbol{\sigma}_{2m}}(\boldsymbol{k}_{2m}) := 0$ if $n \le m$. Moreover, we set $T_n^{\boldsymbol{\sigma}}(\emptyset) = S_n^{\boldsymbol{\sigma}}(\emptyset) := 1$ for convenience. We call (1.7) and (1.8) *alternating multiple T-harmonic sums*, and call (1.9) and (1.10) *alternating multiple S-harmonic sums.*

According to the definitions of alternating multiple T-harmonic sums and alternating multiple S-harmonic sums, we have the following relations

$$T_n^{\sigma_1,\sigma_2,\dots,\sigma_{2m-1}}(k_1,k_2,\dots,k_{2m-1}) = 2\sum_{j=1}^{n} \frac{T_j^{\sigma_2,\sigma_3,\dots,\sigma_{2m-1}}(k_2,k_3,\dots,k_{2m-1})}{(2j-1)^{k_1}}\sigma_1^j, \tag{1.11}$$

$$T_n^{\sigma_1,\sigma_2,\dots,\sigma_{2m}}(k_1,k_2,\dots,k_{2m}) = 2\sum_{j=1}^{n-1} \frac{T_j^{\sigma_2,\sigma_3,\dots,\sigma_{2m}}(k_2,k_3,\dots,k_{2m})}{(2j)^{k_1}}\sigma_1^j, \tag{1.12}$$

$$S_n^{\sigma_1,\sigma_2,\dots,\sigma_{2m-1}}(k_1,k_2,\dots,k_{2m-1}) = 2\sum_{j=1}^{n-1} \frac{S_j^{\sigma_2,\sigma_3,\dots,\sigma_{2m-1}}(k_2,k_3,\dots,k_{2m-1})}{(2j)^{k_1}}\sigma_1^j, \tag{1.13}$$

$$S_n^{\sigma_1,\sigma_2,\dots,\sigma_{2m}}(k_1,k_2,\dots,k_{2m}) = 2\sum_{j=1}^{n} \frac{S_j^{\sigma_2,\sigma_3,\dots,\sigma_{2m}}(k_2,k_3,\dots,k_{2m})}{(2j-1)^{k_1}}\sigma_1^j. \tag{1.14}$$

By direct calculations we obtain

$$\begin{aligned}
&T_n^{\sigma_1,\sigma_2,\dots,\sigma_{2m-1}}(k_1,k_2,\dots,k_{2m-1})\\
&= \prod_{j=1}^{m-1}(\sigma_{2j-1}\sigma_{2j})^{m-j} \sum_{n+m>n_1>n_2>\dots>n_{2m-1}>0} \frac{2^{2m-1}\sigma_1^{n_1}\sigma_2^{n_2}\cdots\sigma_{2m-1}^{n_{2m-1}}}{\prod\limits_{j=1}^{2m-1}(2n_j-2m+j)^{k_j}}\\
&= \sum_{2n>n_1>n_2>\dots>n_{2m-1}>0} \frac{\prod\limits_{j=1}^{m-1}\left(1-(-1)^{n_{2j-1}}\right)\sigma_{2j-1}^{(n_{2j-1}+1)/2}\left(1+(-1)^{n_{2j}}\right)\sigma_{2j}^{n_{2j}/2}}{n_1^{k_1}n_2^{k_2}\cdots n_{2m-2}^{k_{2m-2}}n_{2m-1}^{k_{2m-1}}}\\
&\qquad \times \left(1-(-1)^{n_{2m-1}}\right)\sigma_{2m-1}^{(n_{2m-1}+1)/2},
\end{aligned}\tag{1.15}$$

$$\begin{aligned}
&T_n^{\sigma_1,\sigma_2,\dots,\sigma_{2m}}(k_1,k_2,\dots,k_{2m})\\
&= \prod_{j=1}^{m-1}(\sigma_{2j-1})^{m-j+1}(\sigma_{2j})^{m-j} \sum_{n+m>n_1>n_2>\dots>n_{2m}>0} \frac{2^{2m}\sigma_1^{n_1}\sigma_2^{n_2}\cdots\sigma_{2m}^{n_{2m}}}{\prod\limits_{j=1}^{2m}(2n_j-2m-1+j)^{k_j}}\\
&= \sum_{2n>n_1>n_2>\dots>n_{2m}>0} \frac{\prod\limits_{j=1}^{m}\left(1+(-1)^{n_{2j-1}}\right)\sigma_{2j-1}^{n_{2j-1}/2}\left(1-(-1)^{n_{2j}}\right)\sigma_{2j}^{(n_{2j}+1)/2}}{n_1^{k_1}n_2^{k_2}\cdots n_{2m-1}^{k_{2m-1}}n_{2m}^{k_{2m}}},
\end{aligned}\tag{1.16}$$

$$\begin{aligned}
&S_n^{\sigma_1,\sigma_2,\dots,\sigma_{2m-1}}(k_1,k_2,\dots,k_{2m-1})\\
&= \sigma_1^{m-1}\prod_{j=1}^{m-2}(\sigma_{2j}\sigma_{2j+1})^{m-j-1} \sum_{n+m-1>n_1>n_2>\dots>n_{2m-1}>0} \frac{2^{2m-1}\sigma_1^{n_1}\sigma_2^{n_2}\cdots\sigma_{2m-1}^{n_{2m-1}}}{\prod\limits_{j=1}^{2m-1}(2n_j-2m+j+1)^{k_j}}
\end{aligned}$$

$$= \sum_{2n>n_1>n_2>\cdots>n_{2m-1}>0} \frac{\prod\limits_{j=1}^{m-1} (1+(-1)^{n_{2j-1}})\, \sigma_{2j-1}^{n_{2j-1}/2}\, (1-(-1)^{n_{2j}})\, \sigma_{2j}^{(n_{2j}+1)/2}}{n_1^{k_1} n_2^{k_2} \cdots n_{2m-2}^{k_{2m-2}} n_{2m-1}^{k_{2m-1}}} \times (1+(-1)^{n_{2m-1}})\, \sigma_{2m-1}^{n_{2m-1}/2}, \tag{1.17}$$

$$S_n^{\sigma_1,\sigma_2,\ldots,\sigma_{2m}}(k_1,k_2,\ldots,k_{2m})$$
$$= \prod_{j=1}^{m-1} (\sigma_{2j-1}\sigma_{2j})^{m-j} \sum_{n+m>n_1>n_2>\cdots>n_{2m}>0} \frac{2^{2m} \sigma_1^{n_1} \sigma_2^{n_2} \cdots \sigma_{2m}^{n_{2m}}}{\prod\limits_{j=1}^{2m} (2n_j - 2m + j)^{k_j}}$$
$$= \sum_{2n>n_1>n_2>\cdots>n_{2m}>0} \frac{\prod\limits_{j=1}^{m} (1-(-1)^{n_{2j-1}})\, \sigma_{2j-1}^{(n_{2j-1}+1)/2}\, (1+(-1)^{n_{2j}})\, \sigma_{2j}^{n_{2j}/2}}{n_1^{k_1} n_2^{k_2} \cdots n_{2m-1}^{k_{2m-1}} n_{2m}^{k_{2m}}}. \tag{1.18}$$

Hence, for any composition $\boldsymbol{k} = (k_1, \ldots, k_r)$ and $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \ldots, \sigma_r) \in \{\pm 1\}^r$ with $(s_1, \sigma_1) \neq (1, 1)$, we may define the *alternating MTVs* (AMTVs) and *alternating MSVs* (AMSVs) as follow:

$$T^{\boldsymbol{\sigma}}(\boldsymbol{k}) \equiv T^{\sigma_1,\sigma_2,\ldots,\sigma_r}(k_1,k_2,\ldots,k_r) := \lim_{n\to\infty} T_n^{\sigma_1,\sigma_2,\ldots,\sigma_r}(k_1,k_2,\ldots,k_r), \tag{1.19}$$

$$S^{\boldsymbol{\sigma}}(\boldsymbol{k}) \equiv S^{\sigma_1,\sigma_2,\ldots,\sigma_r}(k_1,k_2,\ldots,k_r) := \lim_{n\to\infty} S_n^{\sigma_1,\sigma_2,\ldots,\sigma_r}(k_1,k_2,\ldots,k_r). \tag{1.20}$$

We may compactly indicate the presence of an alternating sign as before: if $\sigma_j = -1$ then we place a bar over the corresponding component k_j. For example,

$$T(\bar{2}, 3, \bar{1}, 4) = T^{-1,1,-1,1}(2,3,1,4) \quad \text{and} \quad S(\bar{1}, 3, \bar{2}, 4, \bar{2}) = S^{-1,1,-1,1,-1}(1,3,2,4,2).$$

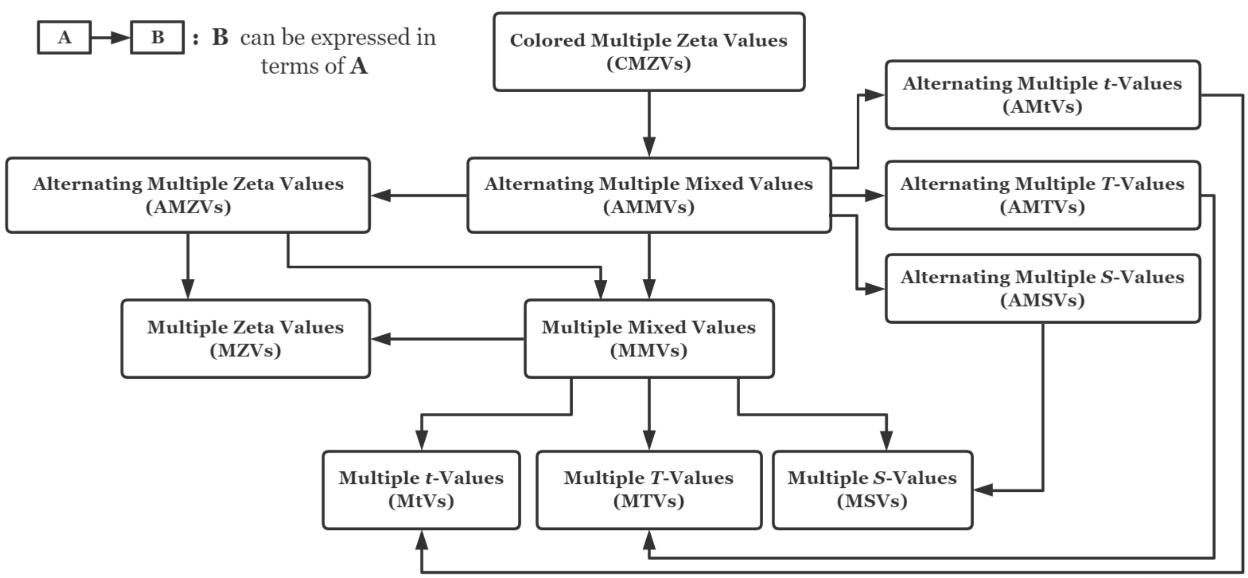
Obviously, we have

- All $\varepsilon_j = 1$ in AMMVs $M_{\boldsymbol{\sigma}}^{\boldsymbol{\varepsilon}}(\boldsymbol{k}) \Longrightarrow$ AMZVs$\times \dfrac{1}{2^{k_1+k_2+\cdots+k_r-r}}$;
- All $\varepsilon_j = -1$ in AMMVs $M_{\boldsymbol{\sigma}}^{\boldsymbol{\varepsilon}}(\boldsymbol{k}) \Longrightarrow$ AMtVs$\times 2^r$;
- All $\varepsilon_j = (-1)^{r+1-j}$ in AMMVs $M_{\boldsymbol{\sigma}}^{\boldsymbol{\varepsilon}}(\boldsymbol{k}) \Longrightarrow$ AMTVs;
- All $\varepsilon_j = (-1)^{r-j}$ in AMMVs $M_{\boldsymbol{\sigma}}^{\boldsymbol{\varepsilon}}(\boldsymbol{k}) \Longrightarrow$ AMSVs.

For the detailed definition and introduction of *alternating MZVs* (AMZVs) and *alternating MtVs* (AMtVs) , please see [16, 38]. Let AMZV_w be the $\mathbb{Q}$-vector space generated by the AMZVs of weight w. Setting $\dim_{\mathbb{Q}} \mathsf{AMZV}_0 = 1$, we have the dimension bound $\dim_{\mathbb{Q}} \mathsf{AMZV}_w \leq F_w$ obtained by Deligne and Goncharov [12] and moreover a set of generators of AMZV_w shown by Deligne [11, Thm. 7.2], where $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. In particular, in a recent paper, S. Charlton [5] proved the AMtVs $t(\{\bar{1}\}_a, 1, \{\bar{1}\}_b)$ $(a, b \in \mathbb{N})$ can be expressed in terms of $\log(2)$, Riemann zeta values and Dirichlet beta values (see (4.1)).

According to the definitions of AMTVs and AMSVs, for any $m \in \mathbb{N}$, we have the following iterated integral expressions:

$$T^{\sigma_1,\sigma_2,\dots,\sigma_{2m-1}}(k_1,k_2,\dots,k_{2m-1}) = M^{\{-1,1\}_{m-1},-1}_{\sigma_1,\sigma_2,\dots,\sigma_{2m-1}}(k_1,k_2,\dots,k_{2m-1})$$
$$= \left\{\prod_{j=1}^{m-1}(\sigma_1\sigma_2\cdots\sigma_{2j})\right\} \int_0^1 w_0^{k_1-1}w_{\sigma_1}^{-1}w_0^{k_2-1}w_{\sigma_1\sigma_2}^{-1}\cdots w_0^{k_{2m-1}-1}w_{\sigma_1\sigma_2\cdots\sigma_{2m-1}}^{-1},$$
$$T^{\sigma_1,\sigma_2,\dots,\sigma_{2m}}(k_1,k_2,\dots,k_{2m}) = M^{\{1,-1\}_m}_{\sigma_1,\sigma_2,\dots,\sigma_{2m}}(k_1,k_2,\dots,k_{2m})$$
$$= \left\{\prod_{j=1}^{m}(\sigma_1\sigma_2\cdots\sigma_{2j-1})\right\} \int_0^1 w_0^{k_1-1}w_{\sigma_1}^{-1}w_0^{k_2-1}w_{\sigma_1\sigma_2}^{-1}\cdots w_0^{k_{2m}-1}w_{\sigma_1\sigma_2\cdots\sigma_{2m}}^{-1},$$
$$S^{\sigma_1,\sigma_2,\dots,\sigma_{2m-1}}(k_1,k_2,\dots,k_{2m-1}) = M^{\{1,-1\}_{m-1},1}_{\sigma_1,\sigma_2,\dots,\sigma_{2m-1}}(k_1,k_2,\dots,k_{2m-1})$$
$$= \left\{\prod_{j=1}^{m-1}(\sigma_1\sigma_2\cdots\sigma_{2j-1})\right\} \int_0^1 w_0^{k_1-1}w_{\sigma_1}^{-1}w_0^{k_2-1}w_{\sigma_1\sigma_2}^{-1}\cdots w_0^{k_{2m-2}-1}w_{\sigma_1\sigma_2\cdots\sigma_{2m-2}}^{-1}w_0^{k_{2m-1}-1}w_{\sigma_1\sigma_2\cdots\sigma_{2m-1}}^{1},$$
$$S^{\sigma_1,\sigma_2,\dots,\sigma_{2m}}(k_1,k_2,\dots,k_{2m}) = M^{\{-1,1\}_m}_{\sigma_1,\sigma_2,\dots,\sigma_{2m}}(k_1,k_2,\dots,k_{2m})$$
$$= \left\{\prod_{j=1}^{m-1}(\sigma_1\sigma_2\cdots\sigma_{2j})\right\} \int_0^1 w_0^{k_1-1}w_{\sigma_1}^{-1}w_0^{k_2-1}w_{\sigma_1\sigma_2}^{-1}\cdots w_0^{k_{2m-1}-1}w_{\sigma_1\sigma_2\cdots\sigma_{2m-1}}^{-1}w_0^{k_{2m}-1}w_{\sigma_1\sigma_2\cdots\sigma_{2m}}^{1}.$$



1.3 Main results

The primary goals of this paper are to study the explicit relations of AMMVs and its special type, and establish some explicit evaluations of the AMMVs and related values via AMMVs of lower depths.

The remainder of this paper is organized as follows.

In Sections 2 and 3 we find the series stuffle relations and integral shuffle relations of AMMVs, and set up the algebraic framework for the regularized double shuffle relations

(DBSFs) of AMMVs.

In Section 4, we first prove four integral identities involving the arctangent function. Then we apply these formulas obtained to establish two explicit relations between AMTVs and AMSVs.

In Section 5, we prove an reducibility theorem for AMMVs at depth three by using the method of contour integration and residue computations. Moreover, we give some specific examples.

In Section 6, we compute the dimensions of AMMVs, AMtVs, AMTVs and AMSVs for weight less than 7, and give some conjectures on relations between AMMVs, AMtVs, AMTVs and AMSVs.

2 Algebraic setup for AMMVs

For every *decorated positive integer* $k \in \mathbb{K}$ (say $k = s, \bar{s}, \check{s}$ or $k = \check{\bar{s}}$ for some $s \in \mathbb{N}$), we define its *absolute value, sign* and *parity* by

$$|k| = s, \qquad \operatorname{sgn}(k) = \begin{cases} 1, & \text{if } k = s \text{ or } \check{s}; \\ -1, & \text{if } k = \bar{s} \text{ or } \check{\bar{s}}, \end{cases} \qquad \operatorname{par}(k) = \begin{cases} 1, & \text{if } k = s \text{ or } \bar{s}; \\ -1, & \text{if } k = \check{s} \text{ or } \check{\bar{s}}, \end{cases}$$

respectively. For any $k \in \mathbb{K}$ define the word of length $|k|$

$$\mathtt{z}_k := \omega_0^{|k|-1} \omega_{\operatorname{sgn}(s)}^{\operatorname{par}(s)}.$$

One can now define an algebra of words as follows.

Definition 2.1. Let X be the alphabet consisting of the letters ω_0 and $\omega_\sigma^\varepsilon$ $(\sigma, \varepsilon = \pm 1)$. A *word* $\mathbf{w}$ is a monomial in the letters in X. Its *weight*, denoted by $|\mathbf{w}|$, is the number of letters contained in $\mathbf{w}$, and its *depth*, denoted by $\operatorname{dep}(\mathbf{w})$, is the number of $\omega_\sigma^\varepsilon$'s contained in $\mathbf{w}$. Define the *AMMV* algebra, denoted by $\mathfrak{A}$, to be the (weight) graded noncommutative polynomial $\mathbb{Q}$-algebra generated by words (including the empty word $\mathbf{1}$) over the alphabet X. Let $\mathfrak{A}^0$ be the subalgebra of $\mathfrak{A}$ generated by words not beginning with $\omega_1^{\pm 1}$ and not ending with ω_0. The words in $\mathfrak{A}^0$ are called *admissible words.*

By Eq. (1.5) every AMMV can be expressed as an iterated integral over the closed interval $[0, 1]$ of an admissible word. Thus for $\mathbf{w} = \mathtt{z}_{\boldsymbol{k}} := \mathtt{z}_{k_1} \dots \mathtt{z}_{k_r} \in \mathfrak{A}^0$, we set

$$\mathtt{M}(\mathbf{w}) := \int_0^1 \mathbf{w}, \qquad M(\mathbf{w}) := M(\boldsymbol{k}).$$

We also extend $\mathtt{M}$ and M to $\mathfrak{A}^0$ by $\mathbb{Q}$-linearity. Hence, if $\boldsymbol{\varepsilon} = (\operatorname{par}(k_1), \dots, \operatorname{par}(k_r))$ and $\boldsymbol{\sigma} = (\operatorname{sgn}(k_1), \dots, \operatorname{sgn}(k_r))$, then from (1.5) and (1.6) one has

$$M(\mathbf{w}) = \mathtt{M}\big(\mathbf{p}(\mathbf{w})\big) := \mathtt{M}\Big(w_0^{k_1-1} w_{\sigma_1}^{\varepsilon_1,\varepsilon_2} w_0^{k_2-1} w_{\sigma_1\sigma_2}^{\varepsilon_2,\varepsilon_3} \cdots w_0^{k_r-1} w_{\sigma_1\sigma_2\cdots\sigma_r}^{\varepsilon_r}\Big), \tag{2.1}$$

$$\begin{aligned} \mathtt{M}(\mathbf{w}) = M\big(\mathbf{q}(\mathbf{w})\big) &:= \operatorname{sgn}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) M_{\sigma_1,\sigma_2\sigma_1,\dots,\sigma_r\sigma_{r-1}}^{\varepsilon_1\cdots\varepsilon_r,\varepsilon_2\cdots\varepsilon_r,\dots,\varepsilon_{r-1}\varepsilon_r,\varepsilon_r}(k_1, k_2, \dots, k_r), \\ &= \operatorname{sgn}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) M\Big(w_0^{k_1-1} w_{\sigma_1}^{\varepsilon_1\cdots\varepsilon_r} w_0^{k_2-1} w_{\sigma_1\sigma_2}^{\varepsilon_2\cdots\varepsilon_r} \cdots w_0^{k_r-1} w_{\sigma_{r-1}\sigma_r}^{\varepsilon_r}\Big). \end{aligned} \tag{2.2}$$

Here for empty word $\mathbf{1}$ we set $M(\mathbf{1}) = \mathtt{M}(\mathbf{1}) = 1$ and $\mathbf{p}(\mathbf{1}) = \mathbf{q}(\mathbf{1}) = \mathbf{1}$.

Let $\mathfrak{A}_{ш}$ be the algebra of $\mathfrak{A}$ where the multiplication is defined by the usual shuffle product ш. Denote the subalgebra $\mathfrak{A}^0$ by $\mathfrak{A}^0_{ш}$ when one considers this shuffle product.

Proposition 2.1. *The map* $\mathtt{M} : \mathfrak{A}^0_{ш} \longrightarrow \mathbb{R}$ *is an algebra homomorphism.*

Proof. Similarly to the proof of the corresponding result for MZVs, this follows easily from Chen's theory of the shuffle product relations of iterated integrals by formula (2.1). We leave the details to the interested reader. □

On the other hand, the AMMVs also satisfy the series stuffle relations. To define the stuffle relations of alternating multiple harmonic sums a double cover $\mathbb{D}$ of $\mathbb{N}$ is defined in [38, Chapter 7]: $\mathbb{D} = \mathbb{N} \cup \bar{\mathbb{N}} = \mathbb{N} \cup \{\bar{n} : n \in \mathbb{N}\}$ which is equipped with a binary operation $\oplus$ as follows. For any $a, b \in \mathbb{N}$, set $a \oplus b = \bar{a} \oplus \bar{b} := a + b$ and $\bar{a} \oplus b = a \oplus \bar{b} := \overline{a+b}$.

For the stuffle relation of AMMVs, we define a double cover $\mathbb{K}$ of $\mathbb{D}$: $\mathbb{K} = \mathbb{D} \cup \check{\mathbb{D}} = \mathbb{D} \cup \{\check{s} : s \in \mathbb{D}\}$ together with a map extending the binary operation $\oplus$, still denoted by $\oplus$, as follows. For any $a, b \in \mathbb{D}$, $\check{a} \oplus \check{b} := (a \oplus b)\check{}$.

Define $M(\emptyset) = 1$. Further, for $d, l \geq 0$ and $\boldsymbol{s} = (s_1, \dots, s_d) \in \mathbb{K}^d, \boldsymbol{t} = (t_1, \dots, t_l) \in \mathbb{K}^l$ it is easy to see that

$$M(\boldsymbol{s})M(\boldsymbol{t}) = M(\boldsymbol{s} * \boldsymbol{t})$$

where the commutative stuffle product $\boldsymbol{s} * \boldsymbol{t}$ is defined recursively by

$$\begin{aligned} \emptyset * \boldsymbol{s} = \boldsymbol{s} * \emptyset =& \boldsymbol{s}, \\ (s_1, \dots, s_d) * (t_1, \dots, t_l) =& \big(s_1, (s_2, \dots, s_d) * (t_1, \dots, t_l)\big) + \big(t_1, (s_1, \dots, s_d) * (t_2, \dots, t_l)\big) \\ &+ \delta(\mathrm{par}(s_1), \mathrm{par}(t_1))\big(s_1 \oplus t_1, (s_2, \dots, s_d) * (t_2, \dots, t_l)\big) \end{aligned}$$

for all $d, l \geq 1$, where $\delta(\varepsilon, \eta) = |\varepsilon + \eta|$.

Definition 2.2. Denote by $\mathfrak{A}^1$ the subalgebra of $\mathfrak{A}$ generated by words $\mathbf{z}^{\sigma}_{k,\varepsilon}$ with $k \in \mathbb{N}$ and $\sigma, \varepsilon = \pm 1$. Equivalently, $\mathfrak{A}^1$ is the subalgebra of $\mathfrak{A}$ generated by words not ending with ω_0. We then define another multiplication ⧢ on $\mathfrak{A}^1$ by

$$\mathbf{u} \text{ш}\!\!\text{ш}\, \mathbf{v} = \mathbf{p}\big(\mathbf{q}(\mathbf{u}) * \mathbf{q}(\mathbf{v})\big), \qquad \forall \mathbf{u}, \mathbf{v} \in \mathfrak{A}^1 \tag{2.3}$$

where $*$ is the ordinary stuffle which is defined by the following: (i) it distributes over addition, (ii) $1 * \mathbf{w} = \mathbf{w} * 1 = \mathbf{w}$, $\forall \mathbf{w} \in \mathfrak{A}^1$, and (iii) $\forall \mathbf{u}, \mathbf{v} \in \mathfrak{A}^1$, $s, t \in \mathbb{N}$ and $\sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 = \pm 1$

$$\mathbf{z}^{\varepsilon_1}_{s,\sigma_1}\mathbf{u} * \mathbf{z}^{\varepsilon_2}_{t,\sigma_2}\mathbf{v} = \mathbf{z}^{\varepsilon_1}_{s,\sigma_1}\big(\mathbf{u} * \mathbf{z}^{\varepsilon_2}_{t,\sigma_2}\mathbf{v}\big) + \mathbf{z}^{\varepsilon_2}_{t,\sigma_2}\big(\mathbf{z}^{\varepsilon_1}_{s,\sigma_1}\mathbf{u} * \mathbf{v}\big) + \delta(\varepsilon_1, \varepsilon_2)(\mathbf{u} * \mathbf{v})\mathbf{z}^{\varepsilon_1}_{s+t,\sigma_1\sigma_2}.$$

This multiplication ⧢ is called the *stuffle product* for AMMVs.

If we denote by $\mathfrak{A}^1_{⧢}$ the algebra $(\mathfrak{A}^1, ⧢)$ then it is not hard to prove the next proposition.

Proposition 2.2. *The polynomial algebra* $\mathfrak{A}^1_{⧢}$ *is a commutative weight graded* $\mathbb{Q}$*-algebra.*

Proof. One can refer to [15, Theorem 3.2] for a similar proof. □

Now we can define the subalgebra $\mathfrak{A}^0_{\sqcap\!\sqcap}$ similar to $\mathfrak{A}^0_{\sqcup\!\sqcup}$ by replacing the shuffle product by the M-stuffle product. Then by the induction on the word lengths and using the series definition one can quickly check that for any $\mathbf{w}_1, \mathbf{w}_2 \in \mathfrak{A}^0_{\sqcap\!\sqcap}$

$$\mathtt{M}(\mathbf{w}_1)\mathtt{M}(\mathbf{w}_2) = \mathtt{M}(\mathbf{w}_1 \sqcap\!\sqcap \mathbf{w}_2).$$

This implies the following result.

Proposition 2.3. *The map* $\mathtt{M} : \mathfrak{A}^0_{\sqcap\!\sqcap} \longrightarrow \mathbb{R}$ *is an algebra homomorphism.*

Proof. One can refer to [15, Theorem 4.2] for a similar proof. □

Definition 2.3. Let w be an integer such that $w \geq 2$. For nontrivial words $\mathbf{w}_1, \mathbf{w}_2 \in \mathfrak{A}^0$ with $|\mathbf{w}_1| + |\mathbf{w}_2| = w$, we say that the equation

$$\mathtt{M}(\mathbf{w}_1 \sqcup\!\sqcup \mathbf{w}_2 - \mathbf{w}_1 \sqcap\!\sqcap \mathbf{w}_2) = 0 \tag{2.4}$$

provides a *finite double shuffle relation* (finite DBSF) of AMMVs of weight w.

3 Regularization for divergent AMMVs

It is well known that the finite DBSFs cannot imply the identity $\zeta(2,1) = \zeta(3)$ even at level one (i.e., the MZV case). However, it is believed that one can remedy this by considering the *regularized double shuffle relation* (regularized DBSF) produced by the following mechanism.

First, combining Prop. 2.1 and Prop. 2.3 and using the algebra structures of $\mathfrak{A}^1_{\sqcap\!\sqcap}$ (resp. $\mathfrak{A}^1_{\sqcup\!\sqcup}$) over $\mathfrak{A}^0_{\sqcap\!\sqcap}$ (resp. $\mathfrak{A}^0_{\sqcup\!\sqcup}$) we can prove the following algebraic result without too much difficulty.

Proposition 3.1. *We have an algebra homomorphism:*

$$\mathtt{M}_{\sqcap\!\sqcap} : (\mathfrak{A}^1_{\sqcap\!\sqcap}, \sqcap\!\sqcap) \longrightarrow \mathbb{R}[T]$$

which is uniquely determined by the properties that they both extend the evaluation map $\mathtt{M} : \mathfrak{A}^0 \longrightarrow \mathbb{R}$ *by sending* $\mathtt{z}_1$ *to* T *and* $\mathtt{z}_{\bar{1}}$ *to* $T + 2\log 2$. *Further, we have an algebra homomorphism:*

$$\mathtt{M}_{\sqcup\!\sqcup} : (\mathfrak{A}^1_{\sqcup\!\sqcup}, \sqcup\!\sqcup) \longrightarrow \mathbb{R}[T]$$

which is uniquely determined by the properties that it extends the evaluation map $\mathtt{M} : \mathfrak{A}^0 \longrightarrow \mathbb{R}$ *by sending* $\mathtt{z}_1$ *to* $T - \log 2$ *and* $\mathtt{z}_{\bar{1}}$ *to* $T + \log 2$.

Proof. The proof is the same as that for the corresponding results of MMVs. See [31, Prop. 2.7 and Prop. 2.8]. □

We can now apply the same mechanism as in [17] to derive the following regularized DBSFs.

Theorem 3.2. *Define an* $\mathbb{R}$*-linear map* $\rho : \mathbb{R}[T] \to \mathbb{R}[T]$ *by*

$$\rho(e^{Tu}) = \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) u^n\right) e^{(T-\log 2)u}, \qquad |u| < 1.$$

Then for any $\mathbf{w} \in \mathfrak{A}^1$ *one has*

$$\mathtt{M}_{\sqcup\!\sqcup}(\mathbf{w}; T) = \rho\big(\mathtt{M}_{\sqcap\!\sqcap}(\mathbf{w}; T)\big). \tag{3.1}$$

Remark 3.3. We know there should be $\mathbb{Q}$-linear relations among Euler sums or AMZVs that are not consequences of the regularized DBSF, see [36, Remark 3.5]. The same should hold for AMMVs.

Example 3.4. We illustrate the above theorem by a simple example. First,

$$\begin{aligned}
\mathtt{M}_{\sqcup\!\sqcup}(\check{1}, \bar{2}) &= \mathtt{M}_{\sqcup\!\sqcup}\big(\mathbf{p}(\omega_1^{-1}\omega_0\omega_{-1}^1)\big) = \mathtt{M}_{\sqcup\!\sqcup}(\omega_1^{-1}\omega_0\omega_{-1}^1) \\
&= \mathtt{M}_{\sqcup\!\sqcup}(\omega_1^{-1} \sqcup\!\sqcup\, \omega_0\omega_{-1}^1 - \omega_0\omega_1^{-1}\omega_{-1}^1 - \omega_0\omega_{-1}^1\omega_1^{-1}) \\
&= \mathtt{M}_{\sqcup\!\sqcup}(\omega_1^{-1})M(\check{2}) - M(\mathbf{q}(\omega_0\omega_1^{-1}\omega_{-1}^1)) - M(\mathbf{q}(\omega_0\omega_{-1}^1\omega_1^{-1}))) \\
&= (T + \log 2)M(\bar{2}) - M(\check{2}, \bar{1}) - M(\check{\bar{2}}, \check{\bar{1}}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathtt{M}_{\sqcap\!\sqcap}(\check{1}, \bar{2}) &= \mathtt{M}_{\sqcap\!\sqcap}\big(\mathbf{p}(\mathbf{z}_{1,1}^{-1}\mathbf{z}_{2,-1}^1)\big) = \mathtt{M}_{\sqcap\!\sqcap}\big(\mathbf{p}(\mathbf{z}_{1,1}^{-1} * \mathbf{z}_{2,-1}^1 - \mathbf{z}_{2,-1}^1\mathbf{z}_{1,1}^{-1})\big) \\
&= \mathtt{M}_{\sqcap\!\sqcap}\big(\mathbf{z}_{1,1}^{-1} \sqcap\!\sqcap \mathbf{z}_{2,-1}^1 - \mathbf{p}(\mathbf{z}_{2,-1}^1\mathbf{z}_{1,1}^{-1})\big) = (T + 2\log 2)M(\bar{2}) - M(\bar{2}, \check{1}),
\end{aligned}$$

since no stuffing can appear. By Theorem 3.2, we get

$$\begin{aligned}
(T + \log 2)M(\bar{2}) - M(\check{2}, \bar{1}) - M(\check{\bar{2}}, \check{\bar{1}}) &= \rho\Big((T + 2\log 2)M(\bar{2}) - M(\bar{2}, \check{1})\Big) \\
&= (T + \log 2)M(\bar{2}) - M(\bar{2}, \check{1}) \\
&\Longrightarrow M(\check{2}, \bar{1}) + M(\check{\bar{2}}, \check{\bar{1}}) = M(\bar{2}, \check{1}).
\end{aligned}$$

We can check this identity numerically that both sides are ≈ -0.7739912.

Theorem 3.5. (Duality Relation) *Let* $\boldsymbol{k} = (k_1, \dots, k_r), \boldsymbol{l} = (l_1, \dots, l_r) \in \mathbb{N}^r$ *and* $\boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \{\pm 1\}^r$ *with* $\varepsilon_1 = -1$. *Then*

$$M\big(\omega_0^{k_r}(\omega_{\sigma_r}^{\varepsilon_r})^{l_r} \cdots (\omega_{\sigma_2}^{\varepsilon_2})^{l_2}\omega_0^{k_1}(\omega_{\sigma_1}^{-1})^{l_1}\big) = M\big((u_{\sigma_1}^{-1})^{l_1}u_0^{k_1}(u_{\sigma_2}^{\varepsilon_2})^{l_2} \cdots (u_{\sigma_r}^{\varepsilon_r})^{l_r}u_0^{k_r}\big), \tag{3.2}$$

where $u_0 = w_{+1}^{-1}$, $u_{+1}^{-1} = w_0$, $u_{-1}^{-1} = w_{-1}^{-1}$, $u_{+1}^{+1} = w_0 + w_{+1}^{+1} - w_{+1}^{-1}$ *and* $u_{-1}^{+1} = w_{+1}^{+1} - w_{+1}^{-1} - w_{-1}^{+1}$.

Proof. Under the substitution $t \to \dfrac{1-t}{1+t}$, we get

$$dt \to \frac{-2dt}{(1+t)^2}, \quad 1 - t^2 \to \frac{4t}{(1+t)^2}, \quad 1 + t^2 \to \frac{2(1+t^2)}{(1+t)^2}.$$

Thus

$$
\begin{aligned}
w_0=\frac{dt}{t}\to -\frac{2dt}{1-t^2}=-w^{-1}_{+1},\quad w^{-1}_{+1}=\frac{2dt}{1-t^2}\to -w_0,\quad w^{-1}_{-1}=\frac{-2dt}{1+t^2}\to -w^{-1}_{-1},\\
w^{+1}_{+1}=\frac{2tdt}{1-t^2}\to \frac{-(1-t)dt}{t(1+t)}=w^{-1}_{+1}-w_0-w^{+1}_{+1},\\
w^{+1}_{-1}=\frac{-2tdt}{1+t^2}\to \frac{2(1-t)dt}{(1+t^2)(1+t)}=w^{-1}_{+1}-w^{+1}_{+1}+w^{+1}_{-1}.
\end{aligned}
$$

The theorem follows immediately. $\square$

Remark 3.6. Note that on the left-hand side of the duality relation (3.2) the last 1-form must be $\omega^{-1}_{\pm1}$. It is not difficult to adopt a regularization procedure to derive the duality relation with ending 1-form equal to $\omega^{+1}_{\pm1}$. We will leave this to the interested readers.

Example 3.7. We have the duality relations

$$
\begin{aligned}
M(\check{\bar{2}},\check{1},\check{\bar{1}})&=\int_0^1 w_0w^{+1}_{-1}w^{+1}_{-1}w^{-1}_{+1}=\int_0^1 u^{-1}_{+1}u^{+1}_{-1}u^{+1}_{-1}u_0\\
&=M(\check{2},\check{1},\check{1})+M(\check{2},1,\check{1})+M(2,\check{\bar{1}},\check{\bar{1}})+M(\bar{2},\bar{1},\check{1})+M(\check{\bar{2}},\check{1},\check{\bar{1}})\\
&\quad -M(2,1,\check{1})-M(\check{2},\check{\bar{1}},\check{\bar{1}})-M(2,\check{1},\check{1})-M(\check{\bar{2}},\check{\bar{1}},\check{1}),\\
M(2,\check{1},\check{\bar{2}})&=\int_0^1 w_0w^{-1}_{+1}w^{+1}_{+1}w_0w^{-1}_{-1}=\int_0^1 u^{-1}_{-1}u_0u^{+1}_{+1}u^{-1}_{+1}u_0\\
&=M(\check{\bar{1}},\bar{1},\check{3})+M(\check{\bar{1}},\bar{1},\check{1},\check{2})+M(\bar{1},\check{\bar{1}},1,\check{2}),\\
M(\bar{2},\bar{3},\check{\bar{1}})&=\int_0^1 w_0w^{+1}_{-1}w_0^2w^{-1}_{+1}w^{-1}_{-1}=\int_0^1 u^{-1}_{-1}u^{-1}_{+1}u_0^2u^{+1}_{-1}u_0\\
&=-M(\bar{1},\check{\bar{2}},1,\check{1},\check{1})-M(\check{\bar{1}},\bar{2},\check{1},1,\check{1})+M(\bar{1},\check{\bar{2}},1,\check{\bar{1}},\check{\bar{1}}),\\
M(\check{\bar{3}},\check{1},\bar{2},\check{1})&=\int_0^1 w_0^2w^{+1}_{-1}w^{-1}_{-1}w_0\left(w^{-1}_{+1}\right)^2=\int_0^1 \left(u^{-1}_{+1}\right)^2u_0u^{-1}_{-1}u^{+1}_{-1}u_0^2\\
&=M(3,\check{\bar{1}},\bar{1},1,\check{1})+M(\check{3},\bar{1},\check{\bar{1}},1,\check{1})-M(3,\check{\bar{1}},1,\bar{1},\check{1}).
\end{aligned}
$$

We end this section by the following parity theorem of AMMVs of arbitrary depth which follows from the general parity result of Panzer on colored MZVs (see [26]).

Theorem 3.8. *Suppose* $\boldsymbol{k}$ *is admissible whose depth* $r\geq 2$ *and weight* w *are of different parity. Then* $M^{\boldsymbol{\varepsilon}}_{\boldsymbol{\sigma}}(\boldsymbol{k})$ *can be expressed as a* $\mathbb{Q}$*-linear combination of alternating multiple* M*-values of lower depths and products of multiple* M*-values whose sum of depths are smaller than* r*.*

4 Some results on AMTVs and AMSVs

In this section, we will establish some explicit relations between alternating multiple T-values and alternating multiple S-values by using the integrals of arctangent function. Recall that $\{l\}_m=(l,\dots,l)$ with l repeating m times.

Lemma 4.1. *(cf. [25, Eqs. (2.2) and (2.3)]) For $p \in \mathbb{N}$, we have*

$$\int_0^{\pi/2} x^p \cot(x)dx = \left(\frac{\pi}{2}\right)^p \left\{ \log(2) + \sum_{k=1}^{[p/2]} \frac{p!(-1)^k(4^k-1)}{(p-2k)!(2\pi)^{2k}}\zeta(2k+1) \right\}$$
$$+ \delta_{[p/2],p/2} \frac{p!(-1)^{p/2}}{2^p}\zeta(p+1),$$
$$\int_0^{\pi/4} x^p \cot(x)dx = \frac{1}{2}\left(\frac{\pi}{4}\right)^p \left\{ \begin{array}{c} \log(2) + \displaystyle\sum_{k=1}^{[p/2]} \frac{p!(-1)^k(4^k-1)}{(p-2k)!(2\pi)^{2k}}\zeta(2k+1) \\ - \displaystyle\sum_{k=1}^{[(p+1)/2]} \frac{p!(-4)^k\beta(2k)}{(p+1-2k)!\pi^{2k-1}} \end{array} \right\}$$
$$+ \delta_{[p/2],p/2} \frac{p!(-1)^{p/2}}{2^p}\zeta(p+1),$$

where δ is the Kronecker symbol, and the Dirichlet beta function $\beta(s)$ is defined by

$$\beta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s} \quad (\Re(s) > 0). \tag{4.1}$$

When $s = 2$, $\beta(2) = G$ is known as Catalan's constant.

Proposition 4.2. *For $p \in \mathbb{N}$, we have*

$$\int_0^1 \arctan^p(x)dx \in \mathbb{Q}[\log(2), \pi, \beta(2), \zeta(3), \beta(4), \zeta(5), \cdots]. \tag{4.2}$$

Proof. Letting $x = \tan(t)$ and using integration by parts, we note that the integral on the left-hand side can be rewritten as

$$\int_0^1 \arctan^p(x)dx = \int_0^{\pi/4} t^p d\tan(t) = \left(\frac{\pi}{4}\right)^p - p\int_0^{\pi/4} t^{p-1}\tan(t)dt$$
$$= \left(\frac{\pi}{4}\right)^p - p\int_{\pi/4}^{\pi/2} \left(\frac{\pi}{2} - u\right)^{p-1} \cot(u)du$$
$$= \left(\frac{\pi}{4}\right)^p - p\frac{\pi^{p-1}}{2^p}\log(2) - p\sum_{k=1}^{p-1}(-1)^k \binom{p-1}{k} \left(\frac{\pi}{2}\right)^{p-1-k} \int_{\pi/4}^{\pi/2} u^k \cot(u)du. \tag{4.3}$$

Thus, applying Lemma 4.1 yields the desired description. □

As some examples, we have

$$\int_0^1 \arctan(x)dx = \frac{\pi}{4} - \frac{1}{2}\log(2),$$
$$\int_0^1 \arctan^2(x)dx = \frac{\pi^2}{16} + \frac{1}{4}\pi\log(2) - G,$$

$$\int_0^1 \arctan^3(x)dx = \frac{\pi^3}{64} + \frac{3}{32}\pi^2\log(2) - \frac{3}{4}\pi G + \frac{63}{64}\zeta(3),$$
$$\int_0^1 \arctan^4(x)dx = \frac{\pi^4}{256} + \frac{\pi^3}{32}\log(2) - \frac{3}{8}\pi^2 G - \frac{9}{64}\pi\zeta(3) + 3\beta(4).$$

Remark 4.3. In [1, p. 122], there is the Maclaurin series expansion

$$\arctan^p(x) = p!\sum_{k_0=1}^{\infty}(-1)^{k_0-1}\frac{x^{2k_0+p-2}}{2k_0+p-2}\prod_{\alpha=1}^{p-1}\left(\sum_{k_\alpha=1}^{k_{\alpha-1}}\frac{1}{2k_\alpha+p-\alpha-2}\right) \tag{4.4}$$

for $p \in \mathbb{N}$. Integrating (4.4) from 0 to 1 results in

$$\begin{aligned}\int_0^1 \arctan^p(x)dx =& p!\sum_{k_0=1}^{\infty}\frac{(-1)^{k_0-1}}{(2k_0+p-1)(2k_0+p-2)}\prod_{\alpha=1}^{p-1}\left(\sum_{k_\alpha=1}^{k_{\alpha-1}}\frac{1}{2k_\alpha+p-\alpha-2}\right)\\ =& p!\sum_{k_0=1}^{\infty}\frac{(-1)^{k_0-1}}{2k_0+p-2}\prod_{\alpha=1}^{p-1}\left(\sum_{k_\alpha=1}^{k_{\alpha-1}}\frac{1}{2k_\alpha+p-\alpha-2}\right)\\ &- p!\sum_{k_0=1}^{\infty}\frac{(-1)^{k_0-1}}{2k_0+p-1}\prod_{\alpha=1}^{p-1}\left(\sum_{k_\alpha=1}^{k_{\alpha-1}}\frac{1}{2k_\alpha+p-\alpha-2}\right).\end{aligned}$$

Therefore, we have

$$\int_0^1 \arctan^p(x)dx = \begin{cases} \dfrac{(-1)^{\frac{p}{2}}p!}{2^p}I_p & \text{if } p \text{ is even;}\\ \dfrac{(-1)^{\frac{p+1}{2}}p!}{2^p}J_p & \text{if } p \text{ is odd,}\end{cases} \tag{4.5}$$

where (setting $q = \frac{p}{2}$ and $q' = \frac{p-1}{2}$)

$$I_p = M_{-1,\{1\}_{p-1}}^{\{\mathrm{ev},\mathrm{od}\}_q}(\{1\}_p) + M_{-1,\{1\}_{p-1}}^{\mathrm{od},\mathrm{od},\{\mathrm{ev},\mathrm{od}\}_{q-1}}(\{1\}_p),$$

$$J_p = M_{-1,\{1\}_{p-1}}^{\mathrm{od},\{\mathrm{ev},\mathrm{od}\}_{q'}}(\{1\}_p) - M_{-1,\{1\}_{p-1}}^{\mathrm{ev},\{\mathrm{ev},\mathrm{od}\}_{q'}}(\{1\}_p).$$

For example,

$$\int_0^1 \arctan(x)dx = -\frac{1}{2}\left(M_{-1}^{\mathrm{od}}(1) - M_{-1}^{\mathrm{ev}}(1)\right),$$
$$\int_0^1 \arctan^2(x)dx = -\frac{1}{2}\left(M_{-1,1}^{\mathrm{ev},\mathrm{od}}(1,1) + M_{-1,1}^{\mathrm{od},\mathrm{od}}(1,1)\right),$$
$$\int_0^1 \arctan^3(x)dx = \frac{3}{4}\left(M_{-1,1,1}^{\mathrm{od},\mathrm{ev},\mathrm{od}}(1,1,1) - M_{-1,1,1}^{\mathrm{ev},\mathrm{ev},\mathrm{od}}(1,1,1)\right),$$
$$\int_0^1 \arctan^4(x)dx = \frac{3}{2}\left(M_{-1,1,1,1}^{\mathrm{ev},\mathrm{od},\mathrm{ev},\mathrm{od}}(1,1,1,1) + M_{-1,1,1,1}^{\mathrm{od},\mathrm{od},\mathrm{ev},\mathrm{od}}(1,1,1,1)\right).$$

For positive integers m and n such that $n \geq m$, we set

$$T_n^{\boldsymbol{\sigma}_{2m-1}} := T_n^{\boldsymbol{\sigma}_{2m-1}}(\{1\}_{2m-1}), \quad T_n^{\boldsymbol{\sigma}_{2m}} := T_n^{\boldsymbol{\sigma}_{2m}}(\{1\}_{2m}),$$
$$S_n^{\boldsymbol{\sigma}_{2m-1}} := S_n^{\boldsymbol{\sigma}_{2m-1}}(\{1\}_{2m-1}), \quad S_n^{\boldsymbol{\sigma}_{2m}} := S_n^{\boldsymbol{\sigma}_{2m}}(\{1\}_{2m}).$$

For any $n \in \mathbb{N}$ we put $T_n^{\boldsymbol{\sigma}_0} = S_n^{\boldsymbol{\sigma}_0} := 1$.

Theorem 4.4. *For positive integers n and m,*

$$\begin{aligned}\int_0^1 x^{2n-2}\arctan^{2m}(x)dx =& \frac{1+(-1)^n}{2n-1}t^{2m}(\bar{1}) + \frac{(-1)^{n+m}(2m)!}{2^{2m}(2n-1)}T_n^{\{1\}_{2m-1},-1} \\ &- \frac{(-1)^n(2m)!}{2n-1}\sum_{u=0}^{m-1}\frac{(-1)^u T_n^{\{1\}_{2u}}}{(2m-2u)!2^{2u}}A(2m-2u) \\ &+ \frac{(-1)^n(2m)!}{2n-1}\sum_{v=1}^{m-1}\frac{(-1)^v t^{2m-2v}(\bar{1})}{2^{2v}(2m-2v)!}\left(T_n^{\{1\}_{2v}} + T_n^{\{1\}_{2v-1},-1}\right) \\ &+ \frac{(-1)^n(2m)!}{2n-1}\sum_{v=0}^{m-1}\frac{(-1)^v t^{2m-2v-1}(\bar{1})}{2^{2v+1}(2m-2v-1)!}\left(S_n^{\{1\}_{2v+1}} - S_n^{\{1\}_{2v},-1}\right),\end{aligned} \tag{4.6}$$

$$\begin{aligned}\int_0^1 x^{2n-2}\arctan^{2m-1}(x)dx =& -\frac{1+(-1)^n}{2n-1}t^{2m-1}(\bar{1}) - \frac{(-1)^{n+m}(2m-1)!}{2^{2m-1}(2n-1)}S_n^{\{1\}_{2m-2},-1} \\ &- \frac{(-1)^n(2m-1)!}{2n-1}\sum_{u=0}^{m-1}\frac{(-1)^u T_n^{\{1\}_{2u}}}{(2m-2u-1)!2^{2u}}A(2m-2u-1) \\ &+ \frac{(-1)^n(2m-1)!}{2n-1}\sum_{v=1}^{m-1}\frac{(-1)^{v+1} t^{2m-2v-1}(\bar{1})}{2^{2v}(2m-2v-1)!}\left(T_n^{\{1\}_{2v}} + T_n^{\{1\}_{2v-1},-1}\right) \\ &+ \frac{(-1)^n(2m-1)!}{2n-1}\sum_{v=1}^{m-1}\frac{(-1)^v t^{2m-2v}(\bar{1})}{2^{2v-1}(2m-2v)!}\left(S_n^{\{1\}_{2v-1}} - S_n^{\{1\}_{2v-2},-1}\right),\end{aligned} \tag{4.7}$$

$$\begin{aligned}\int_0^1 x^{2n-1}\arctan^{2m}(x)dx =& \frac{1-(-1)^n}{2n}t^{2m}(\bar{1}) + \frac{(-1)^{n+m}(2m)!}{2^{2m}(2n)}S_n^{\{1\}_{2m-1},-1} \\ &+ \frac{(-1)^n(2m)!}{2n}\sum_{u=0}^{m-1}\frac{(-1)^u T_n^{\{1\}_{2u+1}}}{(2m-2u-1)!2^{2u+1}}A(2m-2u-1) \\ &+ \frac{(-1)^n(2m)!}{2n}\sum_{v=0}^{m-1}\frac{(-1)^v t^{2m-2v-1}(\bar{1})}{2^{2v+1}(2m-2v-1)!}\left(T_n^{\{1\}_{2v+1}} + T_n^{\{1\}_{2v},-1}\right) \\ &+ \frac{(-1)^n(2m)!}{2n}\sum_{v=1}^{m-1}\frac{(-1)^{v+1} t^{2m-2v}(\bar{1})}{2^{2v}(2m-2v)!}\left(S_n^{\{1\}_{2v}} - S_n^{\{1\}_{2v-1},-1}\right),\end{aligned} \tag{4.8}$$

$$\int_0^1 x^{2n-1}\arctan^{2m-1}(x)dx = -\frac{1-(-1)^n}{2n}t^{2m-1}(\bar{1}) + \frac{(-1)^{n+m}(2m-1)!}{2^{2m-1}(2n)}T_n^{\{1\}_{2m-2},-1}$$
$$-\frac{(-1)^n(2m-1)!}{2n}\sum_{u=1}^{m-1}\frac{(-1)^u T_n^{\{1\}_{2u-1}}}{(2m-2u)!2^{2u-1}}A(2m-2u)$$
$$+\frac{(-1)^n(2m-1)!}{2n}\sum_{v=1}^{m-1}\frac{(-1)^v t^{2m-2v}(\bar{1})}{2^{2v-1}(2m-2v)!}\left(T_n^{\{1\}_{2v-1}} + T_n^{\{1\}_{2v-2},-1}\right)$$
$$+\frac{(-1)^n(2m-1)!}{2n}\sum_{v=1}^{m-1}\frac{(-1)^v t^{2m-2v-1}(\bar{1})}{2^{2v}(2m-2v-1)!}\left(S_n^{\{1\}_{2v}} - S_n^{\{1\}_{2v-1},-1}\right), \tag{4.9}$$

where $A(p) = \int_0^1 \arctan^p(x)dx$*, and it can be explicitly expressed by* (4.3) *and Lemma 4.1.*

Proof. Consider the integral

$$\int_0^1 x^{k-1}\arctan^p(x)dx = \frac{1}{k}\int_0^1 \arctan^p(x)dx^k$$
$$= \frac{(-1)^p t^p(\bar{1})}{k} - \frac{p}{k}\int_0^1 \frac{x^k}{1+x^2}\arctan^{p-1}(x)dx, \tag{4.10}$$

where we use the identity $\arctan(1) = -t(\bar{1}) = \frac{\pi}{4} = -\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1} = \beta(1)$.

When $k = 2n$ then

$$\int_0^1 x^{2n-1}\arctan^p(x)dx = \frac{(-1)^p t^p(\bar{1})}{2n} - \frac{p}{2n}\int_0^1 \frac{x^{2n}}{1+x^2}\arctan^{p-1}(x)dx$$
$$= \frac{(-1)^p t^p(\bar{1})}{2n} - \frac{p}{2n}\int_0^1\left\{\frac{(-1)^n}{1+x^2} + (-1)^n\sum_{k=1}^{n}(-1)^k x^{2k-2}\right\}\arctan^{p-1}(x)dx$$
$$= \frac{(-1)^p t^p(\bar{1})}{2n} - \frac{p}{2n}(-1)^n\int_0^1\frac{\arctan^{p-1}(x)}{1+x^2}dx - \frac{p}{2n}(-1)^n\sum_{k=1}^{n}(-1)^k\int_0^1 x^{2k-2}\arctan^{p-1}(x)dx$$
$$= \frac{1-(-1)^n}{2n}(-1)^p t^p(\bar{1}) - p\frac{(-1)^n}{2n}\sum_{k=1}^{n}(-1)^k\int_0^1 x^{2k-2}\arctan^{p-1}(x)dx. \tag{4.11}$$

When $k = 2n-1$ then

$$\int_0^1 x^{2n-2}\arctan^p(x)dx = \frac{(-1)^p t^p(\bar{1})}{2n-1} - \frac{p}{2n-1}\int_0^1\frac{x^{2n-1}}{1+x^2}\arctan^{p-1}(x)dx$$
$$= \frac{(-1)^p t^p(\bar{1})}{2n-1} - \frac{p}{2n-1}\int_0^1 x\left\{\frac{(-1)^{n-1}}{1+x^2} + (-1)^{n-1}\sum_{k=1}^{n-1}(-1)^k x^{2k-2}\right\}\arctan^{p-1}(x)dx$$

$$= \frac{(-1)^p t^p(\bar{1})}{2n-1} + p\frac{(-1)^n}{2n-1}\int_0^1 \frac{x\arctan^{p-1}(x)}{1+x^2}dx + p\frac{(-1)^n}{2n-1}\sum_{k=1}^{n-1}(-1)^k\int_0^1 x^{2k-1}\arctan^{p-1}(x)dx. \tag{4.12}$$

Integration by parts leads to

$$\begin{aligned}\int_0^1 \frac{x\arctan^{p-1}(x)}{1+x^2}dx &= \int_0^1 x\arctan^{p-1}(x)d(\arctan(x))\\ &= (-1)^p t^p(\bar{1}) - \int_0^1 \arctan^p(x)dt - (p-1)\int_0^1 \frac{x\arctan^{p-1}(x)}{1+x^2}dx,\end{aligned}$$

and hence:

$$p\int_0^1 \frac{x\arctan^{p-1}(x)}{1+x^2}dx = (-1)^p t^p(\bar{1}) - \int_0^1 \arctan^p(x)dx. \tag{4.13}$$

Substituting (4.13) into (4.12), we obtain

$$\begin{aligned}&\int_0^1 x^{2n-2}\arctan^p(x)dx\\ =&\frac{1+(-1)^n}{2n-1}(-1)^p t^p(\bar{1}) - \frac{(-1)^n}{2n-1}\int_0^1 \arctan^p(x)dx + p\frac{(-1)^n}{2n-1}\sum_{k=1}^{n-1}(-1)^k\int_0^1 x^{2k-1}\arctan^{p-1}(x)dx.\end{aligned} \tag{4.14}$$

Using the two recurrence relations (4.11) and (4.14), we can imply the desired evaluations immediately. □

Theorem 4.5. *For positive integer m,*

$$\begin{aligned}\int_0^1 \frac{\arctan^{2m}(x)}{x}dx =& t^{2m-1}(\bar{1})(t(\bar{2})+t(2)) + \frac{(-1)^m(2m-1)!}{2^{2m}}S(2,\{1\}_{2m-2},\bar{1})\\ &+ (2m-1)!\sum_{u=0}^{m-1}\frac{(-1)^u A(2m-2u-1)}{(2m-2u-1)!2^{2u+1}}T(2,\{1\}_{2u})\\ &- (2m-1)!\sum_{v=1}^{m-1}\frac{(-1)^{v+1}t^{2m-2v-1}(\bar{1})}{2^{2v+1}(2m-2v-1)!}\left(T(2,\{1\}_{2v})+T(2,\{1\}_{2v-1},\bar{1})\right)\\ &- (2m-1)!\sum_{v=1}^{m-1}\frac{(-1)^v t^{2m-2v}(\bar{1})}{2^{2v}(2m-2v)!}\left(S(2,\{1\}_{2v-1})-S(2,\{1\}_{2v-2},\bar{1})\right),\end{aligned} \tag{4.15}$$

$$\begin{aligned}\int_0^1 \frac{\arctan^{2m+1}(x)}{x}dx =& -t^{2m}(\bar{1})(t(\bar{2})+t(2)) - \frac{(-1)^m(2m)!}{2^{2m+1}}T(2,\{1\}_{2m-1},\bar{1})\\ &+ (2m)!\sum_{u=0}^{m-1}\frac{(-1)^u A(2m-2u)}{(2m-2u)!2^{2u+1}}T(2,\{1\}_{2u})\end{aligned}$$

$$
\begin{aligned}
&- (2m)! \sum_{v=1}^{m-1} \frac{(-1)^v t^{2m-2v}(\bar{1})}{2^{2v+1}(2m-2v)!} \left(T(2,\{1\}_{2v}) + T(2,\{1\}_{2v-1},\bar{1})\right) \\
&- (2m)! \sum_{v=0}^{m-1} \frac{(-1)^v t^{2m-2v-1}(\bar{1})}{2^{2v+2}(2m-2v-1)!} \left(S(2,\{1\}_{2v+1}) - S(2,\{1\}_{2v},\bar{1})\right),
\end{aligned}
\tag{4.16}
$$

where $t(\bar{k}) = -\beta(k)$ *for* $k \in \mathbb{N}$.

Proof. Multiplying (4.6) and (4.7) by $\frac{(-1)^{n-1}}{2n-1}$ and summing n from 1 to infinity, we can deduce the desired results by (1.11)-(1.14) and the definitions of AMTVs and AMSVs. □

Example 4.6. Let $m = 1, 2$ in (4.15) and (4.16), we can obtain the following examples:

$$
\begin{aligned}
\int_0^1 \frac{\arctan^2(x)}{x} dx =& \frac{\pi}{4} G - \frac{\pi^2}{16}\log(2) - \frac{1}{4} S(2,\bar{1}), \\
\int_0^1 \frac{\arctan^3(x)}{x} dx =& -\frac{\pi^2}{16} G + \frac{\pi^3}{32}\log(2) + \frac{1}{4} T(2,1,\bar{1}) + \frac{\pi}{8}\left(S(2,1) - S(2,\bar{1})\right), \\
\int_0^1 \frac{\arctan^4(x)}{x} dx =& \frac{63\pi^2}{512}\zeta(3) - \frac{5\pi^3}{64} G + \frac{3\pi^4}{256}\log(2) + \frac{3}{8}\log(2) T(2,1,1) + \frac{3\pi}{16} T(2,1,\bar{1}) \\
&+ \frac{3}{8} S(2,1,1,\bar{1}) + \frac{3\pi^2}{64}\left(S(2,1) - S(2,\bar{1})\right), \\
\int_0^1 \frac{\arctan^5(x)}{x} dx =& \frac{3\pi^2}{8}\beta(4) - \frac{9\pi^3}{512}\zeta(3) - \frac{11\pi^4}{256} G + \frac{\pi^5}{256}\log(2) \\
&- \frac{3}{4} T(2,1,1,1,\bar{1}) + \left(\frac{3}{2} G - \frac{3\pi}{8}\log(2)\right) T(2,1,1) + \frac{3\pi^2}{32} T(2,1,\bar{1}) \\
&+ \frac{\pi^3}{64}\left(S(2,1) - S(2,\bar{1})\right) - \frac{3\pi}{8}\left(S(2,1,1,1) - S(2,1,1,\bar{1})\right).
\end{aligned}
$$

Proposition 4.7. *For positive integer* r,

$$
\int_0^1 \frac{\arctan^r(x)}{x} dx = (-1)^{[(r+1)/2]} \frac{r!}{2^r} T(\bar{2},\{1\}_{r-1}). \tag{4.17}
$$

Proof. According to the definition of $\arctan(x)$, we have

$$
\begin{aligned}
\arctan^r(x) &= r! \int_0^x \left(\frac{dt}{1+t^2}\right)^r \\
&= r! \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{(-1)^{n_1 - r} x^{2n_1 - r}}{(2n_1 - r)(2n_2 - r + 1)\cdots(2n_r - 1)}.
\end{aligned}
\tag{4.18}
$$

Hence, multiplying it by $1/x$ and integrating over $(0, 1)$ yields

$$
\int_0^1 \frac{\arctan^r(x)}{x} dx = r! \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{(-1)^{n_1 - r}}{(2n_1 - r)^2(2n_2 - r + 1)\cdots(2n_r - 1)}. \tag{4.19}
$$

Further, from the definition of AMTVs (1.19) give

$$\begin{aligned}
&T(\bar{2},\{1\}_{2m-2})\\
&=(-1)^{m+1}2^{2m-1}\sum_{n_1>n_2>\cdots>n_{2m-1}>0}\frac{(-1)^{n_1}}{(2n_1-2m+1)^2(2n_2-2m+2)\cdots(2n_{2m-1}-1)},\\
&T(\bar{2},\{1\}_{2m-1})\\
&=(-1)^{m}2^{2m}\sum_{n_1>n_2>\cdots>n_{2m-1}>0}\frac{(-1)^{n_1}}{(2n_1-2m)^2(2n_2-2m+1)\cdots(2n_{2m}-1)}.
\end{aligned}$$

Thus, we obtain the desired result with an elementary calculation. □

From [32, Thms. 3.16 and 3.17], for $p\in\mathbb{N}$ we have

$$T(\bar{1},\{1\}_{p-1},\bar{1})=T(\bar{2},\{1\}_{p-1})\in\mathbb{Q}[\beta(1),\zeta(2),\beta(2),\zeta(3),\beta(3),\zeta(4),\ldots].$$

Hence, we deduce the following cases ($\beta(2)=G$):

$$\begin{aligned}
&\int_0^1\frac{\arctan(x)}{x}dx=G,\\
&\int_0^1\frac{\arctan^2(x)}{x}dx=\frac{1}{2}\pi G-\frac{7}{8}\zeta(3),\\
&\int_0^1\frac{\arctan^3(x)}{x}dx=\frac{9}{8}\zeta(2)G-\frac{3}{2}\beta(4),\\
&\int_0^1\frac{\arctan^4(x)}{x}dx=\frac{93}{32}\zeta(5)-\frac{3}{2}\pi\beta(4)+\frac{1}{16}\pi^3G.
\end{aligned}$$

Corollary 4.8. *For positive integer m,*

$$S(2,\{1\}_{2m-2},\bar{1}),T(2,\{1\}_{2m-1},\bar{1})\in\mathbb{Q}[\log(2),\pi,\zeta(2),\beta(2),\zeta(3),\beta(3),\zeta(4),\ldots].\tag{4.20}$$

Proof. The corollary follows immediately from Theorem 4.5 and Proposition 4.7 and notating the fact that $T(2,\{1\}_{m-1})=T(m+1)$ and $S(2,\{1\}_{2m-1})$ can be expressed in terms of a linear combination of products of Riemann zeta values (see [31, Eq. (3.17)]). □

As four examples, we have

$$\begin{aligned}
&S(2,\bar{1})=\frac{7}{2}\zeta(3)-\pi G-\frac{\pi^2}{4}\log(2),\\
&T(2,1,\bar{1})=-6\beta(4)+3\zeta(2)G,\\
&S(2,1,1,\bar{1})=\frac{31}{4}\zeta(5)-\frac{15}{8}\zeta(4)\log(2)-\frac{63}{32}\zeta(2)\zeta(3)-\pi\beta(4),\\
&T(2,1,1,1,\bar{1})=\frac{15}{4}\zeta(4)G+3\zeta(2)\beta(4)-10\beta(6).
\end{aligned}$$

Question 1. From the second and fourth examples, we can find that log(2) does not appear. Is it true that

$$T(2,\{1\}_{2m-1},\bar{1})\in\mathbb{Q}[\pi,\zeta(2),\beta(2),\zeta(3),\beta(3),\zeta(4),\ldots]?$$

Remark 4.9. (i) Recall that the convoluted T-values $T(\boldsymbol{k}\circledast\boldsymbol{l})$ (see [31, Defn. 1.2]) can be regarded as a T-variant of Kaneko–Yamamoto MZVs $\zeta(\boldsymbol{k}\circledast\boldsymbol{l}^\star)$ (see [21]). Similarly, by using the alternating multiple T-harmonic sums and alternating multiple S-harmonic sums, it is possible to define the alternating convoluted T-values so that the first factor in the sum is alternating multiple T-harmonic sums and the second is either alternating multiple S-harmonic sums or alternating multiple T-harmonic sums. (ii) For positive integers k and m, it is possible to establish some explicit relations between $S(k+1,\{1\}_{2m-2},\bar{1})$ and $T(k+1,\{1\}_{2m-1},\bar{1})$ (even more general alternating convoluted T-values) by considering the following iterated integral

$$\int_0^1 \frac{\arctan^p(t)dt}{t}\left(\frac{dt}{t}\right)^m\frac{\arctan^q(t)dt}{t}\quad (p,m\in\mathbb{N}_0,q\in\mathbb{N}).$$

Clearly, the above iterated integral can be expressed in terms of AMMVs.

Applying Au's Mathematica package [2,3], we can find many explicit evaluations of AMTVs and AMSVs. We listed some cases, please see Examples 4.10, 4.11 and 4.13.

Example 4.10. We have

$$\begin{aligned}
T(\bar{2},1,\bar{1}) &= -4G^2+\frac{1}{32}\pi^4,\\
T(\bar{2},1,1,1,\bar{1}) &= \frac{1}{2}\pi^2G^2+\frac{5}{1536}\pi^6-8G\beta(4),\\
S(\bar{2},1,1) &= -2G^2-\frac{53}{1440}\pi^4+2G\pi\log(2)-\frac{1}{6}\pi^2\log^2(2)+\frac{1}{6}\log^4(2)+4\operatorname{Li}_4(1/2),\\
S(\bar{2},1,\bar{1}) &= -2G^2-\frac{61}{5760}\pi^4+G\pi\log(2)-\frac{1}{12}\pi^2\log^2(2)+\frac{1}{12}\log^4(2)+2\operatorname{Li}_4(1/2),\\
S(\bar{2},1,1,1,1) &= \frac{31}{20160}\pi^6-4G\beta(4)-\frac{1}{12}G\pi^3\log(2)+2\pi\beta(4)\log(2)+2\zeta(\bar{5},1)+\frac{3}{2}G\pi\zeta(3)\\
&\quad-\frac{33}{16}\zeta^2(3)-\frac{31}{8}\log(2)\zeta(5),\\
S(\bar{2},1,1,1,\bar{1}) &= \frac{443}{322560}\pi^6-4G\beta(4)-\frac{1}{24}G\pi^3\log(2)+\pi\beta(4)\log(2)+\zeta(\bar{5},1)+\frac{21}{16}G\pi\zeta(3)\\
&\quad-\frac{195}{128}\zeta^2(3)-\frac{31}{16}\log(2)\zeta(5).
\end{aligned}$$

Noting the fact that

$$\zeta(\bar{3},1)=-\frac{1}{48}\pi^4-\frac{1}{12}\pi^2\log^2(2)+\frac{1}{12}\log^4(2)+2\operatorname{Li}_4(1/2)+\frac{7}{4}\log(2)\zeta(3),$$

hence, we would like to conclude this section with the following question.

Question 2. Is it true that for $m\in\mathbb{N}$,

$$T(\bar{2},\{1\}_{2m-1},\bar{1})\in\mathbb{Q}[\pi,\zeta(2),\beta(2),\zeta(3),\beta(3),\zeta(4),\ldots]$$

and $S(\bar{2},\{1\}_{2m})$ and $S(\bar{2},\{1\}_{2m-1},\bar{1})$ can be expressed in terms of a linear combinations of products of $\log(2),\pi,\zeta(\overline{2k+1},1),\zeta(2l+1)$ and $\beta(2p)$, where $k,l,p\in\mathbb{N}$.

Example 4.11. We have

$$
\begin{aligned}
T(\bar{3},1,\bar{1}) &= -\frac{1}{8}\pi^3 G - \frac{7}{32}\pi^2\zeta(3) + \frac{93}{16}\zeta(5),\\
S(\bar{3},1,\bar{1}) &= 2\operatorname{Li}_5(1/2) - \frac{589}{256}\zeta(5) - \frac{7}{8}\zeta(3)\log^2(2) - \frac{1}{60}\log^5(2) + \frac{1}{36}\pi^2\log^3(2)\\
&\quad + \frac{151}{5760}\pi^4\log(2),\\
S(4,1,\bar{1}) &= \zeta(\bar{5},1) + \frac{1}{12}\pi^2\operatorname{Li}_4(1/2) - \frac{83}{128}\zeta^2(3) + \frac{7}{32}\pi^2\zeta(3)\log(2) - \frac{31}{16}\zeta(5)\log(2)\\
&\quad + \frac{227\pi^6}{967680} + \frac{1}{288}\pi^2\log^4(2) - \frac{1}{288}\pi^4\log^2(2),\\
T(\bar{3},2,\bar{1}) &= 12G\beta(4) - \frac{1}{2}\pi^2G^2 - \frac{1}{8}\pi^3G\log(2) - \frac{1}{4}\pi^3\Im\operatorname{Li}_3\left(\frac{1+i}{2}\right) + \pi^2\operatorname{Li}_4(1/2)\log(2)\\
&\quad + \frac{7}{8}\pi^2\zeta(3) - \frac{257}{23040}\pi^6 + \frac{1}{24}\pi^2\log^4(2) - \frac{13}{384}\pi^4\log^2(2).
\end{aligned}
$$

Remark 4.12. For explicit evaluations of more general (alternating) triple T-values and (alternating) triple t-values, please see [30].

Example 4.13. We have

$$
\begin{aligned}
T(\bar{1},2) &= G\pi - \frac{7}{2}\zeta(3),\\
S(\bar{1},1) &= -2G + \pi\log(2),\\
S(\bar{1},\bar{1}) &= -2G + \frac{1}{2}\pi\log(2),\\
T(\bar{1},2,1) &= -\frac{1}{4}G\pi^2 + 6\beta(4) - \frac{7}{8}\pi\zeta(3),\\
T(\bar{1},1,2) &= -6\beta(4) + \frac{7}{4}\pi\zeta(3),\\
S(\bar{1},1,1) &= \frac{1}{4}\pi^2\log(2) - \frac{21}{16}\zeta(3),\\
S(\bar{1},1,\bar{1}) &= \frac{1}{8}\pi^2\log(2) - \frac{7}{8}\zeta(3),\\
T(\bar{1},2,1,1) &= -\frac{1}{24}G\pi^3 + 3\pi\beta(4) - \frac{31}{4}\pi\zeta(5),\\
T(\bar{1},1,1,2) &= \pi\beta(4) + \frac{7}{16}\pi^2\zeta(3) - \frac{31}{4}\zeta(5),\\
T(\bar{1},1,2,1) &= -3\pi\beta(4) - \frac{7}{32}\pi^2\zeta(3) + \frac{93}{8}\zeta(5),\\
S(\bar{1},1,1,1) &= -2\beta(4) - \frac{1}{24}\pi^3\log(2) + \frac{3}{4}\pi\zeta(3),\\
S(\bar{1},1,1,\bar{1}) &= -2\beta(4) - \frac{1}{48}\pi^3\log(2) + \frac{21}{32}\pi\zeta(3),\\
T(\bar{1},2,1,1,1) &= \frac{1}{192}G\pi^4 - \frac{3}{4}\pi^2\beta(4) + 10\beta(6) - \frac{31}{32}\pi\zeta(5),
\end{aligned}
$$

$$T(\bar{1},1,1,1,2) = -10\beta(6) - \frac{7}{96}\pi^3\zeta(3) + \frac{31}{8}\zeta(5),$$
$$T(\bar{1},1,2,1,1) = \frac{3}{4}\pi^2\beta(4) - 20\beta(6) + \frac{31}{8}\pi\zeta(5),$$
$$T(\bar{1},1,1,2,1) = -\frac{1}{4}\pi^2\beta(4) + 20\beta(6) + \frac{7}{192}\pi^3\zeta(3) - \frac{93}{16}\pi\zeta(5),$$
$$S(\bar{1},1,1,1,1) = -\frac{1}{192}\pi^4\log(2) + \frac{3}{16}\pi^2\zeta(3) - \frac{465}{256}\zeta(5),$$
$$S(\bar{1},1,1,1,\bar{1}) = -\frac{1}{384}\pi^4\log(2) + \frac{21}{128}\pi^2\zeta(3) - \frac{217}{128}\zeta(5),$$
$$S(\bar{1},1,1,1,1,1) = -2\beta(6) + \frac{1}{1920}\pi^5\log(2) - \frac{1}{32}\pi^3\zeta(3) + \frac{15}{16}\pi\zeta(5),$$
$$S(\bar{1},1,1,1,1,\bar{1}) = -2\beta(6) + \frac{1}{3840}\pi^5\log(2) - \frac{7}{256}\pi^3\zeta(3) + \frac{465}{512}\pi\zeta(5).$$

From Example 4.13, we end this section by the following three Theorems on the evaluations of AMTVs and AMSVs of special forms.

Theorem 4.14. *For $m \in \mathbb{N}_0$, we have*

$$S(\bar{1},\{1\}_m) \in \mathbb{Q}[\log(2),\pi,\zeta(2),\beta(2),\zeta(3),\beta(3),\zeta(4),\ldots]. \tag{4.21}$$

Proof. According to the iterated integral expressions of AMSVs, one obtain

$$\begin{aligned}
S(\bar{1},\{1\}_m) &= (-1)^{[m/2]}\int_0^1 \underbrace{w_{-1}^{-1}\cdots w_{-1}^{-1}}_{m} w_{-1}^{+1}\\
&= (-1)^{[(m+1)/2]}2^{m+1}\int_0^1 \left(\frac{dt}{1+t^2}\right)^m \frac{tdt}{1+t^2}\\
&= \frac{(-1)^{[(m+1)/2]}2^{m+1}}{m!}\int_0^1 \frac{\left(\frac{\pi}{4}-\arctan t\right)^m tdt}{1+t^2}\\
&= \frac{(-1)^{[(m+1)/2]}2^{m+1}}{m!}\sum_{k=0}^m \binom{m}{k}\left(\frac{\pi}{4}\right)^{m-k}(-1)^k\int_0^1 \frac{(\arctan t)^k t}{1+t^2}dt\\
&= (-1)^{[(m+1)/2]}2^{m+1}\sum_{k=0}^m \frac{\left(\frac{\pi}{4}\right)^{m-k}(-1)^k}{(k+1)!(m-k)!}\left\{\left(\frac{\pi}{4}\right)^{k+1} - \int_0^1 (\arctan t)^{k+1}dt\right\},
\end{aligned}$$

where $[x]$ denotes the greatest integer less than or equal to x.

Finally, applying the Proposition 4.2 yields the desired result. □

Theorem 4.15. *For $a, b \in \mathbb{N}_0$, we have*

$$T(\bar{1},\{1\}_a,2,\{1\}_b) \in \mathbb{Q}[\pi,\zeta(2),\beta(2),\zeta(3),\beta(3),\zeta(4),\ldots]. \tag{4.22}$$

Proof. Applying the iterated integral expression of AMTVs gives

$$T(\overline{k_1},k_2,\ldots,k_r) = (-1)^{[r/2]}\int_0^1 w_0^{k_1-1}w_{-1}^{-1}w_0^{k_2-1}w_{-1}^{-1}\ldots w_0^{k_r-1}w_{-1}^{-1}.$$

Hence, by an elementary calculation yields

$$\begin{aligned}
T(\bar{1},\{1\}_a,2,\{1\}_b) &= (-1)^{[(a+b+2)/2]}\int_0^1 \underbrace{w_{-1}^{-1}\cdots w_{-1}^{-1}}_{a+1} w_0 \underbrace{w_{-1}^{-1}\cdots w_{-1}^{-1}}_{b+1}\\
&= 2^{a+b+2}(-1)^{[(3a+3b)/2]+1}\int_0^1 \left(\frac{dt}{1+t^2}\right)^{a+1}\frac{dt}{t}\left(\frac{dt}{1+t^2}\right)^{b+1}\\
&= \frac{2^{a+b+2}(-1)^{[(3a+3b)/2]+1}}{(a+1)!(b+1)!}\int_0^1 \frac{\left(\frac{\pi}{4}-\arctan t\right)^{a+1}(\arctan t)^{b+1}}{t}dt\\
&= \frac{2^{a+b+2}(-1)^{[(3a+3b)/2]+1}}{(a+1)!(b+1)!}\sum_{k=0}^{a+1}\binom{a+1}{k}\left(\frac{\pi}{4}\right)^{a+1-k}(-1)^k\int_0^1 \frac{(\arctan t)^{k+b+1}}{t}dt.
\end{aligned}$$

Thus, using Proposition 4.7 and noting the fact that

$$\int_0^1 \frac{(\arctan t)^m}{t}dt, T(\overline{2},\{1\}_{m-1}) \in \mathbb{Q}[\beta(1),\zeta(2),\beta(2),\zeta(3),\beta(3),\zeta(4),\ldots] \quad (m\in\mathbb{N}),$$

we arrive at the desired conclusion. □

Theorem 4.16. *For any $m\in\mathbb{N}_0$, we have*

$$S(\bar{1},\{1\}_m,\bar{1}) \in \mathbb{Q}[\log(2),\pi,\zeta(2),\beta(2),\zeta(3),\beta(3),\zeta(4),\ldots]. \tag{4.23}$$

Proof. Similar to the iterated integral of $S(\bar{1},\{1\}_m)$, by definition of AMSVs, we obtain the iterated integral of $S(\bar{1},\{1\}_m,\bar{1})$:

$$S(\bar{1},\{1\}_m,\bar{1}) = \frac{(-1)^{[m/2]+1}2^{m+2}}{(m+1)!}\int_0^1 \frac{\left(\frac{\pi}{4}-\arctan t\right)^{m+1}t}{1-t^2}dt \quad (m\in\mathbb{N}_0).$$

Integration by parts yields

$$S(\bar{1},\{1\}_m,\bar{1}) = \frac{(-1)^{[m/2]}2^{m+1}}{m!}\int_0^1 \frac{\left(\frac{\pi}{4}-\arctan t\right)^{m}\log(1-t^2)}{1+t^2}dt.$$

Setting $t=\tan x$ gives

$$\begin{aligned}
S(\bar{1},\{1\}_m,\bar{1}) &= \frac{(-1)^{[m/2]}2^{m+1}}{m!}\int_0^{\pi/4}\left(\frac{\pi}{4}-x\right)^m \log\left(\frac{\cos(2x)}{\cos^2(x)}\right)dt\\
&= \frac{(-1)^{[m/2]}2^{m+1}}{m!}\int_0^{\pi/4}\left(\frac{\pi}{4}-x\right)^m \log(\cos(2x))dt\\
&\quad - \frac{(-1)^{[m/2]}2^{m+2}}{m!}\int_0^{\pi/4}\left(\frac{\pi}{4}-x\right)^m \log(\cos(x))dt.
\end{aligned}$$

Letting $x = \frac{\pi}{4} - u$ in the first integral and $x = \frac{\pi}{2} - u$ in the second integral, one obtains

$$\begin{aligned}
S(\bar{1}, \{1\}_m, \bar{1}) &= \frac{(-1)^{[m/2]}2^{m+1}}{m!} \left\{ \int_0^{\pi/4} u^m \log(\sin 2u) du - 2 \int_{\pi/4}^{\pi/2} \left(u - \frac{\pi}{4}\right)^m \log(\sin u) du \right\} \\
&= \frac{(-1)^{[m/2]}}{m!} \int_0^{\pi/2} y^m \log(\sin y) dy \\
&\quad - \frac{(-1)^{[m/2]}2^{m+2}}{m!} \sum_{k=0}^m \binom{m}{k} \left(\frac{\pi}{4}\right)^{m-k} \int_{\pi/4}^{\pi/2} u^k \log(\sin u) du,
\end{aligned}$$

where we replaced the $2u$ by y in the last step.

On the other hand, applying Lemma 4.1 and notating the fact that for $p \in \mathbb{N}$,

$$\int_0^{\pi/2} x^p \cot(x) dx = -p \int_0^{\pi/2} x^{p-1} \log(\sin x) dx \in \mathbb{Q}[\log(2), \pi, \zeta(2), \beta(2), \zeta(3), \beta(3), \zeta(4), \ldots]$$

and

$$\int_0^{\pi/4} x^p \cot(x) dx = -p \int_0^{\pi/4} x^{p-1} \log(\sin x) dx \in \mathbb{Q}[\log(2), \pi, \zeta(2), \beta(2), \zeta(3), \beta(3), \zeta(4), \ldots],$$

we deduce the desired result.

□

5 Parity of alternating triple M-values

In [32, Conj. 5.2], we conjectured the following parity result.

Conjecture 5.1. *For composition* $\boldsymbol{k} = (k_1, k_2, \ldots, k_r)$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_r) \in \{\pm 1\}^r$ *and* $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_r) \in \{\pm 1\}^r$ *with* $(k_1, \sigma_1) \neq (1, 1)$,

$$\left\{ 1 - \prod_{j=1}^r \left((-1)^{k_j+1} \operatorname{sign}(\sigma_j + \varepsilon_j + 1) \right) \right\} M_{\boldsymbol{\sigma}}^{\boldsymbol{\varepsilon}}(\boldsymbol{k})$$

can be expressed in terms of products of AMMVs of lower depth.

In [30], we obtain the parity theorems of AMtVs and AMTVs by considering the contour integral. Similar to [30], we will use the contour integral to evaluate the parity of alternating triple M-values. First, we introduce some notations and Lemmas.

Let $A = \{a_k\}_{k \in \mathbb{Z}}$ be a real sequence satisfying $\lim_{k \to \infty} a_k/k = 0$. For convenience, denote A_1 and A_2 as the constant sequence $\{1^k\}$ and the alternating sequence $\{(-1)^k\}$, respectively.

For brevity, the following notations of sums related to the sequence A are used in the sequel. Let $j \geq 1$ be a positive integer. Then for $n \in \mathbb{N}_0$, define

$$D^{(A)}(j) = \sum_{k=1}^\infty \frac{a_k}{k^j}, \quad D^{(A)}(1) = 0,$$

$$E_n^{(A)}(j)=\sum_{k=1}^{n}\frac{a_{n-k}}{k^j}\,,\quad E_0^{(A)}(j)=0\,,\qquad \bar{E}_n^{(A)}(j)=\sum_{k=1}^{n}\frac{a_{k-n-1}}{k^j}\,,\quad \bar{E}_0^{(A)}(j)=0\,,$$

$$\hat{E}_n^{(A)}(j)=\sum_{k=1}^{n}\frac{a_{n-k}}{(k-1/2)^j}\,,\quad \hat{E}_0^{(A)}(j)=0\,,\quad \tilde{E}_n^{(A)}(j)=\sum_{k=1}^{n}\frac{a_{k-n-1}}{(k-1/2)^j}\,,\quad \tilde{E}_0^{(A)}(j)=0\,,$$

and for $n\in\mathbb{Z}$, define

$$F_n^{(A)}(j)=\begin{cases}\sum_{k=1}^{\infty}\frac{a_{k+n}-a_k}{k}\,,\ j=1\,,\\ \sum_{k=1}^{\infty}\frac{a_{k+n}}{k^j}\,,\qquad j>1\,,\end{cases}\qquad \bar{F}_n^{(A)}(j)=\begin{cases}\sum_{k=1}^{\infty}\frac{a_{k-n}-a_k}{k}\,,\ j=1\,,\\ \sum_{k=1}^{\infty}\frac{a_{k-n}}{k^j}\,,\qquad j>1\,,\end{cases}$$

$$\hat{F}_n^{(A)}(j)=\begin{cases}\sum_{k=1}^{\infty}\left(\frac{a_{k+n}}{k-1/2}-\frac{a_k}{k}\right)\,,\ j=1\,,\\ \sum_{k=1}^{\infty}\frac{a_{k+n}}{(k-1/2)^j}\,,\qquad j>1\,,\end{cases}\qquad \tilde{F}_n^{(A)}(j)=\begin{cases}\sum_{k=1}^{\infty}\left(\frac{a_{k-n}}{k-1/2}-\frac{a_k}{k}\right)\,,\ j=1\,,\\ \sum_{k=1}^{\infty}\frac{a_{k-n}}{(k-1/2)^j}\,,\qquad j>1\,.\end{cases}$$

Now, let

$$\begin{aligned}
&G_n^{(A)}(j)=E_n^{(A)}(j)-\bar{E}_{n-1}^{(A)}(j)-\tfrac{a_0}{n^j}\,, && G_0^{(A)}(j)=0\,,\\
&L_n^{(A)}(j)=F_n^{(A)}(j)+(-1)^j\bar{F}_n^{(A)}(j)\,, &&\\
&M_n^{(A)}(j)=E_n^{(A)}(j)+(-1)^jF_n^{(A)}(j)\,, && \bar{M}_n^{(A)}(j)=\bar{F}_n^{(A)}(j)-\bar{E}_{n-1}^{(A)}(j)\,,\\
&N_n^{(A)}(j)=\hat{E}_n^{(A)}(j)+(-1)^j\hat{F}_{n-1}^{(A)}(j)\,, && \bar{N}_n^{(A)}(j)=\tilde{F}_n^{(A)}(j)-\tilde{E}_{n-1}^{(A)}(j)\,,\\
&R_n^{(A)}(j)=G_n^{(A)}(j)+(-1)^jL_n^{(A)}(j)\,, && S_n^{(A)}(j)=N_n^{(A)}(j)+\bar{N}_n^{(A)}(j)-\tfrac{a_0}{(n-1/2)^j}\,,
\end{aligned}$$

and let

$$\hat{t}^{(A)}(j)=\begin{cases}\sum_{k=1}^{\infty}\left(\frac{a_{k-1}}{k-1/2}-\frac{a_k}{k}\right)\,,\ j=1\,,\\ \sum_{k=1}^{\infty}\frac{a_{k-1}}{(k-1/2)^j}\,,\qquad j>1\,,\end{cases}\qquad \widetilde{t}^{(A)}(j)=\begin{cases}\sum_{k=1}^{\infty}\left(\frac{a_k}{k-1/2}-\frac{a_k}{k}\right)\,,\ j=1\,,\\ \sum_{k=1}^{\infty}\frac{a_k}{(k-1/2)^j}\,,\qquad j>1\,,\end{cases}$$

$$\check{t}^{(A)}(j)=(-1)^{j-1}\hat{t}^{(A)}(j)-\widetilde{t}^{(A)}(j)\,.$$

Note that in the above definitions, setting $A=A_1$ or A_2 yields

$$\begin{aligned}
&M_n^{(A_1)}(j)=H_n^{(j)}+(-1)^j\zeta(j)\,,\quad \bar{M}_n^{(A_1)}(j)=\zeta(j)-H_{n-1}^{(j)}\,,\\
&N_n^{(A_1)}(j)=h_n^{(j)}+(-1)^j\widetilde{t}(j)\,,\quad \bar{N}_n^{(A_1)}(j)=\widetilde{t}(j)-h_{n-1}^{(j)}\,,\\
&R_n^{(A_1)}(j)=(1+(-1)^j)\zeta(j)\,,\quad S_n^{(A_1)}(j)=(1+(-1)^j)\widetilde{t}(j)\,,\\
&D^{(A_1)}(j)=\zeta(j)\,,\quad \hat{t}^{(A_1)}(j)=\widetilde{t}^{(A_1)}(j)=\widetilde{t}(j)\,,\quad \check{t}^{(A_1)}(j)=-(1+(-1)^j)\widetilde{t}(j)\,,
\end{aligned}$$

and

$$\begin{aligned}
&M_n^{(A_2)}(j) = (-1)^{n-1}\bar{H}_n^{(j)} + (-1)^{n+j}\zeta(\bar{j}) - \delta_{j,1}\log(2)\,,\\
&\bar{M}_n^{(A_2)}(j) = (-1)^{n}\bar{H}_{n-1}^{(j)} + (-1)^{n}\zeta(\bar{j}) + \delta_{j,1}\log(2)\,,\\
&N_n^{(A_2)}(j) = (-1)^{n-1}\bar{h}_n^{(j)} + (-1)^{n+j-1}\widetilde{t}(\bar{j}) - \delta_{j,1}\log(2)\,,\\
&\bar{N}_n^{(A_2)}(j) = (-1)^{n}\bar{h}_{n-1}^{(j)} + (-1)^{n}\widetilde{t}(\bar{j}) + \delta_{j,1}\log(2)\,,\\
&R_n^{(A_2)}(j) = (-1)^n(1+(-1)^j)\zeta(\bar{j})\,, \quad S_n^{(A_2)}(j) = (-1)^n(1-(-1)^j)\widetilde{t}(\bar{j})\,,\\
&D^{(A_2)}(j) = \zeta(\bar{j}) + \delta_{j,1}\log(2)\,, \quad \hat{t}^{(A_2)}(j) = -\widetilde{t}(\bar{j}) + \delta_{j,1}\log(2)\,,\\
&\widetilde{t}^{(A_2)}(j) = \widetilde{t}(\bar{j}) + \delta_{j,1}\log(2)\,, \quad \check{t}^{(A_2)}(j) = -(1-(-1)^j)\widetilde{t}(\bar{j})\,.
\end{aligned}$$

where $\delta_{n,k}$ is the Kronecker delta and

$$\widetilde{t}(j) = \sum_{n=1}^{\infty} \frac{1}{(n-1/2)^j} = 2^j t(j) \quad \text{and} \quad \widetilde{t}(\bar{j}) := \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1/2)^j}.$$

Note also that we should interpret $\zeta(1) := 0$ and $\widetilde{t}(1) := 2\log(2)$ wherever they occur. Here $H_n^{(r)}$ stand for the *generalized harmonic numbers* defined by

$$H_0^{(r)} = 0 \quad \text{and} \quad H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r}\,, \quad \text{for } n, r \in \mathbb{N}.$$

Let $h_n^{(r)}$ denote the *odd harmonic numbers of order* r defined by

$$h_0^{(r)} = 0 \quad \text{and} \quad h_n^{(r)} = \sum_{k=1}^{n} \frac{1}{(k-1/2)^r}\,, \quad \text{for } n, r \in \mathbb{N}.$$

Set $H_n \equiv H_n^{(1)}$ and $h_n \equiv h_n^{(1)}$. The *alternating harmonic numbers* $\bar{H}_n^{(p)}$ and the *alternating odd harmonic numbers* $\bar{h}_n^{(p)}$ are defined by

$$\bar{H}_0^{(r)} = 0\,, \quad \bar{H}_n^{(r)} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^r}\,, \quad \text{and} \quad \bar{h}_0^{(r)} = 0\,, \quad \bar{h}_n^{(r)} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(k-1/2)^r}\,,$$

respectively, for $n, r \in \mathbb{N}$.

Some sums presented here can also be found in a previous paper [29] of the first named author. Moreover, in [29], the following two definitions were introduced.

Definition 5.1. The *parametric digamma function* $\Psi(-s; A)$ related to the sequence A is defined by

$$\Psi(-s; A) = \frac{a_0}{s} + \sum_{k=1}^{\infty} \left(\frac{a_k}{k} - \frac{a_k}{k-s} \right)\,, \quad \text{for } s \in \mathbb{C} \setminus \mathbb{N}_0\,.$$

Definition 5.2. The *parametric cotangent function* $\cot(\pi s; A)$ related to the sequence A is defined by

$$\pi\cot(\pi s; A) = -\frac{a_0}{s} + \Psi(-s; A) - \Psi(s; A) = \frac{a_0}{s} - 2s\sum_{k=1}^{\infty}\frac{a_k}{k^2 - s^2}\,.$$

Obviously, when $A = A_1$, the parametric digamma function $\Psi(-s; A)$ reduces to the classical digamma function $\psi(-s) + \gamma$, where γ is the Euler-Mascheroni constant. By setting $A = A_1$ or A_2, respectively, the parametric cotangent function $\cot(\pi s; A)$ turns into

$$\cot(\pi s; A_1) = \cot(\pi s) \quad \text{and} \quad \cot(\pi s; A_2) = \csc(\pi s)\,.$$

Using these notations, we now recall a few series expansions appearing in [29] and [30], which will be frequently used in the computation in the sequel. For the parametric digamma function $\Psi(-s; A)$, we have

Lemma 5.2. *(cf. [29, Theorem 2.1]) Let $p \geq 1$ and n be nonnegative integers, if $|s-n| < 1$ with $s \neq n$, then*

$$\frac{\Psi^{(p-1)}(-s; A)}{(p-1)!} = \frac{1}{(s-n)^p}\left\{a_n - \sum_{j=1}^{\infty}(-1)^j\binom{j+p-2}{p-1}M_n^{(A)}(j+p-1)(s-n)^{j+p-1}\right\}. \tag{5.1}$$

Lemma 5.3. *(cf. [29, Theorem 2.2]) For positive integers p and n, if $|s+n| < 1$, then*

$$\frac{\Psi^{(p-1)}(-s; A)}{(p-1)!} = (-1)^p\sum_{j=1}^{\infty}\binom{j+p-2}{p-1}\bar{M}_n^{(A)}(j+p-1)(s+n)^{j-1}\,.$$

Moreover, the next two lemmas hold.

Lemma 5.4. *(cf. [30, Lemma 2.2]) For integers $p \geq 1$ and $n \geq 0$, if $|s-n| < 1$, then*

$$\frac{\Psi^{(p-1)}(\frac{1}{2} - s; A)}{(p-1)!} = \sum_{j=1}^{\infty}(-1)^{j-1}\binom{j+p-2}{p-1}N_n^{(A)}(j+p-1)(s-n)^{j-1}\,, \tag{5.2}$$

and if $|s+n| < 1$, then

$$\frac{\Psi^{(p-1)}(\frac{1}{2} - s; A)}{(p-1)!} = (-1)^p\sum_{j=1}^{\infty}\binom{j+p-2}{p-1}\bar{N}_{n+1}^{(A)}(j+p-1)(s+n)^{j-1}\,. \tag{5.3}$$

Lemma 5.5. *(cf. [30, Lemma 2.3]) For positive integers p and n, if $0 < |s-n+1/2| < 1$, then*

$$\frac{\Psi^{(p-1)}(\frac{1}{2} - s; A)}{(p-1)!}$$
$$= \frac{1}{(s-n+\frac{1}{2})^p}\left(a_{n-1} - \sum_{j=1}^{\infty}(-1)^j\binom{j+p-2}{p-1}M_{n-1}^{(A)}(j+p-1)(s-n+\tfrac{1}{2})^{j+p-1}\right).$$

Setting $n=0$ in (5.2) and (5.3) gives

$$\frac{\Psi^{(p-1)}(\frac{1}{2}-s;A)}{(p-1)!}=(-1)^p\sum_{j=1}^{\infty}\binom{j+p-2}{p-1}\hat{t}^{(A)}(j+p-1)s^{j-1}\,,\quad \text{for } |s|<1\,, \tag{5.4}$$

and setting $n=1$ in Lemma 5.5 yields

$$\frac{\Psi^{(p-1)}(\frac{1}{2}-s;A)}{(p-1)!}=\frac{a_0}{(s-\frac{1}{2})^p}+(-1)^p\sum_{j=1}^{\infty}\binom{j+p-2}{p-1}D^{(A)}(j+p-1)(s-\tfrac{1}{2})^{j-1}\,. \tag{5.5}$$

For the parametric cotangent function $\cot(\pi s;A)$, Definition 5.2 gives

$$\pi\cot(\pi s;A)=\frac{a_0}{s}-2\sum_{j=1}^{\infty}D^{(A)}(2j)s^{2j-1}\,,\quad \text{for } |s|<1\,. \tag{5.6}$$

Since $R_0^{(A)}(j)=(1+(-1)^j)D^{(A)}(j)$, it can be found that the above expansion is the $n=0$ case of the following result.

Lemma 5.6. *(cf. [29, Theorem 2.3]) For integer n, if $0<|s-n|<1$, then*

$$\pi\cot(\pi s;A)=\frac{a_{|n|}}{s-n}-\sum_{j=1}^{\infty}(-\sigma_n)^jR_{|n|}^{(A)}(j)(s-n)^{j-1}\,,$$

where σ_n is defined by

$$\sigma_n=\begin{cases}1\,, & n\geq 0\,,\\ -1\,, & n<0\,.\end{cases}$$

Additionally, we have the next result.

Lemma 5.7. *(cf. [30, Lemma 2.5]) For integers $m\geq 0$ and $n\geq 1$, if $|s-n+1/2|<1$, then*

$$\frac{\mathrm{d}^m}{\mathrm{d}s^m}(\pi\cot(\pi s;A))=(-1)^m m!\sum_{j=1}^{\infty}(-1)^{j-1}\binom{j+m-1}{m}S_n^{(A)}(j+m)(s-n+\tfrac{1}{2})^{j-1}\,.$$

Flajolet and Salvy [13] defined a kernel function $\xi(s)$ with two requirements: 1. $\xi(s)$ is meromorphic in the whole complex plane. 2. $\xi(s)$ satisfies $\xi(s)=o(s)$ over an infinite collection of circles $|s|=\rho_k$ with $\rho_k\to\infty$. Applying these two conditions of kernel function $\xi(s)$, Flajolet and Salvy discovered the following residue lemma.

Lemma 5.8. *(cf. [13]) Let $\xi(s)$ be a kernel function and let $r(s)$ be a rational function which is $O(s^{-2})$ at infinity. Then*

$$\sum_{\alpha\in O}\mathrm{Res}(r(s)\xi(s),\alpha)+\sum_{\beta\in S}\mathrm{Res}(r(s)\xi(s),\beta)=0, \tag{5.7}$$

where S is the set of poles of $r(s)$ and O is the set of poles of $\xi(s)$ that are not poles $r(s)$. Here $\mathrm{Res}(r(s),\alpha)$ denotes the residue of $r(s)$ at $s=\alpha$.

In [13], Flajolet and Salvy used residue computations on large circular contour and specific functions to obtain more independent relations for Euler sums. These functions are of the form $\xi(s)r(s)$, where $r(s) := 1/s^q$ and $\xi(s)$ is a product of cotangent (or cosecant) and polygamma function. Applying Lemmas 5.2-5.8 and residue computations, we can establish some new identities. The main results are as follows.

Theorem 5.9. *For integers $m, p \geq 1$ and $q \geq 2$, the following identity on sums related to the sequences A, B, C holds:*

$$\begin{aligned}
&(-1)^{m+p+q}\sum_{n=1}^{\infty}\frac{a_n\bar{M}_n^{(B)}(m)\bar{N}_{n+1}^{(C)}(p)}{(n-1/2)^q}+\sum_{n=1}^{\infty}\frac{a_{n-1}M_{n-1}^{(B)}(m)N_{n-1}^{(C)}(p)}{(n-1/2)^q}\\
&+(-1)^m\sum_{k=0}^{m}\binom{m+q-k-1}{q-1}\binom{p+k-1}{p-1}\sum_{n=1}^{\infty}\frac{a_{n-1}b_{n-1}N_{n-1}^{(C)}(p+k)}{(n-1/2)^{m+q-k}}\\
&-(-1)^m\sum_{\substack{k_1+k_2+k_3=m-1\\k_1,k_2,k_3\geq 0}}\binom{p+k_2-1}{p-1}\binom{q+k_3-1}{q-1}\sum_{n=1}^{\infty}\frac{b_{n-1}R_{n-1}^{(A)}(k_1+1)N_{n-1}^{(C)}(p+k_2)}{(n-1/2)^{q+k_3}}\\
&-(-1)^p\sum_{\substack{k_1+k_2+k_3=p-1\\k_1,k_2,k_3\geq 0}}\binom{m+k_2-1}{m-1}\binom{q+k_3-1}{q-1}\sum_{n=1}^{\infty}\frac{c_{n-1}S_n^{(A)}(k_1+1)N_n^{(B)}(m+k_2)}{n^{q+k_3}}\\
&=\mathcal{I}_0\,,
\end{aligned}$$

where

$$\mathcal{I}_0=(-1)^{m+p}\sum_{\substack{j_1+j_2+j_3=q-1\\j_1,j_2,j_3\geq 0}}(-1)^{j_1}\binom{m+j_2-1}{m-1}\binom{p+j_3-1}{p-1}\check{t}^{(A)}(j_1+1)\hat{t}^{(B)}(m+j_2)\bar{F}_1^{(C)}(p+j_3).$$

Proof. In this case, use the kernel function

$$\xi(s)=\pi\cot(\pi s;A)\frac{\Psi^{(m-1)}(-s;B)\Psi^{(p-1)}\left(\frac{1}{2}-s;C\right)}{(m-1)!(p-1)!}$$

and the base function $r(s)=(s+1/2)^{-q}$. It is obvious that the function $\mathcal{F}(s)=\xi(s)r(s)$ has simple poles at $s=-n$ for $n\geq 1$, with residues

$$\mathrm{Res}(\mathcal{F}(s),-n)=(-1)^{m+p+q}\frac{a_n\bar{M}_n^{(B)}(m)\bar{N}_{n+1}^{(C)}(p)}{(n-1/2)^q}\,,$$

where Lemmas 5.3, 5.4 and 5.6 are used. Next, $\mathcal{F}(s)$ has poles of order $m+1$ at $s=n$ for $n\geq 0$. By Lemmas 5.2 and 5.6, we find that the residues are

$$\mathrm{Res}(\mathcal{F}(s),n)=(-1)^m\sum_{k=0}^{m}\binom{m+q-k-1}{q-1}\binom{p+k-1}{p-1}\frac{a_nb_nN_n^{(C)}(p+k)}{(n+1/2)^{m+q-k}}$$

$$
\begin{aligned}
&-(-1)^m \sum_{j=1}^{m} \sum_{k=1}^{m-j+1} \binom{m+q-k-j}{q-1} \binom{p+k-2}{p-1} \frac{b_n R_n^{(A)}(j) N_n^{(C)}(p+k-1)}{(n+1/2)^{m+q-k-j+1}} \\
&+ \frac{a_n M_n^{(B)}(m) N_n^{(C)}(p)}{(n+1/2)^q}.
\end{aligned}
$$

Moreover, $\mathcal{F}(s)$ has poles of order p at $s = n - 1/2$ for $n \geq 1$. By Lemmas 5.4, 5.5 and 5.7, we arrive at

$$
\begin{aligned}
&\operatorname{Res}(\mathcal{F}(s), n-1/2) \\
&= (-1)^{p-1} \sum_{\substack{k_1+k_2+k_3=p-1, \\ k_1,k_2,k_3 \geq 0}} \binom{m+k_2-1}{m-1} \binom{q+k_3-1}{q-1} \frac{c_{n-1} S_n^{(A)}(k_1+1) N_n^{(B)}(m+k_2)}{n^{q+k_3}}.
\end{aligned}
$$

Finally, $\mathcal{F}(s)$ has a pole of order q at $s = -1/2$. Noting the facts that

$$
\begin{aligned}
\lim_{s \to -1/2} \frac{\mathrm{d}^m}{\mathrm{d}s^m} \pi \cot(\pi s; A) &= (-1)^{m+1} m! \check{t}^{(A)}(m+1), \\
\lim_{s \to -1/2} \frac{\mathrm{d}^m}{\mathrm{d}s^m} \Psi^{(p-1)}(-s; A) &= (-1)^p (m+p-1)! \hat{t}^{(A)}(m+p), \\
\lim_{s \to -1/2} \frac{\mathrm{d}^m}{\mathrm{d}s^m} \Psi^{(p-1)}(1/2-s; A) &= (-1)^p (m+p-1)! \bar{M}_1^{(A)}(m+p) \\
&= (-1)^p (m+p-1)! \bar{F}_1^{(A)}(m+p),
\end{aligned}
$$

the residue $\operatorname{Res}(\mathcal{F}(s), -1/2)$ is found to be $\mathcal{I}_0$ given in the theorem. Hence, combining these four residue results to Lemma 5.8, we obtain the desired result. □

Theorem 5.10. *For integers $m, p \geq 1$ and $q \geq 2$, the following identity on sums related to the sequences A, B, C holds:*

$$
\begin{aligned}
&(-1)^{m+p+q} \sum_{n=1}^{\infty} \frac{a_n \bar{M}_n^{(B)}(m) \bar{N}_{n+1}^{(C)}(p)}{n^q} + \sum_{n=1}^{\infty} \frac{a_n M_n^{(B)}(m) N_n^{(C)}(p)}{n^q} \\
&+ (-1)^m \sum_{k=0}^{m} \binom{m+q-k-1}{q-1} \binom{p+k-1}{p-1} \sum_{n=1}^{\infty} \frac{a_n b_n N_n^{(C)}(p+k)}{n^{m+q-k}} \\
&- (-1)^m \sum_{\substack{k_1+k_2+k_3=m-1 \\ k_1,k_2,k_3 \geq 0}} \binom{p+k_2-1}{p-1} \binom{q+k_3-1}{q-1} \sum_{n=1}^{\infty} \frac{b_n R_n^{(A)}(k_1+1) N_n^{(C)}(p+k_2)}{n^{q+k_3}} \\
&- (-1)^p \sum_{\substack{k_1+k_2+k_3=p-1 \\ k_1,k_2,k_3 \geq 0}} \binom{m+k_2-1}{m-1} \binom{q+k_3-1}{q-1} \sum_{n=1}^{\infty} \frac{c_{n-1} S_n^{(A)}(k_1+1) N_n^{(B)}(m+k_2)}{(n-1/2)^{q+k_3}} \\
&= -\mathcal{J}_0\,,
\end{aligned}
$$

where

$$
\mathcal{J}_0 = a_0 b_0 (-1)^p \binom{m+p+q-1}{p-1} \hat{t}^{(C)}(m+p+q)
$$

$$
\begin{aligned}
&+ a_0(-1)^{m+p}\sum_{j=1}^{q+1}\binom{m+j-2}{m-1}\binom{p+q-j}{p-1}D^{(B)}(m+j-1)\hat{t}^{(C)}(p+q-j+1)\\
&- 2b_0(-1)^p\sum_{j=1}^{\left[\frac{m+q}{2}\right]}\binom{m+p+q-2j-1}{p-1}D^{(A)}(2j)\hat{t}^{(C)}(m+p+q-2j)\\
&-2(-1)^{m+p}\sum_{\substack{2j_1+j_2+j_3=q+2\\ j_1,j_2,j_3\geq 1}}\binom{m+j_2-2}{m-1}\binom{p+j_3-2}{p-1}\\
&\qquad\qquad\times D^{(A)}(2j_1)D^{(B)}(m+j_2-1)\hat{t}^{(C)}(p+j_3-1)\,.
\end{aligned}
$$

Proof. Similarly to the above results, to obtain this theorem, we apply the kernel function $\xi(s)$ to the base function $r(s) = s^{-q}$, and doing the usual residue computation, where $\mathcal{J}_0 = \operatorname{Res}(\xi(s)r(s), 0)$. □

From these two theorems, we can obtain two kinds of mixed type Euler sums, which involve both the (alternating) harmonic numbers and the (alternating) odd harmonic numbers. For simplicity, denote

$$
\begin{aligned}
&\delta_\sigma^k = 1-\sigma(-1)^k, \quad \mathit{for} \quad \sigma=\pm 1,\\
&\mathcal{I}_1 = -(-1)^m 2^{p+q-2}\delta_{\sigma_p\sigma_q}^{p+q}\zeta(m;\sigma_m)M_{\sigma_q,\sigma_p}^{-1,-1}(q,p) - \sigma_p(-1)^p 2^{m+q-2}\delta_{\sigma_q}^{m+q}\widetilde{t}(p;\sigma_p)M_{\sigma_q,\sigma_m}^{-1,1}(q,m)\\
&\qquad - \sigma_p\sigma_q(-1)^{m+p+q}2^{m+p+q-2}M_{\sigma_p\sigma_q,\sigma_m}^{-1,1}(p+q,m),\\
&\mathcal{I}_2 = \sigma_p\sigma_q(-1)^{m+p+q}\zeta(m;\sigma_m)\widetilde{t}(p+q;\sigma_p\sigma_q) - \sigma_p(-1)^{m+p}\delta_{\sigma_q}^{q-1}\zeta(m;\sigma_m)\widetilde{t}(p;\sigma_p)\widetilde{t}(q;\sigma_q),\\
&\mathcal{I}_3 = -(-1)^m\sum_{k=0}^{m}\binom{p+k-1}{p-1}\binom{m+q-k-1}{q-1}\\
&\qquad\qquad \times\left\{\begin{array}{c}\sigma_p(-1)^{p+k}\widetilde{t}(p+k;\sigma_p)\widetilde{t}(m+q-k;\sigma_q)\\ +2^{m+p+q-2}M_{\sigma_q,\sigma_p}^{-1,-1}(m+q-k,p+k)\end{array}\right\},\\
&\mathcal{I}_4 = (-1)^m\sum_{\substack{k_1+k_2+k_3=m-1\\ k_1,k_2,k_3\geq 0}}\binom{p+k_2-1}{p-1}\binom{q+k_3-1}{q-1}\delta_1^{k_1}\zeta(k_1+1;\sigma_m\sigma_p\sigma_q)\\
&\qquad\qquad \times\left\{\begin{array}{c}\sigma_p(-1)^{p+k_2}\widetilde{t}(p+k_2;\sigma_p)\widetilde{t}(q+k_3;\sigma_q)\\ +2^{p+q+k_2+k_3-2}M_{\sigma_q,\sigma_p}^{-1,-1}(q+k_3,p+k_2)\end{array}\right\}\\
&\quad +\sigma_p\sigma_q(-1)^p\sum_{\substack{k_1+k_2+k_3=p-1\\ k_1,k_2,k_3\geq 0}}\binom{m+k_2-1}{m-1}\binom{q+k_3-1}{q-1}\delta_{\sigma_m\sigma_p\sigma_q}^{k_1}\widetilde{t}(k_1+1;\sigma_m\sigma_p\sigma_q)\\
&\qquad\qquad \times\left\{\begin{array}{c}\sigma_m(-1)^{m+k_2}\widetilde{t}(m+k_2;\sigma_m)\zeta(q+k_3;\sigma_q)\\ +2^{m+q+k_2+k_3-2}M_{\sigma_q,\sigma_m}^{1,-1}(q+k_3,m+k_2)\end{array}\right\},
\end{aligned}
$$

$$\mathcal{I}_5 = (-1)^{m+p} \sum_{\substack{j_1+j_2+j_3=q-1 \\ j_1,j_2,j_3\geq 0}} \binom{m+j_2-1}{m-1}\binom{p+j_3-1}{p-1}$$
$$\times\, \delta^{j_1}_{\sigma_m\sigma_p\sigma_q} \widetilde{t}(j_1+1;\sigma_m\sigma_p\sigma_q)\widetilde{t}(m+j_2;\sigma_m)\zeta(p+j_3;\sigma_p),$$

$$\mathcal{I}_6 = \log 2 \times \Bigg\{ -\delta^{m+q}_{-\sigma_q} 2^{m+q-2} M^{-1,1}_{-\sigma_q,\sigma_m}(q,m) - (-1)^m \delta^{q-1}_{-\sigma_q}\zeta(m;\sigma_m)\widetilde{t}(q;-\sigma_q)$$
$$-(-1)^m \binom{m+q-1}{q-1}\widetilde{t}(m+q;-\sigma_q)$$
$$+(-1)^m \sum_{k=1}^{m} \binom{m+q-k-1}{q-1} \delta^{k-1}_1 \zeta(k;-\sigma_m\sigma_q)\widetilde{t}(m+q-k;-\sigma_q)$$
$$+(-1)^m \sum_{j=1}^{q} \binom{m+q-j-1}{m-1} \delta^{j-1}_{-\sigma_m\sigma_q} \widetilde{t}(j;-\sigma_m\sigma_q)\widetilde{t}(m+q-j;\sigma_m) \Bigg\},$$

where $\zeta(1)$ and $\widetilde{t}(1)$ should be interpreted as 0 and $2\log 2$, respectively. Applying Theorem 5.9, we obtain the following result.

Theorem 5.11. *For positive integers m, p and $q \geq 2$, we have*

$$\delta^{m+p+q-1}_{\sigma_p\sigma_q}\sigma_m\sigma_p \sum_{n=1}^{\infty} \frac{{\sigma_q}^n H^{(m)}_{n-1}(\sigma_m) h^{(p)}_{n-1}(\sigma_p)}{(n-1/2)^q} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5 + \delta_{p,1}\delta_{\sigma_p,-1}\mathcal{I}_6.$$

Note that $\mathcal{I}_i$, for $i = 1, 2, \ldots, 6$, are defined as above, $\sigma_m, \sigma_p, \sigma_q \in \{\pm 1\}$, the numbers $H^{(m)}_n(\sigma)$ represent the classical harmonic numbers $H^{(m)}_n$ for $\sigma = 1$ and the alternating harmonic numbers $\bar{H}^{(m)}_n$ for $\sigma = -1$, the numbers $h^{(p)}_n(\sigma)$ represent the odd harmonic numbers $h^{(p)}_n$ for $\sigma = 1$ and the alternating odd harmonic numbers $\bar{h}^{(p)}_n$ for $\sigma = -1$.

Proof. For integers $m, p \geq 1$ and $q \geq 2$, letting $(A, B, C) \in \{A_1, A_2\}$ in Theorem 5.9 yields the desired result. Note that the $\delta_{m,1}, \delta_{p,1}$ appear when A, B, C are replaced by A_2. By considering the four cases according to whether or not m, p equal 1, and using the expressions of $\widetilde{t}(q, p; -\sigma_q, \sigma_p)$ [30, Eqs. (3.3) and (3.4)], we can only eliminate the combinations of the terms on the Kronecker delta $\delta_{m,1}$. □

Denote

$$\mathcal{J}_1 = -(-1)^m 2^{p+q-2}\delta^{p+q}_{\sigma_p}\zeta(m;\sigma_m)M^{1,-1}_{\sigma_q,\sigma_p}(q,p) - \sigma_p(-1)^p 2^{m+q-2}\delta^{m+q}_1 \widetilde{t}(p;\sigma_p) M^{1,1}_{\sigma_q,\sigma_m}(q,m)$$
$$-2^{m+p+q-2} M^{1,-1}_{\sigma_m\sigma_q,\sigma_p}(m+q,p),$$

$$\mathcal{J}_2 = -\sigma_p(-1)^p \widetilde{t}(p;\sigma_p)\zeta(m+q;\sigma_m\sigma_q) - \sigma_p(-1)^{m+p}\delta^{q-1}_1\zeta(m;\sigma_m)\widetilde{t}(p;\sigma_p)\zeta(q;\sigma_q)$$
$$-\sigma_p(-1)^p \binom{m+p+q-1}{p-1}\widetilde{t}(m+p+q;\sigma_p),$$

$$\mathcal{J}_3 = -(-1)^m \sum_{k=0}^{m} \binom{p+k-1}{p-1}\binom{m+q-k-1}{q-1}$$
$$\times \left\{ \begin{array}{c} \sigma_p(-1)^{p+k}\widetilde{t}(p+k;\sigma_p)\zeta(m+q-k;\sigma_q) \\ +2^{m+p+q-2} M^{1,-1}_{\sigma_q,\sigma_p}(m+q-k,p+k) \end{array} \right\},$$

$$
\begin{aligned}
\mathcal{J}_4 = (-1)^m & \sum_{\substack{k_1+k_2+k_3=m-1\\ k_1,k_2,k_3\geq 0}} \binom{p+k_2-1}{p-1}\binom{q+k_3-1}{q-1}\delta_1^{k_1}\zeta(k_1+1;\sigma_m\sigma_p\sigma_q)\\
&\qquad \times\left\{\begin{matrix}\sigma_p(-1)^{p+k_2}\widetilde{t}(p+k_2;\sigma_p)\zeta(q+k_3;\sigma_q)\\ +2^{p+q+k_2+k_3-2}M^{1,-1}_{\sigma_q,\sigma_p}(q+k_3,p+k_2)\end{matrix}\right\}\\
+\sigma_p(-1)^p & \sum_{\substack{k_1+k_2+k_3=p-1\\ k_1,k_2,k_3\geq 0}} \binom{m+k_2-1}{m-1}\binom{q+k_3-1}{q-1}\delta^{k_1}_{\sigma_m\sigma_p\sigma_q}\widetilde{t}(k_1+1;\sigma_m\sigma_p\sigma_q)\\
&\qquad \times\left\{\begin{matrix}\sigma_m(-1)^{m+k_2}\widetilde{t}(m+k_2;\sigma_m)\widetilde{t}(q+k_3;\sigma_q)\\ +\widetilde{t}(m+q+k_2+k_3;\sigma_m\sigma_q)+2^{m+q+k_2+k_3-2}M^{-1,-1}_{\sigma_q,\sigma_m}(q+k_3,m+k_2)\end{matrix}\right\},\\
\mathcal{J}_5 = 2\sigma_p(-1)^{m+p} & \sum_{\substack{2j_1+j_2+j_3=q+2\\ j_1,j_2,j_3\geq 1}} \binom{m+j_2-1}{m-1}\binom{p+j_3-1}{p-1}\\
&\qquad \times\zeta(2j_1;\sigma_m\sigma_p\sigma_q)\zeta(m+j_2-1;\sigma_m)\widetilde{t}(p+j_3-1;\sigma_p)\\
&+2\sigma_p(-1)^p\sum_{j=1}^{\left[\frac{m+q}{2}\right]}\binom{m+p+q-2j-1}{p-1}\zeta(2j;\sigma_m\sigma_p\sigma_q)\widetilde{t}(m+p+q-2j;\sigma_p)\\
&-\sigma_p(-1)^{m+p}\sum_{j=1}^{q+1}\binom{m+j-2}{m-1}\binom{p+q-j}{p-1}\zeta(m+j-1;\sigma_m)\widetilde{t}(p+q-j+1;\sigma_p),\\
\mathcal{J}_6 = \log 2\times & \Bigg\{\delta_1^{m+q}2^{m+q-2}M^{1,1}_{-\sigma_q,\sigma_m}(q,m)+(-1)^m\delta_1^{q-1}\zeta(m;\sigma_m)\zeta(q;-\sigma_q)\\
&+(1-2\delta_{m+q,even})\zeta(m+q;-\sigma_m\sigma_q)\\
&+(-1)^m\binom{m+q-1}{q-1}\zeta(m+q;-\sigma_q)+(-1)^m\binom{m+q-1}{m-1}\zeta(m+q;\sigma_m)\\
&-2(-1)^m\sum_{k=1}^{\left[\frac{m}{2}\right]}\binom{m+q-2k-1}{q-1}\zeta(2k;-\sigma_m\sigma_q)\zeta(m+q-2k;-\sigma_q)\\
&-2(-1)^m\sum_{j=1}^{\left[\frac{q}{2}\right]}\binom{m+q-2j-1}{m-1}\zeta(2j;-\sigma_m\sigma_q)\zeta(m+q-2j;\sigma_m)\Bigg\}.
\end{aligned}
$$

Then the following result holds.

Theorem 5.12. *For positive integers m, p and $q \geq 2$,*

$$
\delta_{\sigma_p}^{m+p+q-1}\sigma_m\sigma_p\sum_{n=1}^{\infty}\frac{{\sigma_q}^n H_{n-1}^{(m)}(\sigma_m)h_n^{(p)}(\sigma_p)}{n^q}=\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4+\mathcal{J}_5+\delta_{p,1}\delta_{\sigma_p,-1}\mathcal{J}_6.
$$

Note that $\mathcal{J}_i$, for $i = 1, 2, \ldots, 6$, are defined as above.

Proof. For integers $m, p \geq 1$ and $q \geq 2$, letting $(A, B, C) \in \{A_1, A_2\}$ in Theorem 5.10 yields the desired result. Note that the $\delta_{m,1}, \delta_{p,1}$ appear when A, B, C are replaced by A_2. By considering the four cases according to whether or not m, p equal 1, and using the expressions of $T(q, p; -\sigma_q, \sigma_p)$ [30, Eqs. (3.6) and (3.8)], we can only eliminate the combinations of the terms on the Kronecker delta $\delta_{m,1}$. □

Hence, we can obtain the parity theorem of (alternating) triple M-values.

Theorem 5.13. *Let*

$$\tau_r = \begin{cases} 1 & \text{if } \varepsilon_r = 1; \\ \sigma_r & \text{if } \varepsilon_r = -1, \end{cases}$$

then when the weight $w = k_1 + k_2 + k_3$ and the quantity $\frac{1-\tau_1\tau_2\tau_3}{2}$ have the same parity, the (alternating) triple M-values $M^{\varepsilon_1,\varepsilon_2,\varepsilon_3}_{\sigma_1,\sigma_2,\sigma_3}(k_1, k_2, k_3)$, where $k_2 \geq 2$, are reducible to combinations of zeta values, the Dirichlet beta values, and (alternating) double M-values.

Proof. Note that by convention, $(k_1, \sigma_1) \neq (1, 1)$. when $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, 1, 1)$, the quantity $\frac{1-\tau_1\tau_2\tau_3}{2} = \frac{1-1}{2} = 0$ is even, we can rewrite $M^{1,1,1}_{\sigma_1,\sigma_2,\sigma_3}(k_1, k_2, k_3)$ in the form

$$\begin{aligned} M^{1,1,1}_{\sigma_1,\sigma_2,\sigma_3}(k_1, k_2, k_3) = & \sum_{n_1>n_2>n_3>0} \frac{2^3\sigma_1^{n_1}\sigma_2^{n_2}\sigma_3^{n_3}}{(2n_1)^{k_1}(2n_2)^{k_2}(2n_3)^{k_3}} \\ = & \sum_{n=2}^{\infty} \frac{2\sigma_2^n}{(2n)^{k_2}} \sum_{i=1}^{n-1} \frac{2\sigma_3^i}{(2i)^{k_3}} \sum_{j=n+1}^{\infty} \frac{2\sigma_1^j}{(2j)^{k_1}} \\ = & M^1_{\sigma_1}(k_1) M^{1,1}_{\sigma_2,\sigma_3}(k_2, k_3) - 2^{3-k_1-k_2-k_3} \sum_{n=1}^{\infty} \frac{\sigma_2^n}{n^{k_2}} \sum_{i=1}^{n-1} \frac{\sigma_3^i}{i^{k_3}} \sum_{j=1}^{n} \frac{\sigma_1^j}{j^{k_1}} \\ = & 2^{1-k_1}\zeta(k_1; \sigma_1) M^{1,1}_{\sigma_2,\sigma_3}(k_2, k_3) + 2M^{1,1}_{\sigma_2\sigma_3,\sigma_1}(k_2 + k_3, k_1) \\ & + 2^{3-k_1-k_2-k_3}\zeta(k_1 + k_2 + k_3; \sigma_1\sigma_2\sigma_3) \\ & - 2^{3-k_1-k_2-k_3}\sigma_1\sigma_2\sigma_3 \sum_{n=1}^{\infty} \frac{H_n^{(k_1)}(\sigma_1) H_n^{(k_3)}(\sigma_3)}{n^{k_2}} \sigma_2^{n-1}. \end{aligned}$$

According to [29, Corollary 3.7], the (alternating) quadratic Euler sums are reducible to the (alternating) linear Euler sums when $k_1 + k_2 + k_3$ is even. And the (alternating) linear Euler sums are expressible in terms of zeta values and double zeta values by [13, Theorems 7.1 and 7.2]. Hence $M^{1,1,1}_{\sigma_1,\sigma_2,\sigma_3}(k_1, k_2, k_3)$ are reducible to zeta values and double M-values.

When $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1, -1, -1)$, the quantity $\frac{1-\tau_1\tau_2\tau_3}{2} = \frac{1-\sigma_1\sigma_2\sigma_3}{2}$,

$$M^{-1,-1,-1}_{\sigma_1,\sigma_2,\sigma_3}(k_1, k_2, k_3) = 2^3 t(k_1, k_2, k_3; \sigma_1, \sigma_2, \sigma_3).$$

And the triple t-values are reducible to combinations of zeta values, the Dirichlet beta values, double zeta values and double t-values when the weight $w = k_1 + k_2 + k_3$ and the quantity $\frac{1-\sigma_1\sigma_2\sigma_3}{2}$ have the same parity by [30, Theorem 4.4]. According to the definition, double zeta values and double t-values can be rewritten as double M-values.

So $M^{-1,-1,-1}_{\sigma_1,\sigma_2,\sigma_3}(k_1,k_2,k_3)$ are reducible to combinations of zeta values, the Dirichlet beta values, double M-values.

When $(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(1,-1,1)$, the quantity $\frac{1-\tau_1\tau_2\tau_3}{2}=\frac{1-\sigma_2}{2}$,

$$\begin{aligned}
M^{1,-1,1}_{\sigma_1,\sigma_2,\sigma_3}(k_1,k_2,k_3)=&\sum_{n_1>n_2>n_3>0}\frac{2^3\sigma_1^{n_1-1}\sigma_2^{n_2}\sigma_3^{n_3}}{(2n_1-2)^{k_1}(2n_2-1)^{k_2}(2n_3)^{k_3}}\\
=&\sum_{n=2}^{\infty}\frac{2\sigma_2^n}{(2n-1)^{k_2}}\sum_{i=1}^{n-1}\frac{2\sigma_3^i}{(2i)^{k_3}}\sum_{j=n}^{\infty}\frac{2\sigma_1^j}{(2j)^{k_1}}\\
=&M^1_{\sigma_1}(k_1)M^{-1,1}_{\sigma_2,\sigma_3}(k_2,k_3)-2^{3-k_1-k_2-k_3}\sum_{n=1}^{\infty}\frac{\sigma_2^n}{\left(n-\frac{1}{2}\right)^{k_2}}\sum_{i=1}^{n-1}\frac{\sigma_3^i}{i^{k_3}}\sum_{j=1}^{n-1}\frac{\sigma_1^j}{j^{k_1}}\\
=&2^{1-k_1}\zeta(k_1;\sigma_1)M^{-1,1}_{\sigma_2,\sigma_3}(k_2,k_3)\\
&-2^{3-k_1-k_2-k_3}\sigma_1\sigma_2\sigma_3\sum_{n=0}^{\infty}\frac{H_n^{(k_1)}(\sigma_1)H_n^{(k_3)}(\sigma_3)}{\left(n+\frac{1}{2}\right)^{k_2}}\sigma_2^n,
\end{aligned}$$

where $\sum_{n=0}^{\infty}\frac{H_n^{(k_1)}(\sigma_1)H_n^{(k_3)}(\sigma_3)}{\left(n+\frac{1}{2}\right)^{k_2}}\sigma_2^n$ is the (alternating) quadratic Euler R-sums which are defined in [28]. And by [28, Corollary 3.6] and the above equation, the sums $M^{1,-1,1}_{\sigma_1,\sigma_2,\sigma_3}(k_1,k_2,k_3)$ can reduce to combinations of zeta values, the Dirichlet beta values, linear Euler R-sums and double M-values when the weight $w=k_1+k_2+k_3$ and the quantity $\frac{1-\sigma_2}{2}$ have the same parity. Note that linear Euler R-sums can be rewritten as double M-values.

When $(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(-1,1,-1)$, the quantity $\frac{1-\tau_1\tau_2\tau_3}{2}=\frac{1-\sigma_1\sigma_3}{2}$, we have

$$M^{-1,1,-1}_{\sigma_1,\sigma_2,\sigma_3}(k_1,k_2,k_3)=T^{\sigma_1,\sigma_2,\sigma_3}(k_1,k_2,k_3)=\sigma_1\sigma_2T(k_1,k_2,k_3;\sigma_1,\sigma_2,\sigma_3),$$

where $T(k_1,k_2,k_3;\sigma_1,\sigma_2,\sigma_3)$ is defined in [30]. By the parity theorem of the Kaneko-Tsumura triple T-values [30, Theorem 4.9], the triple M-values $M^{-1,1,-1}_{\sigma_1,\sigma_2,\sigma_3}(k_1,k_2,k_3)$ reduce to combinations of zeta values, the Dirichlet beta values, linear Euler R-sums and double T-values. Note that linear Euler R-sums and double T-values can be rewritten as double M-values.

When $(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(1,-1,-1)$, the quantity $\frac{1-\tau_1\tau_2\tau_3}{2}=\frac{1-\sigma_2\sigma_3}{2}$. According to their definition we get

$$\begin{aligned}
M^{1,-1,-1}_{\sigma_1,\sigma_2,\sigma_3}(k_1,k_2,k_3)=&2^{1-k_1}\zeta(k_1;\sigma_1)M^{-1,-1}_{\sigma_2,\sigma_3}(k_2,k_3)\\
&-2^{3-k_1-k_2-k_3}\sigma_1\sigma_3\sum_{n=1}^{\infty}\frac{\sigma_2^nH_{n-1}^{(k_1)}(\sigma_1)h_{n-1}^{(k_3)}(\sigma_3)}{\left(n-\frac{1}{2}\right)^{k_2}}.
\end{aligned}$$

By Theorem 5.11 and the above equation, under the conditions of this theorem, the triple M-values $M^{1,-1,-1}_{\sigma_1,\sigma_2,\sigma_3}(k_1,k_2,k_3)$ are expressible in terms of zeta values, the Dirichlet beta values and double M-values.

When $(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(-1,-1,1)$, the quantity $\frac{1-\tau_1\tau_2\tau_3}{2}=\frac{1-\sigma_1\sigma_2}{2}$. We deduce

$$M^{-1,-1,1}_{\sigma_1,\sigma_2,\sigma_3}(k_1,k_2,k_3)=2^{1-k_1}\widetilde{t}(k_1;\sigma_1)M^{-1,1}_{\sigma_2,\sigma_3}(k_2,k_3)-2M^{-1,1}_{\sigma_1\sigma_2,\sigma_3}(k_1+k_2,k_3)$$

$$- 2^{3-k_1-k_2-k_3}\sigma_1\sigma_3\sum_{n=1}^{\infty}\frac{\sigma_2^n H_{n-1}^{(k_3)}(\sigma_3)h_{n-1}^{(k_1)}(\sigma_1)}{\left(n-\frac{1}{2}\right)^{k_2}}.$$

Thus, Theorem 5.11 and the above equation assert that, when the weight $w = k_1+k_2+k_3$ and the quantity $\frac{1-\sigma_1\sigma_2}{2}$ have the same parity, $M^{-1,-1,1}_{\sigma_1,\sigma_2,\sigma_3}(k_1,k_2,k_3)$ are reducible to zeta values, the Dirichlet beta values and double M-values.

When $(\varepsilon_1,\varepsilon_2,\varepsilon_3) = (-1,1,1)$, the quantity $\frac{1-\tau_1\tau_2\tau_3}{2} = \frac{1-\sigma_1}{2}$. According to their definition we have

$$\begin{aligned}M^{-1,1,1}_{\sigma_1,\sigma_2,\sigma_3}(k_1,k_2,k_3) =& 2^{1-k_1}\widetilde{t}(k_1;\sigma_1)M^{1,1}_{\sigma_2,\sigma_3}(k_2,k_3)\\ &- 2^{3-k_1-k_2-k_3}\sigma_1\sigma_3\sum_{n=1}^{\infty}\frac{\sigma_2^n H_{n-1}^{(k_3)}(\sigma_3)h_{n}^{(k_1)}(\sigma_1)}{n^{k_2}}.\end{aligned}$$

By Theorem 5.12 and the above equation, the triple M-values $M^{-1,1,1}_{\sigma_1,\sigma_2,\sigma_3}(k_1,k_2,k_3)$ are expressible in terms of zeta values, the Dirichlet beta values and double M-values.

When $(\varepsilon_1,\varepsilon_2,\varepsilon_3) = (1,1,-1)$, the quantity $\frac{1-\tau_1\tau_2\tau_3}{2} = \frac{1-\sigma_3}{2}$. We deduce

$$\begin{aligned}M^{1,1,-1}_{\sigma_1,\sigma_2,\sigma_3}(k_1,k_2,k_3) =& 2^{1-k_1}\zeta(k_1;\sigma_1)M^{1,-1}_{\sigma_2,\sigma_3}(k_2,k_3) - 2M^{1,-1}_{\sigma_1\sigma_2,\sigma_3}(k_1+k_2,k_3)\\ &- 2^{3-k_1-k_2-k_3}\sigma_1\sigma_3\sum_{n=1}^{\infty}\frac{\sigma_2^n H_{n-1}^{(k_1)}(\sigma_1)h_{n}^{(k_3)}(\sigma_3)}{n^{k_2}}.\end{aligned}$$

Thus, Theorem 5.12 and the above equation assert that, under the conditions of this theorem, $M^{1,1,-1}_{\sigma_1,\sigma_2,\sigma_3}(k_1,k_2,k_3)$ are reducible to zeta values, the Dirichlet beta values and double M-values.

This completes the proof. □

Example 5.14. We have

$$\begin{aligned}M^{1,-1,-1}_{-1,1,1}(1,2,1) =& -2\log(2)M^{-1,1}_{1,-1}(2,1) - \frac{\pi}{2}M^{1,-1}_{1,-1}(2,1) + M^{-1,1}_{1,-1}(3,1) - 2M^{-1,-1}_{1,1}(3,1)\\ &- M^{-1,-1}_{1,1}(2,2) + \frac{35}{4}\log(2)\zeta(3) - \frac{\pi^2}{2}\log^2(2) - \frac{\pi^4}{16},\\ M^{1,-1,-1}_{1,1,-1}(2,2,1) =& \frac{\pi^2}{8}M^{-1,-1}_{1,-1}(2,1) + \frac{\pi}{2}M^{1,-1}_{1,1}(2,2) + \log(2)M^{-1,1}_{-1,1}(2,2) + M^{-1,-1}_{1,-1}(2,3)\\ &+ M^{-1,1}_{-1,1}(3,2) + 2M^{-1,-1}_{1,-1}(3,2) + 3M^{-1,-1}_{1,-1}(4,1) + 7G\zeta(3) + \frac{7\pi}{2}\log(2)\zeta(3)\\ &- 6\log(2)\beta(4) - \frac{\pi^2}{12}\log(2)G - \frac{\pi^5}{24},\\ M^{1,-1,-1}_{-1,-1,1}(2,3,2) =& -\frac{\pi^2}{8}M^{-1,-1}_{-1,1}(3,2) + \frac{\pi^2}{4}M^{1,-1}_{-1,-1}(3,2) + 3M^{-1,-1}_{-1,1}(3,4) + 6M^{-1,-1}_{-1,1}(4,3)\\ &+ 6M^{-1,-1}_{-1,1}(5,2) + M^{-1,1}_{-1,-1}(5,2) + 21\zeta(3)\beta(4) + \frac{5\pi^2}{32}G\zeta(3) - \frac{173\pi^7}{18432},\\ M^{-1,-1,1}_{-1,-1,-1}(1,2,3) =& \frac{3}{16}\zeta(3)M^{-1,-1}_{-1,-1}(2,1) - \frac{\pi^2}{24}M^{-1,-1}_{-1,-1}(2,2) - \frac{\pi^2}{12}M^{-1,-1}_{-1,-1}(3,1) - \frac{\pi}{2}M^{-1,1}_{-1,-1}(2,3)\end{aligned}$$

$$
\begin{aligned}
&- \frac{\pi}{2} M^{1,-1}_{-1,-1}(2,3) + \log(2) M^{-1,1}_{1,-1}(2,3) - M^{-1,-1}_{-1,-1}(2,4) - 2M^{-1,-1}_{-1,-1}(3,3) \\
&- M^{-1,1}_{1,-1}(3,3) - 3M^{-1,-1}_{-1,-1}(4,2) - 4M^{-1,-1}_{-1,-1}(5,1) - \frac{31}{4}\log(2)\zeta(5) + 16G\beta(4) \\
&+ 6\pi\log(2)\beta(4) + \frac{21}{64}\zeta^2(3) - \frac{19\pi^2}{192}\log(2)\zeta(3) + \frac{\pi^2}{6}G^2 - \frac{3\pi^6}{128}, \\
M^{-1,-1,1}_{1,1,-1}(2,4,2) =& \frac{\pi^2}{4} M^{-1,1}_{1,-1}(4,2) + \frac{\pi^2}{24} M^{-1,-1}_{1,1}(4,2) + \pi M^{1,-1}_{1,-1}(4,3) + 2\pi M^{1,-1}_{1,-1}(5,2) \\
&+ 3M^{-1,-1}_{1,1}(4,4) + 8M^{-1,-1}_{1,1}(5,3) + 10M^{-1,-1}_{1,1}(6,2) - M^{-1,1}_{1,-1}(6,2) \\
&- \frac{217}{8}\zeta(3)\zeta(5) - \frac{3\pi}{2}\zeta(3)\beta(4) - \frac{\pi^3}{16}G\zeta(3) + \frac{13\pi^8}{2880}, \\
M^{-1,1,1}_{-1,1,-1}(1,2,2) =& -\frac{\pi^2}{12} M^{1,-1}_{1,-1}(2,1) - \frac{\pi}{2} M^{1,1}_{1,-1}(2,2) + M^{1,-1}_{1,-1}(2,3) + 2M^{1,-1}_{1,-1}(3,2) \\
&+ 3M^{1,-1}_{1,-1}(4,1) + M^{1,-1}_{-1,-1}(4,1) + \frac{7}{4}G\zeta(3), \\
M^{-1,1,1}_{1,-1,-1}(2,3,3) =& \frac{3}{16}\zeta(3) M^{1,-1}_{-1,1}(3,2) + \frac{\pi^2}{6} M^{1,-1}_{-1,1}(3,3) + \frac{\pi^2}{4} M^{1,1}_{-1,-1}(3,3) - \frac{\pi^2}{4} M^{-1,-1}_{-1,-1}(3,3) \\
&+ \frac{\pi^2}{4} M^{1,-1}_{-1,1}(4,2) - 4M^{1,-1}_{-1,1}(3,5) - 9M^{1,-1}_{-1,1}(4,4) - 12M^{1,-1}_{-1,1}(5,3) \\
&- 10M^{1,-1}_{-1,1}(6,2) + M^{1,-1}_{1,1}(6,2) - \frac{\pi^8}{1920}, \\
M^{1,1,-1}_{-1,-1,1}(1,2,1) =& -2\log(2) M^{1,1}_{1,-1}(2,1) - M^{1,-1}_{-1,1}(2,2) - 2M^{1,-1}_{-1,1}(3,1) - M^{1,-1}_{1,1}(3,1) \\
&- \frac{27}{8}\log(2)\zeta(3) + \frac{\pi^2}{4}\log^2(2), \\
M^{1,1,-1}_{1,-1,1}(2,2,2) =& \frac{\pi^2}{8} M^{1,-1}_{-1,1}(2,2) + \pi M^{-1,-1}_{-1,1}(2,3) + \pi M^{-1,-1}_{-1,1}(3,2) + 3M^{1,-1}_{-1,1}(2,4) \\
&+ 4M^{1,-1}_{-1,1}(3,3) + 2M^{1,-1}_{-1,1}(4,2) + \frac{49}{16}\zeta^2(3) + \frac{7\pi}{2}G\zeta(3) - \frac{5\pi^6}{384}, \\
M^{1,1,-1}_{-1,-1,-1}(1,2,4) =& 2\beta(4) M^{1,1}_{-1,-1}(2,1) - \frac{\pi^3}{16} M^{-1,-1}_{-1,-1}(2,2) - \frac{\pi^3}{8} M^{-1,-1}_{-1,-1}(3,1) - \frac{\pi}{2} M^{-1,-1}_{-1,-1}(2,4) \\
&- \pi M^{-1,-1}_{-1,-1}(3,3) - \frac{3\pi}{2} M^{-1,-1}_{-1,-1}(4,2) - 2\pi M^{-1,-1}_{-1,-1}(5,1) - 4M^{1,-1}_{-1,-1}(2,5) \\
&- 2M^{1,-1}_{-1,-1}(3,4) - M^{1,-1}_{1,-1}(3,4) + 20\log(2)\beta(6) + \frac{13}{8}\zeta(3)\beta(4) \\
&+ 8\pi G\beta(4) + \frac{\pi^3}{4}G^2 - \frac{11\pi^7}{768}, \\
M^{1,-1,1}_{1,1,-1}(2,2,2) =& -\frac{\pi^2}{24} M^{1,-1}_{1,1}(2,2) - \frac{\pi^2}{8} M^{1,-1}_{-1,1}(2,2) - \pi M^{-1,-1}_{1,-1}(2,3) - \pi M^{-1,-1}_{-1,1}(2,3) \\
&- \pi M^{-1,-1}_{1,-1}(3,2) - \pi M^{-1,-1}_{-1,1}(3,2) - 3M^{1,-1}_{1,1}(2,4) - 3M^{1,-1}_{-1,1}(2,4) \\
&- 4M^{1,-1}_{1,1}(3,3) - 4M^{1,-1}_{-1,1}(3,3) - 3M^{1,-1}_{1,1}(4,2) - 3M^{1,-1}_{-1,1}(4,2) \\
&- 7\pi G\zeta(3) + \frac{5\pi^6}{192},
\end{aligned}
$$

$$
\begin{aligned}
M^{1,-1,1}_{1,-1,1}(2,2,3) =& \frac{1}{4}\zeta(3)M^{1,-1}_{1,-1}(2,2)+\frac{\pi^2}{12}M^{1,-1}_{1,-1}(2,3)+\frac{\pi^2}{12}M^{-1,1}_{-1,1}(2,3)+\frac{\pi^2}{24}M^{1,-1}_{1,-1}(3,2)\\
&+\frac{3\pi}{2}M^{-1,-1}_{1,1}(2,4)+2\pi M^{-1,-1}_{1,1}(3,3)+\frac{3\pi}{2}M^{-1,-1}_{1,1}(4,2)-2M^{-1,1}_{-1,1}(2,5)\\
&+4M^{1,-1}_{1,-1}(2,5)+3M^{1,-1}_{1,-1}(3,4)-4M^{1,-1}_{1,-1}(5,2)-\frac{1}{4}G\zeta(5)+\frac{\pi^2}{24}G\zeta(3)\\
&-\frac{49\pi}{8}\zeta^2(3)+\frac{\pi^7}{96},\\
M^{1,-1,1}_{-1,1,-1}(3,2,3) =& -\frac{3}{16}\zeta(3)M^{-1,1}_{1,-1}(2,3)-\frac{3}{8}\zeta(3)M^{1,-1}_{-1,1}(3,2)-\frac{\pi^2}{3}M^{1,-1}_{-1,1}(3,3)-\frac{\pi^2}{2}M^{1,1}_{-1,-1}(3,3)\\
&+\frac{\pi^2}{2}M^{-1,-1}_{-1,-1}(3,3)-\frac{\pi^2}{2}M^{1,-1}_{-1,1}(4,2)-2M^{-1,1}_{1,1}(2,6)+8M^{1,-1}_{-1,1}(3,5)\\
&+18M^{1,-1}_{-1,1}(4,4)+24M^{1,-1}_{-1,1}(5,3)+20M^{1,-1}_{-1,1}(6,2)-2M^{1,-1}_{1,1}(6,2)+\frac{\pi^8}{960}.
\end{aligned}
$$

6 Dimension computation of AMMVs

Let AMMV_w be the $\mathbb{Q}$-vector space generated by all the AMMVs of weight w and denote all its subspaces similarly. Set $\dim_{\mathbb{Q}} V_0 = 1$ for all the subspaces V of AMMV including AMMV itself. Clearly, AMtVs, AMTVs and AMSVs are special cases of AMMVs, and the AMMVs can be written as $\mathbb{Q}[i]$-linear combinations of the CMZVs of level four. In [3], Au found the all $\mathbb{Q}$-linear relations among CMZVs of level four with weight ≤ 6. Hence, applying Au's Mathematica package [2,3], we find the Table 1.

w	0	1	2	3	4	5	6
$\dim_{\mathbb{Q}} \mathsf{AMZV}_w$	1	1	2	3	5	8	13
$\dim_{\mathbb{Q}} \mathsf{AMtV}_w$	1	1	3	6	12	24	48
$\dim_{\mathbb{Q}} \mathsf{AMTV}_w$	1	1	2	4	7	13	24
$\dim_{\mathbb{Q}} \mathsf{AMSV}_w$	1	1	3	6	12	22	42
$\dim_{\mathbb{Q}} \mathsf{AMMV}_w$	1	2	4	8	16	32	64
$\dim_{\mathbb{Q}} \mathsf{CMZV}^4_w$	1	2	4	8	16	32	64

Table 1: Conjectural Dimensions of Various Subspaces of AMMV.

For each weight w, we let $\mathbf{MB}_w$ be the conjectural basis of $\mathbb{Q}$-vector space generated by all AMMVs of weight w. Using Au's Mathematica package [2,3], we obtain

$\mathbf{MB}_1 = \{\pi, \log(2)\}$,

$\mathbf{MB}_2 = \{G, \pi^2, \pi\log(2), \log^2(2)\}$,

$\mathbf{MB}_3 = \left\{\zeta(3), \Im \operatorname{Li}_3\left(\frac{1+i}{2}\right), G\pi, G\log(2), \pi^3, \pi^2\log(2), \pi\log^2(2), \log^3(2)\right\}$,

$$\mathbf{MB}_4=\left\{\begin{array}{c}\mathrm{Li}_4(1/2),\beta(4),\Im\,\mathrm{Li}_4\left(\frac{1+i}{2}\right),\pi\zeta(3),\log(2)\zeta(3),\pi\Im\,\mathrm{Li}_3\left(\frac{1+i}{2}\right),\log(2)\Im\,\mathrm{Li}_3\left(\frac{1+i}{2}\right),\\ G^2,G\pi^2,G\pi\log(2),G\log^2(2),\pi^4,\pi^3\log(2),\pi^2\log^2(2),\pi\log^3(2),\log^4(2)\end{array}\right\}.$$

Further, the set $\mathbf{MB}_w$ $(1\le w\le 4)$ can be taken in the following forms:

$$\mathbf{MB}_1=\left\{M_{-1}^{1}(1),M_{-1}^{-1}(1)\right\},$$
$$\mathbf{MB}_2=\left\{M_{1}^{1}(2),M_{-1}^{-1}(2),M_{-1,1}^{1,1}(1,1),M_{-1,1}^{-1,1}(1,1)\right\},$$
$$\mathbf{MB}_3=\left\{\begin{array}{c}M_{1}^{1}(3),M_{-1}^{-1}(3),M_{1,-1}^{1,1}(2,1),M_{1,-1}^{1,-1}(2,1),\\ M_{-1,1}^{1,-1}(2,1),M_{-1,1}^{-1,1}(2,1),M_{-1,1,1}^{1,1,1}(1,1,1),M_{-1,1,1}^{-1,1,1}(1,1,1)\end{array}\right\},$$
$$\mathbf{MB}_4=\left\{\begin{array}{l}M_{1}^{1}(4),M_{-1}^{-1}(4),M_{1,-1}^{1,1}(3,1),M_{-1,1}^{1,1}(3,1),M_{1,-1}^{1,-1}(3,1),M_{-1,-1}^{1,-1}(3,1),M_{1,-1}^{-1,1}(3,1),\\ M_{-1,1}^{-1,1}(3,1),M_{1,1,-1}^{1,1,1}(2,1,1),M_{1,1,-1}^{1,1,-1}(2,1,1),M_{1,-1,1}^{1,1,-1}(2,1,1),M_{1,-1,1}^{1,-1,1}(2,1,1),\\ M_{-1,1,1}^{1,-1,1}(2,1,1),M_{-1,1,1}^{-1,1,1}(2,1,1),M_{-1,1,1,1}^{1,1,1,1}(1,1,1,1),M_{-1,1,1,1}^{-1,1,1,1}(1,1,1,1)\end{array}\right\}.$$

Of course, we can also obtain the $\mathbf{MB}_w$ $(w=5,6)$ by using Au's Mathematica package:

$$\mathbf{MB}_5=\left\{\begin{array}{l}M_{1}^{1}(5),M_{-1}^{-1}(5),M_{1,1}^{1,1}(4,1),M_{1,-1}^{1,1}(4,1),M_{1,-1}^{1,-1}(4,1),M_{-1,1}^{1,-1}(4,1),M_{-1,-1}^{1,-1}(4,1),\\ M_{1,-1}^{-1,1}(4,1),M_{-1,1}^{-1,1}(4,1),M_{-1,-1}^{-1,1}(4,1),M_{1,1,-1}^{1,1,1}(3,1,1),M_{1,-1,1}^{1,1,1}(3,1,1),\\ M_{1,-1,-1}^{1,1,1}(3,1,1),M_{1,1,-1}^{1,1,-1}(3,1,1),M_{1,-1,-1}^{1,1,-1}(3,1,1),M_{1,-1,1}^{1,-1,1}(3,1,1),M_{1,-1,-1}^{1,-1,1}(3,1,1),\\ M_{1,-1,1}^{1,-1,-1}(3,1,1),M_{1,-1,-1}^{1,-1,-1}(3,1,1),M_{1,1,-1}^{-1,1,1}(3,1,1),M_{1,-1,1}^{-1,1,1}(3,1,1),M_{-1,1,1}^{-1,1,1}(3,1,1),\\ M_{1,1,1,-1}^{1,1,1,1}(2,1,1,1),M_{1,1,1,-1}^{1,1,1,-1}(2,1,1,1),M_{1,1,-1,1}^{1,1,1,-1}(2,1,1,1),M_{1,1,-1,1}^{1,1,-1,1}(2,1,1,1),\\ M_{1,-1,1,1}^{1,1,-1,1}(2,1,1,1),M_{1,-1,1,1}^{1,-1,1,1}(2,1,1,1),M_{-1,1,1,1}^{1,-1,1,1}(2,1,1,1),M_{-1,1,1,1}^{-1,1,1,1}(2,1,1,1),\\ M_{-1,1,1,1,1}^{1,1,1,1,1}(1,1,1,1,1),M_{-1,1,1,1,1}^{-1,1,1,1,1}(1,1,1,1,1)\end{array}\right\},$$

$$\mathbf{MB}_6=\left\{\begin{array}{l}M_{1}^{1}(6),M_{-1}^{-1}(6),M_{1,1}^{1,1}(5,1),M_{1,-1}^{1,1}(5,1),M_{-1,1}^{1,1}(5,1),M_{1,-1}^{1,-1}(5,1),M_{-1,-1}^{1,-1}(5,1),\\ M_{1,-1}^{-1,1}(5,1),M_{-1,1}^{-1,1}(5,1),M_{-1,-1}^{-1,1}(5,1),M_{1,-1}^{-1,-1}(5,1),M_{-1,1}^{1,-1}(4,2),M_{1,1,-1}^{1,1,1}(4,1,1),\\ M_{1,-1,1}^{1,1,1}(4,1,1),M_{1,-1,-1}^{1,1,1}(4,1,1),M_{1,1,-1}^{1,1,-1}(4,1,1),M_{1,-1,1}^{1,1,-1}(4,1,1),M_{1,-1,-1}^{1,1,-1}(4,1,1),\\ M_{-1,1,1}^{1,1,-1}(4,1,1),M_{-1,1,-1}^{1,1,-1}(4,1,1),M_{1,-1,1}^{1,-1,1}(4,1,1),M_{1,-1,-1}^{1,-1,1}(4,1,1),M_{-1,1,1}^{1,-1,1}(4,1,1),\\ M_{-1,-1,1}^{1,-1,1}(4,1,1),M_{1,-1,-1}^{1,-1,-1}(4,1,1),M_{-1,-1,-1}^{1,-1,-1}(4,1,1),M_{1,1,-1}^{-1,1,1}(4,1,1),M_{1,-1,1}^{-1,1,1}(4,1,1),\\ M_{-1,1,1}^{-1,1,1}(4,1,1),M_{-1,1,-1}^{-1,1,1}(4,1,1),M_{-1,-1,-1}^{-1,1,1}(4,1,1),M_{1,1,-1}^{-1,1,-1}(4,1,1),\\ M_{1,1,1,-1}^{1,1,1,1}(3,1,1,1),M_{1,1,-1,1}^{1,1,1,1}(3,1,1,1),M_{1,1,-1,-1}^{1,1,1,1}(3,1,1,1),M_{1,-1,1,1}^{1,1,1,1}(3,1,1,1),\\ M_{1,1,1,-1}^{1,1,1,-1}(3,1,1,1),M_{1,1,-1,-1}^{1,1,1,-1}(3,1,1,1),M_{1,1,-1,1}^{1,1,-1,1}(3,1,1,1),M_{1,1,-1,-1}^{1,1,-1,1}(3,1,1,1),\\ M_{1,1,-1,1}^{1,1,-1,-1}(3,1,1,1),M_{1,1,-1,-1}^{1,1,-1,-1}(3,1,1,1),M_{1,-1,1,1}^{1,-1,1,1}(3,1,1,1),M_{1,-1,1,-1}^{1,-1,1,1}(3,1,1,1),\\ M_{1,-1,1,1}^{1,-1,1,-1}(3,1,1,1),M_{1,-1,1,-1}^{1,-1,1,-1}(3,1,1,1),M_{1,-1,1,1}^{1,-1,-1,1}(3,1,1,1),M_{1,-1,-1,1}^{1,-1,-1,1}(3,1,1,1),\\ M_{1,1,1,-1}^{-1,1,1,1}(3,1,1,1),M_{1,1,-1,1}^{-1,1,1,1}(3,1,1,1),M_{1,-1,1,1}^{-1,1,1,1}(3,1,1,1),M_{-1,1,1,1}^{-1,1,1,1}(3,1,1,1),\\ M_{1,1,1,1,-1}^{1,1,1,1,1}(2,1,1,1,1),M_{1,1,1,1,-1}^{1,1,1,1,-1}(2,1,1,1,1),M_{1,1,1,-1,1}^{1,1,1,1,-1}(2,1,1,1,1),\\ M_{1,1,1,-1,1}^{1,1,1,-1,1}(2,1,1,1,1),M_{1,1,-1,1,1}^{1,1,1,-1,1}(2,1,1,1,1),M_{1,1,-1,1,1}^{1,1,-1,1,1}(2,1,1,1,1),\\ M_{1,-1,1,1,1}^{1,1,-1,1,1}(2,1,1,1,1),M_{1,-1,1,1,1}^{1,-1,1,1,1}(2,1,1,1,1),M_{-1,1,1,1,1}^{1,-1,1,1,1}(2,1,1,1,1),\\ M_{-1,1,1,1,1}^{-1,1,1,1,1}(2,1,1,1,1),M_{-1,1,1,1,1,1}^{1,1,1,1,1,1}(1,1,1,1,1,1),M_{-1,1,1,1,1,1}^{-1,1,1,1,1,1}(1,1,1,1,1,1)\end{array}\right\}.$$

Example 6.1. We have

$$\begin{aligned}
M^{1,1,-1}_{-1,-1,1}(2,1,1) &= -\frac{11}{8}M^{1}_{1}(4) - 12M^{1,1}_{-1,1}(3,1) - 4M^{1,1}_{1,-1}(3,1)\\
&= \frac{91}{1920}\pi^4 + \frac{1}{4}\pi^2\log^2(2) - \frac{1}{4}\log^4(2) - 6\operatorname{Li}_4(1/2) - \frac{7}{2}\log(2)\zeta(3),\\
M^{-1,1,1,-1}_{-1,1,-1,1}(1,1,1,1) &= \frac{1}{2}M^{-1,1}_{-1,1}(3,1) + \frac{1}{2}M^{1,-1}_{-1,-1}(3,1)\\
&= -\frac{1}{12}G\pi^2 - 2\beta(4) + \frac{1}{16}\pi^3\log(2) + \frac{7}{16}\pi\zeta(3),\\
M^{-1,-1,1,-1}_{-1,1,-1,1}(1,1,1,1) &= -\frac{3}{4}M^{-1}_{-1}(4) - \frac{1}{2}M^{-1,1}_{-1,1}(3,1) - \frac{7}{4}M^{1,-1}_{-1,-1}(3,1) + \frac{1}{2}M^{1,-1}_{1,-1}(3,1)\\
&= \frac{1}{3}G\pi^2 + 2\beta(4) - \frac{1}{16}\pi^3\log(2) - \frac{7}{8}\pi\zeta(3),\\
M^{1,1,-1,-1}_{-1,-1,-1,1}(1,1,1,1) &= \frac{9}{4}M^{-1}_{-1}(4) - \frac{1}{2}M^{-1,1}_{-1,1}(3,1) + \frac{5}{4}M^{1,-1}_{-1,-1}(3,1) - \frac{9}{2}M^{1,-1}_{1,-1}(3,1)\\
&\quad - M^{1,1,-1}_{1,1,-1}(2,1,1)\\
&= -\frac{2}{3}G\pi^2 - 30\beta(4) + 32\Im\operatorname{Li}_4\left(\frac{1+i}{2}\right) + \frac{1}{4}\pi^3\log(2)\\
&\quad + 8\Im\operatorname{Li}_3\left(\frac{1+i}{2}\right)\log(2) - \frac{1}{12}\pi\log^3(2) + \frac{21}{8}\pi\zeta(3).\\
M^{1,-1,1,1}_{-1,-1,-1,1}(1,2,1,1) &= \frac{169}{10}M^{-1}_{-1}(5) + \frac{27}{2}M^{-1,1}_{-1,-1}(4,1) - \frac{21}{4}M^{-1,1}_{-1,1}(4,1) + \frac{291}{8}M^{1,-1}_{-1,-1}(4,1)\\
&\quad - \frac{177}{2}M^{1,-1}_{1,-1}(4,1) - 7M^{1,-1,1}_{1,-1,-1}(3,1,1) - \frac{1}{2}M^{1,-1,1}_{1,-1,1}(3,1,1)\\
&\quad - M^{1,1,-1}_{1,-1,-1}(3,1,1) - 2M^{1,1,-1}_{1,1,-1}(3,1,1) - 7M^{1,-1,1,1}_{1,-1,1,1}(2,1,1,1)\\
&\quad - 4M^{1,1,-1,1}_{1,1,-1,1}(2,1,1,1) - 2M^{1,1,1,-1}_{1,1,1,-1}(2,1,1,1),\\
M^{1,-1,-1,-1}_{-1,1,1,1}(3,1,1,1) &= -\frac{1023}{64}M^{1}_{1}(6) - \frac{3}{2}M^{1,-1}_{-1,1}(4,2) - 136M^{1,1}_{-1,1}(5,1) - \frac{95}{8}M^{1,1}_{1,1}(5,1)\\
&\quad - \frac{3}{2}M^{1,-1,1}_{-1,1,1}(4,1,1) + \frac{15}{2}M^{1,1,-1}_{-1,1,1}(4,1,1) - 18M^{1,1,-1}_{1,-1,1}(4,1,1).
\end{aligned}$$

Similarly, for each weight w, we let $\mathbf{tB}_w, \mathbf{TB}_w$ and $\mathbf{SB}_w$ be the conjectural basis of $\mathbb{Q}$-vector space generated by all AMtVs of weight w, all AMTVs of weight w and all AMSVs of weight w, respectively.

For convenience, let

$$\ddot{t}(s_1,\dots,s_r;\sigma_1,\dots,\sigma_r) := M^{\{-1\}_r}_{\sigma_1,\dots,\sigma_r}(s_1,\dots,s_r) = 2^r t(s_1,\dots,s_r;\sigma_1,\dots,\sigma_r),$$

where $(s_1,\sigma_1)\neq(1,1)$ and the AMtVs are denoted by

$$t(s_1,\dots,s_r;\sigma_1,\dots,\sigma_r) := \sum_{n_1>\dots>n_r>0}\frac{\sigma_1^{n_1}\cdots\sigma_r^{n_r}}{(2n_1-1)^{s_1}\cdots(2n_r-1)^{s_r}}.$$

In particular, if $\sigma_1=\dots=\sigma_r=1$, then $t(s_1,\dots,s_r)=t(s_1,\dots,s_r;\{1\}_r)$. Similar to AMZVs, we may compactly indicate the presence of an alternating sign as follows.

Whenever $\varepsilon_j = -1$, we place a bar over the corresponding integer exponent k_j. For example,

$$\ddot{t}(\bar{2},3,\bar{1},4) = \ddot{t}(2,3,1,4;-1,1,-1,1).$$

Using Au's Mathematica package, we have the following results of $\mathbf{tB}_w, \mathbf{TB}_w$ and $\mathbf{SB}_w$ with $1 \le w \le 6$.

$$\mathbf{tB}_1 = \{t(\bar{1})\},$$

$$\mathbf{tB}_2 = \{t(2), t(\bar{2}), t(\bar{1},1)\},$$

$$\mathbf{tB}_3 = \{t(3), t(\bar{3}), t(2,1), t(2,\bar{1}), t(\bar{1},\bar{2}), t(\bar{1},1,1)\},$$

$$\mathbf{tB}_4 = \left\{\begin{array}{l} t(4), t(\bar{4}), t(3,1), t(3,\bar{1}), t(\bar{3},1), t(2,\bar{2}), t(\bar{2},\bar{2}), \\ t(2,1,1), t(2,1,\bar{1}), t(\bar{1},2,\bar{1}), t(\bar{1},1,\bar{2}), t(\bar{1},1,1,1) \end{array}\right\},$$

$$\mathbf{tB}_5 = \left\{\begin{array}{l} t(5), t(\bar{5}), t(4,1), t(4,\bar{1}), t(\bar{4},1), t(\bar{4},\bar{1}), t(\bar{1},\bar{4}), t(3,\bar{2}), t(\bar{3},\bar{2}), t(3,1,1), t(3,1,\bar{1}), \\ t(3,\bar{1},1), t(\bar{3},1,1), t(2,2,\bar{1}), t(\bar{2},2,\bar{1}), t(2,1,\bar{2}), t(2,\bar{1},\bar{2}), t(\bar{1},\bar{2},\bar{2}), t(2,1,1,1), \\ t(2,1,1,\bar{1}), t(\bar{1},2,1,\bar{1}), t(\bar{1},1,2,\bar{1}), t(\bar{1},1,1,\bar{2}), t(\bar{1},1,1,1,1) \end{array}\right\},$$

$$\mathbf{tB}_6 = \left\{\begin{array}{l} t(6), t(\bar{6}), t(5,1), t(5,\bar{1}), t(\bar{5},1), t(\bar{5},\bar{1}), t(\bar{1},5), t(4,\bar{2}), t(\bar{4},\bar{2}), t(2,\bar{4}), t(\bar{2},\bar{4}), \\ t(4,1,1), t(4,1,\bar{1}), t(4,\bar{1},1), t(4,\bar{1},\bar{1}), t(\bar{4},1,1), t(\bar{4},1,\bar{1}), t(\bar{1},4,\bar{1}), t(\bar{1},\bar{4},1), \\ t(\bar{1},1,\bar{4}), t(3,2,\bar{1}), t(\bar{3},2,\bar{1}), t(3,1,\bar{2}), t(3,\bar{1},\bar{2}), t(\bar{3},1,\bar{2}), t(3,1,1,1), t(3,1,1,\bar{1}), \\ t(3,1,\bar{1},1), t(3,\bar{1},1,1), t(\bar{3},1,1,1), t(2,\bar{2},\bar{2}), t(\bar{2},\bar{2},\bar{2}), t(2,2,1,\bar{1}), t(\bar{2},2,1,\bar{1}), \\ t(2,1,2,\bar{1}), t(2,\bar{1},2,\bar{1}), t(2,1,1,\bar{2}), t(2,1,\bar{1},\bar{2}), t(\bar{1},\bar{2},2,\bar{1}), t(\bar{1},2,\bar{1},\bar{2}), t(\bar{1},1,\bar{2},\bar{2}), \\ t(2,1,1,1,1), t(2,1,1,1,\bar{1}), t(\bar{1},2,1,1,\bar{1}), t(\bar{1},1,2,1,\bar{1}), t(\bar{1},1,1,2,\bar{1}), \\ t(\bar{1},1,1,1,\bar{2}), t(\bar{1},1,1,1,1,1) \end{array}\right\};$$

$$\mathbf{TB}_1 = \{T(\bar{1})\},$$

$$\mathbf{TB}_2 = \{T(2), T(\bar{2})\},$$

$$\mathbf{TB}_3 = \{T(3), T(\bar{3}), T(2,\bar{1}), T(\bar{2},1)\},$$

$$\mathbf{TB}_4 = \{T(4), T(\bar{4}), T(3,1), T(3,\bar{1}), T(\bar{3},\bar{1}), T(\bar{1},3), T(2,\bar{1},1)\},$$

$$\mathbf{TB}_5 = \left\{\begin{array}{l} T(5), T(\bar{5}), T(4,1), T(4,\bar{1}), T(\bar{4},1), T(\bar{4},\bar{1}), T(\bar{1},4), T(3,\bar{2}), \\ T(3,1,\bar{1}), T(3,\bar{1},1), T(3,\bar{1},\bar{1}), T(\bar{1},3,1), T(\bar{1},3,\bar{1}) \end{array}\right\},$$

$$\mathbf{TB}_6 = \left\{\begin{array}{l} T(6), T(\bar{6}), T(5,1), T(5,\bar{1}), T(\bar{5},1), T(\bar{5},\bar{1}), T(\bar{1},5), T(\bar{1},\bar{5}), T(\bar{4},2), T(2,\bar{4}), \\ T(4,1,1), T(4,1,\bar{1}), T(4,\bar{1},1), T(4,\bar{1},\bar{1}), T(\bar{4},\bar{1},\bar{1}), T(\bar{1},4,1), T(\bar{1},4,\bar{1}), \\ T(\bar{1},\bar{4},\bar{1}), T(\bar{1},1,\bar{4}), T(3,2,\bar{1}), T(3,\bar{1},1,1), T(3,\bar{1},1,\bar{1}), T(\bar{1},3,\bar{1},\bar{1}), T(\bar{1},\bar{1},3,\bar{1}) \end{array}\right\};$$

$$\mathbf{SB}_1 = \{S(\bar{1})\},$$

$$\mathbf{SB}_2 = \{S(2), S(\bar{1},1), S(\bar{1},\bar{1})\},$$

$$\mathbf{SB}_3 = \{S(3), S(2,1), S(2,\bar{1}), S(\bar{2},1), S(\bar{2},\bar{1}), S(\bar{1},\bar{2})\},$$

$$\mathbf{SB}_4 = \left\{\begin{array}{l} S(4), S(3,1), S(3,\bar{1}), S(\bar{3},1), S(\bar{3},\bar{1}), S(\bar{1},3), S(\bar{1},\bar{3}), S(2,2), \\ S(2,\bar{1},1), S(2,\bar{1},\bar{1}), S(\bar{2},1,1), S(\bar{1},1,\bar{1},1) \end{array}\right\},$$

$$\mathbf{SB}_5 = \left\{\begin{array}{l} S(5), S(4,1), S(4,\bar{1}), S(\bar{4},1), S(\bar{4},\bar{1}), S(\bar{1},4), S(\bar{1},\bar{4}), S(3,2), S(3,\bar{2}), S(\bar{3},2), \\ S(3,1,1), S(3,1,\bar{1}), S(3,\bar{1},1), S(3,\bar{1},\bar{1}), S(\bar{3},\bar{1},1), S(\bar{1},3,1), S(\bar{1},3,\bar{1}), S(\bar{1},\bar{3},\bar{1}), \\ S(2,\bar{1},1,1), S(2,\bar{1},1,\bar{1}), S(\bar{1},2,1,1), S(\bar{1},2,\bar{1},1) \end{array}\right\},$$

$$\mathbf{SB}_6=\left\{\begin{array}{l} S(6),S(5,1),S(5,\bar1),S(\bar5,1),S(\bar5,\bar1),S(\bar1,5),S(\bar1,\bar5),S(4,2),S(4,\bar2),S(\bar4,2),\\ S(\bar4,\bar2),S(2,4),S(4,1,1),S(4,1,\bar1),S(4,\bar1,1),S(4,\bar1,\bar1),S(\bar4,1,1),S(\bar4,\bar1,1),\\ S(\bar4,\bar1,\bar1),S(\bar1,4,1),S(\bar1,4,\bar1),S(\bar1,\bar4,1),S(\bar1,\bar4,\bar1),S(\bar1,\bar1,4),S(3,2,\bar1),S(3,\bar2,1),\\ S(3,1,\bar1,1),S(3,1,\bar1,\bar1),S(3,\bar1,1,1),S(3,\bar1,1,\bar1),S(3,\bar1,\bar1,1),S(3,\bar1,\bar1,\bar1),\\ S(\bar3,1,\bar1,1),S(\bar1,3,1,1),S(\bar1,3,1,\bar1),S(\bar1,3,\bar1,1),S(\bar1,3,\bar1,\bar1),S(\bar1,\bar3,\bar1,1),\\ S(\bar1,\bar3,\bar1,\bar1),S(\bar1,1,3,\bar1),S(2,2,\bar1,1),S(\bar2,2,\bar1,1) \end{array}\right\}.$$

Remark 6.2. In [32], we also studied the $\mathbb{Q}$-vector space generated by the AMTVs of any fixed weight w and provide some evidence for the conjecture that their dimensions $\{d_w\}_{w\geq1}$ form the tribonacci sequence 1, 2, 4, 7, 13, ...

Example 6.3. We have

$$\begin{aligned}
\ddot t(\bar2,\bar1)&=-\frac12\ddot t(3)+\frac12\ddot t(2,1)=\frac14\pi^2\log(2)-\frac74\zeta(3),\\
\ddot t(\bar1,1,\bar1)&=-\ddot t(\bar1,\bar2)-\frac12\ddot t(2,1)=-G\pi+\frac{21}{8}\zeta(3),\\
\ddot t(\bar1,\bar1,\bar1)&=-\frac12\ddot t(3)-\ddot t(\bar1,\bar2)-\frac34\ddot t(2,1)=-G\pi-\frac18\pi^2\log(2)+\frac{35}{16}\zeta(3),\\
\ddot t(\bar2,\bar1,1)&=-\frac58\ddot t(4)-\ddot t(\bar2,\bar2)-\frac14\ddot t(3,1)+\frac34\ddot t(2,1,1)\\
&=-2G^2-\frac{23}{1440}+\frac{5}{24}\pi^2\log^2(2)+\frac16\log^4(2)+4\operatorname{Li}_4(1/2),\\
\ddot t(\bar1,\bar2,\bar1)&=2\ddot t(\bar4)-2\ddot t(2,\bar2)+2\ddot t(\bar3,1)-4\ddot t(3,\bar1)-\ddot t(2,1,\bar1)\\
&=-16\beta(4)+16\Im\operatorname{Li}_4\left(\frac{1+i}{2}\right)+\frac{3}{16}\pi^3\log(2)+\frac1{12}\pi\log^3(2)+\frac78\pi\zeta(3),\\
\ddot t(\bar2,\bar1,\bar1,\bar1)&=\frac1{40}\ddot t(5)+\frac95\ddot t(\bar4,\bar1)-\frac{41}{40}\ddot t(4,1)-\frac18\ddot t(3,1,1)+\frac18\ddot t(2,1,1,1)\\
&=-\frac{91}{5760}\pi^4\log(2)+\frac1{72}\pi^2\log^3(2)+\frac1{60}\log^5(2)-2\operatorname{Li}_5(1/2)\\
&\quad+\frac{29}{128}\pi^2\zeta(3)-\frac{155}{256}\zeta(5),\\
\ddot t(\bar1,\bar1,1,1,1)&=-\frac15\ddot t(5)-\frac{13}{20}\ddot t(\bar4,\bar1)+\frac{51}{80}\ddot t(4,1)+\ddot t(2,\bar1,\bar2)+\frac18\ddot t(3,1,1)\\
&\quad-\ddot t(\bar1,2,1,\bar1)-\frac7{16}\ddot t(2,1,1,1)\\
&=\pi\beta(4)-4\pi\Im\operatorname{Li}_4\left(\frac{1+i}{2}\right)-\frac{19}{576}\pi^4\log(2)-\frac1{144}\pi^2\log^3(2)-\frac1{48}\log^5(2)\\
&\quad+\frac52\operatorname{Li}_5(1/2)-\frac14\pi^2\zeta(3)+\frac{403}{64}\zeta(5).
\end{aligned}$$

Example 6.4. We have

$$\begin{aligned}
T(\bar2,\bar1)&=-2T(\bar3)+2T(2,\bar1)=-\frac3{16}\pi^3+8\Im\operatorname{Li}_3\left(\frac{1+i}{2}\right)+4G\log(2)-\frac14\pi\log^2(2),\\
T(\bar1,1,1,\bar1)&=\frac12T(\bar4)+\frac32T(\bar3,\bar1)=-\frac14G\pi^2+2\beta(4),
\end{aligned}$$

$$
\begin{aligned}
T(\bar{1},1,\bar{1},\bar{1}) &= 2T(\bar{3},\bar{1})+T(3,\bar{1}) = -\frac{1}{4}G\pi^2+6\beta(4)-\frac{7}{8}\pi\zeta(3),\\
T(\bar{3},\bar{1},1) &= \frac{6}{5}T(\bar{5})+2T(3,\bar{2})+6T(\bar{4},\bar{1})-6T(4,\bar{1})+2T(3,\bar{1},\bar{1})\\
&= -\frac{7}{128}\pi^5+2\pi^2\Im\operatorname{Li}_3\left(\frac{1+i}{2}\right)+G\pi^2\log(2)-\frac{1}{16}\pi^3\log^2(2),\\
T(\bar{3},1,1,1) &= -\frac{5}{48}T(6)-T(\bar{5},1)+T(5,1)-\frac{1}{4}T(4,1,1)\\
&= -\frac{509}{322560}\pi^6-\frac{1}{96}\pi^4\log^2(2)+\frac{1}{96}\pi^2\log^4(2)+\zeta(\bar{5},1)+\frac{1}{4}\pi^2\operatorname{Li}_4(1/2)\\
&\quad+\frac{7}{32}\pi^2\log(2)\zeta(3)-\frac{97}{128}\zeta^2(3),\\
T(\bar{1},2,1,1,1) &= -\frac{1}{2}T(\bar{1},\bar{5})+\frac{13}{2}T(\bar{5},\bar{1})+T(5,\bar{1}).
\end{aligned}
$$

Example 6.5. We have

$$
\begin{aligned}
S(\bar{1},2) &= -\frac{2}{3}S(\bar{2},\bar{1})+\frac{1}{3}S(\bar{2},1) = \frac{11}{96}\pi^3-4\Im\operatorname{Li}_3\left(\frac{1+i}{2}\right)-2G\log(2)+\frac{1}{8}\pi\log^2(2),\\
S(\bar{1},\bar{1},2) &= \frac{8}{3}S(\bar{1},\bar{3})-\frac{19}{15}S(\bar{1},3)-\frac{8}{5}S(\bar{3},\bar{1})+\frac{4}{5}S(\bar{3},1)-2S(2,\bar{1},\bar{1})+S(2,\bar{1},1)\\
&= -6\beta(4)+8\Im\operatorname{Li}_4\left(\frac{1+i}{2}\right)+4\Im\operatorname{Li}_3\left(\frac{1+i}{2}\right)\log(2)+G\log^2(2)-\frac{1}{12}\pi\log^3(2),\\
S(\bar{1},1,\bar{1},\bar{1}) &= \frac{15}{16}S(4)+\frac{1}{8}S(2,2)-\frac{1}{4}S(\bar{2},1,1)+\frac{3}{4}S(\bar{1},1,\bar{1},1)\\
&= 2G^2+\frac{9}{64}\pi^4-6\pi\Im\operatorname{Li}_3\left(\frac{1+i}{2}\right)-2G\pi\log(2)+\frac{3}{16}\pi^2\log^2(2)-\frac{7}{4}\log(2)\zeta(3),\\
S(\bar{3},1,\bar{1}) &= -\frac{31}{112}S(5)-\frac{13}{112}S(3,2)-\frac{3}{16}S(4,1)+\frac{1}{4}S(3,1,1),\\
S(\bar{3},1,2) &= \frac{35}{128}S(6)-\frac{25}{192}S(2,4)+\frac{25}{192}S(4,2)-\frac{1}{3}S(4,1,\bar{1})+\frac{1}{6}S(4,1,1)\\
&= \frac{247}{967680}\pi^6+\frac{1}{288}\pi^4\log^2(2)-\frac{1}{288}\pi^2\log^4(2)-\frac{1}{12}\pi^2\operatorname{Li}_4(1/2)\\
&\quad-\frac{7}{96}\pi^2\log(2)\zeta(3)+\frac{7}{16}\zeta^2(3).
\end{aligned}
$$

Theorem 6.6. *For any $w\in\mathbb{N}_0$, we have*

$$\dim_{\mathbb{Q}[i]}\mathsf{AMMV}_w\otimes_{\mathbb{Q}}\mathbb{Q}[i]=\dim_{\mathbb{Q}[i]}\mathsf{CMZV}^4_w\otimes_{\mathbb{Q}}\mathbb{Q}[i].$$

Proof. It is clear that the $\mathsf{AMMV}_w\subset\mathsf{CMZV}^4_w\otimes_{\mathbb{Q}}\mathbb{Q}[i]$ as vector spaces over $\mathbb{Q}[i]$. According to the definition of colored MZVs of level four, we know that $\eta_j\in\{e^{\pi i/2},e^{\pi i},e^{3\pi i/2},e^{2\pi i}\}=\{\pm1,\pm i\}$. Hence, for colored MZVs of level four, we have

$$\eta_j^{n_j}:=\begin{cases}\{\pm1,\pm i(-1)^{(n_j-1)/2}\} & \text{if } n_j \text{ odd};\\ \{1,\pm(-1)^{n_j/2}\} & \text{if } n_j \text{ even},\end{cases}$$

and

$$
\begin{aligned}
Li_{k_1,k_2,\dots,k_r}(\eta_1,\dots,\eta_r) &= \sum_{n_1>\cdots>n_r>0} \frac{\eta_1^{n_1}\dots\eta_r^{n_r}}{n_1^{k_1}\dots n_r^{k_r}} \\
&= \sum_{\delta_1=0}^{1}\cdots\sum_{\delta_r=0}^{1} \sum_{\substack{n_1>\cdots>n_r>0\\ n_j\equiv\delta_j \pmod 2\ \forall j}} \frac{\eta_1^{n_1}\dots\eta_r^{n_r}}{n_1^{k_1}\dots n_r^{k_r}} \in \mathsf{AMMV}\otimes_{\mathbb{Q}}\mathbb{Q}[i].
\end{aligned}
$$

Hence, $\mathsf{CMZV}_w^4 \subset \mathsf{AMMV}_w \otimes_{\mathbb{Q}} \mathbb{Q}[i]$. This concludes the proof of the theorem. □

Conjecture 6.7. *(cf. [32, Conj. 5.2]) We have the following generating function*

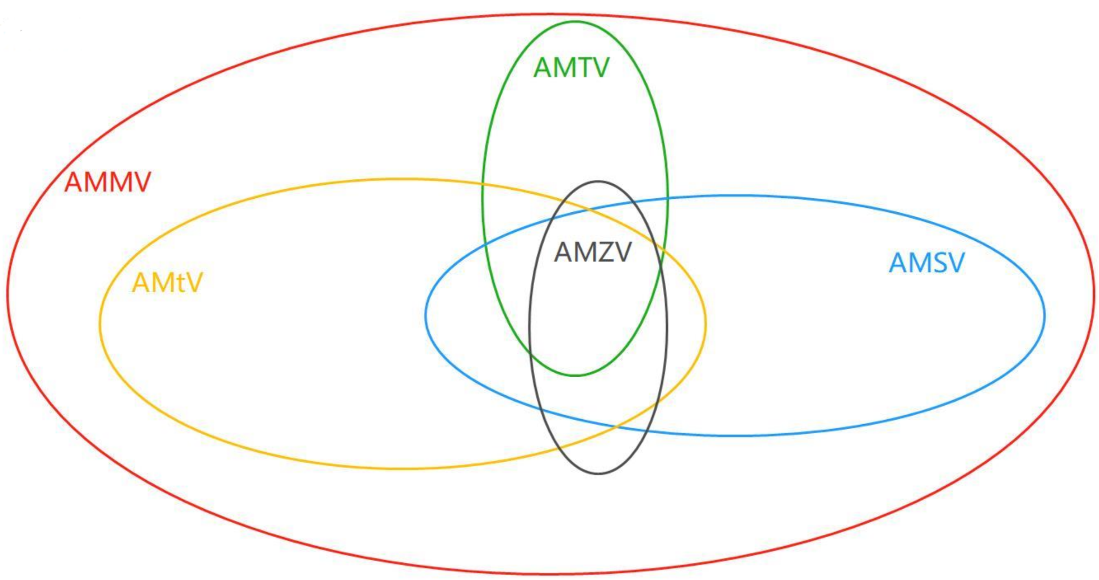
$$\sum_{n=0}^{\infty}(\dim_{\mathbb{Q}}\mathsf{AMTV}_n)t^n = \frac{1}{1-t-t^2-t^3}.$$

Namely, the dimensions form the tribonacci sequence $\{d_w\}_{w\geq 1} = \{1,2,4,7,13,24,\dots\}$, see A000073 at oeis.org.

Conjecture 6.8. *For $n \geq 2$, we have*

$$\dim_{\mathbb{Q}}\mathsf{AMtV}_n = 3\times 2^{n-2}.$$

In summary, we have the following Venn diagram showing relations between all the different variations of MZVs above.



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