Alternative Dynamics and Stability Results in a Standard OLG Model: An Interpretation

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Abstract: In this paper we make a critique of the standard stability results already established in a basic two period OLG model. Under an alternative interpretation (competitive financial markets), we are able to select the test equilibrium in the long run, in such a way that the economy can reach the Golden Rule Steady State, even when at this interest rate, all young agents are creditors. We show that even if the economy starts with only young people —referred to as the Garden of Eden Economy—, it also reaches the Golden Rule Steady State. Additionally, as a by-product of this study, we obtain a complete characterization of the possible equilibria.

Keywords: OLG models, selection of equilibria, Golden Rule Steady State, Samuelson Case, Garden of Eden Economy.

Resumen: En este trabajo presentamos una crítica de algunos resultados existentes en relación con las dinámicas posibles en un modelo de generaciones traslapadas estándar. Bajo hipótesis alternativas (mercados financieros competitivos), es posible seleccionar el mejor equilibrio en el largo plazo, de tal modo que la economía alcanza el equilibrio estacionario de la Regla de Oro, aun cuando, prevaleciendo dicho tipo de interés, todos los agentes son prestamistas. Mostramos que, aun en el caso en que la economía empieza sólo con jóvenes —situación que se suele denominar como El Jardín del Edén—, también se alcanza la Regla de Oro. Adicionalmente, obtenemos una completa caracterización de los equilibrios posibles.

Palabras clave: modelos de generaciones traslapadas, selección de equilibrios, Regla de Oro, caso Samuelson, economía del Jardín del Edén.

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Introduction

In this paper we present an alternative point of view (competitive financial markets) in order to interpret the basic OLG model in the case that the economy, in the aggregate, is saving (called the *Samuelson Case Economy*). This new interpretation allows us to find alternative dynamics and stability results, in such a way that the best steady state is selected (the Golden Rule steady state, the optimal steady state), in sharp contrast to the standard analysis, according to which in a Samuelson Case Economy, the Golden Rule steady state is unstable and the non-trade steady state (which is non-optimal) is asymptotically stable, so that the economy is converging to a non-optimal steady state.

We also have found some new results valid for both our point of view and the standard point of view, specially in relation to the classification of the possible equilibria.

To better understand the motivation of our study, we will review the established results. In the paper by Samuelson (1958), a stationary two period OLG economy with homogeneous agents endowed with one unit of a perishable good in the first period of their lives and nothing in the second period was considered, in order to present a general equilibrium explanation of positive interest rates. As is well known by now, Samuelson found that an equilibrium interest rate exists which is equal to the growth rate of the population, referred to as the *Golden Rule Interest Rate,* and the corresponding constant consumption program is called the *Golden Rule Steady State.*

Four aspects of this model (imposing the market clearing condition, that is, the feasibility condition with equality) have been extensively studied: optimality, stability (dynamics), sensibleness of the model's results and, finally, sensibleness of the model itself. The optimality issue for the simple version studied here is indeed a closed question, namely, that the Golden Rule Steady State is always intertemporally optimal. However, regarding the other steady state corresponding to the constant program given by the endowments, entitled the *Non-Trade Steady State,* we have two possible results. If all young people at the Golden Rule Interest Rate are debtors (referred to as the *Classical Case Economy*), the Non-Trade Steady State is optimal, whereas if all young people at the Golden Rule Interest Rate are creditors (referred to as the Samuelson Case Economy), it is not (see, for example, Malinvaud, 1953; Starret, 1972; and Gale, 1973). Notwithstanding, with respect to the three other aspects, there are still some open questions, which are the subject of this paper.

First, let us consider the issues of stability and sensibleness of the model's results. Gale (1973) showed that there are only two steady states, the Golden Rule Steady State and the Non-Trade Steady State. Furthermore, he showed that in a Classical Case Economy, the Non-Trade Steady State is unstable. On the other hand, in a Samuelson Case Economy, the Non-Trade Steady State is locally stable. In order to draw our interpretation from these results, it is worth clarifying the meaning of the classification Classical Case Economy and Samuelson Case Economy. To fix ideas, consider the extreme case that Samuelson analyzed, that in which the second period endowment is zero. In this situation, the economy is always a Samuelson Case Economy. Now, this, however, is the most reasonable scenario that one can imagine in order to set up a model with almost any analytical objective: Most people in real life can be described as having a unit of the good when young and having nothing (almost nothing) when old. But, surprisingly enough, the model then predicts that there will be no trade at all, or even if there was, the economy will converge to the Non-Trade Steady State, this Non-Trade Steady State being non-optimal. Summarizing: A Samuelson Case Economy —perhaps the most reasonable set up for applications—, if left to work through free markets (imposing the market clearing condition), always ends in a non-optimal situation. Nevertheless, this is not the most paradoxical fact of this result: The model says that even in an extreme situation of necessity (the agents have no units of the good in the second period), they will not use the possibility that the market is giving to them in order to obtain goods when old. Observe that if one interpreted this situation in terms of real life, the agents would be dying when old, but not because of a natural death: They would be dying due to starvation![1] This observation begs the questions: *1)* What are the reasons for these paradoxical results? *2)* Is it possible in this model to present an alternative and reasonable view in such a way that a Samuelson Case Economy reaches or converges to the optimal Golden Rule Steady State? As mentioned above, in this paper we show the reasons for these results and we demonstrate that the answer to the second question is yes. Another

[1] A reader may think that to die due to starvation or to die because of a natural death is a non relevant distinction. Think that agents receive at the beginning of the period of life the corresponding endowments. Then, if the second period endowment is zero, agents have nothing to consume until the end of the second period, that is, they would die young.

point studied in this paper, profoundly connected with the stability issue, is the so called *"Samuelson Impossibility Theorem,"* which asserts that a Garden of Eden Economy can never converge to the Golden Rule Steady State (assertion proven in Gale, 1973). More precisely, the model predicts that a Garden of Eden Economy will forever stay in the non-optimal Non-Trade Steady State, one in which, exactly as in the Samuelson Case Economy, people are deciding a sort of collective suicide, even though the free market structure would provide them a way out of this situation. Once again, this last comment begs the same questions *1)* and *2)* as before. Not surprisingly, the answers are the same as for the general Samuelson Case Economy.

Second, let us consider the model's sensibleness. This point resulted to be in question as a consequence of what Samuelson called the *"infinity paradox revealed,"* which reads as follows: Imagine a Samuelson Case Economy and consider the Golden Rule Steady State, so that, by definition, all the agents in the economy are creditors and, therefore, the question comes, Who is then borrowing? The point is that the model was considered as a "pure loan consumption model." Of course, considered such, the question above is pertinent and, furthermore, has no answer. Instead of that, we will assume that there is a bank that serves as a financial intermediary, and therefore, the infinity paradox revealed disappears as a paradox and the Golden Rule Steady State acquires a clear sense.

As the reader perhaps would be expecting, we are not saying that the previous stability results are false from a mathematical point of view. Our argumentation, instead, relies on the following two points: *1) a priori,* there are no general economic reasons to impose the feasibility condition with equality; *2)* we assume that there are complete markets because there is a financial market (banks) that offer bonds in order to allow for savings: That financial market works competitively. A simple consequence of this fact in this model is that there is no uniqueness of equilibria. Therefore, our equilibrium paths will be just examples among all the possible ones. We argue, then, that they are more reasonable than the equilibrium paths calculated imposing the feasibility condition with equality. In fact, and fundamentally, our equilibrium emerges as a direct consequence of imposing a free competition concept on the financial markets, markets that we introduce in order to give more sensibleness to the model, as commented before.

The intuition of our alternative assumptions and results goes as follows. As it is commented in the precedent paragraph, one of our two main alternative assumptions is the existence of competitive financial markets, which complete the markets. The second is that we do not assume that the market clearing condition holds, that is, we do not impose, *a fortiori,* the feasibility condition with equality. For the first assumption, the usual justification in complete markets theory applies: The *medium* by which a market is completed is a bond or, more generally, an asset, which is, roughly speaking, a piece of paper that promises to pay a defined quantity (of the good or of money) at some moment in the future. For the sake of the exposition, in our case, we will be more concrete. We will assume that there are banks that issue bonds in order to finance its debt and, as in any competitive market of bonds (that is, there are more than one bond in the market), given a level of risk (we assume that there is no risk), the bond that pays the highest interest rate is the bond that absorbs all the demand. In relation to the second assumption, the intuition is drawn from the real life: In productions related with land, it is not too rare to see that, if the price of a good is too low, the producers decide, to say, to throw away goods to the sea, in order to increase the price. Finally, the intuition of the argument used to obtain the results is as follows. Imagine a situation in which the preferences of the agents are such that they prefer to save more than they do, because the actual plan prescribes a consumption for the old age too low, but they have enough endowments when young to save more. That is, if it would be possible, they would prefer to consume less today and more when old to the actual consumption plan. The banks, knowing that, will then offer the highest interest rate possible in the market (because there are more than one bank), which turns out to be the gross rate of the population. Therefore, even if the actual consumption plans of the agents would be such that they use the total offer, they would decide to consume less and to save more.

The rest of the paper is organized as follows. In Section I we comment on some related literature. Section II formally presents the model and in Section III the two main results of this paper are shown. In Section IV we present the argument in favor of the existence of the bank and fully develop the argument in favor of our results in Section III. Lastly, in Section V we add some final remarks.

I. Related Literature

Although we do not present our argument as a formal learning process, our main result, the selection of the test equilibrium among all the possible ones, is closely related with the important issue on learning and the selection of equilibria in OLG models. See, for example, J. Duffy (1994), J. Bullard (1994), and the references therein. However, our result has not been obtained previously.

On the other hand, as far as we know, there are no antecedents of studies dealing with the paradoxical fact of the non-convergence to the Golden Rule if the OLG economy is a Samuelson Case Economy or if it is a Garden of Eden Economy. Gale (1973), instead of considering this fact a sort of collective suicide, provided a logical argument. Now, the optimality property of the Golden Rule has been extensively studied and discussed. Starting with Samuelson (1958), the following series of papers have been written regarding this point: Cass and Yaari (1966), Lerner (1959a, b), Meckling (1960a, b), Samuelson (1959, 1960), Phelps (1961), Thompson (1967) and Starret (1972).[2] This topic, at least the formal statement, is clearly understood by now. Finally, regarding the “infinity paradox revealed,” apart from Samuelson (1958), this problem is clearly highlighted and considered in Cass and Yaari (1966). In their opinion, it does not have a solution, that is, it is not possible to imagine any type of trade in this economy. Actually, they consider the possibility of a financial intermediary, but they reject it on the basis of the following argument:

> The outlook seems rosy until one takes a brief look at the balance sheet of our financial intermediary: The balance sheet as of the end of period t shows zero assets and liabilities of $s(1 + n)^t$,[3] where s is the (stationary) saving ratio of people in their first period of life. This means that at the end of period t the net worth of the intermediary is given by $-s(1 + n)^t$. Now by not doing anything (that is, by shutting down) the intermediary can guarantee itself a net worth of zero and so one might argue that it will never choose to engage in the aforementioned transactions. (Cass and Yaari, 1966, p. 360.)

We offer another panorama of the situation which considers two important points.

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[2] Of course, we do not pretend to present a complete account of this literature.
[3] $(1 + n)$ is the gross population growth.

First, we must see if this business is profitable for the banker. In effect, until the banker is able to receive the new assets, the net worth of the bank is negative. However, it is important to consider that he/she is not forced to pay the liabilities before that moment. On the other hand, at the time the obligations have to be paid, he will be receiving the new assets and thus, will be able to pay them. Therefore, one could think that the bank is making zero profits, as we do. Indeed, this is a one period business, and to analyze if it is profitable, one has to wait until the business is mature, in this case, until the new generation comes and buys new bonds.[4]

Second, we must wonder if such a financial institution would be considered healthy by a monetary authority. The introduction of a monetary authority in this economy may seem artificial, given that money has no natural interpretation here due to that it is difficult to interpret that there is trade between the goods "the good today" and "the good tomorrow" —they trade goods today for bonds—. True, but in any event, if one insists on the existence of an institution in charge of the supervision of the bank's behavior,[5] the situation is not dangerous for society. In any financial institution in real life, one may find particular moments when its net worth is negative, that is, without the ability to cover the deposits. What matters is the bank's ability to pay the debt at the correct time. To the contrary, the monetary authority would force the bank to contract a new debt at a very high interest rate or, in the worst case, would force the institution to shut down. During the day, banks may have a negative net worth, but at the end of the day, if they are in debt, they can go to the overnight market and cover it, or they may even accept being in debt, provided that they have no debt at the end of the natural period of the economy —28 days in many countries in real life—. In our economy something similar is happening. Until the bank receives the new assets, it is in debt, but at the moment the return of the bonds has to be paid, the bank can pay the debt. Period by period —this is the natural period in our economy and the natural period of the bank's business—, the bank is contracting new debts, but period by period is paying the old debt, no consumer would be disappointed. Thus, the monetary authority, if it ex-

[4] Indeed, it is a business as many businesses in real life. Think of working the land. The business man may need funds to begin with. Imagine that he decides to issue bonds, which promise an interest rate for the next year. This business man is in debt during the whole year, but at the moment the interest promised by the bonds has to be paid, if the harvest is good enough, he will pay the debt and the business goes on.

[5] In real life, banks and financial institutions are heavily regulated.

ists, would consider the bank healthy. In other words, one may think that the banks are playing a "Ponzi Game" and, indeed, in real life, many banks are playing a Ponzi Game. Nevertheless, as they are institution with an infinite lifespan, they can play it. Particulars are not allowed to play a Ponzi Game.[6]

II. The Model

We consider a double-ended, stationary overlapping-generations economy with no production where agents live for two periods. There is a single, perishable good each period. At each $t \in \mathbb{Z}$, a generation of agents of size $N_t > 0$ is born. There is an exogenous gross population growth rate $\gamma > 0$, that is, we suppose that $N_{t+1} = \gamma N_t$ for all $t \in \mathbb{Z}$.

The first period of their life, agents receive a fixed endowment, which we normalize to 1. The second period, endowment is denoted by $e > 0$. Agents have a continuous intertemporal utility function

$$u(c_0, c_1)$$

strictly increasing in both arguments, where c_0, c_1 are the consumption levels in the first and second period of their lives, respectively.

The market structure is as follows. There is a bank that works as a financial intermediary. If people want to lend (save), because they have little goods for the old age or they are patient people, they buy a bond that promises to pay $R_t = 1 + r_t$ units of the good tomorrow per unit of the good today, so that the price of the bond is $\frac{1}{1+r_t}$, in terms of the good today. The interest rate r_t is offered by the bank. On the other hand, if people want to borrow, because they have a lot of goods in the old age or they are impatient people, they borrow from the bank, and in this case the bank asks an interest rate $r_t' = r_t$ as well from them (in real life, we have $r_t' > r_t$), so we are assuming that the bank is making zero profits. For each generation t, we denote their consumption decisions by $c(t) = (c_0(t), c_1(t+1))$. Therefore, the consumer's problem is the following

[6] The term "Ponzi Game" is used here, informally, as reference to situations where an agent (infinitely lived) is rolling over a debt forever, that is, at any time, the current debt is paid by making a new borrow, promising to pay it in the future, and so forth. See, for example, Blanchard and Fisher (1989), pp. 49 and 84. Also, see Blanchard and Weil (1992).

$$\left.\begin{array}{ll}\max & u(c_0, c_1) \\ \text{s.t.} & c_0(t) + c_1(t+1)\dfrac{1}{R_t} = 1 + \dfrac{1}{R_t}e \\ & c_0(t), c_1(t+1) \geq 0\end{array}\right\} \tag{1}$$

An economy so described is then given by $\varepsilon = ((1, e), u, \gamma)$.

Definition 1 *An* equilibrium *for the economy* ε *is a pair* $(\mathbf{C}, \mathbf{R})$ *where* $\mathbf{C} = \{(c_0(t), c_1(t+1))\}_{t \in \mathbb{Z}}$ *is a sequence of agents' consumption plans and* $\mathbf{R} = \{R_t\}_{t \in \mathbb{Z}}$ *is a sequence of positive quantities (called interest rates), such that*

(i) For all $t \in \mathbb{Z}$, $(c_0(t), c_1(t+1))$ *solves (1), and;*
(ii) The feasibility condition holds for all $t \in \mathbb{Z}$:

$$N_t c_0(t) + N_{t-1} c_1(t) \leq N_t + eN_{t-1}. \tag{2}$$

An equilibrium is a steady state *if* $(c_0(t), c_1(t+1)) = (c_0, c_1)$ *for all* $t \in \mathbb{Z}$.

Notice that the feasibility condition is equivalent to

$$\gamma c_0(t) + c_1(t) \leq \gamma + e. \tag{3}$$

For completeness, we restate formally the following definitions.

Definition 2 *The* Golden Rule Steady State *is defined as a constant program* $c^{GR}(t) = (c_0^{GR}, c_1^{GR})$ *for* $t \in \mathbb{Z}$ *such that* (c_0^{GR}, c_1^{GR}) *solves*

$$\left.\begin{array}{ll}\max & u(c_0, c_1) \\ s.t. & \gamma c_0 + c_1 \leq \gamma + e \\ & c_0, c_1 \geq 0\end{array}\right\}. \tag{4}$$

It follows at once from the previous definition, that the pair $(\mathbf{C}^{GR}, \mathbf{R}^{GR})$, where $\mathbf{C}^{GR}$ is the constant sequence equal to (c_0^{GR}, c_1^{GR}) and $\mathbf{R}^{GR}$ is the constant sequence equal to g, is an equilibrium.

Finally, we present the following.

Definition 3 *An economy* $e = ((1, e), u, \gamma)$ *is a* Samuelson Case Economy *if* $c_0^{GR} < 1$, *and is a* Classical Case Economy *if* $c_0^{GR} > 1$.

III. The Results

III.1. The General Samuelson Case

Our first result says that, given a Samuelson Case Economy, even in the worst case in which there is an equilibrium that is converging to the non-optimal Non-Trade Steady State, it is possible to find another equilibrium in which the Golden Rule Steady State is reached.[7] Formally, we have:

Theorem 4 *Let* $\varepsilon = ((1, e), u, \gamma)$ *a Samuelson Case Economy. Assume that there exists an equilibrium* $(\{(c_0(t), c_1(t+1))\}_{t \in \mathbb{Z}}, \{R_t\}_{t \in \mathbb{Z}})$ *such that* $(c_0(t), c_1(t+1)) \to (1, e)$. *Therefore, there exists an equilibrium* $(\{(\tilde{c}_0(t), \tilde{c}_1(t+1))\}_{t \in \mathbb{Z}}, \{\tilde{R}_t\}_{t \in \mathbb{Z}})$ *and a natural number T, such that we have* $(\tilde{c}_0(t), \tilde{c}_1(t+1)) = (c_0^{GR}, c_1^{GR})$ *and* $\tilde{R}_t = \gamma$ *for all* $t \geq T$, *that is, there exists an equilibrium in which the Golden Rule Steady State is reached.*

Proof: As we have $(c_0(t), c_1(t+1)) \to (1, e)$, then there exists a $\bar{T}$ such that for all $t \geq \bar{T}$, we have $c_0^{GR} < c_0(t)$ (recall that in a Samuelson Case Economy, we have $c_0^{GR} < 1$). Define

$$T = \min\{t \geq 0 \,|\, c_0^{GR} < c_0(t)\}.^{8}$$

Thus, we have $c_0^{GR} < c_0(T)$. Now observe the feasibility condition at time T (recall (3)):

$$\gamma c_0(T) + c_1(T) \leq \gamma + e.$$

Consequently, we have

$$\gamma c_0^{GR} + c_1(T) < \gamma + e, \tag{5}$$

since $c_0^{GR} < c_0(T)$. Hence, if we take the sequence $(\tilde{R}_t, (\tilde{c}_0(t), \tilde{c}_1(t+1))) = (g, (c_0^{GR}, c_1^{GR}))$ from $t \geq T$, we clearly have that

[7] Recall that the Non-Trade Steady State is locally stable in a Samuelson Case Economy (Gale, 1973). Thus, in most situations we will have in fact that relevant equilibria to study are such that it is converging to the Non-Trade Steady State. If this is not the case, nothing can be said in general.

[8] Notice that we only pay attention on what would happen from $t = 0$. Therefore, we are implicitly assuming that we study the economy from $t = 0$.

$$\gamma\tilde{c}_0(t) + \tilde{c}_1(t) = \gamma c_0^{GR} + c_1^{GR} = \gamma + e,$$

for all $t \geq T + 1$, since (c_0^{GR}, c_1^{GR}) solves (4). Therefore, the sequence $(\tilde{R}_t, (\tilde{c}_0(t), \tilde{c}_1(t+1))) = (\gamma, (c_0^{GR}, c_1^{GR}))$ solves (1) and satisfies (2) for all $t \geq T + 1$, because (c_0^{GR}, c_1^{GR}) solves (4) and because of (5). Now, let us define

$$(\tilde{R}_t, (\tilde{c}_0(t), \tilde{c}_1(t+1))) = \begin{cases} (R_t, (c_0(t), c_1(t+1))) \text{ if } t < T \text{ and} \\ (\gamma, (c_0^{GR}, c_1^{GR})) \quad \text{if } t \geq T \end{cases}. \qquad (6)$$

Now, notice the following facts: First,

$(\tilde{R}_t, (\tilde{c}_0(t), \tilde{c}_1(t+1)))$ solves (1) and satisfies (2) for all $t < T$,[9] because it coincides with $(R_t, (c_0(t), c_1(t+1)))$ if $t < T$. (7)

Second,

at $t = T$ we have that $\gamma\tilde{c}_0(T) + \tilde{c}_1(T) = \gamma c_0^{GR} + c_1(T) < \gamma + e$, due to (5). (8)

Finally, notice that

from $t = T + 1$ we have that $\gamma\tilde{c}_0(T+1) + \tilde{c}_1(T+1) = \gamma c_0^{GR} + c_1^{GR}$, so that $(\tilde{R}_t, (\tilde{c}_0(t), \tilde{c}_1(t+1)))$ solves (1) and satisfies (2) for all $t > T$. (9)

Now, it follows at once from (7), (8) and (9) that the sequence given in (6) is an equilibrium that reaches the Golden Rule from $t = T$. This ends the proof of the theorem. ■

Remark 1 *It is important to notice the following fact. According to the equilibrium found in the previous theorem, at time T the feasibility condition is not binding (8). This situation can be interpreted as that the young generation at this time is throwing away goods. However, this is not an argument that can be used in order to reject our equilibrium: In real life, there are indeed situations in which people decide to throw away goods, for example, if a good has a very low price, the producer many times throw away goods in order to revert the situation.*

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[9] In section 5.2 we justify why the agents at time T prefer (c_0^{GR}, c_1^{GR}) to $(c_0(T), c_1(T+1))$.

Remark 2 *In Geanakoplos (1987), the differences between the Arrow-Debreu model and OLG models are clearly highlighted. For the two period models that we consider here, the most important differences are in relation to the indeterminacy and the non-optimality of the equilibrium. One of his questions was: Why the equilibrium may not be optimal in OLG models? An immediate derived question from that one is the following: Why an optimal equilibrium may not even be locally stable? In general, it does not seem to be easy to find a simple answer. Nevertheless, our point of view suggests the possibility of a simple answer: If we allow for the free disposal property to hold, local stability can be obtained. Thus, the only point to take into account is the feasibility condition.*

III.2. The Garden of Eden Economy

The second technical result of this paper says that in a Garden of Eden Economy the Golden Rule can be achieved, in contrast to the Samuelson Impossibility Theorem, which asserts that, under the extreme situation where $e = 0$ (in this case the economy is always a Samuelson Case Economy), then, if at time zero there are only young people, no equilibrium can converge to the Golden Rule. In a two period case, if we impose the market clearing condition, the result is trivial. For the case of more than two periods, this conjecture was proven in Gale (1973).

Now we give formal definitions. A Garden of Eden Economy is defined as follows. At time $t = 0$ there are only N_0 young people and from this moment on, the population grows at a rate γ, in such a way that at any time $t > 0$ are born $N_t = \gamma^t N_0$ new agents. The rest of the elements of the economy are defined as in the Samuelson general case. The equilibrium then is defined as follows:

Definition 5 *An* equilibrium *for the economy* ε *is a pair* $(\mathbf{C}, \mathbf{R})$ *where* $\mathbf{C} = \{(c_0(t), c_1(t+1))\}_{t \geq 0}$ *is a sequence of agents' consumption plans and* $\mathbf{R} = \{R_t\}_{t \geq 0}$ *is a sequence of positive quantities (called interest rates), such that*

(i) *For all* $t \geq 0$, $(c_0(t), c_1(t+1))$ *solves*

$$\left.\begin{array}{ll} \max & u(c_0, c_1) \\ s.t. & \begin{array}{l} c_0 + \frac{1}{R_t} c_1 = 1 + \frac{1}{R_t} e \\ c_0, c_1 \geq 0 \end{array} \end{array}\right\}, \tag{10}$$

and

(ii) *the feasibility condition holds for all* $t \geq 0$ *that is*:

(a) *at* $t = 0$

$$N_0 c_0(t) \leq N_0, \tag{11}$$

and

(b) *for* $t > 0$

$$N_t c_0(t) + N_{t-1} c_1(t) \leq N_t + eN_{t-1}.$$

Theorem 6 *Let* $e = ((1, e), u, \gamma)$ *a Garden of Eden Economy, where* $c_0^{GR} < 1$ (*that is, it is a Samuelson Case Economy*). *Then, the Golden Rule program is an equilibrium. That is, at time* $t = 0$, *the economy reaches the Golden Rule.*

Proof: Just observe that the young generation at time zero can throw away goods (exactly as the young generation at time T in the previous theorem), and decide to consume c_0^{GR} instead of 1. It follows at once then that the Golden Rule program is an equilibrium. ■

IV. Interpretation and Classification of Equilibria

As we have already commented above, the bank, the financial intermediary, would be making zero profits under our interpretation, in such a way that the model is sensibly consistent with this assumption. The main point in question here is that the model does not describe why there are complete markets. It is assumed, so it is assumed implicitly a mechanism that completes the markets as well. Therefore, the model, if we do not impose the market clearing condition (which need not to be assumed in advance, as we argued before), it does not select an equilibrium among all the possible ones. As we will see in the sequel, this fact is behind of our final argumentation. Our interpretation then will provide a sensible criteria from an economic point of view in order to select an equilibrium. But, before we proceed to give it, we will need the following result which, as a by-product, provides a general classification of the possible equilibria when the clearing market condition is imposed.

IV.1. Classification of Equilibria

Theorem 7 *Let* $e = ((1, e), u, \gamma)$ *be an economy. Then,*

(i) if $((c_0^E(t), c_1^E(t+1)), R_t)_{t\in\mathbb{Z}}$ *denotes any equilibrium in which the feasibility condition holds with equality for all* $t \in \mathbb{Z}$*, that is, the market clearing condition holds for all* $t \in \mathbb{Z}$*, then* $1 - c_0^E(t) > 0$ *for all* $t \in \mathbb{Z}$ *or,* $1 - c_0^E(t) < 0$ *for all* $t \in \mathbb{Z}$ or $1 - c_0^E(t) = 0$ *for all* $t \in \mathbb{Z}$*; in the first case, we call the equilibrium to be of the Samuelson Case, in the second case we call it to be of the Classical Case and, in the third case, we call it to be of a Garden of Eden Case;*

(ii) if $e = ((1, e), u, \gamma)$ *is a Samuelson Case Economy and* $((c_0^E(t), c_1^E(t+1)), R_t)_{t\in\mathbb{Z}}$ *denotes any equilibrium such that* $((c_0^E(t), c_1^E(t+1)), R_t) \rightarrow ((1, e), \rho(NT))$*, where* $\rho(NT)$ *is the gross interest rate corresponding to the Non-Trade Steady State (see Gale,* 1973) *and such that for some* $\bar{t}$*, we have* $1 - c_0^E(\bar{t}) > 0$*, then*

(ii.1) $1 - c_0^E(t) > 0$ *for all* $t \geq \bar{t}$*, and*
(ii.2) $\gamma > R_{t-1}$ *for all t large enough.*

Proof: (i) Dividing (2) by N_{t-1}, we obtain

$$\gamma c_0(t) + c_1(t) \leq \gamma + e. \tag{12}$$

From the budget constraint of problem (1) at time $t - 1$, we have

$$e - c_1^E(t) = -R_{t-1}(1 - c_0^E(t-1))$$

and hence (12) becomes

$$\left.\begin{aligned} 0 &\leq \gamma + e - (\gamma c_0^E(t) + c_1^E(t)) = \\ &= \gamma(1 - c_0^E(t)) + (e - c_1^E(t)) = \\ &= \gamma(1 - c_0^E(t)) - R_{t-1}(1 - c_0^E(t-1)), \end{aligned}\right\} \quad (13)$$

so $\gamma(1 - c_0^E(t)) \geq R_{t-1}(1 - c_0^E(t-1))$ for all $t \in \mathbb{Z}$. Now, if the market condition holds for all $t \in \mathbb{Z}$, it follows from (13) that $\gamma(1 - c_0^E(t)) = R_{t-1}(1 - c_0^E(t-1))$ for all $t \in \mathbb{Z}$ and therefore the claim in (i) follows at once.

(ii.1) From (13) we observe that $\gamma(1 - c_0^E(t)) \geq R_{t-1}(1 - c_0^E(\bar{t})) > 0$ for all $t > \bar{t}$ and the claim is proven.

(ii.2) Provided that $(1 - c_0^E(t)) > 0$ for all $t \geq \bar{t}$, and rearranging (13), we obtain

$$\frac{\gamma}{R_{t-1}} \geq \frac{(1 - c_0^E(t-1))}{(1 - c_0^E(t))} \text{ for all } t > \bar{t}.$$

On the other hand, since $((c_0^E(t), c_1^E(t+1)), R_t) \to ((1, e), \rho(NT))$, we can assure that $c_0^E(t)$ is increasing (see theorem 4, in Gale, 1973), and therefore $\frac{(1 - c_0^E(t-1))}{(1 - c_0^E(t))} > 1$ for all t large enough, so that $\gamma > R_{t-1}$, for all t large enough, and the claim is proven. ■

IV.2. Interpretation

Suppose that $((c_0^E(t), c_1^E(t+1)), R_t)_{t \in \mathbb{Z}}$ denotes any equilibrium such that $((c_0^E(t), c_1^E(t+1)), R_t) \to ((1, e), \rho(NT))$ and such that $1 - c_0^E(t) > 0$ for all $t \in \mathbb{Z}$ (recall footnote 6).[10]

Observe that, in this case, the set of the possible interest rates is bounded by γ (see theorem 3 (ii.2)). Now, in concordance with our hypothesis that there is a bank completing the markets, we assume that there is free entry regarding the possibility of installing a bank, that is, we assume that the financial markets behave competitively. Consequently, the actual interest rate active in the market will obviously be the largest one among the possible ones.

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[10] The case when the equilibrium $((c_0^E(t), c_1^E(t+1)), R_t)_{t \in \mathbb{Z}}$ is such that there exists $\bar{t}$ such that $1 - c_0^E(\bar{t}) < 0$ can be analized exactly as if it were a Garden of Eden Economy: At time $\bar{t}$ the Golden Rule Steady State can be reached.

Given these hypothesis, it follows at once the viability of our equilibria in theorems 1 and 2: Upon the possibility to offer γ (the largest possible interest rate), this will be the active interest rate in the economy, and the Golden Rule equilibrium is achieved in a general free market setting.

Summarizing, our alternative interpretation of the model consists of assuming: *1)* the existence of a bank that completes the markets; *2)* that the financial markets behave competitively; *3)* not to impose the market clearing condition.

Now, our argument in order to assert that the equilibria given in theorems 1 and 2 are more reasonable than the ones imposing the market clearing conditions, directly follows from the points 1 to 3 above and from the continuity of the utility function. Indeed, if the utility function is continuous —as it was assumed—, we have that if $((c_0^E(t), c_1^E(t+1)), R_t)_{t\in\mathbb{Z}}$ denotes any equilibrium such that $((c_0^E(t), c_1^E(t+1)), R_t) \to ((1, e), \mathrm{r}(NT))$, then

$$u((c_0^E(t), c_1^E(t+1)) < u(c_0^{GR}, c_1^{GR}) \text{ for all } t \text{ large enough,}$$

since $u(c_0^{GR}, c_1^{GR}) > u(1, e)$.

Therefore, in the case of theorem 1, taking T large enough in such a way that $u(c_0^E(t), c_1^E(t+1)) < u(c_0^{GR}, c_1^{GR})$ and $\gamma > R_t$ for all $t \geq T$ (recall (ii.2) in theorem 3) and therefore having the banks the option to offer γ (the largest possible interest rate), because of the competition in the financial markets, all banks will offer γ; on the other hand, since the generation T prefer $u(c_0^{GR}, c_1^{GR})$ than $u(c_0^E(T), c_1^E(T+1))$, no consumer will hesitate in accepting γ, so that the Golden Rule steady state becomes the effective equilibrium path in the economy from T on.

The argument in the case of the Garden of Eden Economy is totally analogue and hence it is omitted.

V. Final Remarks

In this paper we consider the important question of selecting equilibria in models with multiple equilibria. The main result of this paper is to select the best equilibrium in a basic OLG model in a Samuelson type economy. Our argument is twofold: *1)* The market clearing condition need not always to hold, as it is indeed in a real world economy; *2)* In order to understand the ultimate sense of the basic OLG model, it needs to be completed describing the mechanism that completes the markets. Our argument is, therefore, to assume the existence of competitive financial markets.

So, Why in the basic model the local stability of the Non-Trade Steady State in a Samuelson Case Economy was obtained? The answer is because it was inappropriately assumed the market clearing condition (our argument 1 above). Our second question: Is it possible to offer a reasonable alternative view in such a way that the Golden Rule equilibrium is obtained? The answer is yes: Assuming the existence of a competitive financial market that completes the markets (our argument 2 above), the Golden Rule is reached as the final part of a path without any intervention of a government or social security system, just competition in a general sense.

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