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Department of Mathematics
Odessa State University
Petra Velikogo 2
270 000 Odessa, Ukraine
E-mail: kolyada@kv.otb.odessa.ua

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Amenability and the second dual of a Banach algebra

by

FRÉDÉRIC GOURDEAU (Québec)

Abstract. Amenability and the Arens product are studied. Using the Arens product, derivations from $\mathcal{A}$ are extended to derivations from $\mathcal{A}^{**}$. This is used to show directly that $\mathcal{A}^{**}$ amenable implies $\mathcal{A}$ amenable.

1. Introduction and preliminaries. The study of cohomological properties of $\mathcal{A}^{**}$ in relation to those of $\mathcal{A}$ goes back to B. E. Johnson's seminal article [9]. Recently, Ghahramani, Loy and Willis [4] have studied the amenability and weak amenability of $\mathcal{A}$ in relation to the same properties for $\mathcal{A}^{**}$, with an emphasis on the Banach algebra $L^1(\mathcal{G})$. One of their result is that the amenability of $\mathcal{A}^{**}$ implies the amenability of A: this result was originally proved in [5] by other methods, but has not been published.

In this article, we show how Arens' construction of a product on the second dual of a Banach algebra enables us to extend derivations from $\mathcal{A}$ into a bimodule $\mathcal{X}$ to derivations from $\mathcal{A}^{**}$ into $\mathcal{X}^{**}$, answering a question raised in [9]. This is then used, along with a criterion for amenability which does not involve duals, to give a simple proof that $\mathcal{A}^{**}$ amenable implies $\mathcal{A}$ amenable.

For basic definitions, the reader is referred to [2]. Let $\mathcal{A}$ be a Banach algebra. Then the second dual of $\mathcal{A}$ can also be made into a Banach algebra, using either the *Arens product* or the *reversed Arens product*. For clarity and completeness, we recall precisely a few definitions related to the Arens product, and regroup properties we shall need in a lemma. The reader who wishes to return to the original is referred to [1].

Let X, Y and Z be Banach spaces and let $m : X \times Y \to Z$ be a bounded bilinear map. Let $x \in X$, $x' \in X^*$ and $x'' \in X^{**}$, where X^* is the Banach space dual of X, with similar notations for Y and Z. From m, we can construct a map $m^{***} : X^{**} \times Y^{**} \to Z^{**}$ in the following manner. For $x \in X$, $x' \in X^*$, $x'' \in X^{**}$, and so on, we have maps:

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$$m^* : Z^* \times X \to Y^* \quad \text{given by} \quad m^*(z', x)(y) = z'(m(x, y));$$
$$m^{**} : Y^{**} \times Z^* \to X^* \quad \text{given by} \quad m^{**}(y'', z')(x) = y''(m^*(z', x));$$
$$m^{***} : X^{**} \times Y^{**} \to Z^{**} \quad \text{given by} \quad m^{***}(x'', y'')(z') = x''(m^{**}(y'', z')).$$

The same construction can be done with the transposed map $m^{\mathrm{T}} : Y \times X \to Z$, to obtain $m^{\mathrm{T}***} : Y^{**} \times X^{**} \to Z^{**}$. Transposing again, we have $m^{\mathrm{T}***\mathrm{T}} : X^{**} \times Y^{**} \to Z^{**}$. How are m^{***} and $m^{\mathrm{T}***\mathrm{T}}$ related?

DEFINITION. The bilinear map m is *Arens regular* if and only if, for all $x'' \in X^{**}$ and all $y'' \in Y^{**}$, $m^{\mathrm{T}***\mathrm{T}}(x'', y'') = m^{***}(x'', y'')$.

Of course, the product of a Banach algebra $\mathcal{A}$ can be seen as a bilinear map $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ given by $m(a_1, a_2) = a_1 a_2$ for all $a_1, a_2 \in \mathcal{A}$. Using the construction above, we can construct two bilinear maps m^{***} and $m^{\mathrm{T}***\mathrm{T}}$ from $\mathcal{A}^{**} \times \mathcal{A}^{**}$ into $\mathcal{A}^{**}$; these maps turn out to give two products on $\mathcal{A}^{**}$.

DEFINITION. The Banach algebra $\mathcal{A}$ is *Arens regular* if the two definitions of a product given above coincide.

The product defined by m^{***} is called the *Arens product*, while the product defined by $m^{\mathrm{T}***\mathrm{T}}$ is called the *reversed Arens product*. The basic properties of the Arens product which we shall need are given in the following lemma.

For convenience, we denote by "$x_\alpha \xrightarrow{*} x''$" that x_α is a bounded net in X which, when seen as a net in X^{**}, converges to x'' in the weak-$*$ topology of X^{**}. We use $y_\beta \xrightarrow{*} y''$ in a similar way.

LEMMA 1.1. *Let $m : X \times Y \to Z$ be a bilinear map.*

(i) *m is Arens regular if and only if, for all $z' \in Z^*$, the bilinear form $z' \circ m : X \times Y \to \mathbb{C}$ is Arens regular;*

(ii) *$m^{***} : X^{**} \times Y^{**} \to Z^{**}$ is weak-$*$ to weak-$*$ continuous in $x'' \in X^{**}$ for a fixed $y'' \in Y^{**}$, and is weak-$*$ to weak-$*$ continuous in y'' for a fixed $x \in X$;*

(iii) *$m^{\mathrm{T}***\mathrm{T}} : X^{**} \times Y^{**} \to Z^{**}$ is weak-$*$ to weak-$*$ continuous in $x'' \in X^{**}$ for a fixed $y \in Y$, and is weak-$*$ to weak-$*$ continuous in y'' for a fixed $x'' \in X^{**}$;*

(iv) *$m^{***}(x'', y'') = \lim_\alpha \lim_\beta m(x_\alpha, y_\beta)$ weak-$*$ in Z^{**} for all $x'' \in X^{**}$, $y'' \in Y^{**}$, and all $x_\alpha \xrightarrow{*} x''$, $y_\beta \xrightarrow{*} y''$;*

(v) *$m^{\mathrm{T}***\mathrm{T}}(x'', y'') = \lim_\beta \lim_\alpha m(x_\alpha, y_\beta)$ weak-$*$ in Z^{**} for all $x'' \in X^{**}$, $y'' \in Y^{**}$, and all $x_\alpha \xrightarrow{*} x''$, $y_\beta \xrightarrow{*} y''$.*

Proof. The proofs of (i)–(iii) can be found in ([1], 2.3 and 3.2). However, we find it easier to prove them after proving (iv) and (v); the author did not find these last two properties stated as such elsewhere in the literature on the subject.

Given any bilinear map $m : X \times Y \to Z$, we have from the construction of m^{***},

$$\begin{aligned} m^{***}(x'', y'')(z') &= \langle m^{**}(y'', z'), x'' \rangle \\ &= \lim_\alpha \langle x_\alpha, m^{**}(y'', z') \rangle \quad \text{where } x_\alpha \xrightarrow{*} x'' \\ &= \lim_\alpha \langle m^*(z', x_\alpha), y'' \rangle \\ &= \lim_\alpha (\lim_\beta \langle y_\beta, m^*(z', x_\alpha) \rangle) \quad \text{where } y_\beta \xrightarrow{*} y'' \\ &= \lim_\alpha \lim_\beta \langle m(x_\alpha, y_\beta), z' \rangle. \end{aligned}$$

Therefore $m^{***}(x'', y'') = \lim_\alpha \lim_\beta m(x_\alpha, y_b)$ in the weak-$*$ topology of Z^{**}.

Similarly, $m^{\mathrm{T}***\mathrm{T}}(x'', y'') = \lim_\beta \lim_\alpha m(x_\alpha, y_\beta)$ in the weak-$*$ topology of Z^{**}. Thus we have proven (iv) and (v).

To prove (i), suppose that m is not Arens regular. From (iv) and (v), we deduce that there are $z' \in Z^*$, $x_\alpha \xrightarrow{*} x''$ and $y_\beta \xrightarrow{*} y''$ such that

$$\begin{aligned} \lim_\alpha \lim_\beta z' \circ m(x_\alpha, y_\beta) &= \lim_\alpha \lim_\beta \langle m(x_\alpha, y_\beta), z' \rangle \\ &\neq \lim_\beta \lim_\alpha \langle m(x_\alpha, y_\beta), z' \rangle = \lim_\beta \lim_\alpha z' \circ m(x_\alpha, y_\beta). \end{aligned}$$

Now, the left-hand side defines $(z' \circ m)^{***}(x'', y'')$ while the right-hand side defines $(z' \circ m)^{\mathrm{T}***\mathrm{T}}(x'', y'')$. This proves the "if" part. The other implication follows similarly.

As for (ii) and (iii), we can deduce them from the proof of (iv) and (v). For instance, it follows from $m^{***}(x'', y'')(z') = \langle m^{**}(y'', z'), x'' \rangle$ that m^{***} is weak-$*$ to weak-$*$ continuous in $x'' \in X^{**}$ for a fixed $y'' \in Y^{**}$. ■

Note that part (iv) of the lemma allows us to *define* the Arens product on $\mathcal{A}^{**}$ as follows. For F and G in $\mathcal{A}^{**}$, let (a_α) and (b_β) be two bounded nets in $\mathcal{A}$ which, when seen as nets in $\mathcal{A}^{**}$, converge in the weak-$*$ topology to F and G respectively. Then the Arens product of FG of F and G is given by $FG = \lim_\alpha \lim_\beta a_\alpha b_\beta$, where the limits are taken in the weak-$*$ topology of $\mathcal{A}^{**}$. The reversed Arens product is given by reversing the order of the limits.

2. Arens product and derivations. From now on, let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule (with module actions denoted by $a \cdot x$ and $x \cdot a$ for $a \in \mathcal{A}$ and $x \in \mathcal{X}$), and $D : \mathcal{A} \to \mathcal{X}$ be a bounded derivation. Also from now on, we let $x \in \mathcal{X}$, $x' \in \mathcal{X}^*$ and $x'' \in \mathcal{X}^{**}$, where $\mathcal{X}^*$ is the Banach space dual of $\mathcal{X}$.

Recall that a Banach algebra $\mathcal{A}$ is *amenable* if all bounded derivations from $\mathcal{A}$ into a dual module are necessarily inner. We shall use the following equivalent criterion for amenability (see [6]).

PROPOSITION 2.1 ([6]). *A Banach algebra $\mathcal{A}$ is amenable if and only if any bounded derivation from $\mathcal{A}$ into any Banach $\mathcal{A}$-bimodule is approximately inner*, *or*, *equivalently*, *weakly approximately inner*.

Here, we say that a derivation $D:\mathcal{A}\to\mathcal{X}$ is *approximately inner* (respectively *weakly approximately inner*) if there exists a bounded net (x_α) in $\mathcal{X}$ such that, for all $a\in\mathcal{A}$, $Da=\lim_{\alpha\to\infty}(a\cdot x_\alpha - x_\alpha\cdot a)$ in norm in $\mathcal{X}$ (respectively, in the weak topology on $\mathcal{X}$).

The study of relations between cohomological properties of $\mathcal{A}$ and those of $\mathcal{A}^{**}$ goes back to [9]. In this paper, Johnson considered the following general question: given $\mathcal{A}$ and an $\mathcal{A}$-bimodule $\mathcal{X}$, for which (if any) Banach $\mathcal{A}^{**}$-bimodule $\mathcal{Y}$ is there a link between $\mathcal{H}^1(\mathcal{A},\mathcal{X})$ and $\mathcal{H}^1(\mathcal{A}^{**},\mathcal{Y})$? The case $\mathcal{Y}=\mathcal{X}^*$, inspired by the need to have a dual module when looking at amenability, is shown not to work in general: $\mathcal{Y}$ cannot, in general, be given a natural $\mathcal{A}^{**}$-bimodule structure arising from the bimodule structure of $\mathcal{X}$. Thus, in [9], Johnson concludes that it does not seem possible to link cohomological properties of $\mathcal{A}$ with those of $\mathcal{A}^{**}$.

Fortunately, we can use a classical construction and Arens' ideas to circumvent this difficulty, and explicitly extend a derivation from $\mathcal{A}$ to one from $\mathcal{A}^{**}$.

Given a Banach $\mathcal{A}$-bimodule $\mathcal{X}$, we let $\mathcal{B}=\mathcal{A}\oplus\mathcal{X}$ be a Banach algebra, with product given by $(a,x)(b,y)=(ab,a\cdot y+x\cdot b)$, where (a,x), $(b,y)\in\mathcal{A}\oplus\mathcal{X}$, together with the norm $\|(a,x)\|=\|a\|_{\mathcal{A}}+\|x\|_{\mathcal{X}}$. Given a bounded derivation $D:\mathcal{A}\to\mathcal{X}$, we define the map $\theta:\mathcal{A}\to\mathcal{A}\oplus\mathcal{X}$ by $\theta(a)=(a,Da)$. It is easy to check that θ is a continuous homomorphism from $\mathcal{A}$ into $\mathcal{A}\oplus\mathcal{X}$.

Let us now consider $\mathcal{B}^{**}$ and $\mathcal{A}^{**}$ with the Arens product. (Note that we can also take the reversed Arens product: all the following results still hold, with the appropriate and straightforward modifications in the proofs.) The bilinear maps θ^{**} and D^{**}, given by dualizing θ and D twice, have the following properties.

LEMMA 2.2. *With the notation above*,

(i) *the map θ^{**} is a continuous homomorphism from $\mathcal{A}^{**}$ into $\mathcal{B}^{**}$, and is given by $\theta^{**}(a'')=(a'',D^{**}(a''))$ for $a''\in\mathcal{A}^{**}$*;

(ii) *the product of elements of the type $(a'',0)$ with those of the type $(0,x'')$ in $\mathcal{B}^{**}$ induces a Banach $\mathcal{A}^{**}$-bimodule structure on $\mathcal{X}^{**}$*;

(iii) *D^{**} is then a bounded derivation.*

Proof. By weak-$*$ continuity of θ^{**} (standard theory) and by Lemma 1.1(iv), we have

$$\theta^{**}(FG)=\lim_\alpha\lim_\beta\theta(a_\alpha b_\beta)=\lim_\alpha\lim_\beta\theta(a_\alpha)\theta(b_\beta)$$
$$=\lim_\alpha\theta(a_\alpha)\lim_\beta\theta(b_\beta)=\theta^{**}(F)\theta^{**}(G)$$

where (a_α) and (b_β) are two bounded nets in $\mathcal{A}$ which, when seen as nets in $\mathcal{A}^{**}$, converge in the weak-$*$ topology to F and G respectively. Thus we get the first statement.

To prove the second, define the actions of $\mathcal{A}^{**}$ on $\mathcal{X}^{**}$ by identifying $\mathcal{A}^{**}$ with $\mathcal{A}^{**}\oplus 0$ and $\mathcal{X}^{**}$ with $0\oplus\mathcal{X}^{**}$ in $\mathcal{B}^{**}$. The properties needed for these actions to be bimodule actions follow immediately from the fact that $\mathcal{B}^{**}$ is a Banach algebra. Another way of stating this definition is: let $a''\cdot x''$ be defined by the relation $(0,a''\cdot x'')=(a'',0)(0,x'')$, where the product is taken in $\mathcal{B}^{**}$, with a similar definition for $x''\cdot a''$.

Finally, that D is a continuous derivation is an easy consequence of (i) together with the definition of the product on $\mathcal{B}^{**}$. ■

We have thus shown that we can extend bimodule actions of $\mathcal{A}$ on $\mathcal{X}$ to bimodule actions of $\mathcal{A}^{**}$ on $\mathcal{X}^{**}$ in such a way that D^{**} is a bounded derivation. This yields the result we were after.

THEOREM 2.3 ([5]). *If $\mathcal{A}^{**}$ is amenable then $\mathcal{A}$ is amenable.*

Proof. Let the notation be as above with $D:\mathcal{A}\to\mathcal{X}$ a bounded derivation. Extending D to D^{**} as above, we deduce, from the hypothesis that $\mathcal{A}^{**}$ is amenable and Proposition 2.1, that D^{**} is approximately inner. Thus there is a bounded net $(x''_\alpha)\in\mathcal{X}^{**}$ such that, in particular,

$$Da=D^{**}a=\lim_{\alpha\to\infty}(a\cdot x''_\alpha-x''_\alpha\cdot a)\quad\text{in norm, for all } a\in\mathcal{A}.$$

Let x'' be a weak-$*$ accumulation point of (x''_α) in $\mathcal{X}^{**}$. Then, by definition of the module actions through the product on $\mathcal{B}^{**}$, for a fixed $a\in\mathcal{A}$, the module action is weak-$*$ continuous on $\mathcal{X}^{**}$ (Lemma 1.1). Thus

$$a\cdot x''-x''\cdot a=\lim_\alpha(a\cdot x''_\alpha-x''_\alpha\cdot a)=Da\quad\text{weak-}*\text{ in }\mathcal{X}^{**}.$$

Also from Lemma 1.1(ii), (iii), for a bounded net (x_λ) in $\mathcal{X}$ which tends to x'' in the weak-$*$ topology when seen as a net in $\mathcal{X}^{**}$,

$$a\cdot x''-x''\cdot a=\lim_\lambda(a\cdot x_\lambda-x_\lambda\cdot a)\quad\text{weak-}*\text{ in }\mathcal{X}^{**}.$$

However, in the last expression both $a\cdot x_\lambda-x_\lambda\cdot a$ and $a\cdot x''-x''\cdot a$ are in $\mathcal{X}$, and therefore

$$Da=\lim_\lambda(a\cdot x_\lambda-x_\lambda\cdot a)\quad\text{weakly in }\mathcal{X}.$$

Thus D is weakly approximately inner, and we conclude from Proposition 2.1 that $\mathcal{A}$ is amenable. ■

3. Conclusion. Given the links made between the amenability of $\mathcal{A}^{**}$ and the amenability of $\mathcal{A}$, one can wonder if Arens regularity is also linked to amenability. Counterexamples to all direct implications between amenability (or non-amenability) and Arens regularity (or non-Arens regularity) can be found in the literature, though not explicitly in all cases. Let us briefly give the references.

A class of Banach algebras which provides counterexamples is $l^1(S,w)$, the class of weighted convolution algebras on a discrete semigroup S. This class of algebras has been studied by Craw and Young for Arens regularity [3] and by Niels Grønbæk for amenability and weak amenability [7, 8].

It is shown in [10] that $l^1(\mathcal{G})$ is Arens regular if and only if $\mathcal{G}$ is finite. It is also well known that $l^1(\mathcal{G})$ is amenable if and only if $\mathcal{G}$ is amenable as a group. Thus $l^1(\mathcal{G})$ is amenable and Arens regular if $\mathcal{G}$ is finite; amenable and non-Arens regular if $\mathcal{G}$ is amenable and infinite; and not amenable and non-Arens regular if $\mathcal{G}$ is not amenable and infinite. For example, $l^1(\mathbb{Z})$ is amenable not being Arens regular, and for the free group $\mathcal{G}$ on two symbols, the algebra $l^1(\mathcal{G})$ is neither Arens regular nor amenable.

As a last example, the Banach algebra $l^1(\mathbb{Z},w)$, with weight $w(n) = 1+|n|$ for $n\in\mathbb{Z}$, is Arens regular and is not amenable. It is Arens regular because

$$\inf_{i\le j}\frac{w(m_i+n_j)}{w(m_i)w(n_j)}=0$$

for all sequences of distinct elements of $\mathbb{Z}$ (see [3]). And it is not amenable because $w(n)$ does not satisfy $\sup_g w(g)w(g^{-1})<\infty$ ([7], Theorem 3.2).

Would it be easier to link weak amenability with Arens regularity? There are no direct links there either. An amenable Banach algebra being weakly amenable, we already have examples of weakly amenable Banach algebras, some of which are Arens regular and some of which are not.

For the other examples, note that it easily follows from [8], Corollary 4.8, that $l^1(\mathbb{Z},w)$ is weakly amenable if and only if $\sup_{n\in\mathbb{Z}}\{|n|/(w(n)w(-n))\} = \infty$.

Thus, with the weight $w(n)=e^{|n|}$, $l^1(\mathbb{Z},w)$ is not weakly amenable: it is not Arens regular either as $w(m+n)/(w(m)w(n))=1$ (see [3]). With the weight $w(n)=1+|n|$, $l^1(\mathbb{Z},w)$ is not weakly amenable as $|n|/(w(n)w(-n)) < 1$, and it is Arens regular as we have seen before.

A question left open is if the amenability of $\mathcal{A}^{**}$ implies the Arens regularity of $\mathcal{A}$. The scarcity of examples of $\mathcal{A}^{**}$ which are amenable makes it difficult to have a clear idea of why this should or should not hold, and we have not found counterexamples to this implication (see [4] for more in this area).

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Département de mathématiques et de statistique
Université Laval
Cité Universitaire
Québec, Canada G1K 7P4
E-mail: frederic.gourdeau@mat.ulaval.ca

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