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## AMALGAMATION OVER UNIFORM MATROIDS

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### INTRODUCTION

In this paper we study glueing two matroids together. It will be assumed that the reader is familiar with elementary concepts of the matroid theory. An appropriate reference is the book by Welsh [14]. In drawing matroids we follow [5].

Let  $M_1(X_1)$  and  $M_2(X_2)$  be matroids on sets  $X_1$  and  $X_2$  respectively, and let the restrictions of  $M_1$  and  $M_2$  to  $X_1 \cap X_2$  be the same. A matroid  $M(X_1 \cup X_2)$  is called an *amalgam of  $M_1$  and  $M_2$* , if  $M$  restricted to  $X_i$  is the matroid  $M_i$  for  $i = 1, 2$ .

Constructions related to amalgamation have been studied by many authors [1], [2], [3], [4], [9]. Geometrical description of amalgamation is given in [6]. A particular type of amalgamation was used by Seymour [12] when giving a characterization of regular matroids. Amalgamation of a greater number of matroids was studied in [7], [10].

The paper is divided into five parts. Section 1 contains basic definitions and examples of amalgams. In Section 2 we give some examples of pairs of matroids for which no amalgam exists. Relations between amalgamation and other matroid operations (as restriction, contraction, Dilworth truncation) are discussed in Section 3. In Section 4 we introduce sticky and  $f$ -sticky matroids defined in [11]. The main result is the characterization of sticky uniform matroids. The last section deals with representable matroids.

The amalgamation is closely related to the construction of Ramsey matroids. The vertex partition property for matroids was proved in [7]. The results of this paper are used in [8], where the Ramsey property for the partition of two-point lines is proved. The paper contains some results first stated in [13].

### SECTION 1

Let  $M(X)$  be a matroid and  $Y$  a subset of  $X$ . The closure of the set  $Y$  in  $M$  will be denoted by  $\bar{Y}^M$  or  $\bar{Y}$ . The restriction of  $M$  to  $Y$  is the matroid  $M \upharpoonright Y$  on the set  $Y$  with the rank function  $r(A) = r_M(A)$  for  $A \subset Y$ . The contraction of  $M$  to  $Y$  is the matroid

$M$  on the set  $Y$  with the rank function

$$r(A) = r_M(A \cup (X - Y)) - r_M(X - Y) \text{ for } A \subset Y.$$

Let  $M(X), N(Y)$  be matroids. An injective mapping  $f : X \rightarrow Y$  is called an *embedding* if  $r_M(A) = r_N(f(A))$  for every  $A \subset X$ .

**Definition.** Let  $M_1(X_1), M_2(X_2)$  and  $M(X)$  be matroids and let  $f_i : X \rightarrow X_i$  be an embedding for  $i = 1, 2$ . A matroid  $N(Z)$  is called an *amalgam of  $M_1$  and  $M_2$  over  $M$  with respect to  $f_1$  and  $f_2$*  if for  $i = 1, 2$  there exist embeddings  $g_i : X_i \rightarrow Z$  satisfying

- (i)  $g_1(X_1) \cup g_2(X_2) = Z$ ,
- (ii)  $g_1 f_1 = g_2 f_2$ ,
- (iii)  $|Z| = |X_1| + |X_2| - |X|$ .

In particular, when all embeddings  $f_1, f_2, g_1$  and  $g_2$  are identity mappings (inclusions), the above definition converts into that given in Introduction.

In general, an amalgam needs neither be unique nor exist.

**Example.** A prism  $P_6$  is a matroid of rank 4 given by three nontrivial hyperplanes  $\{1, 2, 3, 4\}, \{1, 2, 5, 6\}$  and  $\{3, 4, 5, 6\}$ . (See Fig. 1).

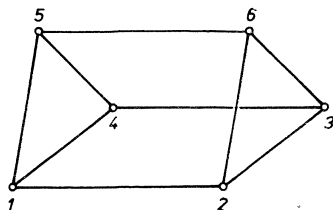


Fig. 1

Let  $M_1(\{1, 2, 3, 4, 5, 6\})$  and  $M_2(\{3, 4, 5, 6, 7, 8\})$  be two copies of the prism  $P_6$  and let  $M(\{3, 4, 5, 6\})$  be the common plane. There is a lot of amalgams of  $M_1$  and  $M_2$ . Three of them are useful examples [5].

Tab. 1

	3-Flats with more than 3 points	4-Flats with more than 4 points	5-Flats with more than 5 points
$V_4$ Vamos rank 4	1234, 1256 3456, 5678, 3478	12345678	
$V_4^+$ rank 4	same as $V_4$ with 1278	12345678	
$V_5$ rank 5	same as $V_4^+$	123456 123478 125678 345678	12345678

In the sequel we shall use the following known fact.

**Lemma.** *If  $V$  is an amalgam of  $M_1$  and  $M_2$  and  $r(V) = 5$  then  $r_V(\{1, 2, 7, 8\}) = 3$ . ■*

## SECTION 2

In this section we give three examples in which no amalgam exists.

**Example 1.** Let  $M_1(\{a, b, c, d, e, g, f\})$  and  $M_2(\{a', b, c, d, e, f, g\})$  be matroids of rank 3 given by Fig. 2. Clearly  $M_1 \upharpoonright \{b, c, d, e, f, g\} = M_2 \upharpoonright \{b, c, d, e, f, g\}$ . Assume that  $M$  is an amalgam of  $M_1$  and  $M_2$ . Denote by  $r$  the rank function of  $M$ . As  $a, a' \in \overline{\{b, e\}} \cap \overline{\{c, f\}}$  we have  $r(\{a, a'\}) = 1$  which contradicts  $r(\{a, d, g\}) = 2$  and  $r(\{a', d, g\}) = 3$ .

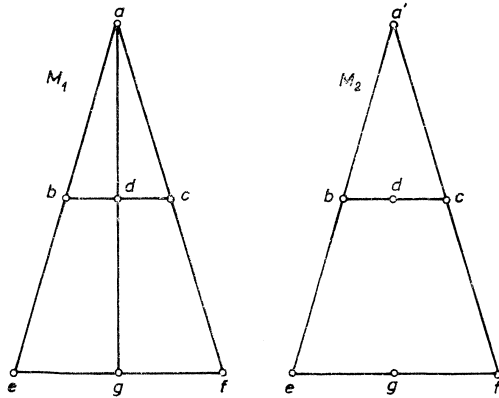


Fig. 2

**Example 2.** Let  $M_1(\{a, b, c, x_1, x_2, x_3, x_4\})$  and  $M_2(\{a', b', c', x_1, x_2, x_3, x_4\})$  be matroids of rank 3 given by Fig. 3. (The matroid  $M_1$  is the Fano matroid and  $M_2$  is the Fano matroid without the line  $\{a', b', c'\}$ .) Clearly  $M_1 \upharpoonright \{x_1, x_2, x_3, x_4\} = M_2 \upharpoonright \{x_1, x_2, x_3, x_4\} = U_4^3$  (uniform matroid). In any amalgam of  $M_1$  and  $M_2$

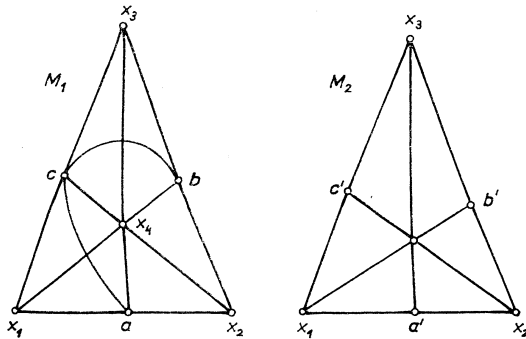


Fig. 3

with a rank function  $r$  the identities  $r(a, a') = r(b, b') = r(c, c') = 1$  must hold, whereas  $r(a, b, c) = 2$  and  $r(a', b', c') = 3$ .

**Example 3.** Let  $M_1(\{1, 2, \dots, 8\})$  be the matroid  $V_5$  and let  $M_2(\{3, 4, 5, 6, 9, 10, 11, 12\})$  be the Vamos matroid  $V_4$  (see Fig. 4).

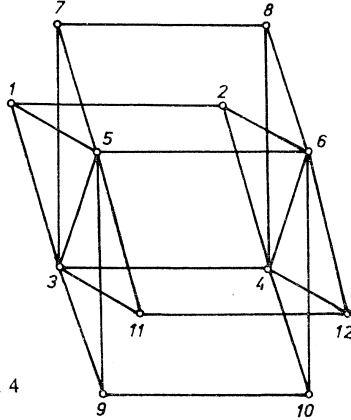


Fig. 4

Clearly  $M_1 \upharpoonright \{3, 4, 5, 6\} = M_2 \upharpoonright \{3, 4, 5, 6\} = U_4^3$ , and moreover,  $\{3, 4, 5, 6\}$  is a flat of both  $M_1$  and  $M_2$ . We prove that there is no amalgam of  $M_1$  and  $M_2$ . Assume that  $r$  is the rank function of an amalgam of  $M_1$  and  $M_2$ . Denote  $a = \{1, 2\}$ ,  $b = \{7, 8\}$ ,  $c = \{9, 10\}$ ,  $d = \{11, 12\}$ ,  $e = \{3, 4, 5, 6\}$ ,  $X = \{1, 2, \dots, 12\}$ . We distinguish three cases and show in each of them that  $r(X) \leq 4$ , which contradicts  $r(M_1) = 5$ .

Case 1.  $(r(a \cup c) = 4 \vee r(a \cup d) = 4) \& (r(b \cup c) = 4 \vee r(b \cup d) = 4)$ . Suppose that e.g.  $r(a \cup c) = 4$  and  $r(b \cup c) = 4$ . It follows from Lemma that  $\overline{a \cup c} \supset \overline{a \cup c \cup e} \supset \overline{a \cup c \cup e \cup d}$  and  $\overline{b \cup d} \supset \overline{b \cup d \cup e} \supset \overline{b \cup d \cup e \cup c}$ . Hence  $r(X) \leq r(a \cup c) + r(b \cup d) - r(\overline{a \cup c} \cap \overline{b \cup d}) \leq 4 + 4 - r(e \cup c \cup d) = 4$ .

Case 2.  $((r(a \cup c) = 4 \vee r(a \cup d) = 4) \& (r(b \cup c) \leq 3 \& r(b \cup d) \leq 3)) \vee ((r(a \cup c) \leq 3 \& r(a \cup d) \leq 3) \& (r(b \cup c) = 4 \vee r(b \cup d) = 4))$ . Suppose that e.g.  $r(a \cup c) = 4 \& r(b \cup c) \leq 3 \& r(b \cup d) \leq 3$ . Then  $r(a \cup b \cup c) \leq r(a \cup b) + r(b \cup c) - r(b) = 4$ . But it follows from Lemma that  $\overline{a \cup b \cup c} \supset \overline{a \cup b \cup c \cup e} \supset X$ .

Case 3.  $r(a \cup c) \leq 3 \& r(a \cup d) \leq 3 \& r(b \cup c) \leq 3 \& r(b \cup d) \leq 3$ . Then  $r(a \cup c \cup d) \leq r(a \cup c) + r(a \cup d) - r(a) \leq 4$  and  $r(b \cup c \cup d) \leq r(b \cup c) + r(b \cup d) - r(b) \leq 4$ . This gives  $r(a \cup b \cup c \cup d) \leq r(a \cup d \cup c) + r(b \cup c \cup d) - r(c \cup d) \leq 4$ . But it follows from the definition of  $M_2$  that  $\overline{a \cup b \cup c \cup d} \supset X$ . ■

SECTION 3

Throughout this section let  $M_1(X_1), M_2(X_2)$  be matroids and  $M(X_1 \cup X_2)$  their amalgam.

**Proposition 1.** *Let  $A \subset X_1 \cup X_2$ . Then  $M \upharpoonright A$  is an amalgam of  $M_1 \upharpoonright A$  and  $M_2 \upharpoonright A$ . ■*

**Proposition 2.** *Let  $Y \subset X_1 \cap X_2$ . Then  $M \cdot ((X_1 \cup X_2) - Y)$  is an amalgam of  $M_1 \cdot (X_1 - Y)$  and  $M_2 \cdot (X_2 - Y)$ .*

*Proof.* Set  $A = (X_1 \cup X_2) - Y$ . We show  $(M \cdot A) \upharpoonright (X_i - Y) = M_i \cdot (X_i - Y)$  for  $i = 1, 2$ . Let  $K \subset X_i - Y$ . Then

- $K$  is independent in  $(M \cdot A) \upharpoonright (X_i - Y)$  iff
- $K$  is independent in  $M \cdot A$  iff
- $K$  is independent in  $M_i \cdot (X_i - Y)$ . ■

If  $M$  is a matroid with the rank function  $r$  and  $k$  is an integer, then  $M^k$  is a matroid with the rank function  $\min(r, k)$ .

**Proposition 3.**  *$M^k$  is an amalgam of  $M_1^k$  and  $M_2^k$  for any integer  $k$ . ■*

If  $M(X)$  is a matroid and  $f: X \rightarrow Y$  is a mapping, then  $f(M)$  is a matroid on the set  $Y$  such that  $B \subset Y$  is independent in  $f(M)$  iff there is an  $A \subset X$  independent in  $M$  and  $f(A) = B$ .

**Proposition 4.** *Let  $f: X_1 \cup X_2 \rightarrow Y$  be a mapping such that  $f(X_1 \cup X_2) = Y$  and let the sets  $f(X_1 \cap X_2)$ ,  $f(X_1 - X_2)$  and  $f(X_2 - X_1)$  be pairwise disjoint. Then the matroid  $f(M)$  is an amalgam of matroids  $f(M_1)$  and  $f(M_2)$ .*

*Proof.* We show  $f(M) \upharpoonright f(X_i) = f(M_i)$  for  $i = 1, 2$ . Let  $B \subset f(X_i)$ , then  $B$  is independent in  $f(M) \upharpoonright f(X_i)$  if and only if there is  $A \subset X_i$ ,  $A$  independent in  $M$  and  $B = f(A)$ . This is equivalent to  $B$  being independent in  $f(M_i)$ . ■

The Dilworth truncation  $M^D$  of a matroid  $M(X)$  is the matroid on the set  $\{F \mid F \text{ is a flat of } M \text{ with rank } \geq 1\}$  given by  $\{F_1, \dots, F_t\}$  is independent in  $M^D$  if for any  $J \subset \{1, \dots, t\}$

$$r(\bigcup_{j \in J} F_j) \geq |J| + \min \{r(F_j) \mid j \in J\} - 1.$$

If  $Y \subset X$  we can identify the vertices of  $(M \upharpoonright Y)^D$  with the vertices of  $M^D$ , namely a flat  $F$  of  $M \upharpoonright Y$  is identified with the flat  $\bar{F}$  of  $M$ . From this point of view the matroid  $(M \upharpoonright Y)^D$  is a restriction of  $M^D$  to the set  $\{F \mid F \text{ is a flat of } M \upharpoonright Y, \text{ rank } F \geq 1\}$ . These considerations immediately yield

**Proposition 5.** *Set  $\mathcal{A} = \{F \mid F \text{ is a flat of } M_1 \text{ or } M_2\}$ . Then  $M^D \upharpoonright \mathcal{A}$  is an amalgam of  $M_1^D$  and  $M_2^D$ . ■*

SECTION 4

A matroid  $M(X)$  is called an *extension of a matroid*  $N(Y)$ ,  $Y \subset X$  if  $M \upharpoonright Y = N$ .

We define a matroid  $M(X)$  to be *sticky* if there exists an amalgam of any two extensions  $M_1(X_1)$  and  $M_2(X_2)$  of  $M$ , with  $X_1 \cap X_2 = X$ .

We define a matroid  $M(X)$  to be *f-sticky* if there exists an amalgam of any two extensions  $M_1(X_1)$ ,  $M_2(X_2)$  of  $M$  such that  $X$  is a flat of both  $M_1$  and  $M_2$ , and  $X_1 \cap X_2 = X$ .

In [11] we conjectured that  $M$  is sticky if and only if it is modular, and we proved (i) if  $M$  is modular then it is sticky, and (ii) if  $M$  is sticky and  $r(M) \leq 3$  then it is modular.

Clearly, (iii) if a matroid  $M$  is sticky then it is *f-sticky* as well, and (iv) if the conjecture is true then both properties are equivalent.

Let  $1 \leq k \leq n$ . A uniform matroid  $U_n^k$  is a matroid with vertices  $\{1, \dots, n\}$  and with the rank function

$$r(A) = \min \{|A|, k\} \quad \text{for } A \subset \{1, 2, \dots, n\}.$$

**Theorem.** *Let  $U_n^k$  be a uniform matroid. Then the following properties are equivalent.*

- a)  $U_n^k$  is modular,
- b)  $U_n^k$  is sticky,
- c)  $U_n^k$  is *f-sticky*,
- d)  $k \leq 2$  or  $k = n$ .

*Proof.* Clearly (a) is equivalent to (d). The facts mentioned above give (a)  $\Rightarrow$  (b) and (b)  $\Rightarrow$  (c). We show in Steps 1–5 that no uniform matroid  $U_n^k$  with  $3 \leq k < n$  is *f-sticky*. This gives (c)  $\Rightarrow$  (d).

1. Let  $Y$  be a flat of  $M(X)$  such that  $M \upharpoonright Y = U_n^k$ . We construct a one-point extension  $M'(X \cup e)$  of  $M$  such that  $M' \upharpoonright (Y \cup e) = U_{n+1}^k$  and  $Y \cup e$  is a flat of  $M'$ . The rank function  $r$  of  $M'$  is given by

$$r(A) = r_M(A), \quad \text{and}$$

$$r(A \cup e) = \begin{cases} r_M(A) & \text{if } Y \subset \bar{A}^M, \\ r_M(A) + 1 & \text{otherwise} \end{cases}$$

for every  $A \subset X$ .

2. Let  $M_1(X_1)$  and  $M_2(X_2)$  be matroids,  $X_1 \cap X_2 = Y$ ,  $M_1 \upharpoonright Y = M_2 \upharpoonright Y = U_n^k$ , let  $Y$  be a flat of both  $M_1$  and  $M_2$ . Denote by  $M'_1(X_1 \cup e)$  and  $M'_2(X_2 \cup e)$  the extensions of  $M_1$  and  $M_2$ , respectively, constructed as in Step 1. If there is no amalgam of  $M_1$  and  $M_2$  then there is no amalgam of  $M'_1$  and  $M'_2$  according to Proposition 1.

3. Let  $Y$  be a flat of  $M(X)$  such that  $M \upharpoonright Y = U_n^k$ . We construct a matroid  $M'(X \cup e)$

such that  $M' \cdot X = M$ ,  $M \mid (Y \cup e) = U_{n+1}^{k+1}$  and  $Y \cup e$  is a flat of  $M'$ . The rank function  $r$  of  $M'$  is given by

$$r(A) = \begin{cases} r_M(A) + 1 & \text{if } A \text{ is dependent in } M, \\ r_M(A) & \text{if } A \text{ is independent in } M, \end{cases}$$

$$r(A \cup e) = r_M(A) + 1$$

for every  $A \subset X$ .

4. Let  $M_1(X_1)$  and  $M_2(X_2)$  be matroids,  $X_1 \cap X_2 = Y$ , let  $M_1 \mid Y = M_2 \mid Y = U_n^k$  be a flat of both  $M_1$  and  $M_2$ . Denote by  $M'_1(X_1 \cup e)$  and  $M'_2(X_2 \cup e)$  the extensions of  $M_1$  and  $M_2$ , respectively, constructed as in Step 3. If there is no amalgam of  $M_1$  and  $M_2$ , then there is also no amalgam of  $M'_1$  and  $M'_2$  according to Proposition 2.

5. Example 3 proves Theorem for  $k = 3$ ,  $n = 4$ . Steps 2 and 4 extend the result for every  $k, n$ ,  $3 \leq k \leq n$ . ■

## SECTION 5.

**Proposition 6.** *Let  $M_1(X_1)$  and  $M_2(X_2)$  be binary matroids and  $M_1 \mid X_1 \cap X_2 = M_2 \mid X_1 \cap X_2$ . Then there is a binary matroid  $M(X_1 \cup X_2)$  which is an amalgam of  $M_1$  and  $M_2$ .*

The proof follows immediately from the unique representation of binary matroids over  $GF(2)$ . Namely, if  $v_1, \dots, v_k$  are vectors over  $GF(2)$  which form a circuit, then  $v_1 + \dots + v_k = 0$ . ■

On the other hand, it is not very difficult to find examples of matroids which are representable over a field  $F$ ,  $F \neq GF(2)$ , but no amalgam of them is representable over  $F$ . Moreover, Example 1 shows matroids which are representable over any field with more than three elements but no amalgam of which exists.

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