Ambiguity Models and the Machina Paradoxes

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Abstract

Machina (2009) introduced two examples that falsify Choquet expected utility, presently one of the most popular models of ambiguity. This article shows that Machina's examples do not only falsify the model mentioned, but also four other popular models for ambiguity of the literature, namely maxmin expected utility, variational preferences, α-maxmin and the smooth model of ambiguity aversion. Thus, Machina's examples pose a challenge to most of the present field of ambiguity. Finally, the paper discusses how an alternative representation of ambiguity-averse preferences manages to accommodate the Machina paradoxes and what drives the results.

Keywords: Ambiguity, Ellsberg Paradox, Multiple Priors;

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Daniel Ellsberg (1961) constructed counterexamples to show the limitations of Leonard J. Savage's (1954) subjective expected utility (SEU). Ellsberg's examples involved a comparison between objective uncertainty (risk), in which probabilities are known, and subjective uncertainty, in which they are not. The prevailing preference for objective over subjective uncertainty, known as ambiguity aversion, raised an important paradox for economic theory. Since Ellsberg's classical work, many models have been developed to generalize SEU, in order to accommodate the preference for objective over subjective uncertainty.

In the same manner as Ellsberg, Mark Machina (2009) proposed two examples that falsify one of the SEU generalizations, David Schmeidler's (1989) Choquet expected utility (CEU). This article shows that the impact of Machina's examples is not restricted to the model initially targeted. His examples pose difficulties not only for CEU, but also for the four other most popular and widely-used models of ambiguity-averse preferences, namely maxmin expected utility (Itzhak Gilboa and Schmeidler, 1989), variational preferences (Fabio Maccheroni, Massimo Marinacci, and Aldo Rustichini, 2006), α-maxmin (Paolo Ghirardato, Maccheroni, and Marinacci, 2004), and the smooth model of ambiguity aversion (Peter Klibanoff, Marinacci and Sujoy Mukerji, 2005). Consequently, the implications for economics are more profound than initially thought.

We also discuss Marciano Siniscalchi's (2009) vector expected utility (VEU) model, which can account for the typical ambiguity-averse preferences in Machina's examples. Finally, we examine how our results are related to the uncertainty aversion axiom, which is assumed in virtually all commonly-used decision models under ambiguity. In particular, we argue that Machina's examples demonstrate that the uncertainty aversion axiom can be overly restrictive in some circumstances.

The article proceeds as follows: Section I presents the four models of ambiguity-averse preferences, that are alternatives to CEU, and which are examined in this paper. In Section II and III, the implications of Machina's examples for these models are presented. Section IV discusses alternative models and the uncertainty aversion axiom.

I Four popular models of ambiguity-averse preferences

A typical decision problem under uncertainty involves a *state space* S that contains all possible states of nature. Only one of these states is (will be) true, but we do not know which one. By $\Delta(S)$ we denote the set of all probability measures (typically denoted p) over S. An *act* is a mapping from the state space S to a set of monetary outcomes. Assuming a *utility function* mapping the outcomes to the reals, $U_p(f)$ refers to the *expected utility* of act f if the probability distribution over S is p. Using this notation, Gilboa and Schmeidler's (1989) *maxmin expected utility* (MEU), also called *multiple priors*, holds if preferences can be represented by

$$MEU(f) = \min_{p \in C} U_p(f), \tag{1}$$

where C is a subset of $\Delta(S)$ and is called the *set of priors.* C need not be equal to $\Delta(S)$, i.e., decision-makers may think that some probability distributions in $\Delta(S)$ are irrelevant or not possible. MEU is the basis of many results in economics and finance. For instance, James Dow and Sergio Ribeiro Da Costa Werlang (1992), Larry G. Epstein and Tan Wang (1994), among many others, have studied the implications of multiple priors in asset pricing. Introducing multiplier preferences, Lars Peter Hansen and Thomas J. Sargent (2001) showed how the robust-control theory applications used to account for model mispecification in macroeconomic modeling are related to MEU.

Maccheroni, Marinacci and Rustichini (2006) proposed a general model, called *variational preferences* (VP), which captures both MEU and multiplier preferences. Under VP, preferences are represented by:

$$VP(f) = \min_{p \in \Delta(S)} \left\{ U_p(f) + c(p) \right\}, \tag{2}$$

where $c(p) : \Delta(S) \to [0, \infty]$ is an index of ambiguity aversion assigned to the probability distribution p. MEU is the special case of VP where $c(p) = 0$ if $p \in C$ and $c(p) = \infty$ otherwise. Hansen and Sargent's (2001) multiplier preferences correspond to a case with c a function of relative entropy.

The *α-maxmin* model (αM), axiomatized by Ghirardato, Maccheroni, and Marinacci (2004), is a linear combination of maxmin expected utility and *maxmax expected utility*, in which not the worst but the best expected utility is considered. This model extends the well-known Hurwicz criterion to ambiguity. αM holds if preferences can be represented by:

$$\alpha M(f) = \alpha \min_{p \in C} U_p(f) + (1 - \alpha) \max_{p \in C} U_p(f). \tag{3}$$

The set of priors C and the parameter α may be interpreted as ambiguity and ambiguity attitude respectively. Consider the case of Ellsberg's three-color urn (an urn with 30 red balls and 60 balls that are either yellow or black in unknown proportion and where one ball is to be drawn at random) and an αM decision-maker who strictly prefers to bet on red rather than on yellow and also strictly prefers to bet on red rather than on black. A decision-maker of this type is clearly ambiguity averse and violates SEU. It can be shown that in such a case, α must be higher than $1/2$.

The fourth model was introduced by Klibanoff, Marinacci and Mukerji (2005). Their approach is slightly different from the previous ones. Their *smooth model of ambiguity aversion* (KMM) involves a two-stage decomposition of the decision process into risk and ambiguity. Each stage uses an expected-utility-like functional form. Preferences are represented by

$$KMM(f) = \int_{\Delta(S)} \phi\left(U_p(f)\right) d\mu(p), \tag{4}$$

where μ is a subjective probability measure over $\Delta(S)$, that is, the measure of the subjective relevance of $p \in \Delta(S)$ to be the 'right' probability. Ambiguity attitude is captured by ϕ. More precisely, concavity of ϕ implies ambiguity aversion. For instance, a decision-maker preferring to bet on red rather than on yellow and to bet on red rather than on black in Ellsberg's urn cannot have a convex ϕ. Klibanoff, Marinacci and Mukerji (2005) defined ambiguity aversion as aversion to mean preserving spreads in terms of expected utility values and their model deals with ambiguity aversion as expected utility does with risk aversion. Hence, it is particularly convenient for applications (e.g., in macroeconomics, Hansen, 2007; in health and environmental policy, Nicolas Treich, 2010; in finance, Christian Gollier, 2009).

In the next two sections, we show precisely how Machina's examples pose difficulties for each of the four models presented above.

II The 50:51 example

The first example proposed by Machina (2009) is based on an urn with 101 balls. Fifty balls are marked with either 1 or 2 and 51 balls are marked with either 3 or 4. Each ball is equally likely to be drawn. E_n denotes the event "a ball marked with a n is drawn". Table 1 displays the outcomes assigned to each event by four acts. These outcomes are expressed in utility units. We use the flexibility of outcomes in Machina's examples to choose outcomes that are equally-spaced on the utility scale.[1] This adaptation of Machina's original example enables us to derive particularly clear counter-examples for MEU, αM, and VP, but is not needed for the KMM model. The specific numbers 0, 101, 202, 303 are proposed for convenience, in order to simplify some formulas (they are multiples of the number of balls in the urn). These numbers do not constitute any further restriction since as under all the models we are dealing with, utility is defined up to unit and level.

	50 balls		51 balls	
Acts	E_1	E_2	E_3	E_4
f_1	202	202	101	101
f_2	202	101	202	101
f_3	303	202	101	0
f_4	303	101	202	0

Table 1: The 50:51 example

In the 50:51 example, f_1 and f_2, as well as f_3 and f_4, differ only in whether they offer the higher prize 202 on the event E_2 or E_3. If a decision-maker is sufficiently ambiguity averse, he/she will prefer f_1 to f_2, as argued by Machina. Indeed, f_1 is clearly unambiguous whereas f_2 is ambiguous but benefits from a slight advantage due to the 51^{th} ball that may yield 202. There is thus a tradeoff between the advantage offered

[1]Eliciting outcomes that are equally-spaced in terms of utility units can easily be done using Wakker and Deneffe's (1996) tradeoff method under risk. For lotteries involving only objective probabilities, their method is compatible with MEU, αM, VP, KMM, and CEU.

by f_2 and the absence of ambiguity offered by f_1. Such a tradeoff is less clear in the choice between f_3 and f_4. Like f_2, f_4 benefits from the 51^{th} ball but f_3 does not offer a particular informational advantage.[2] There are two conflicting principles in this example. A SEU maximizer assuming a uniform distribution over the balls should prefer f_2 and f_4. Yet, a decision-maker who values unambiguous information may prefer f_1 to f_2 and may be indifferent between f_3 and f_4. The informational advantage of f_1 can more than offset its Bayesian disadvantage with respect to f_2 whereas f_3 benefits from no clear informational advantage that could compensate its Bayesian disadvantage with respect to f_4. This would lead to $f_1 \succ f_2$ and $f_3 \prec f_4$. If the 50:51 assignment of balls does not supply the "right advantage", or if it supplies "too much advantage", this can be adjusted, either by changing to 100:101 (if it is necessary to reduce the advantage) or to 25:26 (if it is necessary to increase the advantage).[3] However, Machina showed that under CEU, $f_1 \succ f_2$ if and only if $f_3 \succ f_4$. We show that $f_1 \succ f_2$ also implies $f_3 \succ f_4$ if the decision-maker's preferences are represented by MEU, VP, αM, or KMM with ϕ concave.

Throughout this section, any possible probability distribution over the state space is fully characterized by a pair of numbers (i, j) where i denotes the number of balls marked with a 1 and j denotes the number of balls marked with a 3. There are $50-i$ balls marked with a 2 and $51 - j$ balls marked with a 4. $\Delta(S) = [0, 50] \times [0, 51]$ refers to the set of all possible distributions. For instance, $(25, 25.5)$ is the prior such that E_1 and E_2 are equally-likely with probability $\frac{25}{101}$ each, and such that E_3 and E_4 are equally-likely with probability $\frac{25.5}{101}$ each. $U_{(i,j)}(f)$ denotes the expected utility of act f if the distribution is characterized by (i, j). In what follows, we often suppress f in $U_{(i,j)}(f)$.

First consider MEU. For f_2 and f_3, increasing i or j by 1 increases $U_{(i,j)}$ by 1. For f_4, they increase $U_{(i,j)}$ by 2. As a consequence, a MEU decision-maker will take into account the minimum of $i + j$ for f_2, f_3, and f_4. The same prior can thus be applied to evaluate the four acts, the prior having no impact on the evaluation of the unambiguous act f_1. With the same prior for the four acts, we are back to SEU. Hence, MEU implies the same

[2] Following Klibanoff, Marinacci and Mukerji (2005, pp. 1875-1876), f_1 can be interpreted as a risky asset and f_2 as an ambiguous asset in a management portfolio problem. Asset f_3 (f_4) could be obtained by buying 2 units of f_1 (f_2) and one unit of f_2 (f_1), and by selling a riskless asset yielding 303 for sure.

[3] We are grateful to a referee for bringing this point to our attention.

restriction as CEU (and SEU): $f_1 \succ f_2$ if and only if $f_3 \succ f_4$

Under αM, it can easily be shown that the same result holds. As with MEU, the priors that are used to evaluate f_1, f_2, f_3 and f_4 are the same. In the 50:51 example, αM corresponds to SEU with a specific probability distribution: α times the distribution that minimizes $i + j$ plus $(1 - \alpha)$ times the distribution that maximizes $i + j$ (over the set of priors C).

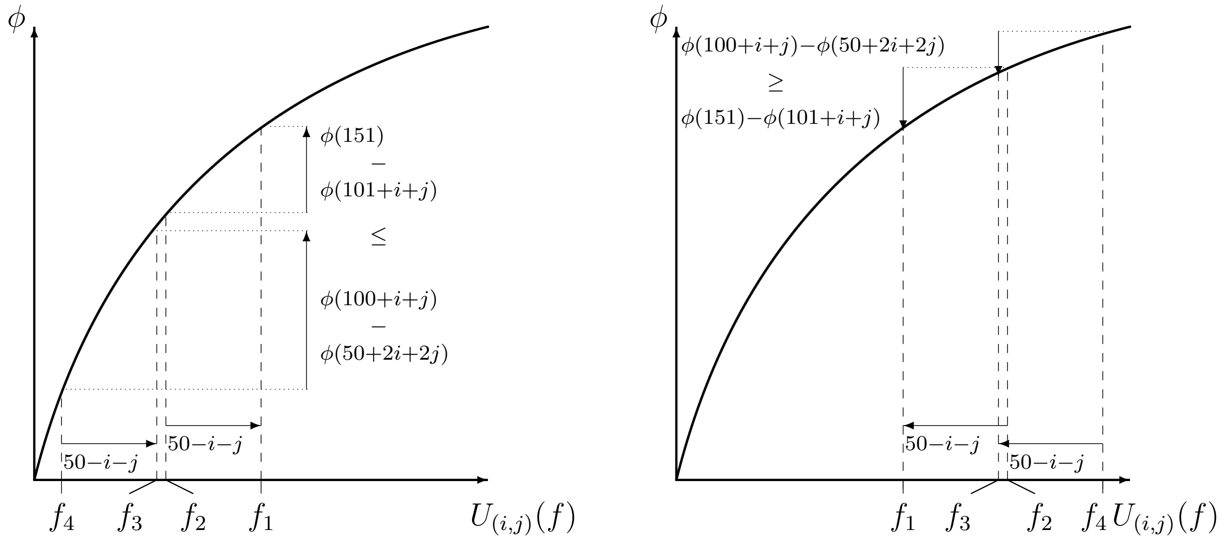
Result 1 in the Appendix establishes that an ambiguity-averse decision-maker, who prefers f_1 to f_2, will violate[4] VP if $f_3 \prec f_4$. A similar result can be derived for KMM with ϕ concave. Using the functional given by (4), the values of the acts in the 50:51 example are:

$$KMM(f_1) = \phi(151),$$
$$KMM(f_2) = \int_{\Delta(S)} \phi(i + j + 101) d\mu(i, j),$$
$$KMM(f_3) = \int_{\Delta(S)} \phi(i + j + 100) d\mu(i, j), \text{ and}$$
$$KMM(f_4) = \int_{\Delta(S)} \phi(2i + 2j + 50) d\mu(i, j).$$

Figure 1 represents the impact of the concavity of ϕ on the evaluation of the acts for $i+j < 50$ and $i+j > 50$. The case $i+j = 50$ is straightforward: it implies $\phi(151)-\phi(101+i+j) = \phi(100+i+j)-\phi(50+2i+2j)$. For $i+j < 50$, $U_{(i,j)}(f_1) > U_{(i,j)}(f_2) > U_{(i,j)}(f_3) > U_{(i,j)}(f_4)$. Moreover, the difference between $U_{(i,j)}(f_1)$ and $U_{(i,j)}(f_2)$ on the one hand and $U_{(i,j)}(f_3)$ and $U_{(i,j)}(f_4)$ on the other hand is the same. Figure 1(a) shows how concavity of ϕ implies that $\phi(151) - \phi(101+i+j) \leq \phi(100+i+j) - \phi(50+2i+2j)$ for all $(i+j) < 50$. The same result holds if $i + j > 50$ as can be seen in Figure 1(b). Under KMM, a preference for both f_1 over f_2 and f_4 over f_3 implies $KMM(f_1) - KMM(f_2) > KMM(f_3) - KMM(f_4)$ which is not possible because $\phi(151) - \phi(101 + i + j) \leq \phi(100 + i + j) - \phi(50 + 2i + 2j)$ for all (i, j). This leads to a contradiction. A decision-maker with ϕ concave cannot exhibit both $f_1 \succ f_2$ and $f_3 \prec f_4$. Note that this result can easily be extended to outcomes that are not equally spaced in terms of utility units, the proof being very similar.

To conclude this section, preferences that reflect the tradeoff between ambiguity and Bayesian advantages ($f_1 \succ f_2$ and $f_3 \prec f_4$) can be represented by none of the models

[4]Note that unlike under CEU, MEU, and αM, $f_1 \prec f_2$ and $f_3 \succ f_4$ may both hold under VP and KMM. These preferences however, are not plausible under natural ambiguity aversion.



(a) $i + j < 50$

(b) $i + j > 50$

Figure 1: Impact of the concavity of ϕ on the evaluation of the acts

examined.

III The reflection example

The second example proposed by Machina (2009), the *reflection example,* entails a slight modification of the previous urn; not 51 but 50 balls are marked with a 3 or a 4. Table 2 describes four acts assigning outcomes evaluated in terms of utility to the four events (with $0 < \pi < 1$). Unlike the previous example, this example does not require the outcomes to be equally-spaced on the utility scale.

	50 balls		50 balls	
Acts	E_1	E_2	E_3	E_4
f_5	100π	**100**	100π	0
f_6	100π	100π	**100**	0
f_7	0	**100**	100π	100π
f_8	0	100π	**100**	100π

Table 2: The reflection example

III.1 Decision criteria and experimental results

E_1 and E_2 (E_3 and E_4) are *informationally symmetric*: there is no more evidence in favor of one event or the other. Moreover, the two events $E_1 \cup E_2$ and $E_3 \cup E_4$ are equally likely. This is why Machina (2009) argues that f_8 is an *(informationally symmetric) left-right reflection* of f_5, and f_7 is a left-right reflection of f_6. As a consequence, there is no reason to prefer f_8 to f_7 if one prefers f_6 to f_5. We will say that preferences should be *reflected.* Machina shows that under CEU, $f_5 \prec f_6$ is equivalent to $f_7 \prec f_8$ and thus preferences should not be reflected, unless indifference holds. Hence, CEU can only account for reflected preferences through indifference ($f_5 \sim f_6$ and $f_7 \sim f_8$). However, in an experimental study of the reflection example, L'Haridon and Placido (2010) showed that such indifferences are rejected (over 90% of the subjects expressed strict preferences when indifference was allowed) while reflected preferences hold for more than 70% of subjects. Their data thus reject CEU.

More can be said about the pattern of preferences over these acts. Maximizing expected utility assuming a uniform distribution over the four events implies indifference between the four acts, because they lead to the same expected utility. Decision-makers may have to find other criteria unless they accept to be indifferent. On the one hand, $f_5 \prec f_6$ and $f_7 \succ f_8$ can be justified in the light of Ellsberg, because f_6 and f_7 assign known probabilities to at least one outcome (100π). Furthermore, f_6 and f_7 are less exposed to ambiguity than f_5 and f_8.[5] On the other hand, assuming some symmetry between E_2 and E_3, $f_5 \succ f_6$ and $f_7 \prec f_8$ will hold for decision-makers who want to avoid mean preserving spreads in expected utility values (see Result 2 in the Appendix).

The aforementioned arguments do not allow us to clearly predict what the preferences should be. However, we can still let the data speak. Up to now, the only experimental test of the reflection example we are aware of was conducted by L'Haridon and Placido (2010). The typical preference pattern they found was $f_5 \prec f_6$ and $f_7 \succ f_8$ (46% of the participants), even if 28% of the subjects exhibited $f_5 \succ f_6$ and $f_7 \prec f_8$. Furthermore, the experimenters replicated the Ellsberg paradox and found that $f_5 \prec f_6$ and $f_7 \succ f_8$ was still the most common pattern when only the subjects that are clearly ambiguity averse according to the Ellsberg paradox are considered. This confirms that ambiguity averse decision-makers tend to have this pattern of preferences. As a consequence, one might expect that a model of ambiguity aversion can account for $f_5 \prec f_6$ and $f_7 \succ f_8$. This is what we will check for in the four models under consideration in this paper.

III.2 Analysis of the reflection example

In what follows, (k, t) denotes any possible probability distribution over the state space with k the number of balls marked with a 2 and t the number of balls marked with a 3. Therefore, there are $50-k$ balls with a 1 and $50-t$ balls with a 4. $\Delta(S) = [0, 50] \times [0, 50]$ is the set of all possible (k, t) distributions.

[5] Consider an act assigning 100π to E_1, 100 to both E_2 and E_3, and 0 to E_4, and the following choice: remove $100(1-\pi)$ from E_2 (yielding f_6) or remove the same amount from E_3 (yielding f_5). The former completely removes an exposure to ambiguity while the latter only decreases a previously-existing exposure to ambiguity. A similar reasoning applies to f_7 and f_8.

First consider MEU. It can be shown that MEU will minimize some linear combinations of k and t in f_5 and f_8 whereas it minimizes only t in f_6 and only k in f_7. It is thus impossible for both f_6 and f_7 to be preferred to f_5 and f_8 respectively (see Result 4 in the Appendix). However, $f_5 \succ f_6$ and $f_7 \prec f_8$ may hold. MEU predicts that, if preferences are reflected, an ambiguity averse decision-maker will prefer the acts in which none of the outcomes are associated with a known probability. It cannot represent what L'Haridon and Placido (2010) found as being the prevailing ambiguity averse preferences. VP also fail to account for these preferences. The derivation of this result follows the same steps as in the case of MEU (Result 3 and Result 5 in the Appendix).

αM can explain the reflection example for any α as soon as $C \neq \Delta(S)$ and $\alpha \neq 1$. If the set of priors equates the set of all possible distributions ($C = \Delta(S)$), indifference should hold between the four acts. Assume now, for instance, $C = \Delta(S) - (49, 50] \times (49, 50]$ and $\alpha \neq 1$. Note that $k = 50$ or $t = 50$ are still possible independently. The maximum expected utility $(50 + 50\pi)$ is still possible for f_6 and f_7 but not for f_5 and f_8. Assume that $\pi \geq 1/2$. The valuations of the acts are: $\alpha M(f_5) = \alpha M(f_8) = 50\pi + (1-\alpha)(49+\pi)$, which is smaller than $\alpha M(f_6) = \alpha M(f_7) = 50\pi + (1 - \alpha)50$. Assume that $\pi < 1/2$. In such a case: $\alpha M(f_5) = \alpha M(f_8) = 50\pi + (1 - \alpha)(50 - \pi)$, which is also smaller than $\alpha M(f_6) = \alpha M(f_7) = 50\pi + (1 - \alpha)50$. Thus, $f_5 \prec f_6$ and $f_7 \succ f_8$ can both hold.

However, this result relies on a choice of priors that does not seem consistent with the information provided in the (thought) experiment. One may think that the informational symmetry of the decision problem should be present in the set of priors. We will say that the set of priors *replicates* the informational symmetry of the decision problem if $(k, t) \in C$ implies $(50 - k, t) \in C$, $(k, 50 - t) \in C$, and $(t, k) \in C$. If C ($C \neq \Delta(S)$) replicates the informational symmetry,[6] $f_5 \prec f_6$ ($f_7 \succ f_8$) implies $\alpha < 1/2$ (see Result 6). As a consequence, either C does not replicate the informational symmetry or $\alpha < 1/2$ (or both). In other words, decision-makers exhibiting $f_5 \prec f_6$ and $f_7 \succ f_8$ must change their preferences for some permutations of E_1 with E_2, E_3 with E_4, or (E_1, E_2) with (E_3, E_4) (if C does not replicate the informational symmetry, the numbers 1, 2, 3 and 4 must matter) or they must prefer to bet on the yellow and on the black balls rather than on

[6]This excludes the above example $C = \Delta(S) - (49, 50] \times (49, 50]$.

the red balls in the Ellsberg urn[7] (otherwise, α cannot be smaller than 1/2).

Finally, let us study Klibanoff, Marinacci, and Mukerji's smooth model of ambiguity. The preferences $f_5 \prec f_6$ and $f_7 \succ f_8$ imply that a KMM decision-maker cannot have a concave ϕ (see Result 7), no matter what μ is, i.e., whatever a KMM decision-maker thinks about the relevance of each probability distribution. Moreover, if this preference pattern does not depend on the outcomes under consideration, then ϕ must be convex. On the other hand, if the Ellsberg paradox holds whatever the color and the outcomes, then ϕ must be concave. This leads to a contradiction.

To summarize our results, KMM with ϕ concave, VP, and MEU cannot represent the attraction most people seem to feel for acts including outcomes with objective probabilities. αM can accommodate such behavior but, to do so, it must violate either informational symmetry or Ellsberg preferences.

IV Implications of Machina's examples for other models

Up to now, we have focused on four models of ambiguity-averse preferences. Ehud Lehrer (2007a) analyzed the impact of the reflection example for two other models: Lehrer's (2009) concave integral for capacities and Lehrer's (2007b) expected utility maximization w.r.t partially-specified probabilities. In both cases, he found that $f_5 \succ f_6$ and $f_7 \prec f_8$, but not the opposite preferences that were experimentally found. As a consequence, these two models have the same prediction as MEU and VP for the reflection example. Kin Chung Lo (2009) showed similar results for Klibanoff's (2001) version of MEU based on an unpublished example proposed by Machina in an earlier draft. Lo's (2009) results are consistent with ours.

Siniscalchi (2009) proposed VEU, which is able to account for both the 50:51 and the reflection examples. His model is decomposed into an expected utility term and an adjustment term capturing attitude towards ambiguity. Complementarities among

[7] Recall that the Ellsberg paradox involves an urn with 30 red balls and 60 balls that are either black or yellow.

ambiguous events (in the above examples, E_1 and E_2 on the one hand, and E_3 and E_4 on the other hand, have such complementarities) are represented by adjustment factors. The second term of the VEU model is a function defined over these adjustment factors. It is negative if Alain Chateauneuf and Jean-Marc Tallon's (2002) diversification axiom holds. Furthermore, it is negative and concave if Schmeidler's (1989) uncertainty aversion axiom holds. This axiom, which is necessary for VP and MEU, implies that "'smoothing' [...] utility distributions makes the decision-maker better off" (Schmeidler, 1989, p.582). Siniscalchi (2009) showed that VEU can handle the preference patterns considered in the present paper with an adjustment function that is negative but not concave,[8] meaning that the diversification axiom holds, but the uncertainty aversion axiom does not.[9]

A natural conjecture[10] is that the uncertainty aversion axiom drives most of the results in this paper. Without imposing further structure on preferences, this conjecture is false: Result 8 in the Appendix shows that some general preferences, satisfying the uncertainty aversion axiom, can accommodate Machina's paradoxes. However, the example we provide does not seem particularly intuitive; moreover, it is inconsistent with expected utility under risk.

As an alternative way to study the conjecture, we can impose more structure on preferences, for instance by assuming the standard independence axiom for expected utility under risk. Simone Cerreia-Vioglio, Maccheroni, Marinacci and Luigi Montrucchio (2009) consider complete, transitive, monotonic and continuous preferences that satisfy uncertainty aversion and the independence axiom; they provide a representation for such preferences, which we shall call "uncertainty averse representation" (UAR). Cerreia-Vioglio et al. show that MEU, VP, and KMM with ϕ concave are all special cases of UAR. We

[8]In both the 50:51 example and the reflection example, we can write the VEU value of act f as $VEU(f) = \sum_{i=1}^{4} p_{E_i} x_{E_i} + A(p_{E_1} x_{E_1} - p_{E_2} x_{E_2}, p_{E_3} x_{E_3} - p_{E_4} x_{E_4})$, where x_{E_i} denotes the utility value on event E_i and p_{E_i} denotes the (baseline) probability that a VEU decision-maker assigns to E_i. The higher $|p_{E_1} x_{E_1} - p_{E_2} x_{E_2}|$ or $|p_{E_3} x_{E_3} - p_{E_4} x_{E_4}|$, the more ambiguity the decision-maker perceives (these terms are 0 under risk). With $A(\nu_0, \nu_1) = -\frac{\sqrt{1+|\nu_0|}-1}{10} - \frac{\sqrt{1+|\nu_1|}-1}{10}$ and uniform baseline probabilities, we can derive both $f_1 \succ f_2$ and $f_3 \prec f_4$ in the 50:51 example and $f_5 \prec f_6$ and $f_7 \succ f_8$ in the reflection example.

[9]Because a VEU representation satisfying the uncertainty aversion axiom is a VP representation.

[10]We thank a referee for this conjecture.

show in the Appendix that UAR can accommodate the 50:51 example (Result 9), but not the reflection example (Result 3). In consequence, Machina's reflection example calls for going beyond the class of uncertainty averse representations.

Appendix

Result 1. *In the 50:51 example, VP imply* $f_1 \succ f_2 \Rightarrow f_3 \succ f_4$.

We can define (i_h, j_h) as any element of $argmin_{(i,j)\in\Delta(S)}\{U_{(i,j)}(f_h) + c(i,j)\}$. As a consequence, $VP(f_1) = 151 + c(i_1, j_1)$, $VP(f_2) = 101 + i_2 + j_2 + c(i_2, j_2)$, $VP(f_3) = 100 + i_3 + j_3 + c(i_3, j_3)$ and $VP(f_4) = 50 + 2i_4 + 2j_4 + c(i_4, j_4)$.
First, suppose that $f_1 \succ f_2$ and $f_3 \prec f_4$. Hence, $50+c(i_1, j_1) > i_2+j_2+c(i_2, j_2)$. Replacing i_4 and j_4 by i_3 and j_3 in $VP(f_4)$, because this can only increase the evaluation of the act, we obtain $i_3 + j_3 > 50$. By definition of (i_1, j_1), $c(i_1, j_1) \leq c(i_3, j_3)$. The sum of these inequalities gives $50 + c(i_1, j_1) < i_3 + j_3 + c(i_3, j_3)$. As $i_2 + j_2 + c(i_2, j_2) = i_3 + j_3 + c(i_3, j_3)$ must hold, we have $50 + c(i_1, j_1) < i_2 + j_2 + c(i_2, j_2)$. This leads to a contradiction.

Result 2. *Assuming* $\eta(k,t) = \eta(t,k)$ $\forall(k,t) \in \Delta(S)$ *(where* η *is a density defined over* $\Delta(S)$*),* f_6 *(*f_7*) can be derived from* f_5 *(*f_8*) by a series of mean preserving spreads in terms of expected utility values.*

Let η_{f_h} be the density function over the expected utility values induced by f_h and η $(h \in \{5,6\})$. We assume that $\eta(k,t) = \eta(t,k)$. Note that:

$$U_{(k,t)}(f_5) = k + (50 - k + t)\pi \tag{5}$$

$$U_{(k,t)}(f_6) = t + 50\pi \tag{6}$$

For all (k,t) such that $k = t$, both η_{f_5} and η_{f_6} assign $\eta(k,t)$ to $k + 50\pi$.

Let us now consider each $(k,t) \in \Delta(S)$ such that $t < k$ and its symmetric distribution (t,k) (we are thus dealing with every case satisfying $k \neq t$). If $1/2 \leq \pi < 1$:

$$U_{(k,t)}(f_6) < U_{(k,t)}(f_5) \leq U_{(t,k)}(f_5) < U_{(t,k)}(f_6). \tag{7}$$

Otherwise $(0 < \pi < 1/2)$:

$$U_{(k,t)}(f_6) < U_{(t,k)}(f_5) < U_{(k,t)}(f_5) < U_{(t,k)}(f_6). \tag{8}$$

Therefore, for all $t < k$ and no matter what π is, η_{f_5} assigns $\eta(k,t)$ and $\eta(t,k)$ (which, by assumption, are equal) to intermediate values while η_{f_6} assigns them to extreme values, moving density from the center to the tails of the distribution. Moreover, the mean expected utility has not changed because $U_{(k,t)}(f_5) + U_{(t,k)}(f_5) = U_{(k,t)}(f_6) + U_{(t,k)}(f_6)$. This corresponds to Michael Rothschild and Joseph E. Stiglitz's (1970) definition of a mean preserving spread. The same result can be obtained for f_7 and f_8 by symmetry.

Result 3. *UAR cannot accommodate $f_5 \prec f_6$ and $f_7 \succ f_8$.*

Cerreia-Vioglio et al. (2009) defined UAR as:

$$UAR(f) = \min_{p \in \Delta(S)} G(U_p(f), p), \tag{9}$$

where G is quasiconvex and non-decreasing in the first argument. We can define (k_h, t_h) as any element of $argmin_{(k,t)\in\Delta(S)} \left\{G\left(U_{(k,t)}(f_h), (k,t)\right)\right\}$. We must have $UAR(f_5) = G\left(k_5 + (50 - k_5 + t_5)\pi, (k_5, t_5)\right)$, $UAR(f_6) = G\left(t_6 + 50\pi, (k_6, t_6)\right)$, $UAR(f_7) = G\left(k_7 + 50\pi, (k_7, t_7)\right)$, and $UAR(f_8) = G\left(t_8 + (50 - t_8 + k_8)\pi, (k_8, t_8)\right)$. $f_5 \prec f_6$ implies $G\left(k_5 + (50 - k_5 + t_5)\pi, (k_5, t_5)\right) < G\left(t_6 + 50\pi, (k_6, t_6)\right)$ and $f_7 \succ f_8$ implies $G\left(t_8 + (50 - t_8 + k_8)\pi, (k_8, t_8)\right) < G\left(k_7 + 50\pi, (k_7, t_7)\right)$. By definition of (k_6, t_6) and (k_7, t_7), we can infer:

$$G\left(k_5 + (50 - k_5 + t_5)\pi, (k_5, t_5)\right) < G\left(t_5 + 50\pi, (k_5, t_5)\right), \tag{10}$$

$$G\left(t_8 + (50 - t_8 + k_8)\pi, (k_8, t_8)\right) < G\left(k_8 + 50\pi, (k_8, t_8)\right), \tag{11}$$

$$G\left(k_5 + (50 - k_5 + t_5)\pi, (k_5, t_5)\right) < G\left(t_8 + 50\pi, (k_8, t_8)\right), \tag{12}$$

$$G\left(t_8 + (50 - t_8 + k_8)\pi, (k_8, t_8)\right) < G\left(k_5 + 50\pi, (k_5, t_5)\right). \tag{13}$$

Recall that $0 < \pi < 1$. G must be non-decreasing in its first argument. Therefore, Eq. 10 implies $k_5 < t_5$ and Eq. 11 implies $t_8 < k_8$. These two implications, Eqs. 12 and 13, and G being non-decreasing in its first argument imply:

$$G\left(k_5 + 50\pi, (k_5, t_5)\right) < G\left(t_8 + 50\pi, (k_8, t_8)\right), \tag{14}$$

and

$$G\left(t_8 + 50\pi, (k_8, t_8)\right) < G\left(k_5 + 50\pi, (k_5, t_5)\right). \tag{15}$$

The two inequalities (14) and (15) contradict each other.

Result 4. *In the reflection example, MEU preferences $f_5 \prec f_6$ and $f_7 \succ f_8$ cannot both hold.*

This is implied by Result 3 with $G\left(U_{(k,t)}(f_h),(k,t)\right) = U_{(k,t)}(f_h)$.

Result 5. *In the reflection example, VP imply that $f_5 \prec f_6$ and $f_7 \succ f_8$ cannot both hold.*

This is implied by Result 3 with $G\left(U_{(k,t)}(f_h),(k,t)\right) = U_{(k,t)}(f_h) + c(k,t)$.

Result 6. *If C ($C \neq \Delta(S)$) replicates the informational symmetry of the decision problem, $f_5 \prec f_6$ ($f_7 \succ f_8$) implies $\alpha < 1/2$*

We say that the set of priors replicates the informationally symmetric left-right reflection whenever $(k,t) \in C$ implies $(50-k,t) \in C$, $(k,50-t) \in C$, and $(t,k) \in C$. Note that this definition also implies $(50-k,50-t) \in C$. Assume that $(k',t') \in argmin_{(k,t)\in C} U_{(k,t)}(f_5)$ and $(k'',t'') \in argmin_{(k,t)\in C} U_{(k,t)}(f_6)$. As a consequence of the structure of C and by reflection, $(t',k') \in argmin_{(k,t)\in C} U_{(k,t)}(f_8)$ and $(t'',k'') \in argmin_{(k,t)\in C} U_{(k,t)}(f_7)$.

It also implies that $(50-k',50-t') \in argmax_{(k,t)\in C} U_{(k,t)}(f_5)$, $(50-k'',50-t'') \in argmax_{(k,t)\in C} U_{(k,t)}(f_6)$, $(50-t',50-k') \in argmax_{(k,t)\in C} U_{(k,t)}(f_8)$ and $(50-t'',50-k'') \in argmax_{(k,t)\in C} U_{(k,t)}(f_7)$. $f_5 \prec f_6$ (or $f_7 \succ f_8$) implies:

$$\alpha\left(k' + (t'-k')\pi\right) + (1-\alpha)\left(50 - k' + (k'-t')\pi\right) < \alpha t'' + (1-\alpha)(50 - t'') \tag{16}$$

$$\alpha\left(k' - t'' + (t'-k')\pi\right) < (1-\alpha)\left(k' - t'' + (t'-k')\pi\right) \tag{17}$$

By definition of k', t' and t'' and because of the symmetry of C, $t'' \leq t'$ and $t'' \leq k'$. As a consequence, $(k' - t'' + (t'-k')\pi) \geq (k'-t'')(1-\pi) \geq 0$. If $(k' - t'' + (t'-k')\pi) = 0$, indifference should hold. It must thus be strictly positive. Hence, $\alpha < 1-\alpha$ and therefore, $\alpha < 1/2$.

Result 7. *If a KMM decision-maker has a concave ϕ, $f_5 \prec f_6$ and $f_7 \succ f_8$ cannot both hold.*

This is implied by Result 3 and by the fact that preferences that can be represented by KMM with ϕ being concave belong to UAR. It can also be proved in the following way. Values of the acts are:

$$KMM(f_5) = \int_{\Delta(S)} \phi(k + (50 + t - k)\pi) d\mu(k,t),$$
$$KMM(f_6) = \int_{\Delta(S)} \phi(t + 50\pi) d\mu(k,t),$$

$$KMM(f_7) = \int_{\Delta(S)} \phi(k + 50\pi) d\mu(k,t), \text{ and}$$
$$KMM(f_8) = \int_{\Delta(S)} \phi(t + (50 + k - t)\pi) d\mu(k,t).$$

A preference for both f_6 and f_7 against f_5 and f_8 implies

$\int_{\Delta(S)} [\phi(t + (50 + t - k)\pi) - \phi(t + 50\pi) + \phi(k + (50 + k - t)\pi) - \phi(k + 50\pi)] d\mu(k,t) < 0.$

However, if ϕ is concave, for all (k,t):

$\phi(t + (50 + t - k)\pi) - \phi(t + 50\pi) + \phi(k + (50 + k - t)\pi) - \phi(k + 50\pi) \geq 0.$

To prove this, let us define $a(k,t) = \phi(t + (50 + k - t)\pi) - \phi(t + 50\pi)$, and

$b(k,t) = \phi(k + (50 + t - k)\pi) - \phi(k + 50\pi)$.

Assume $t \geq k$; hence, $a(k,t) \leq 0$ and $b(k,t) \geq 0$. Note that $t + 50\pi - (t + (50 + k - t)\pi) = k + (50 + t - k)\pi - (k + 50\pi) = (t - k)\pi > 0$. Consequently, the same increase (i.e., $(t-k)\pi$) of the argument of ϕ is applied to two different levels: $k + 50\pi$ and $t + (50 + k - t)\pi$. If $t \geq k$, $k + 50\pi \leq t + (50 + k - t)\pi$. ϕ being increasing and concave, the impact of an increase of the arguments in terms of ϕ units should be lower for the highest argument. As a consequence, $b(k,t) \geq -a(k,t)$.

The opposite case $k \geq t$ is obtained by symmetry.

Result 8. *Preferences satisfying the uncertainty aversion axiom are not incompatible with $f_1 \succ f_2$ and $f_3 \prec f_4$, and with $f_5 \prec f_6$ and $f_7 \succ f_8$.*

Consider a preference relation represented by:

$$V(x_{E_1}, x_{E_2}, x_{E_3}, x_{E_4}) = \ln(3 + x_{E_1}) \cdot \ln(3 + x_{E_3}) + a \ln(3 + x_{E_2}) \cdot \ln(3 + x_{E_4}), \qquad (18)$$

with $a > 0$. V is defined on $\mathbb{R}^4_+$ and $(x_{E_1}, x_{E_2}, x_{E_3}, x_{E_4})$ represents the utility values associated with E_1, E_2, E_3, and E_4 respectively. V has a negative-definite Hessian matrix on $\mathbb{R}^4_+$ and therefore, V is strictly concave. This implies that the preference relation is complete and transitive (because the representation exists), monotone (because the function is monotone), and, above all, convex (because the function is concave). Convexity (with respect to utility values) implies that the uncertainty aversion axiom, as defined by Schmeidler (1989), holds. It is then sufficient to note that at $a = \frac{3}{2}$, $V(f_1) \approx 61.81$, $V(f_2) \approx 60.69$, $V(f_3) \approx 35.35$, and $V(f_4) \approx 38.12$ to see that convex preferences (satisfying monotonicity and weak ordeing) can accommodate the 50:51 example. Finally, at $a = 1$, $V(f_5) = V(f_8) = (\ln(3 + 100\pi))^2 + \ln(103)\ln(3)$ is less than

$V(f_6) = V(f_7) = \ln(3 + 100\pi)\ln(3) + \ln(3 + 100\pi)\ln(103)$ for all $0 < \pi < 1$.

Result 9. *UAR can accommodate $f_1 \succ f_2$ and $f_3 \prec f_4$.*

Starting from (9), we define $G(U_{(i,j)}(f), (i,j)) = G^*(U_{(i,j)}(f), i+j)$ with G^* such that:

$$G^*(x,y) = \begin{cases} 3 & \text{if } y \leq 49.5 \text{ or } y \geq 51 \\ & \text{or } (49.5 < y < 51 \text{ and } x \geq 151) \\ |4y - 201| & \text{if } 49.5 < y < 51 \text{ and } x \leq 100 + y \\ |4y - 201| + \frac{3-|4y-201|}{151-(100+y)} \times (x - (100 + y)) & \text{if } 49.5 < y < 51 \\ & \text{and } 100 + y < x < 151 \end{cases}$$

G^* (and hence, G) is weakly increasing in its first variable, (continuous) and quasiconvex. We get $UAR(f_1) = 3$, $UAR(f_2) \approx 2.8$, $UAR(f_3) = 0$, and $UAR(f_4) \approx 0.9$.

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