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Ambiguity and Decision Modeling: A Preference-Based Approach

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Abstract

In this article, we develop a model that permits a decision maker's preferences to depend on the decision maker's ambiguity about the probability of an event that is relevant for decision-making purposes. We deal with ambiguity through preference modeling, with ambiguity leading to modifications in the utilities of outcomes. The behavior of ambiguity premiums and probability premiums as the payoffs are varied depends on the nature of the modifications in utilities. Particular forms of the model that arise under different sets of assumptions about preferences include additive, bilinear, and ratio forms. We conclude with a brief example and some thoughts about potential generalizations and implications of the model.

Distinctions between cases in which probabilities are "known" and cases in which we feel ambiguous or vague about probabilities date back at least to Knight (1921), with his risk vs. uncertainty dichotomy. Such distinctions have no role in normative decision theories based on the Savage (1954) and de Finetti (1937) approaches to subjective probability. Indeed, such distinctions suggest that "true" probabilities for events exist, a notion that de Finetti denies in no uncertain terms with his statement that "probability does not exist" (de Finetti, 1974, p. x). However, the famous Ellsberg paradox (Ellsberg, 1961) demonstrates empirically that ambiguity does matter to many individuals. It appears, at least descriptively, that ambiguity about probabilities can affect decision-making behavior.

One version of the Ellsberg paradox involves two urns. Urn I (the ambiguous urn) contains 100 balls, each of which is either red or black, but you do not know the proportion of each color; urn II (the unambiguous urn) contains exactly 50 red and 50 black balls. Betting on Red_I means that a ball will be drawn at random from urn I and you will win a prize if the ball is red and nothing if it is black; Black_I, Red_II, and Black_II are

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defined similarly. Most people are indifferent between Red_{I} and Black_{I} and likewise for Red_{II} and Black_{II}. In choosing between Red_{I} and Red_{II}, however, many take Red_{II}, preferring to deal with the unambiguous urn.

Under the standard expected utility model, the expected utility of a bet on event E (e.g., Red_{I}) is simply $pu(\text{prize}) + (1 - p)u(\text{no prize})$, where p is the individual's subjective probability that E will occur and u represents the individual's utility function. Indifference between Red_{I} and Black_{I} implies that each color has probability one half of occurring in a draw from urn I, and similarly for urn II. But then Red_{I} and Red_{II} both have probability one half, and the model prescribes indifference between urns.

We distinguish between two approaches that might be taken in attempting to modify the standard model to allow for strict preference between ambiguous and unambiguous urns in the Ellsberg situation. One approach is to modify the probabilities in some manner to allow for ambiguity about the probabilities. We call this approach probability-based because of its focus on the probabilities or modified probabilities, which are often called decision weights and which may not obey the usual rules of probability. In contrast, we consider a preference-based approach in which the utilities are modified to allow for any preference an individual may have for or against ambiguous situations.

If probabilities and utilities are treated simply as functions without some basic underlying interpretation, then it may be a matter of taste whether ambiguity aversion is explained through modified utilities or through modified probabilities (decision weights). In a prescriptive setting, however, the distinction between the two approaches is crucial. If in a class of decision situations, the decision maker is suspicious of the probability of winning (e.g., see Bordley and Hazen, 1991; Kadane, 1992), then the appropriate model may indeed be to modify probabilities. In contrast, when a decision maker avoids ambiguity because of regret or self-blame, a utility-modification approach may be more suitable in a prescriptive analysis. In our view the two approaches are complementary, and together they permit a wider application of decision analysis in settings where it is felt that psychological attributes such as regret should be incorporated in an analysis.

It is important to note that by appropriate redefinition, probability-modification and utility-modification approaches can be made to look alike. For example, in prospect theory the decision weight function $\pi(p)$ could be labeled a utility modifier by defining modified utility to be $[\pi(p)/p]V(x)$. The modified utility is now lottery dependent, but this reformulation provides no insight or useful interpretation in a prescriptive analysis. In fact, it complicates decision analysis by destroying the separation of probabilities from utilities. The original interpretation of $\pi(p)$ as a decision weight and $V(x)$ as a value function in prospect theory is more appropriate from an intuitive perspective as well as for any prescriptive purposes.

Our objective in this article is to develop a model that permits a decision maker's preferences to depend on the ambiguity about the probability of an event that is relevant for decision-making purposes. We deal with ambiguity through preference modeling, with ambiguity leading to modifications of the utilities of outcomes. The key idea is that utilities depend not only on the payoff that is received but also on the payoff that might have been received but was not.

This article is organized as follows. In section 1 we briefly summarize some models, both descriptive and normative, that are capable of dealing with ambiguity. The basic structure of our preference-based model is presented in section 2. Particular forms of the model that arise under different sets of assumptions about preferences are developed in sections 3 to 5. An example is given in section 6, followed in section 7 by a summary and some concluding comments.

1. Approaches to modeling ambiguity

Most efforts to allow for ambiguity formally in decision-making models have focused on the probabilities. These probability-based models have generally led to modified probabilities, or decision weights, that do not necessarily obey the rules of probability. In particular, the modified probabilities are typically not additive. We distinguish between models that are descriptive in orientation and those that are normative in nature. Also, since our concern is with models for dealing with ambiguity, we do not attempt to review the large body of literature that discusses ambiguity or presents experimental results related to ambiguity but does not get into serious modeling efforts.

On the descriptive side, Einhorn and Hogarth (1985) suggest a model designed specifically to deal with ambiguity (see also Einhorn and Hogarth, 1986; Kunreuther and Hogarth, 1989; Hogarth and Einhorn, 1990). Their model involves an anchoring and adjustment process in which an individual anchors on an estimate of a probability and then adjusts by imagining other possible values of the probability. The resulting adjusted probabilities are not additive. Kahneman and Tversky (1979) do not focus on ambiguity in their development of decision weights, but they mention that ambiguity could influence decision weights, which are not additive.

Other models with modified probabilities have been developed from a normative perspective, usually by modifying the standard axioms of decision theory. For example, Schmeidler (1989) weakens the independence axiom of Anscombe and Aumann (1963) and generates a model with nonadditive probabilities. Gilboa (1987) takes a similar tack based on the Savage (1954) axioms instead of those of Anscombe and Aumann (see also Wakker, 1989; Sarin and Wakker, 1992). Segal (1987) builds on Quiggin (1982) and Yaari (1987), using anticipated utility theory and relaxing the reduction of compound lotteries to arrive at a model with probabilities that do not necessarily obey the standard rules. Fishburn (1983, 1986) also develops models with nonadditive probabilities (see Fishburn, 1988, chapter 8 for a discussion of a variety of generalizations of additive expected utility, including theories with nonadditive probabilities). In contrast, Hazen (1987) develops an axiomatized subjectively weighted linear utility model in which probabilities are allowed to depend on the consequences associated with events; the resulting "effective probabilities" do obey the standard rules (see also Hazen, 1989; Hazen and Lee, 1989). In the dual bilinear utility model of Luce and Narens (1985), the probability of an event takes on one of three possible values depending on the consequences. Finally, as noted above, Bordley and Hazen (1991) and Kadane (1992) discuss the modification of probabilities as a result of suspicion, which can be handled within the usual probability rules by appropriate conditioning.

Yet another approach is to allow indeterminate probabilities in response to ambiguity instead of requiring sharp probabilities. Nau (1990, 1992) develops a normative theory of confidence-weighted probabilities (and utilities) in this vein by relaxing completeness. Examples of earlier axiomatic models of interval-valued probabilities can be found in Smith (1961) and Walley (1982) (see also Walley, 1991). Others, such as Good (1962), have represented uncertainty about probabilities in terms of second- or higher-order probabilities.

In many decision situations, it seems more natural to model the effects of ambiguity through adjustments in utility than through adjustments in probability. Intuitively, ambiguity about the probability of an event seems to be akin to an increase in the riskiness of a course of action with consequences related to that event. Reactions to ambiguity might thus be viewed as related to attitudes toward risk, which are modeled in the expected utility approach through utilities, not probabilities. Fellner (1961) notes that ambiguity can be thought of in terms of modifications of utility as well as modifications of probability, but he does not pursue this notion. Roberts (1963, p. 332) also comments on this line of attack:

> Ellsberg's analysis of his paradox is based on the assumption that the utilities of outcomes are a function only of the monetary consequences. But this is not necessarily the assumption that the subject was making. As just one illustration (for another in the same vein, the subject may fear that Urn I might contribute to an ulcer, regardless of what a rational analysis of other aspects of the problem may suggest), the subject might feel that his choice of Red_{I} could lead to unpleasant second guesses by someone who observed the experiment: he could be criticized, however unfairly, for not taking an apparently "safe" course of action (Red_{II}) if he lost by taking an "unsafe" one (Red_{I}). An analysis of utility, formal or informal, might reveal the reason for the subject's unwillingness to pay more for Red_{II} than Red_{I}, and similarly for Black_{II} over Black_{I}.

Smith (1969, p. 325) agrees, noting that subjects

> merely violate the axioms, without the necessity for the probabilistic interpretation. . . . As I see it, it is much more plausible to say that violators in "nonstandard process" contingencies, such as the stock price example, suffer utility losses (or gains) relative to what is experienced in less controversial "standard process" contingencies, such as dice games . . . . an individual may have a low psychological tolerance for the "ambiguity" associated with nonstandard process events, which we can very reasonably and naturally describe in terms of "utility losses."

Smith goes on to develop a simple model with the utility $u(x)$ of a consequence x reduced in the presence of ambiguity by $\lambda(x)$, a "utility loss due to ambiguity." This approach is appealing intuitively, but it has some undesirable implications, such as violations of dominance. In this article, we develop a more general model with the same motivation as Smith's model. In our model, we preserve the separation between probabilities and utilities. Probabilities are quantified by "relative likelihood judgments" (see DeGroot, 1970), and utilities are based on preferences for both the consequence obtained and the regret or rejoicing due to the consequence that could have been received.

As Raiffa (1985, p. 113) notes with respect to prescriptive analysis, "The analyst can query the decision maker about what are his or her real concerns and if these cognitive concerns loom large, they can be incorporated into the analysis." To the extent that such effects are preference related, models developed in this article represent a step in the direction of improving the sophistication of decision analysis in dealing with psychological concerns.

A natural question that arises is why we do not treat consequences and regret as two attributes and simply use existing results from multiattribute utility theory to obtain the desired effects. The problem is that standard assumptions such as a marginality condition or utility independence (see Keeney and Raiffa, 1976) cannot be operationalized in our setting. This presumably was the difficultly Bell (1982, 1985) also faced in developing his models. We therefore use an indirect approach in which verifiable assumptions on the decision maker's preferences over bets lead to a model that is interpretable as a two-attribute model.

2. A preference-based model

We will work within the framework of a simple bet B that yields monetary payoff x if event E occurs and y if E does not occur. Without loss of generality, we assume that $x > y$ by defining the event associated with a higher payoff as E. Suppose that a decision maker's assessed probability for E is p. Then the expected utility of B to the decision maker is

$$U(B) = pu(x) + (1 - p)u(y), \tag{1}$$

where u represents the decision maker's von Neumann–Morgenstern (1947) utility function.

Suppose, however, that the decision maker feels ambiguous about p. Then, in our preference-based model, the utilities are replaced with "modified utilities" $v[u(x) \mid u(y)]$ and $v[u(y) \mid u(x)]$. The modified utility function v depends on the degree of ambiguity in the event probability. Notationally, this dependence of v on E is suppressed. To simplify the notation, we express the modified utilities as $v(x \mid y)$ and $v(y \mid x)$, respectively, except where the full notation is needed for clarity in proofs. The expectation of the modified utilities is

$$V(B) = pv(x \mid y) + (1 - p)v(y \mid x). \tag{2}$$

In contrast to probability-based models, the modifications due to ambiguity are reflected not in the probabilities but in the utilities. We assume throughout this article that $V(B)$ is defined as in equation (2).

The modified utilities differ from u in two important ways. First, each v term depends not only on the payoff that is received but also on the payoff that might have been received but was not. We anticipate that the concern about ambiguity is related to both potential payoffs. In particular, as $x - y$ approaches zero, $v(x \mid y)$ approaches $u(x)$ and

$v(y \mid x)$ approaches $u(y)$; when the payoffs are very close, ambiguity about the probability of E really does not matter. If x and y are quite different, on the other hand, ambiguity may be relevant to the decision maker. The decision maker, in receiving the lower payoff y, feels not just the utility $u(y)$ but also a sense of regret because of a feeling that the assessed probability p might have been too optimistic. (With a Bayesian revision using a second-order distribution on the probability of E, this notion could be formalized.) When the higher payoff x is received, there may be some sense of elation at having come through the ambiguous situation in good shape. To some degree, such reactions might be felt by decision makers in any situations, but they would seem to be heightened and especially salient when ambiguity is present.

In some ways, the notions of regret and elation as used here are related in spirit to ideas of regret and disappointment developed in Bell (1982, 1985) and Loomes and Sugden (1982). But there are important differences as well. Regret as defined by Bell and Loomes and Sugden compares what is received with what could have been received if a different action had been taken and the same event occurred, whereas we are working only with a single action at a time and comparing what is received with what could have been received with the same action if a different event occurred. The effect resulting from ambiguity seems closer in nature to Bell's disappointment, which involves comparisons of alternative payoffs under a single action.

A second way in which the modified utilities differ from u is that they will be influenced by the degree of ambiguity. For instance, as the degree of ambiguity is reduced and B approaches an unambiguous lottery, $v(x \mid y)$ and $v(y \mid x)$ will approach $u(x)$ and $u(y)$, respectively. The greater the degree of ambiguity, the greater the potential impact on v.

Because of the ambiguity about the probability of E, we must be careful about the assessment of this probability. The separation of probability elicitation from utilities is a tricky issue (see Kadane and Winkler, 1988). If ambiguity influences choices, which is the premise underlying our model, then probability assessments obtained by using procedures involving lotteries, scoring rules, or bets can be influenced by the ambiguity. We assume that p is assessed not through techniques that involve choices and payoffs, but through techniques using comparisons of events to see which one is more likely. DeGroot (1970, p. 75) recommends "making, whenever possible, a direct comparison of the relative likelihoods of two events without considering consequences which depend on their occurrence or nonoccurrence." We assume that p is assessed in this manner.

3. An additive model

The basic structure of our preference-based model is given by equation (2). The form of $V(B)$, $v(x \mid y)$, and $v(y \mid x)$ under various assumptions of interest, just as the form of single-attribute or multiattribute utility functions under certain assumptions (risk premium constant as wealth changes, additive independence, etc.) has received considerable attention. In this section we develop an additive form, and in the following sections we present assumptions that lead to multilinear and ratio forms.

Fishburn (1991) develops a theory with the ambiguity relation as a primitive. Along these lines, two events E and F belong to the same equivalence class for a decision maker if E and F are judged to be equally ambiguous. We focus on an event E, and we assume throughout this article that the equivalence class containing E has at least one other element F such that $p(E) = p(F)$. Moreover, bets involving different events in an equivalence class are assumed to be evaluated by the same modified utilities v. For example, bets for the equivalence class containing all nonambiguous lotteries are evaluated via the standard utility function u.

Some assumptions of interest involve comparisons of ambiguous lotteries with unambiguous lotteries. Therefore, we assume the existence of an external device such as a roulette wheel (see Pratt, Raiffa, and Schlaifer, 1964) that can be used to specify any unambiguous probabilities. Our development relies on comparisons of roulette wheel lotteries with bets contingent on events. Thus, the roulette wheel is employed conceptually as a measuring device. We denote the bet B yielding payoff x if event E occurs and y if E does not occur by $(x\,E\,y)$. A roulette wheel lottery that yields x with probability w and y with probability $1 - w$ is represented as (x, w, y). Finally, we use $\sim$ to denote indifference. With this notation, our first assumption is as follows:

Assumption 1. If $(x\,E\,y) \sim (x, w, y)$ for some $x > y$, then $(x'\,E\,y') \sim (x', w, y')$ for any (x', y') such that $u(x') - u(y') = u(x) - u(y)$.

To interpret assumption 1, suppose that the assessed probability of event E is p. Then $p - w$ may be interpreted as a probability premium for ambiguity, the amount of probability the decision maker is willing to give up (if $p - w$ is positive) or insists on gaining (if $p - w$ is negative) to avoid the ambiguity associated with event E in the context of bet B. In these terms, assumption 1 states that the probability premium for ambiguity associated with bet B is unchanged if the amounts x and y are changed, as long as the difference between x and y in utility terms remains constant. To obtain (x', y') such that $u(x') - u(y') = u(x) - u(y)$, we can choose any x' and then find y' such that $(x, .5, y') \sim (x', .5, y)$. In the special case of a risk-neutral decision maker, assumption 1 implies that $(x + \Delta\; E\; y + \Delta) \sim (x + \Delta, w, y + \Delta)$ for all Δ provided $(x\,E\,y) \sim (x, w, y)$.

Theorem 1. Assumption 1 implies that $v(x \mid y)$ and $v(y \mid x)$ are of the form

$$\begin{aligned} &v(x \mid y) = u(x) + f[u(x) - u(y)] \\ \text{and}\quad & \\ &v(y \mid x) = u(y) + f[u(y) - u(x)], \end{aligned} \tag{3}$$

where $f(0) = 0$.

In theorem 1, assuming the functions f and u are continuous, as $y \to x$, $v(x \mid y) \to u(x)$, and as $x \to y$, $v(y \mid x) \to u(y)$; therefore, $V(B) \to U(B)$. The function f depends on the payoffs x and y as well as the degree of ambiguity about the probability of E.

In equations (3) above, the assessment of the function f is considerably simplified if we assume it to be linear. That is,

$$f[u(x) - u(y)] = c[u(x) - u(y)]$$

and (4)

$$f[u(y) - u(x)] = d[u(y) - u(x)].$$

The following assumption requires that the probability premium $p - w$ associated with bet B be independent of the payoffs x and y. Interestingly, the linear form (4) for the regret function f implies the stronger assumption 2. This relationship is formalized in theorem 2.

Assumption 2. If $(xEy) \sim (x, w, y)$ for some $x > y$, then $(xEy) \sim (x, w, y)$ for all $x > y$.

Theorem 2. Assumption 2 implies that

$$V(B) = pu(x) + (1 - p)u(y) + [pc - (1 - p)d][u(x) - u(y)]. \quad (5)$$

An alternative way to write equation (5) provides an intuitive decision-weight interpretation:

$$V(B) = \pi(p)u(x) + [1 - \pi(p)]u(y), \quad (6)$$

where

$$\pi(p) = p + pc - (1 - p)d. \quad (7)$$

It is easy to see that the model given by equation(5) is also satisfied if equation (4) holds (i.e., the regret/elation function f is linear).

In many decision-weight models (e.g., see Einhorn and Hogarth, 1985; Schmeidler, 1989), the ambiguity premium does not depend on payoffs. In contrast, models such as those of Hazen (1987) and Hogarth and Einhorn (1990) permit dependence on payoffs of ambiguity premiums. Allowing dependence on the payoffs seems reasonable. For instance, even if ambiguity matters to a decision maker when x is very favorable and y is very unfavorable, we would be surprised if it would matter as much when we let $x - y$ approach zero. Forcing the attitude toward ambiguity to be the same as long as the difference between x and y in utility terms remains constant seems much less restrictive.

If one thinks of $f[u(x) - u(y)]$ and $f[u(y) - u(x)]$ as measuring some type of "regret," then the attributes "utility of payoff received" and "regret" are additive in the models developed in this section. In the next section we examine forms for $v(x \mid y)$ and $v(y \mid x)$ that permit interaction between these two attributes.

4. A bilinear model

Here we present assumptions that lead to bilinear forms, which provide greater flexibility in accommodating preferences in decision-making situations with ambiguity present. For simplicity in exposition, we assume throughout this section that $u(x) = x$. We also introduce exchanges between bets to reflect strength of preference (see Dyer and Sarin, 1979). Further, the strength of preference function and the utility function are assumed to be identical (see Sarin, 1982). Loosely speaking, we may think of an indifference between exchanging B for B' and B'' for B''' as being willing to pay the same price to switch from B to B' as to switch from B'' to B'''.

Assumption 3. For any x, y, x', and y' such that $x - y = x' - y'$ and for any Δ, an exchange of $(x' E y')$ for $(x E y)$ is indifferent to an exchange of $(x' + \Delta E y' + \Delta)$ for $(x + \Delta E y + \Delta)$.

Theorem 3. Assumption 3 implies that $v(x \mid y)$ and $v(y \mid x)$ in equation (2) are of the form

$$v(x \mid y) = x[1 + g(x - y)] + f(x - y)$$
and
$$v(y \mid x) = y[1 + g(y - x)] + f(y - x), \quad (8)$$

where $f(0) = g(0) = 0$.

Theorem 4. If Assumption 3 is satisfied for *any* (x', y') (not requiring $x - y = x' - y'$), then $v(x \mid y)$ and $v(y \mid x)$ are given by equations (3).

Thus, assumption 3 leads to a general bilinear form. Furthermore, with a stronger version of assumption 3, we obtain equations (3) (with the linear utility assumed in this section) as a special case of equations (8). Note that in equations (8), there are two free functions f and g. In order to obtain the more conventional bilinear form, we impose an additional assumption.

Assumption 4. The preference ordering between exchanges of $(x E y)$ for $(x E y')$ and $(x E y'')$ for $(x E y''')$ does not depend on x.

Theorem 5. Assumptions 3 and 4 imply that

$$v(x \mid y) = x + f(x - y) + kxf(x - y)$$
and
$$v(y \mid x) = y + f(y - x) + kyf(y - x), \quad (9)$$

where $f(0) = 0$.

If we further strengthen assumption 4, then we obtain a linear f in equations (9).

Assumption 5. For any x, y, and y' and any Δ, an exchange of $(x\,E\,y)$ for $(x\,E\,y')$ is indifferent to an exchange of $(x\,E\,y + \Delta)$ for $(x\,E\,y' + \Delta)$.

Theorem 6. Assumptions 3 and 5 imply the form of equations (9) with f linear.

The results of this section permit the dependence of the "probability equivalent" w of an ambiguous probability p of an event E on payoffs and "regret" so that the two attributes are nonadditive. Intuitively, this seems to be a useful generalization of the additive model. Of course, whether such nonadditivity is useful in predicting an individual's behavior toward ambiguity is an open empirical question.

5. A ratio form

The key strategy employed in obtaining several of the results in sections 3 and 4 is to make some assumptions about the invariance of the probability equivalent w with respect to *additions* in payoffs. Now, suppose the payoffs are *multiplied* by a constant and we assume invariance of w with respect to the multiplicative transformation of payoffs.

Assumption 6. If $(x\,E\,y) \sim (x, w, y)$, then $(x'\,E\,y') \sim (x'\,w, y')$ for any (x', y') such that $u(x')/u(y') = u(x)/u(y)$, where u is strictly positive.

Theorem 7. Assumption 6 implies that $v(x \mid y)$ and $v(y \mid x)$ in equation (2) are of the form

$$\begin{aligned} &v(x \mid y) = u(x)f[u(x)/u(y)] \\ \text{and} \\ &v(y \mid x) = u(y)f[u(y)/u(x)], \end{aligned} \tag{10}$$

where $f(1) = 1$.

We emphasize that assumption 6 is particularly strong in that it requires that u be scaled so that it is strictly positive and that ratio comparisons of utilities can be made. We present the ratio form to note that behaviorally, "regret" is governed by the difference $u(x) - u(y)$ in the additive form and by the ratio $u(x)/u(y)$ in equations (10).

6. An example

Since the Ellsberg paradox has stimulated much of the interest in decision making under ambiguity and should be familiar to many readers, our example involves the version of the Ellsberg situation described in this article's opening paragraphs. In this situation, $y = 0$, and we assume that red and black are judged to be equally likely in the ambiguous urn. Without loss of generality, suppose that the prize x is contingent upon the ball being red. The choice of the ambiguous urn can therefore be expressed as a bet $(x\,E\,0)$, where $E =$ "the ball drawn is red" and $p = 0.5$. The unambiguous urn can be viewed as a lottery with $w = 0.5$: $(x, 0.5, 0)$.

To isolate ambiguity concerns as opposed to standard risk concerns, we assume that $u(x) = x$. Moreover, suppose that the decision maker's attitude toward ambiguity is such that assumption 2 is applicable. Using equation (1), we have

$$U(\text{Unambiguous Urn}) = 0.5x + 0.5(0) = 0.5x.$$

From theorem 2,

$$V(\text{Ambiguous Urn}) = 0.5x + 0.5(c - d)x.$$

The ambiguity premium for the ambiguous urn is

$$U(\text{Unambiguous Urn}) - V(\text{Ambiguous Urn}) = -0.5(c - d)x,$$

and the probability premium is $0.5 - w$, where w is the solution to $0.5x + 0.5(c - d)x = wx$. This solution is $w = 0.5(1 + c - d)$, and the probability premium associated with the ambiguous urn is therefore $-0.5(c - d)$. If $c - d < 0$, then ambiguity-averse behavior consistent with the Ellsberg paradox is implied by the model. Ambiguity-preferring behavior is suggested if $c - d > 0$, and $c - d = 0$ implies ambiguity-neutral behavior.

To investigate the implications of a bilinear model in this situation, suppose that assumptions 3 and 4 apply instead of assumption 2. From theorem 5, then, $v(x \mid y)$ and $v(y \mid x)$ are given by equations (9), and

$$\begin{aligned} V(\text{Ambiguous Urn}) &= 0.5[x + f(x) + kxf(x)] + 0.5[0 + f(-x) + k(0)f(-x)] \\ &= 0.5[x + f(x) + f(-x) + kxf(x)]. \end{aligned}$$

Suppose further that

$$f(x - y) = 1 - e^{-0.001(x-y)}$$

and

$$f(y - x) = -1 + e^{-0.001(x-y)}.$$

Then

$$V(\text{Ambiguous Urn}) = 0.5[x + kx(1 - e^{-0.001x})].$$

The ambiguity premium for the ambiguous urn when compared with the unambiguous urn for this risk-neutral decision maker is

$$U(\text{Unambiguous Urn}) - V(\text{Ambiguous Urn}) = -0.5kx(1 - e^{-0.001x}).$$

The probability premium associated with the unambiguous urn can be found by equating V(Ambiguous Urn) with wx, the expected utility of $(x, w, 0)$, and solving for w, in which case the probability premium is $0.5 - w$. In our example,

$$\text{probability premium} = -0.5k(1 - e^{-0.001x}).$$

Plots of the ambiguity premium and probability premium as a function of x for selected values of k are shown in Figures 1 and 2, respectively. Note that for a given k, both the ambiguity premium and probability premium start at zero when $x = 0$ and shift

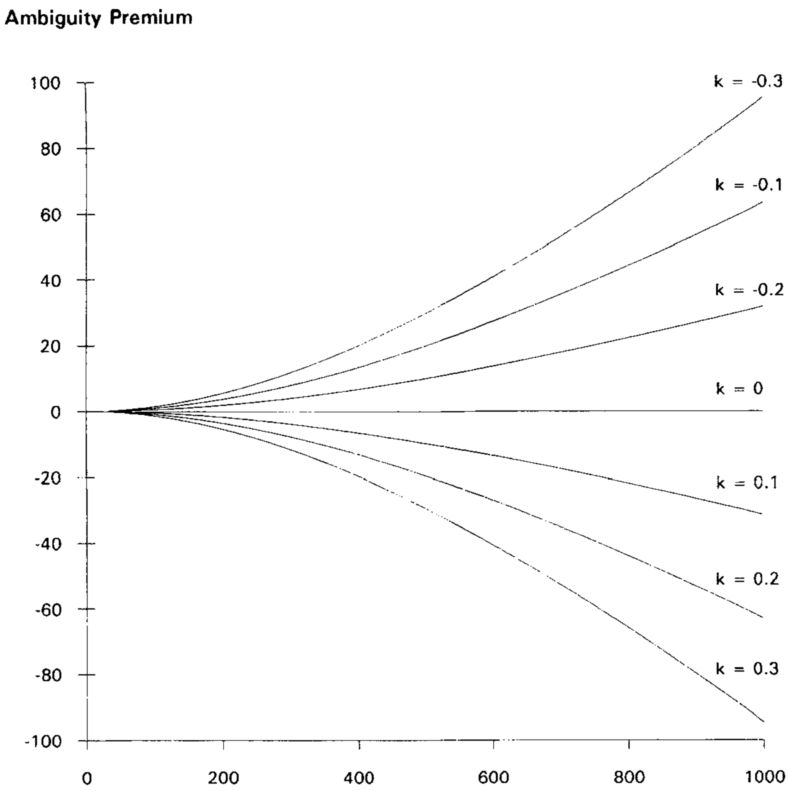

Figure 1.

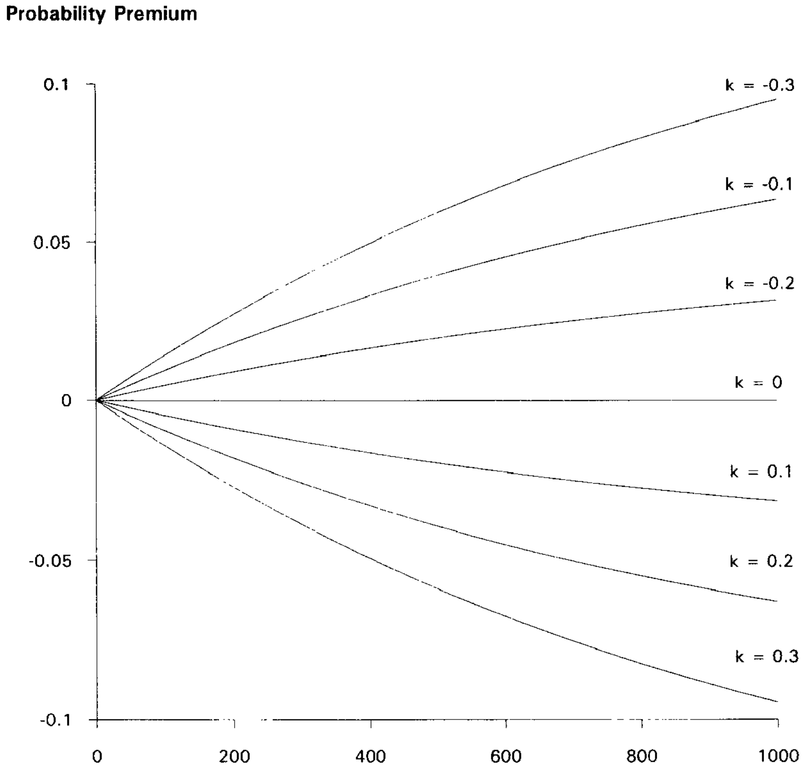

Figure 2.

monotonically as x increases, approaching $-0.5kx$ and $-0.5k$, respectively, as $x \to \infty$. The signs of both the ambiguity premium and the probability premium are the opposite of the sign of k. A negative k implies ambiguity-averse behavior in the example, whereas a positive k implies ambiguity-preferring behavior.

We speculate that the increasing magnitude (in a positive sense for ambiguity-averse decision makers and in a negative sense for ambiguity-preferring decision makers) of the ambiguity premium and the probability premium as x increases is reasonable for many decision makers. Some preliminary support for this conjecture is provided in Hogarth and Einhorn (1990). This question should be further investigated experimentally from a descriptive standpoint. In any event, different forms for the function f would lead to different shapes for the ambiguity premium and probability premium as functions of x. As we saw above, under the very strong assumption 2 the ambiguity premium is linear in x and the probability premium associated with the ambiguous urn does not depend on x. Moreover, different assumptions would lead to different forms for $v(x \mid y)$ and $v(y \mid x)$ and hence for V(Ambiguous Urn).

7. Summary

The general model developed in this article permits a decision maker's preferences to depend on the ambiguity about the probability of an event that is relevant for decision-making purposes. The idea behind our approach is that the influence, if any, of ambiguity on an individual's decisions often has its basis in the individual's preferences. For example, feelings of anxiety and discomfort that might be associated with ambiguous situations are, in a very real sense, part of the overall set of consequences. Therefore, we model ambiguous decision-making situations through modifications in the utilities rather than through modifications in the decision maker's probabilities.

Our model allows the ambiguity premium and the probability premium associated with ambiguity to depend on the payoffs. The behavior of the ambiguity and probability premiums as the payoffs are varied depends on the nature of the modifications in utilities. Particular forms of the model that arise under different sets of assumptions about preferences are developed in sections 3 to 5, and an example with ambiguity and probability premiums increasing in magnitude as the stakes increase in the Ellsberg situation is shown in section 6.

The primary contribution of this article is to argue that ambiguity effects are frequently preference related and to present a model that allows such effects to be expressed formally through the modeling of preferences. Like many probability-based approaches to decision making under ambiguity, our model represents a first step in the sense of dealing with simple (e.g., single-event) situations and requires further generalization to be valuable as a practical tool in the decision analyst's kit. First, it is event specific, a restriction that might be relaxed by developing a procedure to measure the ambiguity felt by a person about a probability in an uncertain situation. Modifications in utilities might then be modeled as a function of the degree of ambiguity. Special cases that index modifications to values of the probability p might also be useful. Second, it deals only with single-probability situations, where the uncertainty is about an event and its complement. Extensions to multiple-event situations would greatly enlarge the potential scope of applicability of the model. Third, any hope of dealing with ambiguity in dynamic decision making rests on procedures to deal with the propagation of ambiguity in sequential events, something more complicated than ambiguity in individual single-event or multiple-event situations.

Our model raises a variety of descriptive questions that could be studied empirically or experimentally. In the spirit of Camerer (1988), the ability of our model to explain actual choices can be investigated. Assumptions such as those presented in sections 3 to 5 are empirically testable. Such descriptive work would not only provide information about the descriptive validity of our model and potential assumptions that lead to special cases, but would also shed more light on the nature of decision makers' reactions to ambiguity.

With respect to normative considerations, we caution that modifications of utilities should certainly not be taken lightly. Further work is needed on the specific impact of ambiguity on decision makers in realistic situations (as opposed to the artificial Ellsberg situation, for example). But to the extent that ambiguity has tangible effects (e.g., anxiety and sleeplessness for ambiguity avoiders, thrills and excitement for ambiguity lovers), it may be reasonable for models of decision making under ambiguity to attempt to allow for these effects. Furthermore, since the effects are often preference related, one logical strategy for considering them in modeling is through the preference side of decision modeling. Trying to incorporate a feature like ambiguity in decision analysis raises some difficult questions (see Winkler, 1991), much as the consideration of equity does in decisions involving multiple actors (e.g., Fishburn, 1984; Keeney and Winkler, 1985; Sarin, 1985). There seem to be compelling reasons for taking account of equity, and although the case is not as compelling with ambiguity, evidence suggests that some individuals would like to incorporate ambiguity considerations in some manner in normative/prescriptive models of their decision-making problems. Our model has some limitations (e.g., single-event, single-stage situations), but it represents a step in this direction.

Appendix

Proof of theorem 1. Suppose that $(x\,E\,y) \sim (x, w, y)$. Then, from equation (2),

$$pv[u(x) \mid u(y)] + (1 - p)v[u(y) \mid u(x)] = wu(x) + (1 - w)u(y). \tag{A1}$$

By assumption 1, $(x'\,E\,y') \sim (x', w, y')$. Hence, from equation (2) again,

$$pv[u(x') \mid u(y')] + (1 - p)v[u(y') \mid u(x')] = wu(x') + (1 - w)u(y'). \tag{A2}$$

Subtracting equation (A1) from equation (A2) yields

$$p(v[u(x') \mid u(y')] - v[u(x) \mid u(y)]) + (1 - p)(v[u(y') \mid u(x')] - v[u(y) \mid u(x)]) = \delta, \tag{A3}$$

where $\delta = u(x') - u(x) = u(y') - u(y)$ by assumption 1.

Since the equivalence class containing E has at least one other event with probability $q = p$, equation (A3) holds with q replacing p. Therefore,

$$v[u(x') \mid u(y')] - v[u(x) \mid u(y)] = \delta,$$

or

$$v[u(x) + \delta \mid u(y) + \delta] - v[u(x) \mid u(y)] = \delta.$$

From Aczel (1966, p. 231),

$$v[u(x) \mid u(y)] = u(x) + f[u(x) - u(y)].$$

Similarly, $v[u(y) \mid u(x)] = u(y) + f[u(y) - u(x)]$.

Proof of Theorem 2. By assumption 2,

$$V(B) = pu(x) + pf[u(x) - u(y)] + (1 - p)u(y) + (1 - p)f[u(y) - u(x)]$$
$$= wu(x) + (1 - w)u(y) \text{ for all } x > y.$$

Rearranging and setting $u(x) - u(y) = t$ yields $pf(t) + (1 - p)f(-t) = \alpha t$, where $\alpha = w - p$. Substituting in $V(B)$ gives the desired result.

Proof of theorem 3. Set $x' = x + d$ and $y' = y + d$. By assumption 3,

$$p[v(x + d \mid y + d) - v(x \mid y)] + (1 - p)[v(y + d \mid x + d) - v(y \mid x)]$$
$$= pv(x + d + \Delta \mid y + d + \Delta) - v(x + \Delta \mid y + \Delta)]$$
$$+ (1 - p)[v(y + d + \Delta \mid x + d + \Delta) - v(y + \Delta \mid x + \Delta).$$

By a similar argument as in the proof of theorem 1,

$$v(x + d \mid y + d) - v(x \mid y) = v(x + d + \Delta \mid y + d + \Delta) - v(x + \Delta \mid y + \Delta). \quad (A4)$$

Let $x - y = t$, and define $v(x \mid y) = v(x, t)$. Thus, equation (A4) can be rewritten as

$$v(x + d, t) - v(x, t) = v(x + d + \Delta, t) - v(x + \Delta, t). \quad (A5)$$

For a fixed t, the conditional function $v(., t)$ is linear by equation (A5). Alternatively, one may interpret the first component to be utility independent of the second. Thus,

$$\frac{v(x,t) - v(0,t)}{v(1,t) - v(0,t)} = x, \text{ where } v(1, 0) = 1 \text{ and } v(0, 0) = 0.$$

Therefore,

$$v(x, t) = x[v(1, t) - v(0, t)] + v(0, t),$$

which can be expressed as

$$v(x, t) = x[1 + g(t)] + f(t),$$

or

$$v(x \mid y) = x[1 + g(x - y)] + f(x - y),$$

as desired. Similarly,

$$v(y \mid x) = y[1 + g(y - x)] + f(y - x).$$

Proof of theorem 4. From equation (A4) in the proof of theorem 3, we obtain

$$v(x' + \Delta \mid y' + \Delta) - v(x' \mid y') = v(x + \Delta \mid y + \Delta) - v(x \mid y). \tag{A6}$$

Since equation (A6) holds for *any* (x', y'), as assumed in theorem 4,

$$v(x + \Delta \mid y + \Delta) - v(x \mid y) = h(\Delta). \tag{A7}$$

Setting $x = y$ and observing that $v(x \mid x) = x$, we obtain $h(\Delta) = \Delta$. Now, equation (A7) will yield the additive form as shown in the proof of theorem 1.

Proof of theorem 5. As shown in the proof of theorem 3, $v(x,t) = x[v(1,t) - v(0,t)] + v(0, t)$. By assumption 4, $v(x, t) - v(x, t') = v(x, t'') - v(x, t''')$ for all x. Thus, $v(x, t)$ is utility independent of x. Hence, $v(1,t) = a + bv(0,t)$, where $a = 1$ since $v(1,0) = 1$ and $v(0,0) = 0$. Thus,

$$\begin{aligned} v(x,t) &= x[1 + (b - 1)v(0,t)] + v(0,t) \\ &= x[1 + kf(t)] + f(t) \\ &= x + f(x - y) + kxf(x - y), \text{ as desired.} \end{aligned}$$

Proof of theorem 6. By assumption 5,

$$\begin{aligned} &[x + f(x - y) + kxf(x - y)] - [x + f(x - y') + kxf(x - y')] \\ &\quad = [x + f(x - y - \Delta) + kxf(x - y - \Delta)] - [x + f(x - y' - \Delta) \\ &\quad + kxf(x - y' - \Delta)]. \end{aligned}$$

Let $x - y = t$ and $x - y' = t'$; then $f(t) - f(t') = f(t - \Delta) - f(t' - \Delta)$, which implies that $f(t) = t$.

Proof of theorem 7. Assumption 6 implies that

$$pv[u(x) \mid u(y)] + (1 - p)v[u(y) \mid u(x)] = wu(x) + (1 - w)u(y),$$

and

$$pv[u(x') \mid u(y')] + (1 - p)v[u(y') \mid u(x')] = wu(x') + (1 - w)u(y').$$

Dividing the above two equations and using a similar argument as in the proof of theorem 1, we obtain

$$\frac{v[u(x') \mid u(y')]}{v[u(x) \mid u(y)]} = \frac{u(x')}{u(x)} = \frac{u(y')}{u(y)} = r.$$

Thus,

$$v[u(x') \mid u(y')] = rv[u(x) \mid u(y)],$$

or

$$v[ru(x) \mid ru(y)] = rv[u(x) \mid u(y)].$$

Now, from Aczel (1966, p. 229),

$$v[u(x) \mid u(y)] = u(x)f[u(x)/u(y)] \text{ for } u(x), u(y) > 0.$$

Similarly,

$$v[u(y) \mid u(x)] = u(y)f[u(y)/u(x)].$$

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