Ambiguity and Second-Order Belief*

Kyoungwon Seo†
University of Rochester

September 22, 2006

Abstract

Anscombe and Aumann (1963) is a classic characterization of subjective expected utility theory. This paper employs the same domain for preference and a closely related (but weaker) set of axioms to characterize preferences that use second-order beliefs (beliefs over probability measures). Such preferences are of interest because they accommodate Ellsberg-type behavior.

Keywords: ambiguity, Ellsberg Paradox, second-order belief.

*I am indebted to Larry G. Epstein for his illuminating guidance and invaluable advice. I thank D. Ahn, P. Barelli, P. Ghirardato, F. Gul, I. Kopylov, M. Marinacci, K. Nehring and P. Wakker for helpful discussions and suggestions. I also thank audiences at University of Rochester, the Canadian Economic Theory Conference and the RUD'06 Workshop on Risk, Utility and Decision. All remaining errors are mine.

†Dept. of Economics, U. Rochester, Rochester, NY 14627, kseo@troi.cc.rochester.edu.

1 Introduction

The Ellsberg (1961) Paradox has raised questions about the subjective expected utility model and has stimulated development of a number of more general theories. In one version of the paradox (Ellsberg (2001, p.151)), there is an urn known to contain 200 balls of 4 colors - R_I, B_I, R_{II} and B_{II}. R_I and R_{II} denote two different shades of red; similarly, B_I and B_{II} denote two different shades of blue. The urn is known to contain 50 R_{II} balls and 50 B_{II} balls. But the number of R_I (or B_I) balls is unknown. One ball is to be drawn from the urn. Consider the following 6 bets on the color of the ball that is drawn.

	100		50	50
	R_I	B_I	R_{II}	B_{II}
A	\$100	\$0	\$0	\$0
B	\$0	\$100	\$0	\$0
C	\$0	\$0	\$100	\$0
D	\$0	\$0	\$0	\$100
AB	\$100	\$100	\$0	\$0
CD	\$0	\$0	\$100	\$100

Bet A gives \$100 if the drawn ball is R_I and \$0 otherwise. The other bets are interpreted similarly. Many subjects rank $C \sim D \succ A \sim B$ and $AB \sim CD$. Subjective Expected Utility (SEU) cannot accommodate this behavior.

One explanation of this behavior is that the DM has in mind a second-order belief, or a probability measure on probability measures. The DM subjectively forms a belief on the proportion of the R_I balls, or the type of the urn. Klibanoff, Marinacci and Mukerji (2005) (henceforth KMM) and Ergin and Gul (2004) propose a utility representation of preference involving a second-order belief, that can accommodate the above ranking.[1]

Models of preference typically model the ranking not only of bets, but also of all other acts - an *act* over a state space S is a (measurable) function from S into the set of outcomes. In the Ellsberg case, the natural state space is

$$S_E = \{R_I, B_I, R_{II}, B_{II}\}$$

and bets are binary acts over S_E. I use the Ellsberg setting to highlight a feature of these models - the domain of preference - that distinguishes them from the model in this paper.

[1] Nau (2006) adopts a domain similar to that of Ergin and Gul. The remarks below apply also to his paper.

KMM assume two subdomains and two corresponding preferences. One subdomain consists of acts on S_E and a preference is given over this set of acts. For the other subdomain, they introduce another state space $\Delta(S_E)$, the set of all probability measures over S_E. Each probability measure over S_E corresponds to a particular number of R_I balls in the Ellsberg urn. KMM call an act over $\Delta(S_E)$ a second-order act. They assume that the preference over second-order acts is an SEU preference, which leads immediately to second-order belief.

Ergin and Gul permit issue preference. They assume two issues and their state space is a product space. In the Ellsberg context, one issue is which ball is drawn and the other is what color is each ball. The second issue determines the type of the urn and hence a probability measure over S_E. Given preference on acts over the product state space, they prove a representation involving a second-order belief.

Therefore, both KMM and Ergin-Gul assume state spaces bigger than S_E. They presume that the analyst can observe more than just the ranking of acts over the color of the drawn ball - the ranking of acts over the "type of the urn" must also be observable. Similar remarks apply to their model in general (not only Ellsbergian) settings.

The importance of the domain assumption can be illustrated in the context of an asset market. Consider a simple model where the asset price may go up (H) or go down (L). In this setting, a bet on H corresponds to buying the asset and a bet on L to selling the asset - decisions that are observed in many data sets. On the other hand, a second-order act (or a bet on the second issue) is a bet on the true nature of the market - the *probability* that the price goes up. But we do not observe bets on the true probability; that is, the payoffs of real-world securities depend on realizations of prices, and not separately on the mechanism that generates these realizations.

This paper adopts a domain consisting of lotteries over acts defined over a basic state space - which is S_E in the Ellsberg case. Arguably, this domain is closer to the set of choices involved in the Ellsberg Paradox than are the domains of KMM and Ergin-Gul. Besides, the domain in this paper is the same as that in Anscombe and Aumann (1963), one of the classic papers on SEU. Frequently, "the Anscombe-Aumann domain" is taken to be the set of all acts whose prizes are lotteries (see Kreps (1988), for example). Note, however, that in their paper, Anscombe-Aumann use the set of all lotteries over such acts.

The model in this paper, referred to as Second Order Subjective Expected Utility (SOSEU), has the following representation:[2]

$$V(P)=\int U(f)dP(f) \text{ and } U(f)=\int_{\Delta(S)} v\left(\int_S u(f)d\mu\right)dm(\mu),$$

where P is a lottery over acts, f is an act and m is a second-order belief (a probability measure on $\Delta(S)$). Degenerate lotteries can be identified with acts and thus V induces the utility function U over acts. When v is linear V collapses to Anscombe-Aumann's SEU.

SOSEU has different axiomatic foundations from SEU. In their characterization of SEU, Anscombe and Aumann assume Order, Continuity, Independence, Reversal of Order and Dominance. I drop Reversal of Order (and modify Dominance) to characterize SOSEU.

The domain in this paper makes it possible to analyze attitudes toward ambiguity and two-stage lotteries at the same time. Specifically, SOSEU has the property that if the DM reduces two-stage lotteries into one-stage lotteries in the usual way, then he does not exhibit Ellsberg-type behavior. This prediction is confirmed in the experiment by Halevy (2005). He claims that a descriptive theory of ambiguity aversion "should account - at the same time - for violation of reduction of compound *objective* lotteries."

The violation of Reduction and the recursive structure of utility in the present model bring to mind the closely related model of Kreps and Porteus (1978). They provide axiomatic foundations for recursive expected utility with *objective* temporal lotteries. Their model not only has a similar functional form to SOSEU, but also takes a similar approach - Kreps and Porteus assume Independence at each stage and relax Reduction. However, precise probabilities are not given in most real world problems. Klibanoff and Ozdenoren (2005) incorporate subjective uncertainty to characterize subjective recursive expected utility, which does not deal with ambiguity. SOSEU is also defined on a domain involving subjective uncertainty but features second-order beliefs that can accommodate Ellsbergian behavior.

The paper is organized as follows: Section 2 introduces the setup. In Section 3, Anscombe and Aumann's axioms and theorem are presented. Section 4 motivates dropping their axiom Reversal of Order, and modifying Dominance. This leads to the SOSEU representation theorem. Section 5 examines the connection between nonindifference to ambiguity and violation of reduction of two-stage lotteries. Proofs are contained in appendices.

[2] Technical details are provided later.

2 The Setup

For any topological space X, let $\Delta(X)$ be the set of all Borel probability measures on X, and let $C_b(X)$ be the set of all bounded continuous functionals on X. Endow $\Delta(X)$ with the weak convergence topology, i.e., for $\nu_n, \nu \in \Delta(X)$, $\nu_n \to \nu$ if $\int \eta d\nu_n \to \int \eta d\nu$ for every $\eta \in C_b(X)$. If X is a separable metric space, so is $\Delta(X)$. (See Aliprantis and Border (1999, p.482); these authors are henceforth AB.) Let $\mathcal{B}_X$ denote the Borel σ-algebra on X and denote by $\delta_x \in \Delta(X)$ a point mass on X, defined by $\delta_x(A) = 0$ if $x \notin A$ and $\delta_x(A) = 1$ if $x \in A$.

Let $S = \{s_1, s_2, ..., s_{|S|}\}$ be a finite set of states. Let Z denote a set of outcomes or prizes, where Z is a separable metric space. An act f is a function from S into $\Delta(Z)$. Let $\mathcal{H}$ be the set of all acts endowed with the product topology. Preference $\succeq$ is defined on $\Delta(\mathcal{H})$.

I refer to an element in $\Delta(Z)$ as a one-stage lottery and to an element in $\Delta(\Delta(Z))$ as a two-stage lottery (or a compound lottery). A constant act (an act taking the same value for every $s \in S$) is viewed also as a one-stage lottery. Moreover, any act f is identified with δ_f. Then, it is immediate that $\Delta(Z) \subset \mathcal{H} \subset \Delta(\mathcal{H})$ and hence $\Delta(\Delta(Z)) \subset \Delta(\mathcal{H})$. Therefore the preference $\succeq$ induces rankings on $\mathcal{H}$, $\Delta(Z)$ and $\Delta(\Delta(Z))$.

Typical elements in $\Delta(\mathcal{H})$ are denoted by P, Q and R. I use f, g and h for elements in $\mathcal{H}$. In addition, $\bar{P}$, $\bar{Q}$ and $\bar{R}$ are typical elements for $\Delta(\Delta(Z))$ and, p, q and r for $\Delta(Z)$. Denote by $(x_1, \alpha_1; ...; x_n, \alpha_n)$ a lottery that gives x_1 with probability α_1 and so on, where $x_1, x_2, ..., x_n$ can be outcomes, lotteries or acts.

A typical object P in $\Delta(\mathcal{H})$ is depicted in Fig. 1.

3 The Anscombe-Aumann Model

Preference having an SEU form on $\Delta(\mathcal{H})$ is characterized by Anscombe and Aumann (1963) (henceforth AA). Using the notations and definitions of this paper, AA's axioms and theorem can be restated.[3]

Axiom 1 *(Order) $\succeq$ is complete and transitive.*

[3] Actually, they didn't state the first 4 axioms - Order, Continuity, Second-Stage Independence and First-Stage Independence. Instead, they assumed expected utility functions on $\Delta(\mathcal{H})$ and $\Delta(Z)$, respectively.

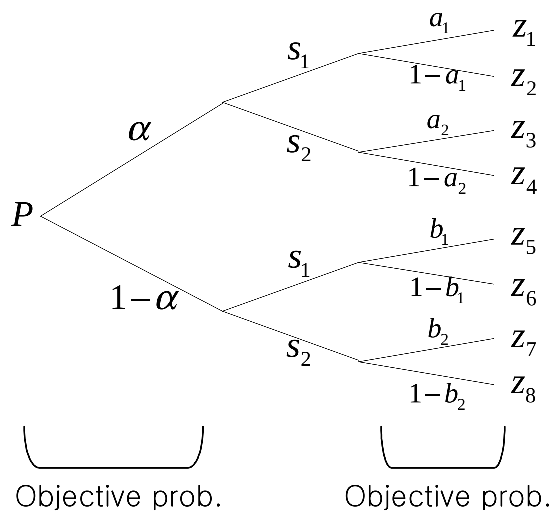

Figure 1: A typical element in $\Delta(\mathcal{H})$. The first and the last nodes are governed by objective probabilities α, a_1, a_2, b_1 and b_2. The second node is selected according to the realized state s_1 or s_2.

Axiom 2 *(Continuity)* $\succeq$ *is continuous.*

Definition 1 *For* $f, g \in \mathcal{H}$ *and* $\alpha \in [0,1]$, $\alpha f \oplus (1-\alpha)g \in \mathcal{H}$ *is a component-wise mixture, i.e., for every* $s \in S$ *and every* $B \in \mathcal{B}_Z$, $(\alpha f \oplus (1-\alpha)g)(s)(B) = \alpha f(s)(B) + (1-\alpha)g(s)(B)$. *This operation is referred to as a second-stage mixture.*

Axiom 3 *(Second-Stage Independence) For any* $\alpha \in (0,1]$ *and one-stage lotteries* $p, q, r \in \Delta(Z)$,

$$\alpha p \oplus (1-\alpha)r \succeq \alpha q \oplus (1-\alpha)r \Longleftrightarrow p \succeq q.$$

Consider two lotteries $\alpha p \oplus (1-\alpha)r$ and $\alpha q \oplus (1-\alpha)r$. Both give the same prize r with probability $(1-\alpha)$. The two lotteries differ only in the α-probability event. So it is intuitive that the DM's ranking between them depends only on the ranking between p and q, regardless of the common prize r.

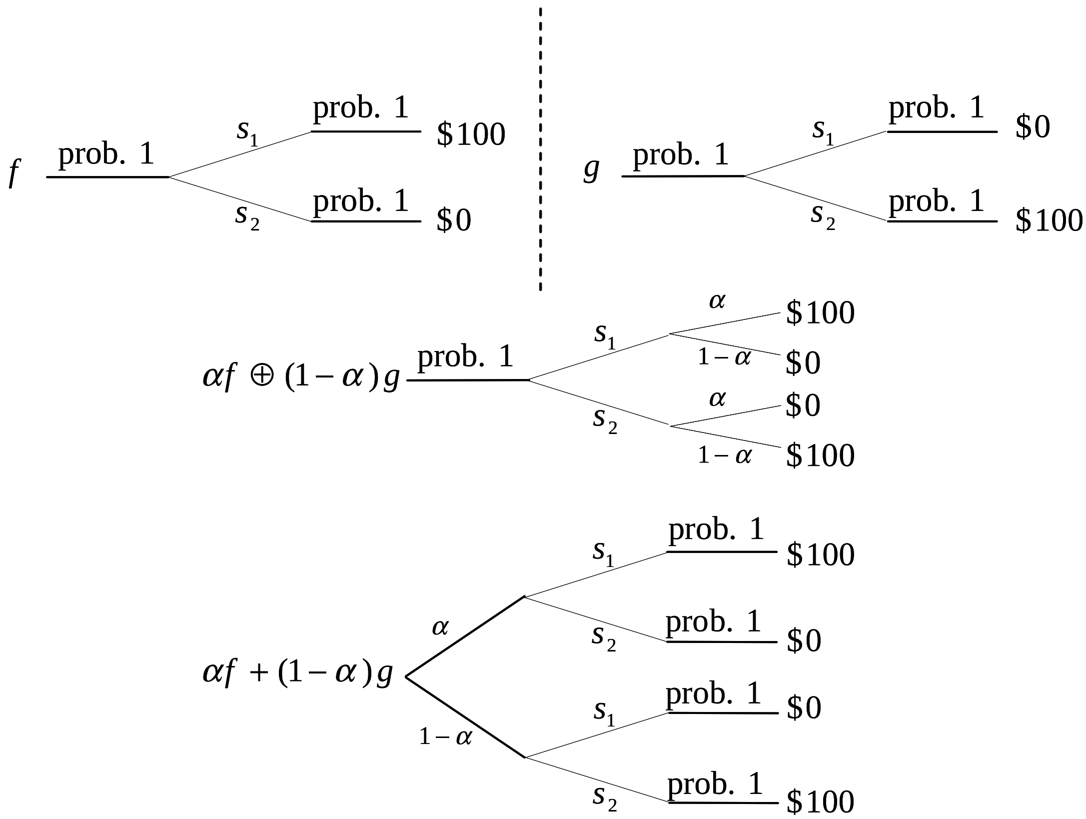

Figure 2: Examples of mixture operations: $f \in \mathcal{H}$ gives \$100 if s_1 is realized, and \$0 if s_2 is realized; $g \in \mathcal{H}$ gives \$0 for s_1 and \$100 for s_2. The second-stage mixture $\alpha f \oplus (1-\alpha) g \in \mathcal{H}$ is an act that gives the lottery $(\$100, \alpha; \$0, 1-\alpha)$ for s_1 and the lottery $(\$0, \alpha; \$100, 1-\alpha)$ for s_2. The first-stage mixture $\alpha f + (1-\alpha) g \in \Delta(\mathcal{H})$ is the lottery $(f, \alpha; g, 1-\alpha)$.

Definition 2 *For $P, Q \in \Delta(\mathcal{H})$ and $\alpha \in [0,1]$, $\alpha P + (1-\alpha) Q \in \Delta(\mathcal{H})$ is a lottery such that $(\alpha P + (1-\alpha) Q)(B) = \alpha P(B) + (1-\alpha) Q(B)$ for $B \in \mathcal{B}_{\mathcal{H}}$. This operation is called a first-stage mixture. For simplicity, I write $\alpha f + (1-\alpha)\, g$ instead of $\alpha \delta_f + (1-\alpha)\, \delta_g$ for any act f and g.*

Axiom 4 *(First-Stage Independence) For any $\alpha \in (0,1]$ and lotteries $P, Q, R \in \Delta(\mathcal{H})$,*

$$\alpha P + (1-\alpha) R \succeq \alpha Q + (1-\alpha) R \Longleftrightarrow P \succeq Q.$$

First-Stage Independence can be interpreted in a way similar to Second-Stage Independence.

Axiom 5 *(Reversal of Order) For every* $f, g \in \mathcal{H}$ *and* $\alpha \in [0,1]$, $\alpha f \oplus (1-\alpha) g \sim \alpha f + (1-\alpha) g$.

Reversal of Order assumes that the DM is not concerned about whether the mixture operation is taken before or after the realization of the state. Later, I will discuss an argument against this axiom.

Axiom 6 *(AA-Dominance) Let* $f, g \in \mathcal{H}$ *and* $s \in S$. *If* $f(s') = g(s')$ *for all* $s' \neq s$ *and* $f(s) \succeq g(s)$, *then* $f \succeq g$.

This axiom says that when two acts give the identical prizes except in one state s, the prizes in state s determine the DM's ranking between the two acts.

Definition 3 *An SEU representation is a bounded continuous mixture linear function* $u : \Delta(Z) \to \mathbb{R}$ *and a probability measure* $\mu \in \Delta(S)$ *such that* V^{AA} *represents* $\succeq$ *on* $\Delta(\mathcal{H})$, *where*

$$V^{AA}(P) = \int U^{AA}(f) dP(f) \text{ and } U^{AA}(f) = \int u(f) d\mu.$$

AA's theorem can be restated.[4]

Theorem 3.1 *(AA (1963)) Preference* $\succeq$ *on* $\Delta(\mathcal{H})$ *satisfies Order, Continuity, Second-Stage Independence, First-Stage Independence, Reversal of Order and AA-Dominance if and only if it has an SEU representation.*

An SEU representation cannot accommodate Ellsberg-type behavior. Therefore, I proceed to develop a generalization of this model.

[4]Under Reversal of Order, one of the two Independence axioms is redundant. I leave both of them to compare with the next section.

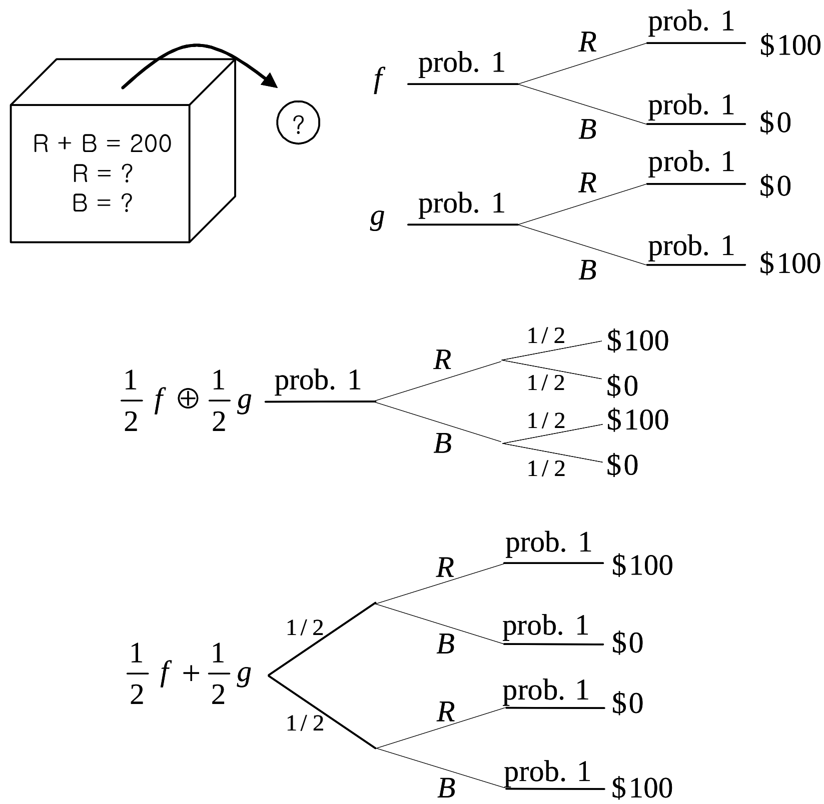

Figure 3: One ball is randomly drawn from the Ellsberg urn which contains 200 balls that are either red or blue. The exact number of red (or blue) balls is unknown. An act f is a bet on red and g is a bet on blue. The second-stage mixture $\frac{1}{2}f \oplus \frac{1}{2}g$ is unambiguous but the first-stage mixture is not.

4 Main Representation Theorem

Here I show that by dropping Reversal of Order and modifying AA-Dominance, one obtains a model of preference that can accommodate nonindifference to ambiguity.

Consider the following example that illustrates that Reversal of Order is problematic given ambiguity. In the Ellsberg example described in the Introduction, let f be the act that gives \$100 if the chosen ball is red (R_I or R_{II}), and nothing otherwise; g gives \$100 if the ball drawn is blue (B_I or B_{II}), and nothing otherwise. Let p be $(\$100, 1/2; \$0, 1/2)$. As Ellsberg predicted and later experiments confirmed, many people feel indifferent between f and g, but strictly prefer p to f and p to g.

Compare $\frac{1}{2}f+\frac{1}{2}g$ and $\frac{1}{2}f\oplus\frac{1}{2}g$ (see Figure 3). The first-stage mixture $\frac{1}{2}f+\frac{1}{2}g$ gives ambiguous acts f or g. If the DM strictly prefers p to f and p to g, it is reasonable to assume that he strictly prefers p to $\frac{1}{2}f+\frac{1}{2}g$ by the intuition of First-Stage Independence. On the other hand, the second-stage mixture $\frac{1}{2}f\oplus\frac{1}{2}g$ has no ambiguity and can be identified with p because it yields the lottery p whichever state is realized. Therefore the DM will strictly prefer $\frac{1}{2}f\oplus\frac{1}{2}g$ to $\frac{1}{2}f+\frac{1}{2}g$. Under Reversal of Order, $\frac{1}{2}f\oplus\frac{1}{2}g$ and $\frac{1}{2}f+\frac{1}{2}g$ must be indifferent. This illustrates the intuition against adopting Reversal of Order.[5]

However one may think in a different way. For any number of blue balls in the urn, the final probability of getting \$100 is 1/2 not only for $\frac{1}{2}f\oplus\frac{1}{2}g$ but also for $\frac{1}{2}f+\frac{1}{2}g$. Hence the DM may be indifferent between $\frac{1}{2}f\oplus\frac{1}{2}g$ and $\frac{1}{2}f+\frac{1}{2}g$, while preferring $\frac{1}{2}f\oplus\frac{1}{2}g$ to f and g. Implicit in this argument is that $\frac{1}{2}f+\frac{1}{2}g$ becomes a two-stage lottery when the number of blue balls is given and that the DM reduces the two-stage lottery to a one-stage lottery.[6]

The preceding argument supporting Reversal of Order is normatively appealing. But Halevy (2005) reports that most people who reduce compound lotteries are ambiguity neutral (see the next section). Since the argument to maintain Reversal of Order requires reduction, it may not be acceptable at a descriptive level. In this paper, I drop Reversal of Order and suggest a descriptive model to explain Ellsberg-type behavior.

Recall that AA-Dominance deals only with $\mathcal{H}$, not with $\Delta(\mathcal{H})$. Under Reversal of Order, stating properties on $\mathcal{H}$ is enough to describe properties on $\Delta(\mathcal{H})$. Since I drop Reversal of Order, AA-Dominance must be modified.

Each $f\in\mathcal{H}$ and $\mu\in\Delta(S)$ induce a one-stage lottery, namely $\Psi(f,\mu)\equiv\mu(s_1)f(s_1)\oplus\mu(s_2)f(s_2)\oplus\ldots\oplus\mu(s_{|S|})f(s_{|S|})\in\Delta(Z)$. For $P\in\Delta(\mathcal{H})$, define $\Psi(P,\mu)\in\Delta(\Delta(Z))$ by $\Psi(P,\mu)(B)=P(\{f\in\mathcal{H}:\Psi(f,\mu)\in B\})$ for each $B\in\mathcal{B}_{\mathcal{H}}$. See Figure 4.

Axiom 7 *(Dominance) For any $P,Q\in\Delta(\mathcal{H})$, if $\Psi(P,\mu)\succeq\Psi(Q,\mu)$ for all $\mu\in\Delta(S)$, then $P\succeq Q$.*

To interpret Dominance, consider a DM who is not certain of the true probability law over states, but who believes that there is a *true law*. Now suppose that $\Psi(P,\mu)\succeq\Psi(Q,\mu)$ for

[5] The preceding intuition translates to the present setting Gilboa and Schmeidler(1989)'s rationale for their axiom "Uncertainty Aversion", namely, that "hedging" across ambiguous states can increase utility.

[6] See Ellsberg (2001, p.230) for a similar argument by Pratt and Raiffa.

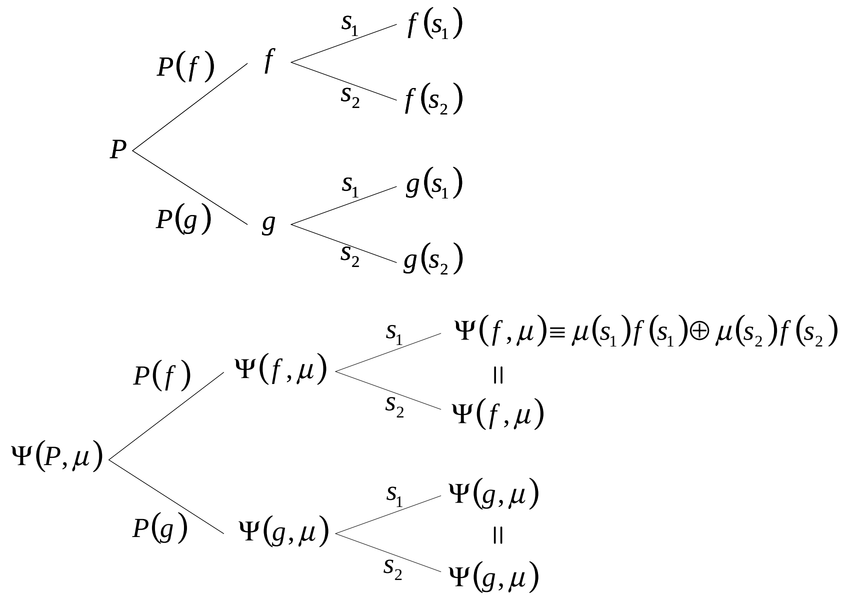

Figure 4: If μ is assumed to be the true probability law, the DM translates P to $\Psi(P,\mu)$.

every $\mu \in \Delta(S)$, that is, for *every* probability law, he prefers the two-stage lottery induced by P to the one induced by Q. Then he must prefer P to Q.[7]

It is instructive to compare Dominance with AA-Dominance. Since the latter deals only with acts in $\mathcal{H}$, restrict Dominance to degenerate lotteries f and g. Under AA-Dominance, $f \succeq g$ if $\Psi(f,\mu) \succeq \Psi(g,\mu)$ for all Dirac measures $\mu = \delta_s$, $s \in S$. Dominance posits the stronger hypothesis that $\Psi(f,\mu) \succeq \Psi(g,\mu)$ for all measures μ in $\Delta(S)$. Thus (when restricted to acts) Dominance is weaker than AA-Dominance.

A more complete and formal comparison of Dominance and AA-Dominance is provided in the next lemma.

Lemma 4.1 *(i) Order, Continuity, Reversal of Order and AA-Dominance imply Dominance.*
(ii) Dominance and Second-Stage Independence imply AA-Dominance.

The main utility representation is defined.

[7] When the probability law is given, one may interpret the object as a three-stage lottery which is out of the domain. Dominance implicitly assumes that the DM reduces the three-stage lottery to a two-stage lottery.

Definition 4 *A Second Order Subjective Expected Utility (SOSEU) representation is a probability measure* $m \in \Delta(\Delta(S))$*, a bounded continuous mixture linear function* $u : \Delta(Z) \to \mathbb{R}$*, and a bounded continuous and strictly increasing function* $v : u(\Delta(Z)) \to \mathbb{R}$*, such that* V *represents* $\succeq$ *on* $\Delta(\mathcal{H})$*, where*

$$V(P) = \int U(f)\, dP(f) \ \text{ and } U(f) = \int v\left(\int u(f)\, d\mu\right) dm(\mu).$$

The probability measure m *is called a second-order belief.*

SOSEU can accommodate nonindifference to ambiguity. When the second-order belief m is nondegenerate and v is nonlinear, the implied behavior cannot be explained by a unique (subjective) probability on S. Instead the DM behaves as though he has multiple priors on S and assigns a probability to each prior on S. SEU is the special case when v is linear.[8]

The new representation theorem follows.

Theorem 4.2 *Preference* $\succeq$ *on* $\Delta(\mathcal{H})$ *satisfies Order, Continuity, Second-Stage Independence, First-Stage Independence and Dominance if and only if it has an SOSEU representation.*

Appendix A provides a sketch of the proof and also some examples to demonstrate the tightness of the theorem. The complete proof is in Appendix B.

Lemma 4.1 suggests that Reversal of Order is the crucial difference between an SEU representation and an SOSEU representation. This is summarized in the next corollary.

Corollary 4.3 *Preference* $\succeq$ *on* $\Delta(\mathcal{H})$ *has an SEU representation if and only if it has an SOSEU representation and satisfies Reversal of Order.*

AA assume Reversal of Order. Under Reversal of Order, the DM does not care when the objective uncertainty is resolved and he collapses the two objective uncertainties into

[8] The functional form of an SOSEU representation is much similar to that of KMM(2005). Many properties of the functional form are investigated in their paper.

one objective uncertainty. Thus the above corollary says that if the DM collapses the two objective probabilities into one, he also collapses the second-order belief (on $\Delta(S)$) into the belief (on S).

Finally, consider briefly uniqueness of the representation in Theorem 4.2. It is easy to show that u and $v \circ u$ are unique up to a positive affine transformation (see Appendix C). The second-order belief m is unique in some special cases - for example, if $v(z) = \exp(z)$, the representation has a similar form to a moment generating function and m is unique. However, m is not unique in general. For example, suppose that v is linear. Then, any second-order belief that has the same first moment will show the same behavior. Similarly, a polynomial v of degree n implies that if two second-order beliefs, m and m', represent the same preference, they have the same moments up to n-th order. See Appendix C for a characterization of the uniqueness class of measures for any given u and v.

5 Ambiguity and Compound Lotteries

Here I discuss the relations between ambiguity attitude and a two-stage lottery. A two-stage lottery deals only with objective probabilities and ambiguity attitude deals with the situation where objective probabilities are unknown. The two may seem conceptually distinct, but in an SOSEU representation, they are closely related.

An axiom on compound lotteries is introduced.

Axiom 8 *(Reduction of Compound Lotteries or ROCL) For any $p, q \in \Delta(Z)$ and $\alpha \in [0,1]$, $\alpha p \oplus (1-\alpha)q \sim \alpha p + (1-\alpha)q$.*

Since p and q are one-stage lotteries, $\alpha p + (1-\alpha)q$ constitutes a two-stage lottery. Observe that $\alpha p \oplus (1-\alpha)q$ and $\alpha p + (1-\alpha)q$ have the same final outcome distribution. Thus, under ROCL, the DM considers only the final distribution and he does not care about the timing of risk resolution.

An SOSEU representation does not satisfy ROCL unless v is linear. When v is nonlinear,

$$\begin{aligned} V(\alpha p + (1-\alpha) q) &= \alpha v(u(p)) + (1-\alpha) v(u(q)) \\ &\neq v(\alpha u(p) + (1-\alpha) u(q)) \\ &= V(\alpha p \oplus (1-\alpha) q). \end{aligned}$$

Under SOSEU, the utility of any act f is given by $U(f) = \int v\left(\int u(f)d\mu\right)dm(\mu)$, which suggests the interpretation that the DM processes an act in a two-stage fashion. This suggests further a connection between the evaluations of acts and two-stage lotteries. In the following, I will show that, given other axioms, ROCL is equivalent to Reversal of Order and that ROCL implies neutrality to ambiguity.

Lemma 5.1 *ROCL and Reversal of Order are equivalent under Dominance.*

Proof. Since $\Delta(Z) \subset \mathcal{H}$, it is straightforward that Reversal of Order implies ROCL.

Conversely, assume ROCL. Then, for any $\mu \in \Delta(S)$,

$$\begin{aligned}\Psi(\alpha f + (1-\alpha) g, \mu) &= \alpha\Psi(f,\mu) + (1-\alpha)\Psi(g,\mu) \\ &\sim \alpha\Psi(f,\mu) \oplus (1-\alpha)\Psi(g,\mu) \\ &= \Psi(\alpha f \oplus (1-\alpha) g, \mu).\end{aligned}$$

Applying Dominance leads to $\alpha f + (1-\alpha) g \sim \alpha f \oplus (1-\alpha) g$. ■

Corollary 5.2 *Preference $\succeq$ has an SEU representation if and only if it has an SOSEU representation and satisfies ROCL.*

Proof. *By Lemma 5.1 and Corollary 4.3, this is straightforward.* ■

An SOSEU representation reduces to SEU if and only if ROCL is satisfied. In particular, ROCL implies neutrality to ambiguity. This is consistent with Halevy's (2005) experimental findings.

Halevy designs the following experiment. There are 3 urns, each containing 10 balls which can be red or black[9]. One ball is to be drawn. Urn 1 contains 5 red balls and 5 black balls. In Urn 2, the proportion is unknown. For Urn 3, a ticket is drawn from a bag containing 11 tickets with numbers 0 to 10 written on them. The number on the drawn ticket determines the number of red balls in Urn 3. Each participant is asked to place a bet on the color of the drawn ball from each urn. Before any ball is drawn, the participant is given the option to sell each bet. The subject is asked the minimal price at which he/she is willing to sell the bet. Let V_i be the reservation price for urn i, $i = 1, 2, 3$.

[9] In his experiment, there were 4 urns. The fourth urn is omitted here because it is not relevant to my point.

Ambiguity neutrality implies $V_1 = V_2$ and ROCL implies $V_1 = V_3$. In Halevy's experiment, 18 subjects set $V_1 = V_3$ and 17 out of them set $V_1 = V_2$. Moreover, out of 86 subjects who show $V_1 \neq V_3$, 80 show $V_1 \neq V_2$. He concludes that "there is a very tight association between ambiguity neutrality and reduction of compound lotteries" and that "a descriptive theory that accounts for ambiguity aversion should account - at the same time - for violation of reduction of compound *objective* lotteries." The domain in this paper includes both acts and two-stage lotteries, and an SOSEU representation relates ambiguity attitude to ROCL.[10]

A Appendix : Proof Sketch and Examples

This section sketches the sufficiency proof of Theorem 4.2 and provides examples to demonstrate the tightness of the theorem.

Proof sketch : First-Stage Independence implies that preference can be represented by $V(P) = \int U(f)\, dP(f)$. Let $\widehat{U}$ be the restriction of U to $\Delta(Z)$. By Second-Stage Independence, $\widehat{U} = v \circ u$ where u is a mixture linear function on $\Delta(Z)$ and v is a strictly increasing function on $u(\Delta(Z))$. The key part of the proof is to construct the second-order belief m.

It suffices to show that there exists $m \in \Delta(\Delta(S))$ satisfying

$$\int_{\Delta(S)} v \circ u \circ \Psi(f, \mu)\, dm(\mu) = U(f) \text{ for all } f \in \mathcal{H}.$$

For intuition about existence of such a measure m, consider the discretized version, $Am = b$ where

$$\begin{aligned} A &= \begin{pmatrix} v \circ u \circ \Psi(f_1, \mu_1) & \cdots & v \circ u \circ \Psi(f_1, \mu_k) \\ \vdots & \ddots & \vdots \\ v \circ u \circ \Psi(f_n, \mu_1) & \cdots & v \circ u \circ \Psi(f_n, \mu_k) \end{pmatrix} \\ m &= \begin{pmatrix} m_1 \\ \vdots \\ m_k \end{pmatrix}, b = \begin{pmatrix} U(f_1) \\ \vdots \\ U(f_n) \end{pmatrix}. \end{aligned}$$

[10] Klibanoff, Marinacci and Mukerji(2005) deal with acts but not with compound lotteries. Segal's (1990) model has two-stage lotteries but no acts.

By Farkas' Lemma, $Am = b$ has a nonnegative solution m if and only if, for all $y \in \mathbb{R}^n$,

$$\begin{aligned} & A^T y \geq 0 \\ \Longrightarrow \quad & b^T y \geq 0 \, . \end{aligned}$$

By the infinite dimensional version of Farkas' Lemma (see Theorem B.1), it suffices to show that, for all $t' \in ca\,(\mathcal{H})$,

$$\int v \circ u \circ \Psi\,(f, \mu)\, dt'\,(f) \geq 0 \text{ for all } \mu \in \Delta(S) \tag{A.1}$$

$$\Longrightarrow \quad \int U\,(f)\, dt' \geq 0 \, . \tag{A.2}$$

Finally, show that under Dominance, (A.1) implies (A.2). Note that t' can be decomposed into $\alpha P - \beta Q$ where $P, Q \in \Delta\,(\mathcal{H})$ are in the domain of objects of choice and $\alpha, \beta \geq 0$. Then, rearranging (A.1) gives

$$\alpha \int (v \circ u)\, d\Psi\,(P, \mu) \geq \beta \int (v \circ u)\, d\Psi\,(Q, \mu) \; \text{ for all } \mu \in \Delta(S). \tag{A.3}$$

Normalize U such that $\int U d\bar{R} = 0$ for some $\bar{R} \in \Delta(\Delta(Z))$. Consider the case $\alpha > \beta \geq 0$. Other cases can be proved similarly. Recall that $\bar{P} \longmapsto \int (v \circ u)\, d\bar{P}$ represents preference on $\Delta(\Delta(Z))$. Then, (A.3) implies

$$\begin{aligned} \Psi\,(P, \mu) \; & \succeq \; \frac{\beta}{\alpha} \Psi\,(Q, \mu) + \left(1 - \frac{\beta}{\alpha}\right) \Psi\left(\bar{R}, \mu\right) \\ & = \; \Psi\left(\frac{\beta}{\alpha} Q + \left(1 - \frac{\beta}{\alpha}\right) \bar{R}, \mu\right) \; \text{ for all } \mu \in \Delta(S). \end{aligned}$$

Now apply Dominance to get

$$P \succeq \frac{\beta}{\alpha} Q + \left(1 - \frac{\beta}{\alpha}\right) \bar{R}.$$

Since $V\,(P) = \int U\,(f)\, dP\,(f)$ represents preference, (A.2) follows and thus a second-order belief m exists.

Examples for the tightness of Theorem 4.2 : Each example satisfies all but one of the axioms characterizing an SOSEU representation.

Example 1 *All but Second-Stage Independence : let*

$$V(P) = \int \left(\int u(f) d\mu \right) dP(f)$$

for some fixed $\mu \in \Delta(S)$ *and a bounded continuous but non-mixture-linear* $u : \Delta(Z) \to \mathbb{R}$.

Example 2 *All but First-Stage Independence : let*

$$V(P) = \min_{\mu \in C} \int \left(\int u(f) \, d\mu \right) dP(f)$$

where u *is bounded, continuous, mixture-linear and* $C \subset \Delta(S)$ *is a closed subset. To show Dominance, note that*

$$\begin{aligned}
& \Psi(P, \mu) \succeq \Psi(Q, \mu) \text{ for all } \mu \in \Delta(S) \\
\Longrightarrow \quad & \int \left(\int u(f) \, d\mu \right) dP(f) \geq \int \left(\int u(f) \, d\mu \right) dQ(f) \text{ for all } \mu \in \Delta(S) \\
\Longrightarrow \quad & \min_{\mu \in C} \int \left(\int u(f) \, d\mu \right) dP(f) \geq \min_{\mu \in C} \int \left(\int u(f) \, d\mu \right) dQ(f) \, .
\end{aligned}$$

Example 3 *All but Dominance : modify Example 2 by taking*

$$V(P) = \int \left(\min_{\mu \in C} \int u(f) \, d\mu \right) dP(f).$$

This violates (only) Dominance. Let $S = \{1, 2\}$, $P = \frac{1}{2} f + \frac{1}{2} g$, $Q = \delta_h$, $u(f(1)) = 1$, $u(f(2)) = 2$, $u(g(1)) = 1$, $u(g(2)) = 0$, $u(h(1)) = 1$, $u(h(2)) = 1$ *and* $C = \Delta(\Delta(S))$. *Then,* $V(\Psi(P, \mu)) = 1 = V(\Psi(Q, \mu))$ *for all* $\mu \in \Delta(S)$ *but* $V(P) = 1/2 < 1 = V(Q)$.

B Appendix : Proofs

B.1 Preliminaries

Notations and definitions follow AB (1999) and Craven and Koliha (1977).

For any real vector space $\mathcal{M}$, let $\mathcal{M}^{\#}$ be the algebraic dual of $\mathcal{M}$, i.e., the set of all linear functionals on $\mathcal{M}$. Denote by $\langle m, m^{\#} \rangle$ an evaluation of $m^{\#} \in \mathcal{M}^{\#}$ at the point $m \in \mathcal{M}$. Suppose that $A : \mathcal{M} \to \mathcal{T}$ is a linear map between two vector spaces $\mathcal{M}$ and $\mathcal{T}$. The algebraic adjoint $A^{\#} : \mathcal{T}^{\#} \to \mathcal{M}^{\#}$ of A is the linear map satisfying

$$\langle m, A^{\#} t^{\#} \rangle = \langle Am, t^{\#} \rangle \text{ for all } m \in \mathcal{M} \text{ and } t^{\#} \in \mathcal{T}^{\#}.$$

A dual pair is a pair $\langle \mathcal{M}, \mathcal{M}' \rangle$ of two vector spaces together with a function $(m, m') \longmapsto \langle m, m' \rangle$, from $\mathcal{M} \times \mathcal{M}'$ into $\mathbb{R}$, satisfying 1) $m \longmapsto \langle m, m' \rangle$ is linear, 2) $m' \longmapsto \langle m, m' \rangle$ is linear, 3) if $\langle m, m' \rangle = 0$ for each $m' \in \mathcal{M}'$, then $m = 0$, and 4) if $\langle m, m' \rangle = 0$ for each $m \in \mathcal{M}$, then $m' = 0$. I will refer to 3) and 4) as separation properties. Given a dual pair $\langle \mathcal{M}, \mathcal{M}' \rangle$, the weak topology on $\mathcal{M}$ is denoted by $\sigma(\mathcal{M}, \mathcal{M}')$. Under $\sigma(\mathcal{M}, \mathcal{M}')$, a sequence $m_n \in \mathcal{M}$ converges to $m \in \mathcal{M}$ if and only if $\langle m_n, m' \rangle \to \langle m, m' \rangle$ for all $m' \in \mathcal{M}'$. It is well known that the topological dual of $(\mathcal{M}, \sigma(\mathcal{M}, \mathcal{M}'))$ may be identified with $\mathcal{M}'$. In other words, for each $\sigma(\mathcal{M}, \mathcal{M}')$-continuous linear functional ϕ on $\mathcal{M}$, there is a unique $m' \in \mathcal{M}'$ such that $\phi(m) = \langle m, m' \rangle$ for all $m \in \mathcal{M}$. The weak topology $\sigma(\mathcal{M}', \mathcal{M})$ is defined symmetrically for $\mathcal{M}'$. From now on, for any dual pair $\langle \mathcal{M}, \mathcal{M}' \rangle$, $\mathcal{M}$ and $\mathcal{M}'$ are topological vector spaces equipped with the weak topologies.

Given dual pairs $\langle \mathcal{M}, \mathcal{M}' \rangle$ and $\langle \mathcal{T}, \mathcal{T}' \rangle$, the continuity of a linear mapping $A : \mathcal{M} \to \mathcal{T}$ can be checked by using $A^{\#}$; A is continuous if and only if $A^{\#}(\mathcal{T}') \subset \mathcal{M}'$. The restriction A' of $A^{\#}$ to $\mathcal{T}'$ is called the topological adjoint of A with respect to $\langle \mathcal{M}, \mathcal{M}' \rangle$ and $\langle \mathcal{T}, \mathcal{T}' \rangle$, or simply the adjoint of A.

A nonempty set $K \subset \mathcal{M}$ is called a convex cone if $K + K \subset K$ and $\alpha K \subset K$ for every $\alpha \geq 0$. The polar cone $K' \subset \mathcal{M}'$ of the convex cone $K \subset \mathcal{M}$ is defined as $K' = \{m' : \langle m, m' \rangle \geq 0 \text{ for all } m \in K\}$.

The main tool used in the paper is the following result from Craven and Koliha (1977, Theorem 2):

Theorem B.1 *(Generalized Farkas Theorem) Let $\langle \mathcal{M}, \mathcal{M}' \rangle$ and $\langle \mathcal{T}, \mathcal{T}' \rangle$ be dual pairs, let K be a convex cone in $\mathcal{M}$, and let $A : \mathcal{M} \to \mathcal{T}$ be a continuous linear map. Let $A(K)$ be*

closed and $\tau \in \mathcal{T}$. Then the following conditions are equivalent.[11]
(a) The equation $Am = \tau$ has a solution $m \in K$.
(b) $A't' \in K' \Longrightarrow \langle \tau, t' \rangle \geq 0$.

B.2 Proof of Theorem 4.2

Lemma B.2 *The map $(f, \mu) \longmapsto \Psi(f, \mu)$ from $\mathcal{H} \times \Delta(S)$ into $\Delta(Z)$ is continuous.*

Proof. Suppose that (f_n, μ_n) converges to (f, μ) in the product space $\mathcal{H} \times \Delta(S)$. Note that S is finite. Then, for any $\eta \in C_b(Z)$,

$$\begin{aligned}
\int_Z \eta d\Psi(f_n, \mu_n) &= \int_Z \eta d \left[\mu_n(s_1) f_n(s_1) \oplus \mu_n(s_2) f_n(s_2) \oplus ... \oplus \mu_n(s_{|S|}) f_n(s_{|S|})\right] \\
&= \sum_{s \in S} \left(\mu_n(s) \int_Z \eta d f_n(s) \right) \\
&\rightarrow \sum_{s \in S} \left(\mu(s) \int_Z \eta d f(s) \right) = \int_Z \eta d\Psi(f, \mu).
\end{aligned}$$

■

Proof. *Necessity:* Completeness, transitivity and continuity are clear.

Second-Stage Independence : For $p \in \Delta(Z)$, $V(p) = v(u(p))$ because p does not depend on the probability measure $m \in \Delta(\Delta(S))$. Since v is strictly increasing, preference on $\Delta(Z)$ is represented by u. Thus Second-Stage Independence is satisfied because u is mixture linear.

First-Stage Independence : Let $\alpha \in (0, 1]$ and $P, R \in \Delta(\mathcal{H})$. Then, it is easy to see that

$$V(\alpha P + (1 - \alpha) R) = \alpha V(P) + (1 - \alpha) V(R).$$

First-Stage Independence is clear.

Dominance : Let P be any element in $\Delta(\mathcal{H})$. By Lemma B.2 and continuity of $v \circ u$, $v[u(\Psi(f, \mu))]$ is jointly continuous on $\mathcal{H} \times \Delta(S)$ and hence $P \times m$-measurable. Since $v \circ u$

[11] It is easy to see (a)⇒(b). Suppose that $Am = \tau$, $m \in K$ and $A't' \in K'$. Then $\langle \tau, t' \rangle = \langle Am, t' \rangle = \langle m, A't' \rangle \geq 0$, because $A't' \in K'$.

is bounded, $v\left[u(\Psi(f,\mu))\right]$ is $P\times m$-integrable. Then, apply the Fubini Theorem (AB (1999, p.411)) to get

$$\begin{aligned} V(P) &= \int_{\mathcal{H}}\int_{\Delta(S)} v\left(\int u(f)d\mu\right) dm(\mu)dP(f) \\ &= \int_{\mathcal{H}}\int_{\Delta(S)} v\left[u(\Psi(f,\mu))\right] dm(\mu)dP(f) \\ &= \int_{\Delta(S)}\int_{\mathcal{H}} v\left[u(\Psi(f,\mu))\right] dP(f)dm(\mu). \end{aligned}$$

Note that by the Change of Variables Theorem (AB (1999, p.452)),

$$\int_{\mathcal{H}} v\left[u\left(\Psi\left(f,\mu\right)\right)\right] dP\left(f\right) = \int_{\Delta(Z)} v\circ u\left(p\right) d\Psi\left(P,\mu\right)\left(p\right) = V\left(\Psi\left(P,\mu\right)\right).$$

Thus,

$$V\left(P\right) = \int_{\Delta(S)} V\left(\Psi\left(P,\mu\right)\right) dm(\mu).$$

Since m is a nonnegative measure, this completes the proof. ■

Turn to sufficiency. When $P\sim Q$ for all $P,Q\in\Delta(\mathcal{H})$, the representation is trivial. Thus assume that $\succeq$ satisfies :

Axiom 9 *(Nondegeneracy) $P\succ Q$ for some $P,Q\in\Delta(\mathcal{H})$.*

Follow these Lemmas to prove sufficiency.

Lemma B.3 *(i) Preference $\succeq$ restricted to $\Delta(Z)$ is represented by a bounded continuous mixture linear function $u:\Delta\left(Z\right)\to\mathbb{R}$. Moreover, u is unique up to positive affine transformation.*
(ii) Preference $\succeq$ is represented on $\Delta\left(\mathcal{H}\right)$ by

$$V(P) = \int U(f)dP(f)$$

for $P\in\Delta(\mathcal{H})$, where $U:\mathcal{H}\to\mathbb{R}$ is a bounded continuous function and unique up to positive affine transformation.

Proof. (i) Since $\mathcal{H}$ is a metric space, the mapping $f \mapsto \delta_f$ from $\mathcal{H}$ into $\Delta(\mathcal{H})$ is an embedding (AB (1999, p.480)). Moreover, $\mathcal{H}$ is a product space of $\Delta(Z)$'s. Thus, the weak convergence topology on $\Delta(Z)$ coincides with the relative topology on $\Delta(Z)$ induced by $\Delta(\mathcal{H})$. Hence, Continuity implies that the restriction of $\succeq$ to $\Delta(Z)$ is continuous under the weak convergence topology on $\Delta(Z)$.

Moreover, preference $\succeq$ restricted to $\Delta(Z)$ satisfies Order and (Second-Stage) Independence. Therefore (i) holds (see Grandmont (1972), for example).

(ii) can be proved in a similar way. ■

Let $\widehat{U}$ be the restriction of U to $\Delta(Z)$.

Lemma B.4 *There exist $p, q \in \Delta(Z)$ such that $p \succ q$. Consequently, there is a one-stage lottery p such that $\widehat{U}(p) \neq 0$.*

Proof. Suppose that $p \sim q$ for all p and q in $\Delta(Z)$. This means that $\bar{P} \sim \bar{Q}$ for all $\bar{P}$ and $\bar{Q}$ in $\Delta(\Delta(Z))$, by Lemma B.3(ii). Then, for any $P, Q \in \Delta(\mathcal{H})$ and $\mu \in \Delta(S)$, $\Psi(P,\mu) \sim \Psi(Q,\mu)$ because $\Psi(P,\mu), \Psi(Q,\mu) \in \Delta(\Delta(Z))$. By Dominance axiom, $P \sim Q$ for all $P, Q \in \Delta(\mathcal{H})$, contradicting to Nondegeneracy. ■

In order to apply the Generalized Farkas Theorem, let

$$\begin{aligned}\mathcal{M} &= ca(\Delta(S)),\ \mathcal{M}' = C_b(\Delta(S)) \text{ and} \\ \mathcal{T} &= C_b(\mathcal{H}),\ \mathcal{T}' = ca(\mathcal{H})\end{aligned}$$

where $ca(X)$ denotes the set of all Borel signed measures on X having bounded variation. Both of $\langle \mathcal{M}, \mathcal{M}' \rangle$ and $\langle \mathcal{T}, \mathcal{T}' \rangle$ are dual pairs with bilinear operations $\langle m, m' \rangle = \int m' dm$ and $\langle t, t' \rangle = \int t dt'$ for $m \in \mathcal{M}$, $m' \in \mathcal{M}'$, $t \in \mathcal{T}$ and $t' \in \mathcal{T}'$ (AB (1999, p.475)). Let

$$K = ca_+(\Delta(S))$$

be the subset of $\mathcal{M}$ consisting of all nonnegative Borel measures on $\Delta(S)$. K is clearly a convex cone. Recall that $\widehat{U}$ is the restriction of U to $\Delta(Z)$ and define a linear mapping A from $\mathcal{M}$ into the set of all functionals on $\mathcal{H}$ by

$$(Am)(f) = \int_{\Delta(S)} \widehat{U} \circ \Psi(f,\mu) dm(\mu) \text{ for } f \in \mathcal{H}. \tag{B.1}$$

The premises of the Generalized Farkas Theorem will be verified.

Lemma B.5 *The mapping A is a linear mapping from $\mathcal{M}$ into $\mathcal{T}$.*

Proof. It suffices to show that $A(\mathcal{M}) \subset \mathcal{T} = C_b(\mathcal{H})$. Let $m \in \mathcal{M}$ and assume that $f_n \to f$ for $f_n, f \in \mathcal{H}$. Note that $\widehat{U}$ is bounded and $\widehat{U} \circ \Psi(f_n, \mu) \to \widehat{U} \circ \Psi(f, \mu)$ by Lemma B.2. By the Lebesgue Dominated Convergence Theorem, $\int \widehat{U} \circ \Psi(f_n, \mu) dm(\mu) \to \int \widehat{U} \circ \Psi(f, \mu) dm(\mu)$. Hence $f \mapsto (Am)(f)$ is continuous. Boundedness of $f \mapsto (Am)(f)$ comes from boundedness of $\widehat{U}$. ■

Lemma B.6 *The mapping A is continuous.*

Proof. It suffices to show that $A^{\#}(\mathcal{T}') \subset \mathcal{M}'$. Let $t' \in \mathcal{T}'$. Then $A^{\#}t'$ lies in $\mathcal{M}^{\#}$, i.e., $A^{\#}t'$ is a linear functional on $\mathcal{M}$. Hence,

$$\begin{aligned} \left\langle m, A^{\#}t' \right\rangle &= \langle Am, t' \rangle = \int_{\mathcal{H}} (Am)(f) dt'(f) \\ &= \int_{\Delta(S)} \int_{\mathcal{H}} \widehat{U} \circ \Psi(f, \mu) dt'(f) dm(\mu) = \langle m, m' \rangle \end{aligned}$$

where $m' \in \mathcal{M}^{\#}$ is defined by $m'(\mu) = \int \widehat{U} \circ \Psi(f, \mu) dt'(f)$. The order of integration has changed in the third equality by the Fubini Theorem. Since $\widehat{U} \circ \Psi$ is bounded continuous, $\widehat{U} \circ \Psi$ is $t' \times m$-integrable and the Fubini Theorem can be applied.

Now, it suffices to show that $m' \in \mathcal{M}' = C_b(\Delta(S))$. Since $\widehat{U} \circ \Psi$ is bounded, m' is bounded. To see continuity, let $\mu_n \to \mu$ for $\mu_n, \mu \in \Delta(S)$. Since $\mu \mapsto \widehat{U} \circ \Psi(f, \mu)$ is continuous for each $f \in \mathcal{H}$, it follows that $\widehat{U} \circ \Psi(f, \mu_n) \to \widehat{U} \circ \Psi(f, \mu)$. Observing that $\widehat{U} \circ \Psi$ is bounded,

$$m'(\mu_n) = \int_{\mathcal{H}} \widehat{U} \circ \Psi(f, \mu_n) dt'(f) \to \int_{\mathcal{H}} \widehat{U} \circ \Psi(f, \mu) dt'(f) = m'(\mu)$$

by the Lebesgue Dominated Convergence Theorem. Hence $m' \in \mathcal{M}'$. ■

Lemma B.7 *$A(K)$ is closed.*

Proof. Suppose that $\theta_n = A(\lambda_n m_n) \in A(K)$ converges to $\theta \in \mathcal{T}$, where $\lambda_n \in \mathbb{R}_+$ and $m_n \in \Delta(\Delta(S))$.

Step 1. m_n has a subsequence $m_{k(n)}$ that converges to some $m \in \Delta(\Delta(S))$: Since S is finite, $\Delta(S)$ is a compact metric space and so is $\Delta(\Delta(S))$ (AB (1999, p.482)). Hence, m_n has a converging subsequence.

Step 2. $\left\langle Am_{k(n)}, t' \right\rangle \to \langle Am, t' \rangle$ *for any* $t' \in T'$ *:* By Step 1 and the continuity of A, $Am_{k(n)} \to Am$. Thus this step is proved.

Step 3. $\lambda_{k(n)}$ *converges to some* $\lambda \geq 0$ *:* By Lemma B.4, take $p \in \Delta(Z)$ such that $\widehat{U} \circ \Psi(p, \mu) = \widehat{U}(p) \neq 0$ for any $\mu \in \Delta(S)$. Then,

$$(Am)(p) = \int_{\Delta(S)} \widehat{U} \circ \Psi(p, \mu) dm(\mu) = \widehat{U}(p) \neq 0,$$

which implies that $Am \neq 0$. Therefore, by the separation property of a dual pair, $\langle Am, \bar{t}' \rangle \neq 0$ for some $\bar{t}' \in \mathcal{T}'$. Note that $\lambda_{k(n)} \left\langle Am_{k(n)}, \bar{t}' \right\rangle = \left\langle \theta_{k(n)}, \bar{t}' \right\rangle \to \langle \theta, \bar{t}' \rangle$. Then by Step 2, it follows that $\lambda_{k(n)} \to \lambda \equiv \langle \theta, \bar{t}' \rangle / \langle Am, \bar{t}' \rangle$. Since $\lambda_n \geq 0$ for all n, $\lambda \geq 0$.

Step 4. $\theta \in A(K)$ *:* For all $t' \in \mathcal{T}'$, $\left\langle \theta_{k(n)}, t' \right\rangle = \lambda_{k(n)} \left\langle Am_{k(n)}, t' \right\rangle \to \lambda \langle Am, t' \rangle = \langle A(\lambda m), t' \rangle$. Moreover, by the hypothesis, $\left\langle \theta_{k(n)}, t' \right\rangle \to \langle \theta, t' \rangle$ for all $t' \in \mathcal{T}'$. Note that $\left\langle \theta_{k(n)}, t' \right\rangle$ is a sequence in $\mathbb{R}$ and converges to at most one point. Thus, $\langle A(\lambda m), t' \rangle = \langle \theta, t' \rangle$ for all $t' \in \mathcal{T}'$, and $\theta = A(\lambda m) \in A(K)$ by the separation property of a dual pair. ■

The following Lemma uses the Generalized Farkas Theorem to prove the existence of second-order belief.

Lemma B.8 *There exists* $m \in \Delta(\Delta(S))$ *such that* $\int_{\Delta(S)} \widehat{U} \circ \Psi(f, \mu) dm(\mu) = U(f)$ *for all* $f \in \mathcal{H}$.

Proof. It is enough to show that $Am = U$ for some $m \in \Delta(\Delta(S))$, where A is defined in (B.1).

First, I will prove that there exists $m \in K = ca_+(\Delta(S))$ solving $Am = U$. I have already shown in Lemmas B.5 - B.7, that the premises of the Generalized Farkas Theorem are satisfied. Therefore, it suffices to show that if $\langle m, A't' \rangle \geq 0$ for all $m \in K$, then $\langle U, t' \rangle \geq 0$.

Assume that $\langle m, A't' \rangle \geq 0$ for all $m \in K$ and show that

$$\langle U, t' \rangle \geq 0. \tag{B.2}$$

By the hypothesis, $\langle m, A't' \rangle = \langle Am, t' \rangle = \int Am dt' = \int \int \widehat{U} \circ \Psi(f, \mu) dm(\mu) dt'(f) \geq 0$ for all $m \in K$. Since $\delta_\mu \in K$ for each $\mu \in \Delta(S)$, it follows that

$$\int \widehat{U} \circ \Psi(f, \mu) dt'(f) \geq 0 \text{ for all } \mu \in \Delta(S). \tag{B.3}$$

Let $t' = \alpha P - \beta Q$ by the Hahn Decomposition Theorem, where $\alpha, \beta \geq 0$ and $P, Q \in \Delta(\mathcal{H})$. Let $\alpha \geq \beta$. The other case, $\alpha < \beta$, can be proved similarly. If $\alpha = 0$, the statement (B.2) is trivial because $\alpha = \beta = 0$.

Let $\alpha > 0$. Note that (B.3) implies

$$\int \widehat{U} \circ \Psi(f,\mu) dP(f) \geq \gamma \int \widehat{U} \circ \Psi(f,\mu) dQ(f) \text{ for all } \mu \in \Delta(S), \tag{B.4}$$

where $\gamma = \beta/\alpha$.

Recall Lemma B.3(ii) that U is unique up to positive affine transformation. Normalize U such that $\int U d\bar{R} = 0$ for some $\bar{R} \in \Delta(\Delta(Z))$. Since $\widehat{U}$ is the restriction of U, $\int \widehat{U} d\bar{R} = 0$. Observe that, for all $B \in \mathcal{B}_{\mathcal{H}}$ and $\mu \in \Delta(S)$,

$$\begin{aligned} \Psi(\bar{R},\mu)(B) &= \bar{R}\left(\{f \in \mathcal{H} : \Psi(f,\mu) \in B\}\right) \\ &= \bar{R}\left(\{p \in \Delta(Z) : p \in B\}\right) \\ &= \bar{R}\left(B \cap \Delta(Z)\right) = \bar{R}(B). \end{aligned}$$

The second equality comes from the fact that $\bar{R}$ assigns zero probability outside of $\Delta(Z)$. Thus, $\bar{R} = \Psi(\bar{R},\mu)$ and $\int \widehat{U} d\Psi(\bar{R},\mu) = 0$ for all $\mu \in \Delta(S)$. Then (B.4) implies[12]

$$\int \widehat{U} d\Psi(P,\mu) \geq \gamma \int \widehat{U} d\Psi\left(Q,\mu\right) + (1-\gamma) \int \widehat{U} d\Psi(\bar{R},\mu) \text{ for all } \mu \in \Delta(S).$$

Hence by Lemma B.3(ii), it follows that

$$\Psi(P,\mu) \succeq \gamma \Psi(Q,\mu) + (1-\gamma)\Psi(\bar{R},\mu) \text{ for all } \mu \in \Delta(S). \tag{B.5}$$

Moreover, for any $B \in \mathcal{B}_{\mathcal{H}}$,

$$\begin{aligned} & \left[\gamma \Psi(Q,\mu) + (1-\gamma)\Psi(\bar{R},\mu)\right](B) \\ = \; & \gamma \Psi(Q,\mu)(B) + (1-\gamma)\Psi(\bar{R},\mu)(B) \\ = \; & \gamma \cdot Q\left(\{f \in \mathcal{H} : \Psi(f,\mu) \in B\}\right) + (1-\gamma) \cdot \bar{R}\left(\{f \in \mathcal{H} : \Psi(f,\mu) \in B\}\right) \\ = \; & (\gamma Q + (1-\gamma)\bar{R})\left(\{f \in \mathcal{H} : \Psi(f,\mu) \in B\}\right) \\ = \; & \Psi(\gamma Q + (1-\gamma)\bar{R},\mu)(B). \end{aligned}$$

[12] Recall that $\Psi(P,\mu)(B) = P\left(\{f \in \mathcal{H} : \Psi(f,\mu) \in B\}\right)$ for $B \in \mathcal{B}_{\mathcal{H}}$. Thus, by the Change of Variables Theorem,

$$\int_{\Delta(\Delta(Z))} \widehat{U}\left(p\right) d\Psi\left(P,\mu\right)(p) = \int_{\mathcal{H}} \widehat{U} \circ \Psi(f,\mu) dP(f).$$

The same is true for Q.

Therefore, by (B.5),

$$\Psi(P,\mu) \succeq \Psi(\gamma Q + (1-\gamma)\bar{R}, \mu) \text{ for all } \mu \in \Delta(S).$$

By Dominance, it follows,

$$P \succeq \gamma Q + (1-\gamma)\bar{R}.$$

Therefore, by Lemma B.3(ii),

$$\int_{\mathcal{H}} U dP \geq \int_{\mathcal{H}} U d\left(\gamma Q + (1-\gamma)\bar{R}\right) = \gamma \int_{\mathcal{H}} U dQ. \tag{B.6}$$

Then, by (B.6),

$$\langle U, t' \rangle = \int U dt' = \int U d\left[\alpha\left(P - \gamma Q\right)\right] \geq 0.$$

This completes the proof of (B.2).

Now, apply the Generalized Farkas Theorem to obtain $m \in K = ca_+(\Delta(S))$ satisfying the equation $Am = U$, or equivalently

$$\int_{\Delta(S)} \widehat{U} \circ \Psi(f,\mu) dm(\mu) = U\left(f\right), \text{ for each } f \in \mathcal{H}. \tag{B.7}$$

To prove that m is a probability measure, let $p \in \Delta(Z)$ be such that $\widehat{U}(p) \neq 0$ as in Lemma B.4 and let f be the constant act giving p in every state. Since $U\left(p\right) = \widehat{U}(p) \neq 0$, (B.7) becomes

$$\int_{\Delta(S)} dm(\mu) = 1.$$

■

Now, I will show a general property about utility representation.

Lemma B.9 *Let X be a connected topological space. If two bounded continuous functions $u : X \to \mathbb{R}$ and $w : X \to \mathbb{R}$ represent the same preference on X, then there exists a continuous and strictly increasing function $v : u(X) \to \mathbb{R}$ such that $w = v \circ u$.*

Proof. Define v on $u\left(X\right)$ by

$$v\left(y\right) = w\left(x\right) \text{ if } u\left(x\right) = y.$$

Then v is well-defined, strictly increasing and $w = v \circ u$.

In order to show the continuity of v, note that X is connected and w is continuous. Hence, $v\left(u\left(X\right)\right) = w\left(X\right)$ is connected. Since v is (strictly) increasing, it must be continuous. ■

Lemma B.10 *There exists a bounded continuous and strictly increasing function* $v : u(\Delta(Z)) \to \mathbb{R}$ *such that* $\widehat{U} = v \circ u$.

Proof. Observe that u and $\widehat{U}$ represents the same preference on $\Delta(Z)$. By Lemma B.9, a continuous and strictly increasing function $v : u(\Delta(Z)) \to \mathbb{R}$ exists such that $\widehat{U} = v \circ u$. Boundedness comes from the fact that $\widehat{U}$ is bounded. ■

Finally, by Lemma B.3(ii), $V(P) = \int_{\mathcal{H}} U(f)dP(f)$ represents $\succeq$ on $\Delta(\mathcal{H})$ and by Lemmas B.3(i), B.8 and B.10, it follows that

$$\begin{aligned} U(f) &= \int_{\Delta(S)} \widehat{U} \circ \Psi(f,\mu)\, dm(\mu) = \int_{\Delta(S)} v \circ u \circ \Psi(f,\mu)\, dm(\mu) \\ &= \int_{\Delta(S)} v\left(\int_S u(f)\, d\mu\right) dm(\mu). \end{aligned}$$

B.3 Proof of Lemma 4.1

Proof. (i) Suppose that $\succeq$ satisfies Order, Continuity, Reversal of Order and AA-Dominance.

For any $P \in \Delta(\mathcal{H})$, let $\Pi(P)$ be the act obtained by collapsing all the objective probabilities into $\Delta(Z)$, i.e., Π is a function from $\Delta(\mathcal{H})$ into $\mathcal{H}$ such that for every $B \in \mathcal{B}_Z$ and $s \in S$, $\Pi(P)(s)(B) = \int_{\mathcal{H}} f(s)(B)dP(f)$. In order for Π to be well-defined, $f(s)(B)$ must be P-integrable as a function of f.

Step 1. Π *is well-defined* : Since Z is metrizable, the function $p \mapsto p(B)$ from $\Delta(Z)$ into $\mathbb{R}$ is measurable (AB (1999, p.485)). Moreover the function $f \mapsto f(s)$ is measurable. Thus, $f \mapsto f(s)(B)$ is measurable. Since $f(s)(B)$ is bounded, $f \mapsto f(s)(B)$ is P-integrable.

Step 2. $\int_Z \eta(z)\, d\Pi(P)(s)(z) = \int_{\mathcal{H}} \int_Z \eta(z)\, df(s)(z)\, dP(f)$ *for any* $s \in S$, $\eta \in C_b(Z)$ *and* $P \in \Delta(\mathcal{H})$: When η is a measurable step function (i.e., it has a finite image), this is clear. For any $\eta \in C_b(Z)$, take a sequence η_n of step functions such that $\eta_n(z)$ converges to $\eta(z)$ for each $z \in Z$. Then, by the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned} \int_Z \eta(z)\, d\Pi(P)(s)(z) &= \lim \int_Z \eta_n(z)\, d\Pi(P)(s)(z) \\ &= \lim \int_{\mathcal{H}} \int_Z \eta_n(z)\, df(s)(z)\, dP(f) \\ &= \int_{\mathcal{H}} \int_Z \eta(z)\, df(s)(z)\, dP(f). \end{aligned}$$

Step 3. Π *is continuous*: Fix $s \in S$. Suppose that $P_n \to P$. Note that $f \mapsto \int_Z \eta(z)\, df(s)(z)$ is continuous. Then, by Step 2,

$$\begin{aligned}\int_Z \eta(z)\, d\Pi(P_n)(s)(z) &= \int_{\mathcal{H}} \left(\int_Z \eta(z)\, df(s)(z)\right) dP_n(f) \\ &\to \int_{\mathcal{H}} \left(\int_Z \eta(z)\, df(s)(z)\right) dP(f) = \int_Z \eta(z)\, d\Pi(P)(s)(z).\end{aligned}$$

Thus, $P \mapsto \Pi(P)(s)$ is continuous for every $s \in S$. Therefore Π is continuous.

Step 4. $\Pi(P) \sim P$ *for any* $P \in \Delta(\mathcal{H})$: Reversal of Order implies that $\Pi(P) \sim P$ when P has a finite support. Since $\mathcal{H}$ is metrizable, the set of all probability measures on $\mathcal{H}$ with finite support is dense in $\Delta(\mathcal{H})$ (AB (1999, p.481)). For any $P \in \Delta(\mathcal{H})$, take $P_n \in \Delta(\mathcal{H})$ with finite support such that $P_n \to P$. Then $\Pi(P_n) \sim P_n$ for all n. By Continuity and Step 3,

$$\Pi(P) = \lim \Pi(P_n) \sim \lim P_n = P.$$

Step 5. $\Pi(\Psi(P,\mu)) = \Psi(\Pi(P),\mu)$ *for any* $P \in \Delta(H)$ *and* $\mu \in \Delta(S)$: For any $B \in \mathcal{B}_Z$,

$$\begin{aligned}\Pi(\Psi(P,\mu))(s)(B) &= \int_{\mathcal{H}} f(s)(B) d\Psi(P,\mu)(f) = \int_{\Delta(Z)} p(B)\, d\Psi(P,\mu)(p) \\ &= \int_{\mathcal{H}} \Psi(f,\mu)(B) dP(f) \\ &= \int_{\mathcal{H}} \left[\mu(s_1) f(s_1) \oplus ... \oplus \mu(s_{|S|}) f(s_{|S|})\right](B) dP(f) \\ &= \int_{\mathcal{H}} \mu(s_1)\left[f(s_1)(B)\right] + ... + \mu(s_{|S|})\left[f(s_{|S|})(B)\right] dP(f) \\ &= \mu(s_1)\int_{\mathcal{H}} f(s_1)(B) dP(f) + ... + \mu(s_{|S|})\int_{\mathcal{H}} f(s_{|S|})(B) dP(f) \\ &= \mu(s_1)\left[\Pi(P)(s_1)(B)\right] + ... + \mu(s_{|S|})\left[\Pi(P)(s_{|S|})(B)\right] \\ &= \left[\mu(s_1)\cdot \Pi(P)(s_1) \oplus ... \oplus \mu(s_{|S|})\cdot \Pi(P)(s_{|S|})\right](B) \\ &= \Psi(\Pi(P),\mu)(B).\end{aligned}$$

The third equality is obtained by the Change of Variables Theorem.

Step 6. $\succeq$ *satisfies Dominance* : Suppose that $\Psi(P,\mu) \succeq \Psi(Q,\mu)$ for all $\mu \in \Delta(S)$. By Steps 4 and 5,

$$\Psi(P,\mu) \sim \Pi(\Psi(P,\mu)) = \Psi(\Pi(P),\mu).$$

Therefore $\Psi(\Pi(P),\mu) \succeq \Psi(\Pi(Q),\mu)$ for all $\mu \in \Delta(S)$. Since $\Psi(\Pi(P),\delta_s) = \Pi(P)(s)$, it follows that $\Pi(P)(s) \succeq \Pi(Q)(s)$ for all $s \in S$. For $k = 0, 1, ..., |S|$, define $h_k \in \mathcal{H}$ by

$$h_k(s) = \begin{cases} \Pi(P)(s) & \text{if} \quad s > k \\ \Pi(Q)(s) & \text{if} \quad s \leq k \end{cases}.$$

Then, by AA-Dominance and Step 4,

$$P \sim \Pi(P) = h_0 \succeq h_1 \succeq ... \succeq h_{|S|} = \Pi(Q) \sim Q,$$

which completes the proof of (i).

(ii) Let $f, g \in \mathcal{H}$ and suppose that $f(s) = g(s)$ for all $s \neq s'$ and $f(s') \succeq g(s')$ for some $s' \in S$. By Second-Stage Independence, $\Psi(f,\mu) \succeq \Psi(g,\mu)$ for any $\mu \in \Delta(S)$. Dominance implies $f \succeq g$. ■

C Appendix : Uniqueness of the SOSEU representation

The following Lemma provides some uniqueness properties.

Lemma C.1 *Suppose that $P \succ Q$ for some $P, Q \in \Delta(\mathcal{H})$ and let the two triples (u, v, m) and (u', v', m') represent $\succeq$ on $\Delta(\mathcal{H})$. Then*
(i) u and u' are the same up to positive affine transformation and so are $v \circ u$ and $v' \circ u'$
(ii) $\int \varphi dm = \int \varphi dm'$ for all $\varphi \in D$ where

$$\begin{aligned} D &= \left\{ \varphi \in C(\Delta(S)) : \exists \lambda \in ca(T) \ \textit{such that}\ \varphi(\mu) = \int v(\mu \cdot t)\, d\lambda(t) \ \textit{for all}\ \mu \right\} \\ \textit{and}\ T &= [u(\Delta(Z))]^{|S|} \subset \mathbb{R}^{|S|}. \end{aligned}$$

Proof. (i) : Note that u and u' represent the same preference on $\Delta(Z)$, and so do $\bar{P} \mapsto \int v \circ u d\bar{P}$ and $\bar{P} \mapsto \int v' \circ u' d\bar{P}$ on $\Delta(\Delta(Z))$.

(ii) : Note that $u' = au + b$ for some $a > 0$ and $b \in \mathbb{R}$, and $v' \circ u' = cv \circ u + d$ for some $c > 0$ and $d \in \mathbb{R}$. Thus, $v'(ax + b) = cv(x) + d$ for any $x \in u(\Delta(Z))$. Then,

$$\begin{aligned} \int v'\left(\int u'(f)d\mu\right) dm'(\mu) &= \int v'\left(a \int u(f)d\mu + b\right) dm'(\mu) \\ &= c \int v\left(\int u(f)d\mu\right) dm'(\mu) + d. \end{aligned}$$

Since $\int v'\left(\int u'(f)d\mu\right)dm'(\mu)$ and $\int v\left(\int u(f)d\mu\right)dm(\mu)$ represent the same preference,

$$\int v\left(\int u(f)d\mu\right)dm(\mu)=\int v\left(\int u(f)d\mu\right)dm'(\mu) \text{ for all } f\in\mathcal{H}.$$

Since S is finite, it follows that

$$\int v\left(\mu\cdot t\right)dm(\mu)=\int v\left(\mu\cdot t\right)dm'(\mu) \text{ for all } t\in u\left(\Delta\left(Z\right)\right)^{|S|}.$$

Integrating both sides gives

$$\int\int v\left(\mu\cdot t\right)dm(\mu)d\lambda\left(t\right)=\int\int v\left(\mu\cdot t\right)dm'(\mu)d\lambda\left(t\right)$$

for any $\lambda\in ca\left(u\left(\Delta\left(Z\right)\right)^{|S|}\right)$. Observe that $(\mu,t)\mapsto v\left(\mu\cdot t\right)$ is jointly continuous and bounded and hence $m\times\lambda$-integrable. By the Fubini Theorem,

$$\int\int v\left(\mu\cdot t\right)d\lambda\left(t\right)dm(\mu)=\int\int v\left(\mu\cdot t\right)d\lambda\left(t\right)dm'(\mu) \text{ for all } \lambda\in ca\left(T\right),$$

for $T=\left[u\left(\Delta\left(Z\right)\right)\right]^{|S|}$, which completes the proof. ■

By the above Lemma, characterizing D is crucial in determining the class of m that represents the same preference.

References

[1] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis*, Springer, Berlin, 1999.

[2] F. J. Anscombe, R. J. Aumann, A Definition of Subjective Probability, *Ann. Math. Statist.* 34 (1963), 199-205.

[3] B. D. Craven and J. J. Koliha, Generalizations of Farkas' Theorem, *SIAM J. Math. Anal.* 8 (1977), 983-997.

[4] D. Ellsberg, Risk, Ambiguity and the Savage Axioms, *Quarterly J. of Econ.* 75 (1961), 643-669.

[5] D. Ellsberg, *Risk, Ambiguity and Decision*, Gerland Publishing, New York, 2001.

[6] H. Ergin and F. Gul, A Subjective Theory of Compound Lotteries, working paper 2004.

[7] I. Gilboa and D. Schmeidler, Maxmin Expected Utility with Non-unique Prior, *J. Math. Econ.* 18 (1989), 141-153.

[8] J. Grandmont, Continuity Properties of a von Neumann-Morgenstern Utility, *J. Econ. Theory* 4 (1972), 45-57.

[9] Y. Halevy, Ellsberg Revisited: an Experimental Study, working paper 2005.

[10] P. Klibanoff, M. Marinacci, and S. Mukerji, A Smooth Model of Decision Making Under Ambiguity, *Econometrica* 73 (2005), 1849-1892.

[11] P. Klibanoff and E. Ozdenoren, Subjective Recursive Expected Utility, *Economic Theory*, forthcoming 2005.

[12] D. M. Kreps, *Notes on the Theory of Choice*, Westview Press, 1988.

[13] D. M. Kreps and E. L. Porteus, Temporal Resolution of Uncertainty and Dynamic Choice Theory, *Econometrica* 46 (1978), 185-200.

[14] R. F. Nau, Uncertainty Aversion with Second-Order Utilities and Probabilities, *Management Science* 52 (2006), 136-145.

[15] U. Segal, Two-Stage Lotteries without the Reduction Axiom, *Econometrica* 58 (1990), 349-377.