AMENABILITY FOR FELL BUNDLES

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Abstract. Given a Fell bundle $\mathcal{B}$, over a discrete group Γ, we construct its reduced cross sectional algebra $C_r^*(\mathcal{B})$, in analogy with the reduced crossed products defined for C^*-dynamical systems. When the reduced and full cross sectional algebras of $\mathcal{B}$ are isomorphic, we say that the bundle is amenable. We then formulate an approximation property which we prove to be a sufficient condition for amenability.

A theory of Γ-graded C^*-algebras possessing a conditional expectation is developed, with an eye on the Fell bundle that one naturally associates to the grading. We show, for instance, that all such algebras are isomorphic to $C_r^*(\mathcal{B})$, when the bundle is amenable.

We also study induced ideals in graded C^*-algebras and obtain a generalization of results of Strătilă and Voiculescu on AF-algebras, and of Nica on quasi-lattice ordered groups. A brief comment is made on the relevance, to our theory, of a certain open problem in the theory of exact C^*-algebras.

An application is given to the case of an $\mathbf{F}_n$–grading of the Cuntz–Krieger algebras $\mathcal{O}_A$, recently discovered by Quigg and Raeburn. Specifically, we show that the Cuntz–Krieger bundle satisfies the approximation property, and hence is amenable, for all matrices A with entries in $\{0, 1\}$, even if A does not satisfy the well known property (I) studied by Cuntz and Krieger in their paper.

1. Introduction. A possible definition of Fell bundles (also called C^*-algebraic bundles [**14**]), for the special case of discrete groups, states that these are given by a collection $\mathcal{B} = \big(B_t\big)_{t\in\Gamma}$ of closed subspaces of a C^*-algebra B, indexed by a discrete group Γ, satisfying $B_t^* = B_{t^{-1}}$ and $B_tB_s \subseteq B_{ts}$ for all t and s in Γ. If, in addition, the B_t's are linearly independent and their direct sum is dense in B, then B is said to be a graded C^*-algebra.

Fell bundles and graded algebras occur in an increasing number of situations in the theory of C^*-algebras, often without the perception that they are there. The example in which the Fell bundle structure is the most conspicuous, is that of the well known crossed

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product construction associated to a C^*-dynamical system [**18**], recently extended to the case of twisted partial dynamical systems [**11**]. See also [**9, 16**].

In the most tractable cases, these bundles have a commutative unit fiber algebra, that is, B_e (where e denotes the unit group element). In that case, by the main result of [**11**], one can say that, up to stabilization, the complexity of the bundle resides in three distinct compartments, namely, the topological structure of the spectrum of B_e, certain homeomorphisms between open sets in that spectrum, and a certain two cocycle. See [**11**] for more details.

Among the examples in which the Fell bundle structure is not so striking to the eye, lie some of the most intensely studied C^*-algebras of the past couple of decades. These include all the AF-algebras [**4,22,10**], the Cuntz–Krieger algebras $\mathcal{O}_A$ [**5,20**], algebras generated by Wiener-Hopf operators [**17**], non-commutative Heisenberg manifolds [**21,1**], the quantum SU_2 groups [**24**] (here one may use one of several easily available circle actions to obtain many interesting gradings, as done in [**9**]), the soft tori A_ε [**7, 8, 6**], and many others.

However intriguing this may be, most of the examples cited possess a commutative unit fiber algebra, and hence the comment above applies. But, there is a catch. The bundle structure alone is not enough to characterize the algebra. The point is that non-isomorphic graded algebras may possess identical associated Fell bundles, as in the case of the reduced and full group C^*-algebras of non-amenable discrete groups (see [**14**, VIII.16.12]).

It is the main purpose of this work to study this point in detail. The crux of the matter is thus to determine conditions on a Fell bundle $\mathcal{B}$, such that all graded C^*-algebras, whose associated Fell bundle coincides with $\mathcal{B}$, are isomorphic to each other. After we show that all such algebras lie in between the full and reduced cross sectional algebras of the bundle, that is $C^*(\mathcal{B})$ and $C^*_r(\mathcal{B})$, respectively, this is equivalent to saying that the left regular representation of $C^*(\mathcal{B})$ is faithful.

Inspired by the work of Andu Nica [**17**], we call such bundles *amenable*. Our main contribution is to formulate an approximation property for Fell bundles, which we prove to be a sufficient condition for amenability. This condition is strongly influenced by the work of Claire Anantharaman-Delaroche and, to a certain extent, could be thought of as an attempt at a generalization of a similar condition studied in [**2**]. We do not claim to have taken the analogy to its limits, as the role of the center of the unit fiber algebra, played in [**2**], is yet to be understood in the Fell bundle situation.

Our major application is to the case of the recently discovered bundle structure, over the free group $\mathbf{F}_n$, of the Cuntz–Krieger algebras $\mathcal{O}_A$, obtained by Quigg and Raeburn in [**20**], in terms of a co-action of $\mathbf{F}_n$. To study this example, we show that the Cuntz–Krieger relations, that is, the relations that define the algebras $\mathcal{O}_A$, give rise to a partial representation [**13**] of the free group. By a partial representation of a group Γ on a Hilbert space H, we mean a unital map $\sigma\colon\Gamma\to\mathcal{L}(H)$, such that, for all $t,r\in\Gamma$, one has $\sigma(t^{-1})=\sigma(t)^*$, and $\sigma(t)\sigma(r)\sigma(r^{-1})=\sigma(tr)\sigma(r^{-1})$. See [**13**] for more details.

In turn, given any partial representation of a discrete group, we construct an associated Fell bundle. This construction ascribes the Fell bundle related to $\mathcal{O}_A$, mentioned above, to the partial representation arising from any universal representation of the Cuntz–Krieger relations.

Irrespective of the main hypothesis imposed on the matrix A in [**5**], namely, condition (I), we show that the Cuntz–Krieger bundle satisfies the approximation property and hence is amenable. In fact, we prove that this holds for the Fell bundle associated to any semi-saturated (see below) partial representation σ of $\mathbf{F}_n$, which satisfies

$$\sum_{i=1}^{n} \sigma(g_i)\sigma(g_i)^* = 1,$$

where $g_1, \ldots, g_n$ are the generators of the free group.

On another front, we study induced ideals in graded algebras, where we are able to mimic Nica's work in [**17**] and obtain results very similar to his. In fact, the work of Nica has been an inspiration for us all along, as he also treats questions related to the approximation property. Other than merely formal generalizations, we seem to have gotten a little further, because our proof of the amenability of the Cuntz–Krieger bundle is given independently of the work of Cuntz and Krieger, themselves, namely, the uniqueness of the C^*-algebra generated by non-trivial representations of the Cuntz–Krieger relations, when property (I) is present [**5**].

Still under the heading of induced ideals, we point out a curious relationship with an open problem in the theory of exact C^*-algebras, stated in [**23**, 2.5.3]. The question of whether $C^*_r(\Gamma)$ is an exact C^*-algebra for any countable discrete group Γ, seems to be related to what we do here.

The author is indebted to several people who contributed in many ways for the evolution of this work. Among these he would like to express his thanks to Claire Anantharaman-Delaroche for a brief, but fruitful discussion during his short visit to Orleans, to Marcelo Laca for pointing out important references and also for several interesting conversations, and to Cristina Cerri for having carefully gone over the paper [**2**] at the operator algebra seminar in São Paulo and for many discussions as well.

2. Reduced Cross Sectional Algebras. Throughout this section, Γ will denote a discrete group and $\mathcal{B} = \{B_t\}_{t\in G}$ will be a fixed C^*-algebraic bundle over Γ, as defined in [**14**]. In recent years it has been customary to refer to C^*-algebraic bundles as *Fell bundles*, a terminology we think is quite appropriate.

Our specialization to the case of discrete groups has the primary purpose of avoiding technical details which occur in the theory of Fell bundles over continuous groups. With some more work, we believe that much, if not all, that we do here can be extended to the general case. Another reason for us being satisfied with the discrete group case is that our main application is for $\Gamma = \mathbf{F}_n$, the free group on n generators.

By a *section of $\mathcal{B}$* we mean any function

$$f\colon \Gamma \to \bigcup_{t\in\Gamma} B_t,$$

such that $f(t) \in \mathcal{B}_t$ for all $t \in \Gamma$.

Let $C_c(\mathcal{B})$ denote the set of all finitely supported sections of $\mathcal{B}$. We shall regard $C_c(\mathcal{B})$ as a right module over $\mathcal{B}_e$, which, when equipped with the B_e–valued inner product

$$\langle \xi, \eta \rangle = \sum_{t\in\Gamma} \xi(t)^* \eta(t), \quad \xi, \eta \in C_c(\mathcal{B}),$$

becomes a pre-Hilbert module in the sense of [**15**, 1.1.1]. The completion of $C_c(\mathcal{B})$ can be shown to consist of all cross sections ξ of $\mathcal{B}$ such that the series

$$\sum_{t\in\Gamma} \xi(t)^* \xi(t)$$

converges unconditionally. Incidentally, we say that a series $\sum_{i\in I} x_i$ in a Banach space E, indexed by any index set I, is unconditionally summable to $x \in E$ if, for any $\varepsilon > 0$, there exists a finite subset $I_0 \subseteq I$ such that, for all finite subsets $J \subseteq I$, with $I_0 \subseteq J$, one has

$$\|x - \sum_{i\in J} x_i\| < \varepsilon.$$

As we said before, we believe in the possibility of generalizing the results in this work to Fell bundles over continuous group. If that is to be attempted, then the unconditional summability, which is such a pervasive ingredient here, is likely to be replaced by the concept of unconditional integrability of [**12**].

We denote by $l_2(\mathcal{B})$ the completion of $C_c(\mathcal{B})$, so that $l_2(\mathcal{B})$ becomes a right Hilbert $\mathcal{B}_e$–module. Given ξ and η in $l_2(\mathcal{B})$ one can show that

$$\langle \xi, \eta \rangle = \sum_{t\in\Gamma} \xi(t)^* \eta(t),$$

where the series is unconditionally summable, as described above.

As in [**15**, 1.1.7], we will denote by $\mathcal{L}_{B_e}(l_2(\mathcal{B}))$, or simply by $\mathcal{L}(l_2(\mathcal{B}))$, the C^*-algebra of all adjointable operators on $l_2(\mathcal{B})$.

For each $t \in \Gamma$ and each $b_t \in B_t$ (the subscript in b_t is not absolutely necessary but it will be used to remind us that b_t belongs to the fiber B_t), define

$$\Lambda(b_t)\xi|_s = b_t \xi(t^{-1}s), \quad \xi \in l_2(\mathcal{B})), \quad s \in \Gamma.$$

It is not difficult to show that $\Lambda(b_t)\xi$ does belong to $l_2(\mathcal{B})$, and that $\|\Lambda(b_t)\xi\| \leq \|b_t\|\|\xi\|$.

2.1. Proposition. *$\Lambda(b_t)$ is an adjointable operator on $l_2(\mathcal{B})$ and $\Lambda(b_t)^* = \Lambda(b_t^*)$.*

Proof. Given $\xi, \eta \in l_2(\mathcal{B})$ we have

$$\langle \xi, \Lambda(b_t)\eta\rangle = \sum_{s\in\Gamma} \xi(s)^* b_t \eta(t^{-1}s) = \sum_{s\in\Gamma} \xi(ts)^* b_t \eta(s) = \sum_{s\in\Gamma} (b_t^* \xi(ts))^* \eta(s) = \ldots$$

Observe that $b_t^* \in B_{t^{-1}}$, hence $\Lambda(b_t^*)\xi|_s = b_t^*\xi(ts)$. So, the above equals

$$\ldots = \sum_{s\in\Gamma} (\Lambda(b_t^*)\xi|_s)^* \eta(s) = \langle \Lambda(b_t^*)\xi, \eta\rangle. \qquad \square$$

2.2. Proposition. *The map*

$$\Lambda\colon \bigcup_{t\in\Gamma} B_t \to \mathcal{L}(l_2(\mathcal{B}))$$

is a representation of $\mathcal{B}$ in the sense that for all $t, s \in \Gamma$ (compare [**14**, VIII.9.1]*)*

i) *Λ is a continuous linear map from B_t to $\mathcal{L}(l_2(\mathcal{B}))$,*

ii) *for $b_t \in B_t$ and $c_s \in B_s$ one has $\Lambda(b_t c_s) = \Lambda(b_t)\Lambda(c_s)$,*

iii) *$\Lambda(b_t^*) = \Lambda(b_t)^*$.*

Proof. Since (i) is easy and (iii) is already demonstrated, let us prove (ii). Take $\xi \in l_2(\mathcal{B})$, then, since $b_t c_s \in B_{ts}$,

$$\Lambda(b_t c_s)\xi|_r = b_t c_s \xi(s^{-1}t^{-1}r) = b_t\big(\Lambda(c_s)\xi|_{t^{-1}r}\big) = \Lambda(b_t)\Lambda(c_s)\xi|_r. \qquad \square$$

From now on we shall refer to Λ as the *left regular representation* of $\mathcal{B}$.

2.3. Definition. *The reduced cross sectional algebra of $\mathcal{B}$, denoted $C_r^*(\mathcal{B})$, is the sub C^*-algebra of $\mathcal{L}(l_2(\mathcal{B}))$ generated by the range of the left regular representation of $\mathcal{B}$.*

We recall that the (full) cross sectional algebra of $\mathcal{B}$, denoted $C^*(\mathcal{B})$ [**14**, VIII.17.2], is the enveloping C^*-algebra of $l_1(\mathcal{B})$. By the universal property (see [**14**, VIII.16.12]) of $C^*(\mathcal{B})$ one sees that there is a canonical epimorphism of C^*-algebras, which we call the *left regular representation* of $C^*(\mathcal{B})$ and, by abuse of language, still denote by Λ,

$$\Lambda\colon C^*(\mathcal{B}) \to C_r^*(\mathcal{B}).$$

We would now like to develop the rudiments of a Fourier analysis for $C_r^*(\mathcal{B})$, and, in particular, to define Fourier coefficients for elements of $C_r^*(\mathcal{B})$. For each $t \in \Gamma$ let

$$j_t\colon B_t \to l_2(\mathcal{B})$$

be the "inclusion" map, given by

$$j_t(b_t)|_s = \begin{cases} b_t & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases}$$

for $b_t \in B_t$ and $s \in \Gamma$

We shall regard each B_t as a right Hilbert B_e–module under the obvious right module structure and the B_e–valued inner product given by

$$\langle b_t, c_t \rangle = b_t^* c_t, \quad b_t, c_t \in B_t.$$

This said, one can easily prove that each j_t is an adjointable map and that for $\xi \in l_2(\mathcal{B})$, one has $j_t^*(\xi) = \xi(t)$. It follows that $j_t^* j_t$ is the identity map on B_t, and hence that j_t is an isometry. This allows us to identify B_t and its image $\bar{B}_t = j_t(B_t)$ within $l_2(\mathcal{B})$.

Incidentally, this shows the very subtle fact that any adjointable map from $l_2(\mathcal{B})$ into any Hilbert B_e–module, remains an adjointable map if restricted to $\bar{B}_t$, because restricting is the same as composing with j_t.

2.4. Proposition. *Let $t, s \in \Gamma$, $b_t \in B_t$ and $c_s \in B_s$. Then $\Lambda(b_t) j_s(c_s) = j_{ts}(b_t c_s)$. Therefore $\Lambda(b_t)\bar{B}_s \subseteq \bar{B}_{ts}$. In addition, the map*

$$b_t \in B_t \mapsto \Lambda(b_t)|_{\bar{B}_e} \in \mathcal{L}(\bar{B}_e, \bar{B}_t)$$

is isometric.

Proof. We have

$$\Lambda(b_t) j_s(c_s)|_r = b_t\big(j_s(c_s)|_{t^{-1}r}\big) = \delta_{t^{-1}r,s}\ b_t c_s = j_{ts}(b_t c_s)|_r,$$

where $\delta_{t^{-1}r,s}$ is the Kronecker symbol. This proves the first statement. With respect to the isometric property of the map above, all we are saying is that

$$\sup\{\|b_t a\| \colon a \in B_e, \|a\| \leq 1\} = \|b_t\|.$$

This follows from the fact that any approximate unit for B_e acts as an approximate unit for B_t also [**14**, VIII.16.3]. □

2.5. Corollary. *For each $t \in \Gamma$ and each $b_t \in B_t$, one has that $\|\Lambda(b_t)\| = \|b_t\|$ and hence each B_t may be identified with its image in $C_r^*(\mathcal{B})$.*

Proof. Follows immediately from (2.4). □

The following is the crucial step in defining Fourier coefficients.

2.6. Proposition. *For each x in $C_r^*(\mathcal{B})$ and each t in Γ there exists a unique b_t in B_t such that $j_t^* x j_e(a) = b_t a$ for every $a \in B_e$. In addition we have $\|b_t\| \leq \|x\|$.*

Proof. Uniqueness follows from the fact that if $b_t a = 0$ for all $a \in B_e$, then $b_t = 0$. As for the existence part, the easiest case, by far, is when B_e has a unit. In this case b_t is just $j_t^* x j_e(1)$. If no unit is available suppose, to start with, that $x = \sum_{s\in\Gamma} \Lambda(b_s)$, where $b_s \in B_s$ and $b_s = 0$ except for finitely many group elements s. We then have

$$j_t^* x j_e(a) = \sum_{s\in\Gamma} j_t^* \Lambda(b_s) j_e(a) = b_t a.$$

This says that $j_t^* x j_e$, viewed as an element of $\mathcal{L}(B_e, B_t)$, coincides with $\Lambda(b_t)|_{B_e}$, and hence lies in the isometric copy of B_t within $\mathcal{L}(B_e, B_t)$ provided by (2.4). Now, since the set of all $x's$ considered is dense in $C_r^*(\mathcal{B})$, we have obtained the existence part for all $x \in C_r^*(\mathcal{B})$.

Next observe that

$$\|b_t\| = \|\Lambda(b_t)|_{B_e}\| = \|j_t^* x j_e\| \le \|x\|. \qquad \square$$

2.7. Definition. *For x in $C_r^*(\mathcal{B})$ and $t \in \Gamma$ the t^{th} Fourier coefficient of x is the unique element $\hat{x}(t) \in B_t$ such that $j_t^* x j_e(a) = \hat{x}(t)a$ for all a in B_e. The Fourier transform of x is the cross section of $\mathcal{B}$ defined by $t \mapsto \hat{x}(t)$.*

Given the left regular representation of $C^*(\mathcal{B})$

$$\Lambda\colon C^*(\mathcal{B}) \to C_r^*(\mathcal{B}),$$

we can easily define the Fourier coefficients for elements of $C^*(\mathcal{B})$ as well, that is, if $y \in C^*(\mathcal{B})$ we put $\hat{y}(t) = \widehat{\Lambda(y)}(t)$.

From the proof of (2.6) we see that, if x is the finite sum $x = \sum_{t\in\Gamma} \Lambda(b_t)$ with $b_t \in B_t$, then $\hat{x}(t) = b_t$. Also it may be worth insisting that (2.6) yields $\|\hat{x}(t)\| \le \|x\|$.

2.8. Proposition. *For x in $C_r^*(\mathcal{B})$, $t, s \in \Gamma$ and $b_t \in B_t$ one has*

$$j_s^* x j_t(b_t) = \hat{x}(st^{-1})b_t.$$

Proof. By continuity it is enough to consider finite sums $x = \sum_{r\in\Gamma} \Lambda(c_r)$ as above. For such an x we have

$$j_s^* x j_t(b_t) = \sum_{r\in\Gamma} j_s^* \Lambda(c_r) j_t(b_t) = \sum_{r\in\Gamma} j_s^* j_{rt}(c_r b_t) = c_{st^{-1}} b_t = \hat{x}(st^{-1})b_t.$$

We have used that $j_s^* j_r = 0$ when $s \ne r$, a fact that is easy to see. $\square$

The Fourier coefficient $\hat{x}(e)$ has special properties worth mentioning.

2.9. Proposition. *The map $E\colon C_r^*(\mathcal{B}) \to B_e$ given by $E(x) = \hat{x}(e)$ is a positive, contractive conditional expectation.*

Proof. We first note that we are tacitly identifying B_e and its sibling $\Lambda(B_e)$, as permitted by (2.5). We have already seen that E is a contractive map. If $x = \Lambda(b_e)$ with $b_e \in B_e$, then we saw that $E(x) = b_e$, so E is idempotent.

If $x = \sum_{t\in\Gamma} \Lambda(b_t)$ as before, then

$$x^* x = \sum_{t,s\in\Gamma} \Lambda(b_t)^* \Lambda(b_s) = \sum_{t,s\in\Gamma} \Lambda(b_t^* b_s) = \sum_{t,r\in\Gamma} \Lambda(b_t^* b_{tr}) = \sum_{r\in\Gamma} \Lambda\big(\sum_{t\in\Gamma} b_t^* b_{tr}\big).$$

Now, since $\sum_{t\in\Gamma} b_t^* b_{tr}$ is in B_r, we have that $E(x^*x) = \sum_{t\in\Gamma} b_t^* b_t \ge 0$, so E is positive. Finally, if $x = \sum_{t\in\Gamma} \Lambda(b_t)$ is a finite sum, and if $a \in B_e$, then

$$E(ax) = E\big(\sum_{t\in\Gamma} \Lambda(ab_t)\big) = ab_e = aE(x)$$

and similarly, $E(xa) = E(x)a$. $\square$

2.10. Proposition. *For $x \in C_r^*(\mathcal{B})$ one has that the sum*

$$\sum_{t\in\Gamma} \hat{x}(t)^*\hat{x}(t)$$

is unconditionally convergent, and hence $\xi_x = \big(\hat{x}(t)\big)_{t\in\Gamma}$ represents an element of $l_2(\mathcal{B})$. Also, for any a in B_e we have $xj_e(a) = \xi_x a$.

Proof. Suppose that $x = \sum_{t\in\Gamma} \Lambda(b_t)$ is a finite sum, with $b_t \in B_t$. Then, obviously $\xi_x = \sum_{t\in\Gamma} j_t(b_t)$ belongs to $l_2(\mathcal{B})$ and we have

$$xj_e(a) = \sum_{t\in\Gamma} \Lambda(b_t)j_e(a) = \sum_{t\in\Gamma} j_t(b_t a) = \sum_{t\in\Gamma} j_t(b_t)a = \xi_x a.$$

Also

$$\|\xi_x\|^2 = \|\sum_{t\in\Gamma} b_t^* b_t\| = \|E(x^*x)\| \le \|x^*x\| = \|x\|^2.$$

Therefore the relation $x \mapsto \xi_x \in l_2(\mathcal{B})$ defines a bounded map, which so far is defined only for the $x's$ as above, but which may be extended to the whole of $C_r^*(\mathcal{B})$ by continuity.

Again by continuity we have $xj_e(a) = \xi_x a$, for any x in $C_r^*(\mathcal{B})$ and any a in B_e. Next observe that, for $t \in \Gamma$,

$$j_t^* x j_e(a) = j_t^*(\xi_x a) = j_t^*(\xi_x)a.$$

which implies, by (2.6), that $j_t^*(\xi_x) = \hat{x}(t)$. This says that $\xi_x = \big(\hat{x}(t)\big)_{t\in\Gamma}$ and the proof is therefore concluded. □

2.11. Corollary. *For $x \in C_r^*(\mathcal{B})$ one has*

$$E(x^*x) = \sum_{t\in\Gamma} \hat{x}(t)^*\hat{x}(t).$$

Proof. For $a, b \in B_e$ we have

$$a^*E(x^*x)b = \langle a, \widehat{x^*x}(e)b\rangle = \langle a, j_e^* x^* x j_e(b)\rangle = \langle xj_e(a), xj_e(b)\rangle =$$

$$= \langle \xi_x a, \xi_x b\rangle = a^*\langle \xi_x, \xi_x\rangle b = a^* \sum_{t\in\Gamma} \hat{x}(t)^*\hat{x}(t)b,$$

from which the conclusion follows. □

With this we arrive at an important result, which says, basically that E is a faithful conditional expectation on $C_r^*(\mathcal{B})$.

2.12. Proposition. *For $x \in C_r^*(\mathcal{B})$ the following are equivalent*

i) $E(x^*x) = 0$

ii) $\hat{x}(t) = 0$ *for every* $t \in \Gamma$

iii) $x = 0$.

Proof. The equivalence of (i) and (ii) is a consequence of (2.11). That (iii) implies (ii) is obvious, so let us prove that (ii) implies (iii). Assume $\hat{x}(t) = 0$ for all $t \in \Gamma$. Then, by (2.8), it follows that $j_s^* x j_t = 0$ for all t and s in Γ.

Now, note that any $\xi \in l_2(\mathcal{B})$ is the sum of the unconditionally convergent series $\xi = \sum_{t\in\Gamma} j_t j_t^*(\xi)$. So

$$x\xi = \sum_{s\in\Gamma}\sum_{t\in\Gamma} j_s j_s^* x j_t j_t^* \xi = 0,$$

that is, $x = 0$. □

3. Graded C^*-algebras. In this section we will study the relationship between graded C^*-algebras and Fell bundles. Graded algebras occur in a great number of different contexts in the theory of operator algebras, as in the theory of co-actions of discrete groups [**19**]. See also [**11**], [**17**], [**13**].

The following concept is taken from [**14**, VIII.16.11].

3.1. Definition. *Let B be a C^*algebra, Γ be a discrete group and let $\big(B_t\big)_{t\in\Gamma}$ be a collection of closed linear subspaces of B. We say that $\big(B_t\big)_{t\in\Gamma}$ is a grading for B if, for each t, s in Γ one has*

i) $B_t^* = B_{t^{-1}}$

ii) $B_t B_s \subseteq B_{ts}$

iii) The subspaces B_t are independent and B is the closure of the direct sum $\bigoplus_{t\in\Gamma} B_t$.

In that case we say that B is a Γ–graded C^*-algebra. Each B_t is called a grading subspace.

The primary example of graded C^*-algebras is offered by the theory of Fell bundles.

3.2. Proposition. *Let $\mathcal{B}$ be a Fell bundle over the discrete group Γ. Then $C_r^*(\mathcal{B})$ is a graded C^*-algebra via the fibers B_t of $\mathcal{B}$.*

Proof. The only slightly non trivial axiom to be proved regards the independence of the B_t's. Assume, for that purpose, that $x = \sum_{t\in\Gamma} \Lambda(b_t)$ is a finite sum with $b_t \in B_t$ and that $x = 0$. Then $b_t = \hat{x}(t) = 0$. □

A similar reasoning shows that the full cross sectional algebra is also naturally graded. Conversely, suppose we are given a graded C^*-algebra $B = \overline{\bigoplus_{t\in\Gamma} B_t}$ (by this notation we wish to say that all of the conditions of (3.1) are verified). One can then construct a Fell bundle over Γ, by taking the fibers of the bundle to be the grading subspaces. The multiplication and adjoint operations, required on a Fell bundle, are defined by restricting the corresponding operations on B.

In fact there is a great similarity between the formal definitions of Fell bundles and that of a graded C^*-algebras. However, there are important conceptual differences, better illustrated by the example provided by the full and reduced group C^*-algebras of a non-amenable discrete group. These are non-isomorphic graded C^*-algebras whose associated Fell bundles are indistinguishable from each other. See also [**14**, VIII.16.11].

Suppose we are given a Γ–graded C^*-algebra $B = \overline{\bigoplus_{t\in\Gamma} B_t}$. Let us denote by $\mathcal{B}$ its associated Fell bundle. Then, by the universal property of $C^*(\mathcal{B})$ [**14**, VIII.16.11], there is a unique C^*-algebra epimorphism

$$\Phi: C^*(\mathcal{B}) \to B,$$

which is the identity on each B_t (identified both with a subspace of $C^*(\mathcal{B})$ and of B in the natural way). This says that $C^*(\mathcal{B})$ is, in a sense, the biggest C^*-algebra whose associated Fell bundle is $\mathcal{B}$. Our next result will show that the reduced cross sectional algebra is on the other extreme of the range. It is also a very economical way to show a C^*-algebra to be graded.

3.3. Theorem. *Let B be a C^*-algebra and assume that for each t in a discrete group Γ, there is associated a closed linear subspace $B_t \subseteq B$ such that, for all t and s in Γ one has*

i) $B_t B_s \subset B_{ts}$,
ii) $B_t^* = B_{t^{-1}}$,
iii) the linear span of $\bigcup_{t\in\Gamma} B_t$ *is dense in* B.

Assume, in addition, that there is a bounded linear map

$$F: B \to B_e,$$

such that F is the identity map on B_e and that F vanishes on each B_t, for $t \neq e$. Then

(a) The subspaces B_t are independent and hence $\big(B_t\big)_{t\in\Gamma}$ is a grading for B.
(b) F is a positive, contractive, conditional expectation.
(c) If $\mathcal{B}$ denotes the associated Fell bundle, then there exists a C^*-algebra epimorphism

$$\lambda: B \to C_r^*(\mathcal{B}),$$

such that $\lambda(b_t) = \Lambda(b_t)$ for each t in Γ and each b_t in B_t.

Proof. If $x = \sum_{t\in\Gamma} b_t$ is a finite sum with $b_t \in B_t$, then

$$x^*x = \sum_{t,s\in\Gamma} b_t^* b_s = \sum_{r\in\Gamma}\Big(\sum_{t\in\Gamma} b_t^* b_{tr}\Big),$$

so $F(x^*x) = \sum_{t\in\Gamma} b_t^* b_t$.

Therefore, if $x = 0$ then each $b_t = 0$, which shows the independence of the B_t's, and also that F is positive. Given a in B_e, it is easy to see that $F(ax) = aF(x)$ and $F(xa) = F(x)a$. So, apart from contractivity, (b) is proven.

Define a pre right Hilbert B_e–module structure on B via the B_e–valued inner product

$$\langle x, y\rangle = F(x^*y), \quad x, y \in B.$$

For $b, x \in B$ we have, by the positivity of F that

$$\langle bx, bx\rangle = F(x^*b^*bx) \le \|b\|^2 F(x^*x) = \|b\|^2\langle x, x\rangle.$$

So, the left multiplication operators

$$L_b \colon x \in B \mapsto bx \in B$$

are bounded and hence extend to the Hilbert module completion X of B (after moding out vectors of norm zero). It is then easy to show that

$$L \colon b \in B \mapsto L_b \in \mathcal{L}(X)$$

is a C^*-algebra homomorphism.

Let $x = \sum_{t\in\Gamma} b_t$ and $y = \sum_{t\in\Gamma} c_t$ be finite sums with $b_t, c_t \in B_t$ and regard both x and y as elements of X. We then have

$$\langle x, y\rangle = F(\sum_{t,s\in\Gamma} b_t^* c_s) = F(\sum_{r\in\Gamma}\sum_{t\in\Gamma} b_t^* c_{tr}) =$$

$$= \sum_{t\in\Gamma} b_t^* c_t = \langle \sum_{t\in\Gamma} j_t(b_t), \sum_{t\in\Gamma} j_t(c_t)\rangle,$$

where the last inner product is that of $l_2(\mathcal{B})$. This is the key ingredient in showing that the formula

$$U(\sum_{t\in\Gamma} b_t) = \sum_{t\in\Gamma} j_t(b_t)$$

can be used to define an isometry of Hilbert B_e–modules

$$U \colon X \to l_2(\mathcal{B}).$$

Here it is important to remark that the continuity of F ensures that the set of finite sums $\sum_{t\in\Gamma} b_t$ is not only dense in B but also in X.

For b_t in B_t and c_s in B_s we have

$$UL_{b_t}(c_s) = U(b_t c_s) = j_{ts}(b_t c_s) = \Lambda(b_t) j_s(c_s) = \Lambda(b_t)U(c_s).$$

Since the finite sums $\sum_{s\in\Gamma} c_s$ are dense in X, as observed above, we conclude that

$$UL_{b_t}U^* = \Lambda(b_t).$$

This implies, for all b in B, that the operator UL_bU^* belongs to $C_r^*(\mathcal{B})$. This defines a map

$$\lambda \colon b \in B \mapsto UL_bU^* \in C_r^*(\mathcal{B}),$$

which satisfies the requirements in (c).

The only remaining task, that is, the proof of contractivity of F, now follows easily because $F = E\lambda$, where E is the conditional expectation of (2.9). □

The map λ, above, should be thought of as a generalized left regular representation of B.

From now on we will be mostly interested in graded algebras possessing a conditional expectation, and so we make the following:

3.4. Definition. *A grading $\left(B_t\right)_{t\in\Gamma}$ on the C^*-algebra B is said to be a topological grading if there exists a conditional expectation of B onto B_e, as in (3.3).*

Recalling our discussion about $C^*(\mathcal{B})$ being the biggest graded algebra for a given Fell bundle, we now see that $C^*_r(\mathcal{B})$ is the smallest such, at least among topologically graded algebras.

3.5. Corollary. *Let $B = \overline{\bigoplus}_{t\in\Gamma} B_t$ be a topologically graded C^*-algebra. Then, for every t in Γ, there exists a contractive linear map*

$$F_t\colon B \to B_t,$$

such that, for all finite sums $x = \sum_{t\in\Gamma} b_t$, with $b_t \in B_t$ one has $F_t(x) = b_t$.

Proof. Simply define $F_t(b) = \widehat{\lambda(b)}(t)$, where λ is as in (3.3). ☐

3.6. Proposition. *Let $B = \overline{\bigoplus}_{t\in\Gamma} B_t$ be a topologically graded C^*-algebra, with conditional expectation F, and let λ be its left regular representation, as in (3.3). Then*

$$Ker(\lambda) = \{b \in B\colon F(b^*b) = 0\}.$$

Proof. Observe, initially, that $F = E\lambda$ and hence

$$F(b^*b) = E(\lambda(b)^*\lambda(b)),$$

from where we see that $F(b^*b) = 0$ if and only if $\lambda(b) = 0$, because E is faithful, as proved in (2.12). ☐

This can be employed to give a useful characterization of $C^*_r(\mathcal{B})$ among graded algebras:

3.7. Proposition. *Let $B = \overline{\bigoplus}_{t\in\Gamma} B_t$ be a topologically graded C^*-algebra with faithful conditional expectation. Then B is naturally isomorphic to $C^*_r(\mathcal{B})$.*

Proof. The left regular representation $\lambda\colon B \to C^*_r(\mathcal{B})$ of (3.3) will be an isomorphism by (3.6). ☐

As an example, let us briefly treat the case of semi-direct product bundles, as defined in [**14**, VIII.4] (see also [**11**]). For this let (A, Γ, α) be a discrete C^*-dynamical system, i.e., A is a C^*-algebra and α is an action of the discrete group Γ on A. The *semi-direct product bundle* associated to (A, Γ, α) is defined to be the Cartesian product $A \times \Gamma$, with the operations

i) $z(a,t)+(b,t)=(za+b,t)$
ii) $(a,r)(b,s)=(a\alpha_r(b),rs)$
iii) $(a,t)^*=(\alpha_t^{-1}(a^*),t^{-1})$
iv) $\|(a,t)\|=\|a\|$

for any $a,b\in A$, any $r,s,t\in\Gamma$ and any complex number z.

3.8. Proposition. *The reduced cross sectional algebra of the semi-direct product bundle above is naturally isomorphic to the reduced crossed product algebra $A\rtimes_{\alpha,r}\Gamma$.*

Proof. $A\rtimes_{\alpha,r}\Gamma$ is graded and the conditional expectation is faithful [**18**, 7.11.3]. The result then follows from (3.7). □

Let us now conduct a study of induced ideals in graded algebras, inspired, among others, by the work of Strătilă and Voiculescu on AF-algebras [**22**] and the work of Nica on quasi-lattice ordered groups [**17**].

3.9. Theorem. *Let J be a closed, two-sided ideal in a topologically graded C^*-algebra $B=\overline{\bigoplus}_{t\in\Gamma}B_t$. Denote by F_t the projections onto each B_t, provided by (3.5). Define*

i) $J_1=\langle J\cap B_e\rangle$, that is, the ideal generated by $J\cap B_e$,
ii) $J_2=\{b\in B: F_t(b)\in J,\forall t\in\Gamma\}$,
iii) $J_3=\{b\in B: F_e(b^*b)\in J\}$.

Then $J_1\subseteq J_2=J_3$.

Proof. Let $\lambda\colon B\to C_r^*(\mathcal{B})$ be as in (3.3), and recall from (2.11) that

$$E(x^*x)=\sum_{t\in\Gamma}\hat{x}(t)^*\hat{x}(t),$$

for all x in $C_r^*(\mathcal{B})$. Now, since we have $F_t(b)=\widehat{\lambda(b)}(t)$, we conclude that

$$F_e(b^*b)=\sum_{t\in\Gamma}F_t(b)^*F_t(b),\quad b\in B.$$

Therefore, recalling that ideals are hereditary sub-algebras, we see that $F_e(b^*b)\in J$ if and only if $F_t(b)^*F_t(b)\in J$ for all t in Γ, which is the same as saying that $F_t(b)\in J$ for all t. This proves that $J_2=J_3$. Now, observe that the ideal generated by $J\cap B_e$ contains, as a dense set, the collection of finite sums of the form

$$x=\sum_i b_{r_i}x_ic_{s_i},$$

where $r_i,s_i\in\Gamma$, $b_{r_i}\in B_{r_i}$, $c_{s_i}\in B_{s_i}$ and x_i is in $J\cap B_e$. For x as above, note that

$$F_t(x)=\sum_{r_is_i=t}b_{r_i}x_ic_{s_i},$$

so $F_t(x)\subseteq J$. This proves that $x\in J_2$ and, by the density of the set of x's considered, that $J_1\subseteq J_2$. □

Later, after concluding our study of amenable bundles, we shall return to the question of the equality between J_1 and J_2.

3.10. Definition. *We say that a closed, two-sided ideal J, in a graded C^*-algebra $B = \overline{\bigoplus}_{t\in\Gamma} B_t$, is an induced ideal if $J = \langle J \cap B_e \rangle$.*

Before closing this section let us consider quotients of graded algebras.

3.11. Proposition. *Let J be an induced ideal in a topologically graded C^*-algebra $B = \overline{\bigoplus}_{t\in\Gamma} B_t$. Then the quotient C^*-algebra B/J is topologically graded by the spaces $\pi(B_t)$, where $\pi\colon B \to B/J$ is the quotient map.*

Proof. By (3.9), J is invariant under F_t. Therefore F_t gives a well defined bounded map on B/J, namely

$$\tilde{F}_t(x + J) = F_t(x) + J, \quad x \in B.$$

We claim that $\pi(B_t)$ is a closed subspace of B/J, for each t. In fact, let $x \in B_t$ and $y \in J$. Then

$$\|x - y\| \geq \|F_t(x - y)\| = \|x - F_t(y)\|.$$

In view of the fact that $F_t(y) \in J \cap B_t$, we conclude that

$$\|x + J\| = \inf\{\|x - y\| \colon y \in J\} = \inf\{\|x - y\| \colon y \in J \cap B_t\}.$$

The last term above gives the norm, in $B_t/(J \cap B_t)$, of the element $x + (J \cap B_t)$. In other words, the natural map

$$B_t/(J \cap B_t) \to B/J$$

is an isometry. Therefore, $\pi(B_t)$, being isometric to $B_t/(J \cap B_t)$, is a Banach space and hence closed in B/J.

It is now immediate to verify that the collection $\big(\pi(B_t)\big)_{t\in\Gamma}$ satisfies (3.3.i–iii), and that $\tilde{F}_e$ fills in the rest of the hypothesis there to allow us to conclude that $\big(\pi(B_t)\big)_{t\in\Gamma}$ is, in fact, a grading for B/J. That this is a topological grading follows from (3.3.b). This concludes the proof. □

4. Amenability for Fell bundles. We return to the case of a general Fell bundle $\mathcal{B}$ over a discrete group Γ. Initially, we would like to adopt a convention designed to simplify our notation. That is, we will enforce, to the fullest extent, the identification of the fibers B_t of $\mathcal{B}$ with its isomorphic image in any Γ–graded C^*-algebra in sight, whose associated Fell bundle is identical to $\mathcal{B}$. Of course, this applies to $C^*_r(\mathcal{B})$ and $C^*(\mathcal{B})$. An excellent point of view is, in fact, to regard all such graded C^*-algebras as completions of the *-algebra $\bigoplus_{t\in\Gamma} B_t$ under various norms.

The result in (3.3), together with the remark preceding it, tells us that the topologically graded algebras whose associated Fell bundles coincide with a given Fell bundle $\mathcal{B}$, are to be found among the quotients of $C^*(\mathcal{B})$ by ideals contained in the kernel of the left regular representation $\Lambda\colon C^*(\mathcal{B}) \to C^*_r(\mathcal{B})$ mentioned after (2.3).

It is therefore crucial to understand the kernel of Λ and, in particular, to determine conditions to ensure that Λ is injective. While we believe a full comprehension of the problem is beyond the present methods, we intend to give a sufficient condition for the injectivity of Λ, which is general enough to be applicable to an interesting Fell bundle appearing in the theory of Cuntz–Krieger algebras. First some terminology.

4.1. Definition. *The Fell bundle $\mathcal{B}$ is said to be amenable if the left regular representation*

$$\Lambda: C^*(\mathcal{B}) \to C^*_r(\mathcal{B}).$$

is injective.

An immediate consequence of the results in the previous section is:

4.2. Proposition. *Let $\mathcal{B}$ be an amenable Fell bundle. Then all topologically graded C^*-algebras, whose associated Fell bundles coincide with $\mathcal{B}$, are isomorphic to each other.*

One of the main tools in the technique employed below is a study of functions $a:\Gamma \to B_e$ such that $\sum_{t\in\Gamma} a(t)^* a(t)$ converges unconditionally. The reason is that it allows us to construct "wrong way maps" $\Psi: C^*_r(\mathcal{B}) \to C^*(\mathcal{B})$, as we will now see.

4.3. Lemma. *Let $a:\Gamma \to B_e$ be such that $\sum_{t\in\Gamma} a(t)^* a(t)$ converges unconditionally. Then there exists a bounded, completely positive map*

$$\Psi: C^*_r(\mathcal{B}) \to C^*(\mathcal{B})$$

such that

$$\Psi(b_t) = \sum_{r\in\Gamma} a(tr)^* b_t a(r), \quad b_t \in B_t.$$

In addition $\|\Psi\| \le \|\sum_{t\in\Gamma} a(t)^* a(t)\|$.

Proof. Assume, as we may, that $C^*(\mathcal{B})$ is faithfully represented as an algebra of operators on a Hilbert space H. In fact, in order to lighten up our notation, we will think of $C^*(\mathcal{B})$ as a subalgebra of $\mathcal{L}(H)$. This also gives us a representation of $\mathcal{B}$ on H, as in [**14**, VIII.9.1]. Define a new representation of $\mathcal{B}$ on $H \otimes l_2(\Gamma)$ by letting

$$\pi(b_t) = b_t \otimes \lambda_t, \quad t \in \Gamma, \quad b_t \in B_t,$$

where $\lambda: \Gamma \to \mathcal{L}(l_2(\Gamma))$ is the usual left regular representation of Γ. Let B be the C^*-algebra of operators on $H \otimes l_2(\Gamma)$ generated by all the $\pi(b_t)$.

Denoting the canonical basis of $l_2(\Gamma)$ by $(\delta_t)_{t\in\Gamma}$ consider the isometries

$$j_t: \xi \in H \to \xi \otimes \delta_t \in H \otimes l_2(\Gamma).$$

For each finite sum $b = \sum_{t\in\Gamma} \pi(b_t)$, with $b_t \in B_t$, one can easily verify that $j_t^* b j_s = b_{ts^{-1}}$. In particular, this shows that $b = 0$ causes each b_t to vanish. It follows that $(B_t \otimes \lambda_t)_{t\in\Gamma}$ is a grading for B. Moreover, we claim that the natural conditional expectation

$$F: b \in B \to (j_e^* b j_e) \otimes \lambda_e \in B_e \otimes \lambda_e$$

is faithful. In fact suppose that $F(b^*b) = 0$, for a given b in B. Then $j_e^* b^* b j_e = 0$ and hence $bj_e = 0$. Consider the blown up right regular representation of Γ, that is, the (anti) representation of Γ on $H \otimes l_2(\Gamma)$ given by

$$\rho_t(\xi \otimes \delta_s) = \xi \otimes \delta_{st}, \quad s, t \in \Gamma, \quad \xi \in H.$$

It is easy to see that each ρ_t belongs to the commutant of B and also that

$$\rho_t j_s = j_{st}, \quad s, t \in \Gamma.$$

Now, since we have $bj_e = 0$, we will also have $bj_t = b\rho_t j_e = \rho_t b j_e = 0$. This implies that b vanishes on each $H \otimes \delta_t$ and hence that $b = 0$. With our claim proven we may apply (3.7) to conclude that B is naturally isomorphic to $C_r^*(\mathcal{B})$.

Consider the operator

$$V\colon H \to H \otimes l_2(\Gamma)$$

given by the formula $V(\xi) = \sum_{t\in\Gamma} a(t)\xi \otimes \delta_t$, for $\xi \in H$. This is well defined because

$$\sum_{t\in\Gamma} \|a(t)\xi\|^2 = \sum_{t\in\Gamma} \langle a(t)^* a(t)\xi, \xi\rangle \leq \|\sum_{t\in\Gamma} a(t)^* a(t)\| \|\xi\|^2,$$

which also informs us that V is bounded with $\|V\| \leq \|\sum_{t\in\Gamma} a(t)^* a(t)\|^{1/2}$.

Define the completely positive map

$$\Psi\colon \mathcal{L}(H \otimes l_2(\Gamma)) \to \mathcal{L}(H)$$

by $\Psi(x) = V^* x V$, for each x in $\mathcal{L}(H \otimes l_2(\Gamma))$.

Observe that for every b_t in B_t and ξ in H one has

$$\Psi(b_t \otimes \lambda_t)\xi = V^*(b_t \otimes \lambda_t)\Big(\sum_{r\in\Gamma} a(r)\xi \otimes \delta_r\Big) = V^*\Big(\sum_{r\in\Gamma} b_t a(r)\xi \otimes \delta_{tr}\Big) = \sum_{r\in\Gamma} a(tr)^* b_t a(r)\xi,$$

where we have used the fact that

$$V^*(\eta \otimes \delta_t) = a(t)^*\eta, \quad t \in \Gamma, \quad \eta \in H,$$

which the reader will have no difficulty in proving. Summarizing, have that

$$\Psi(b_t \otimes \lambda_t) = \sum_{r\in\Gamma} a(tr)^* b_t a(r).$$

Recall that B is generated by the $b_t \otimes \lambda_t$. So, we see that Ψ maps B into $C^*(\mathcal{B})$. Therefore, if we identify $C_r^*(\mathcal{B})$ with B, we may restrict Ψ to $C_r^*(\mathcal{B})$ and get the desired map.

Finally

$$\|\Psi\| \leq \|V\|^2 \leq \|\sum_{t\in\Gamma} a(t)^* a(t)\|. \qquad \square$$

This puts us in position to describe an important concept.

4.4. Definition. *We say that the Fell bundle $\mathcal{B}$ has the approximation property if there exists a net $\big(a_i\big)_{i\in I}$ of functions $a_i\colon \Gamma \to B_e$, which is uniformly bounded in the sense that*

$$\sup_{i\in I} \|\sum_{t\in\Gamma} a_i(t)^* a_i(t)\| < \infty,$$

and such that for all b_t in each B_t one has

$$\lim_{i\to\infty} \sum_{r\in\Gamma} a_i(tr)^* b_t a_i(r) = b_t.$$

An equivalent, although apparently stronger form of the approximation property is given by our next result:

4.5. Proposition. *Suppose $\mathcal{B}$ has the approximation property. Then there exists a net $\big(a_i\big)_{i\in I}$ of finitely supported functions $a_i\colon \Gamma \to B_e$, satisfying the conditions of (4.4).*

Proof. The approximation property implies that there exists a constant $M > 0$ such that for all finite sets

$$X = \{b_{t_1}, b_{t_2}, \ldots, b_{t_n}\} \subseteq \bigcup_{t\in\Gamma} B_t$$

and any $\varepsilon > 0$, there exists a function $a\colon \Gamma \to B_e$ such that $\|\sum_{t\in\Gamma} a(t)^* a(t)\| < M$ and $\|b_{t_k} - \sum_{r\in\Gamma} a(t_k r)^* b_{t_k} a(r)\| < \varepsilon, \quad k = 1, \ldots, n$. By taking the restriction of a to a sufficiently large set, we may assume that a is finitely supported. Once this is done, denote such an a by $a_{X,\varepsilon}$ and think of it as a net, indexed by the set of pairs (X, ε), where X is any finite subset of $\bigcup_{t\in\Gamma} B_t$ and ε is a positive real. It is often the case that the best nets are indexed by the most awful looking sets.

If these pairs are ordered by saying that $(X_1, \varepsilon_1) \le (X_2, \varepsilon_2)$ if and only if $X_1 \subseteq X_2$ and $\varepsilon_1 \ge \varepsilon_2$ then the net $\big(a_{X,\varepsilon}\big)_{(X,\varepsilon)}$ satisfies the required properties. ☐

The relationship between the approximation property and the amenability of Fell bundles is one of our main results:

4.6. Theorem. *If a Fell bundle $\mathcal{B}$ has the approximation property, then it is amenable.*

Proof. Let $\big(a_i\big)_{i\in I}$ be as above. For each $i \in I$ consider the map $\Psi_i\colon C^*_r(\mathcal{B}) \to C^*(\mathcal{B})$ provided by (4.3), regarding each a_i. Define $\Phi_i\colon C^*(\mathcal{B}) \to C^*(\mathcal{B})$ to be the composition $\Phi_i = \Psi_i \Lambda$ and observe that, by hypothesis $\lim_{i\to\infty} \Phi_i(b_t) = b_t$, for every b_t in each B_t. Because the b_t's span a dense subspace of $C^*(\mathcal{B})$, and because the Φ_i's are uniformly bounded, we conclude that $\lim_{i\to\infty} \Phi_i(x) = x$ for every x in $C^*(\mathcal{B})$.

Now, if $x \in C^*(\mathcal{B})$ is in the kernel of the left regular representation, that is, if $\Lambda(x) = 0$, then

$$x = \lim_{i\to\infty} \Phi_i(x) = \lim_{i\to\infty} \Psi_i(\Lambda(x)) = 0,$$

which proves that Λ is injective. ☐

A somewhat trivial example of this situation is that of bundles over amenable groups.

4.7. Theorem. *Let $\mathcal{B}$ be a Fell bundle over the discrete amenable group Γ. Then $\mathcal{B}$ satisfies the approximation property and hence is amenable.*

Proof. According to [**18**, 7.3.8], there exists a bounded net $\big(f_i\big)_{i\in I}\subseteq l_2(\Gamma)$ such that

$$\lim_{i\to\infty}\sum_{r\in\Gamma}\overline{f_i(tr)}f_i(r)=1,\quad t\in\Gamma.$$

Let $\big(u_j\big)_{j\in J}$ be an approximate unit for B_e. One then checks that the doubly indexed net $\big(a_{i,j}\big)_{(i,j)\in I\times J}$ defined by $a_{i,j}(t)=f_i(t)u_j$ satisfies the definition of the approximation property. □

A very important consequence of the approximation property is the existence of an analogue of Cesaro sums, which we would now like to discuss.

4.8. Definition. *A bounded linear map $\Phi\colon B\to B$, where $B=\overline{\bigoplus}_{t\in\Gamma}B_t$ is a graded C^*-algebra, is said to be a summation process if, for all x in B*

i) $\sum_{t\in\Gamma}\Phi(\hat{x}(t))$ *is unconditionally convergent,*

ii) $\Phi(x)=\sum_{t\in\Gamma}\Phi(\hat{x}(t))$,

iii) $\Phi(B_t)\subseteq B_t,\quad t\in\Gamma$.

4.9. Proposition. *Let $B=\overline{\bigoplus}_{t\in\Gamma}B_t$ be a topologically graded C^*-algebra. Assume that the Fell bundle $\mathcal{B}$, associated to B, has the approximation property. Then there exists a bounded net $\big(\Phi_i\big)_{i\in I}$ of summation processes, converging to the identity map of B in the pointwise topology.*

Proof. Since $\mathcal{B}$ is amenable, by (4.2), B is isomorphic to $C^*_r(\mathcal{B})$. We may then assume that, in fact, B is the reduced cross sectional algebra $C^*_r(\mathcal{B})$, itself.

Let $\big(a_i\big)_{i\in I}$ be a finitely supported net, as in (4.5). Denote by Ψ_i the completely positive map $\Psi\colon C^*_r(\mathcal{B})\to C^*(\mathcal{B})$ given by (4.3), regarding each a_i, and let $\Phi_i=\Psi_i\Lambda$, following the first steps of the proof of (4.6). So, we have

$$\lim_{i\to\infty}\Phi_i(x)=x,\quad x\in C^*(\mathcal{B}).$$

For each $i\in I$, let S_i be the finite support of a_i, so

$$\Phi_i(b_t)=\sum_{r\in\Gamma}a_i(tr)^*b_ta_i(r)=\sum_{r\in S_i}a_i(tr)^*b_ta_i(r),\quad b_t\in B_t.$$

One can see, from this, that $\Phi_i(b_t)=0$ unless $t\in S_iS_i^{-1}$. This says that the series appearing in (4.8.i) is in fact a finite series, hence convergent.

Also, (4.8.iii) follows from the above formula for Φ_i, since each $a_i(r)\in B_e$.

Finally, if x is a finite sum $x=\sum_{t\in\Gamma}b_t$, with $b_t\in B_t$, then $\hat{x}(t)=b_t$, that is $x=\sum_{t\in\Gamma}\hat{x}(t)$. This implies (4.8.ii) for such an x and hence also for all $x\in B$, by continuity of both sides of (4.8.ii), now that we know that the sum involved is finite. □

Recalling our discussion of induced ideals in (3.9) we may now add the following:

4.10. Proposition. *Let $B = \overline{\bigoplus}_{t\in\Gamma} B_t$ be a topologically graded C^*-algebra. Assume, in addition to the hypothesis of (3.9), that the Fell bundle $\mathcal{B}$, associated to B, has the approximation property. Then $J_1 = J_2 = J_3$.*

Proof. Let $\left(\Phi_i\right)_{i\in I}$ be a net of summation processes, converging pointwise to the identity map of B, as in (4.9).

Next, let x be in J_2, so that $\hat{x}(t) \in J \cap B_t$ for all $t \in \Gamma$ and, consequently $\hat{x}(t)^*\hat{x}(t) \in J \cap B_e$. Recall that any element x, in any C^*-algebra, satisfies $x = \lim_{n\to\infty} x(x^*x)^{1/n}$. This implies that our $\hat{x}(t)$ belongs to the ideal generated by $J \cap B_e$, that is, J_1.

On the other hand, since

$$\Phi_i(\hat{x}(t)) = \sum_{r\in\Gamma} a_i(tr)^*\hat{x}(t)a(r),$$

we have that $\Phi_i(\hat{x}(t))$ is in J_1, and that $\Phi_i(x) = \sum_{t\in\Gamma} \Phi_i(\hat{x}(t))$ is also in J_1. Finally, since $x = \lim_{i\to\infty} \Phi_i(x)$, we have that $x \in J_1$. □

The result above can definitely fail in the absence of the approximation property. A counter-example is provided by the zero ideal in the full group C^*-algebra of a non-amenable group. In this case $J_1 = \{0\}$ while J_3 is the non-trivial kernel of the left regular representation, by (3.6).

It is then natural to ask whether the result of (4.10) holds, without the approximation property, but for the special case of the reduced cross sectional algebra $C_r^*(\mathcal{B})$. This turns out to be a very delicate question, relating to the theory recently developed by E. Kirchberg of exact C^*-algebras. See [**23**] for details. It still seems to be unknown whether the reduced group C^*-algebra of a countable discrete group is exact [**23**, 2.5.3].

Suppose, for a moment, that there exists a discrete group Γ, for which $C_r^*(\Gamma)$ is not exact and let, therefore

$$0 \to I \to A \to B \to 0$$

be an exact sequence of C^*-algebras for which

$$0 \to I \otimes C_r^*(\Gamma) \to A \otimes C_r^*(\Gamma) \to B \otimes C_r^*(\Gamma) \to 0$$

is not exact, where we are taking the spatial tensor product.

Consider the trivial group action of Γ on A and recall that the reduced cross sectional algebra of the semi-direct product bundle (A, Γ, id), i.e., the reduced crossed product $A \rtimes_{id,r} \Gamma$, coincides with the spatial tensor product $A \otimes C_r^*(\Gamma)$. Let J be the image, in $A \otimes C_r^*(\Gamma)$, of $I \otimes C_r^*(\Gamma)$. Referring to the sets J_1, J_2 and J_3 of (3.9), one may now prove that $J_1 = J = I \otimes C_r^*(\Gamma)$, while J_2 is the kernel of the map $A \otimes C_r^*(\Gamma) \to B \otimes C_r^*(\Gamma)$, which is strictly bigger than J_1, by hypothesis.

We would now like to explore an example of a Fell bundle over the non-amenable free group which will be shown to satisfy the approximation property. The last two sections of the present work will be devoted to this.

5. The Cuntz–Krieger bundle. The main goal of this section will be to describe a Fell bundle over the free group $\mathbf{F}_n$ on n generators, which arises from a grading of the Cuntz–Krieger algebras $\mathcal{O}_A$. This grading was discovered by Quigg and Raeburn [**20**] in the form of a co-action of the free group on $\mathcal{O}_A$.

We will, however, prefer to give a description of this bundle in terms of partial group representations because, on the one hand, this is a framework in which a significant generalization is possible and, on the other, the algebraic manipulations seem especially well suited to treat the problem at hand.

Let $A = \big(a_{i,j}\big)_{i,j}$ be an $n \times n$ matrix in which $a_{i,j} \in \{0,1\}$, and suppose, furthermore, that no row or column of A is identically zero. In [**5**] Cuntz and Krieger studied n–tuples $(S_1, S_2, \ldots, S_n)$ of operators on a Hilbert space H satisfying what we shall call the Cuntz–Krieger relations, namely

(CK0) $S_i S_i^* S_i = S_i$, that is, each S_i is a partial isometry,

(CK1) $S_i^* S_j = 0$, for $i \neq j$,

(CK2) $\sum_{i=1}^n S_i S_i^* = 1$,

(CK3) $S_i^* S_i = \sum_{j=1}^n a_{i,j} S_j S_j^*$.

The Cuntz–Krieger algebra $\mathcal{O}_A$ can be defined as being the universal [**3**] C^*-algebra generated by n symbols, $S_1, S_2, \ldots, S_n$, subject to the Cuntz–Krieger relations. One of the main results in [**5**] is that, when A satisfies a certain condition, referred to as condition (I) in [**5**], then any n–tuple of nonzero operators $(S_1, S_2, \ldots, S_n)$, satisfying (CK0–3), generate a C^*-algebra that is isomorphic of $\mathcal{O}_A$.

In our treatment of $\mathcal{O}_A$ we will make no special requirements on A and so we must insist on the definition of $\mathcal{O}_A$ as a universal C^*-algebra, observing that, in the absence of condition (I), one may find C^*-algebras generated by partial isometries satisfying the above relations which are not isomorphic to $\mathcal{O}_A$.

The Cuntz–Krieger relations are intimately related to the concept of partial representations for $\mathbf{F}_n$, as we would like to show. Recall from [**13**], that a partial representation of a group Γ on a Hilbert space H is a map $S\colon \Gamma \to \mathcal{L}(H)$ satisfying, for all $t, r \in \Gamma$,

(PR1) $S(e) = I$, the identity operator on H,

(PR2) $S(t^{-1}) = S(t)^*$, and

(PR3) $S(t)S(r)S(r^{-1}) = S(tr)S(r^{-1})$.

Let us fix, for the time being, an n–tuple $(S_1, S_2, \ldots, S_n)$ of operators on $\mathcal{L}(H)$ satisfying (CK0–3). We will construct, based on these, a partial representation of $\mathbf{F}_n$. For this, let $G = \{g_1, g_2, \ldots, g_n\}$ be a set of (free) generators of $\mathbf{F}_n$. It is well known that every element t of $\mathbf{F}_n$ has a unique decomposition (called its reduced decomposition, or reduced form)

$$t = x_1 x_2 \ldots x_k,$$

where $x_i \in G \cup G^{-1}$ and $x_{i+1} \neq x_i^{-1}$ for all i.

In this case, the integer k will be referred to as the *length* of t and we will write $|t| = k$. It is elementary to show that $|rs| \leq |r| + |s|$.

If $t = x_1 x_2 \ldots x_k$ is as above, define

$$S(t) = S(x_1)S(x_2)\ldots S(x_k),$$

where we put $S(x) = S_j$ if $x = g_j$ or $S(x) = S_j^*$ if $x = g_j^{-1}$.

One should not expect S to be a multiplicative map, that is $S(tr)$ will often differ from $S(t)S(r)$. A significant case in which these coincide is when the reduced forms of t and r

$$t = x_1 x_2 \ldots x_k$$

$$r = y_1 y_2 \ldots y_k$$

satisfy $x_k \neq y_1^{-1}$ because, then, the reduced form of tr will be $t = x_1 x_2 \ldots x_k y_1 y_2 \ldots y_k$ and we will have $S(tr) = S(t)S(r)$, by definition.

Note that the above condition on the pair (t, r) is equivalent to saying that $|tr| = |t| + |r|$. In fact, this property will acquire some relevance below, so we believe it justifies a special notation.

5.1. Definition. *If $t, r \in \mathbf{F}_n$ are such that $|tr| = |t| + |r|$, then the product tr will be denoted by $t \cdot r$.*

Of course $t \cdot r$ is not supposed to indicate any new operation on $\mathbf{F}_n$ as it means nothing but the usual product of t and r. However, it reminds us that no cancelation is taking place when the reduced forms of t and r are written side by side, in that order.

The first relevant use of our new notation is, thus, in the formula

$$S(t \cdot r) = S(t)S(r)$$

which holds whenever $|tr| = |t| + |r|$.

Let P be the subset of $\mathbf{F}_n$ consisting of elements $\alpha = x_1 x_2 \ldots x_k$ where each x_i is in G (as opposed to $G \cup G^{-1}$). Such elements will be called *positive* and will usually be denoted by letters taken from the beginning of the Greek alphabet. For each natural number k we will denote by P_k the set of positive elements of length k. By convention, $P_0 = \{e\}$, the singleton containing the identity element of $\mathbf{F}_n$.

If $t = x_1 x_2 \ldots x_k$ is in reduced form and if, for some i, we have $x_i \in G^{-1}$ and $x_{i+1} \in G$ then, by (CK1) we will have $S(x_i)S(x_{i+1}) = 0$ and hence $S(t) = 0$. Therefore, the only case when $S(t)$ has a chance of not vanishing is when $t = \alpha\beta^{-1}$ for some $\alpha, \beta \in P$ (although $S(t)$ could vanish even in this case).

5.2. Theorem. *The map $S \colon t \in \mathbf{F}_n \to S(t) \in \mathcal{L}(H)$ is a partial representation of $\mathbf{F}_n$.*

Proof. Properties (PR1–2) are trivial, so we concentrate on (PR3). Our proof will consist of a series of claims, the first of which is:

CLAIM 1: For $\alpha \in P$ and $g_j \in G$ one has $S(\alpha g_j)^* S(\alpha g_j) = \varepsilon S(g_j)^* S(g_j)$ where ε is either zero or one.

This is obvious if $|\alpha| = 0$ so, using induction, assume $\alpha = \beta g_i$ with $\beta \in P$ and $g_i \in G$. Then

$$S(\alpha g_j)^* S(\alpha g_j) = S(\beta g_i \cdot g_j)^* S(\beta g_i \cdot g_j) = S(g_j)^* S(\beta g_i)^* S(\beta g_i) S(g_j) =$$

$$= \varepsilon S(g_j)^* S(g_i)^* S(g_i) S(g_j) = \varepsilon S(g_j)^* \sum_{l=1}^{n} a_{ik} S(g_k) S(g_k)^* S(g_j) =$$

$$= \varepsilon a_{ij} S(g_j)^* S(g_j) S(g_j)^* S(g_j) = \varepsilon a_{ij} S(g_j)^* S(g_j)$$

proving our first claim.

This shows, in particular, that for $\beta \in P$ one has that $S(\beta)^* S(\beta)$ is an idempotent. Now, every operator T, such that T^*T is idempotent, must be a partial isometric operator and, thus $S(\beta)$ is a partial isometry.

CLAIM 2: For every α and β in P, if $|\alpha| = |\beta|$ and $S(\alpha)^* S(\beta) \neq 0$ then $\alpha = \beta$.

Let $m = |\alpha| = |\beta|$. If $m = 1$ then the claim is a consequence of (CK1). If $m \geq 1$ write $\alpha = \tilde{\alpha} g_i$ and $\beta = \tilde{\beta} g_j$ with $\tilde{\alpha}, \tilde{\beta} \in P$ and $g_i, g_j \in G$. Then

$$0 \neq S(\tilde{\alpha} \cdot g_i)^* S(\tilde{\beta} \cdot g_j) = S(g_i)^* S(\tilde{\alpha})^* S(\tilde{\beta}) S(g_j).$$

So, in particular $S(\tilde{\alpha})^* S(\tilde{\beta}) \neq 0$ and, by induction, we have $\tilde{\alpha} = \tilde{\beta}$. By claim (1) it follows that $S(\tilde{\alpha})^* S(\tilde{\alpha}) = \varepsilon S(g_k)^* S(g_k)$ for some $g_k \in G$. Therefore

$$S(\alpha)^* S(\beta) = \varepsilon S(g_i)^* S(g_k)^* S(g_k) S(g_j) =$$

$$= \varepsilon S(g_i)^* \sum_{l=1}^{n} a_{kl} S(g_l) S(g_l)^* S(g_j) = \varepsilon a_{kj} S(g_i)^* S(g_j) S(g_j)^* S(g_j) =$$

$$= \varepsilon a_{kj} S(g_i)^* S(g_j),$$

which can be nonzero only if $g_i = g_j$.

CLAIM 3: For every α and β in P, if $S(\alpha)^* S(\beta) \neq 0$ then $\alpha^{-1}\beta \in P \cup P^{-1}$.

Without loss of generality assume $|\alpha| \leq |\beta|$ and write $\beta = \tilde{\beta}\gamma$ with $|\tilde{\beta}| = |\alpha|$ and $\tilde{\beta}, \gamma \in P$. Then

$$0 \neq S(\alpha)^* S(\tilde{\beta} \cdot \gamma) = S(\alpha)^* S(\tilde{\beta}) S(\gamma),$$

which implies that $S(\alpha)^* S(\tilde{\beta}) \neq 0$ and hence, by claim (2), that $\alpha = \tilde{\beta}$. So $\alpha^{-1}\beta = \gamma \in P \subseteq P \cup P^{-1}$.

Given any t in $\mathbf{F}_n$ let us denote by $E(t) = S(t)S(t)^*$. Since we already know that $S(\alpha)$ is a partial isometry for $\alpha \in P$ we see that $E(\alpha)$ is a self-adjoint idempotent operator.

CLAIM 4: For all α and β in P the operators $E(\alpha)$ and $E(\beta)$ commute.

In the case that $\alpha^{-1}\beta \notin P \cup P^{-1}$ we have

$$E(\alpha)E(\beta) = S(\alpha)S(\alpha)^* S(\beta)S(\beta)^* = 0$$

by claim (3), and similarly $E(\beta)E(\alpha) = 0$.

If, on the other hand , $\alpha^{-1}\beta \in P \cup P^{-1}$, without loss of generality write $\alpha^{-1}\beta = \gamma \in P$ and note that

$$\begin{aligned} E(\alpha)E(\beta) = S(\alpha)S(\alpha)^* S(\alpha\cdot\gamma)S(\alpha\cdot\gamma)^* = S(\alpha)S(\alpha)^* S(\alpha)S(\gamma)S(\gamma)^* S(\alpha)^* = \\ = S(\alpha)S(\gamma)S(\gamma)^* S(\alpha)^* = S(\alpha)S(\gamma)S(\gamma)^* S(\alpha)^* S(\alpha)S(\alpha)^* = \\ = S(\beta)S(\beta)^* S(\alpha)S(\alpha)^* = E(\beta)E(\alpha). \end{aligned}$$

CLAIM 5: For all t and r in $\mathbf{F}_n$ the operators $E(t)$ and $E(r)$ commute.

Recall that $S(t) = 0$, unless $t = \alpha\beta^{-1}$ for some $\alpha, \beta \in P$, which we suppose also satisfy $|t| = |\alpha| + |\beta|$. Then $S(t) = S(\alpha)S(\beta)^*$ and

$$E(t) = S(\alpha)S(\beta)^* S(\beta)S(\alpha)^*.$$

According to claim (1), $S(\beta)^* S(\beta) = \varepsilon S(g_i)^* S(g_i)$ where ε is either zero or one and $g_i \in G$. So

$$E(t) = \varepsilon S(\alpha)S(g_i)^* S(g_i)S(\alpha)^* = \varepsilon \sum_{j=1}^{n} a_{ij} S(\alpha)S(g_j)S(g_j)^* S(\alpha)^* =$$

$$= \varepsilon \sum_{j=1}^{n} a_{ij} S(\alpha\cdot g_j)S(\alpha\cdot g_j)^*,$$

which belongs to the linear span of the set of all $E(\gamma)$, with $\gamma \in P$; a commutative set by claim (4). This proves our last claim and allows us to address the statement of the Theorem under scrutiny, that is, the proof that

$$S(t)S(r)S(r^{-1}) = S(tr)S(r^{-1})$$

for $t, r \in \mathbf{F}_n$.

To do this we use induction on $|t| + |r|$. If either $|t|$ or $|r|$ is zero, there is nothing to prove. So, write $t = \tilde{t}\cdot x$ and $r = y \cdot \tilde{r}$, where $x, y \in G \cup G^{-1}$. In case $x \neq y^{-1}$ we have $|tr| = |t| + |r|$ and hence $S(t\cdot r) = S(t)S(r)$ as observed some time ago. If, on the other hand, $x = y^{-1}$ we have

$$S(t)S(r)S(r^{-1}) = S(\tilde{t}\cdot x)S(x^{-1}\cdot\tilde{r})S(\tilde{r}^{-1}x) = S(\tilde{t})S(x)S(x^{-1})S(\tilde{r})S(\tilde{r}^{-1})S(x) = \ldots$$

With an application of claim (5) and the use of the induction hypothesis we conclude that the above equals

$$\begin{aligned} \ldots = S(\tilde{t})S(\tilde{r})S(\tilde{r}^{-1})S(x)S(x^{-1})S(x) = S(\tilde{t}\tilde{r})S(\tilde{r}^{-1})S(x) = \\ = S(tr)S(\tilde{r}^{-1}\cdot x) = S(tr)S(r^{-1}). \end{aligned}$$ □

Let us briefly introduce an abstract concept, applicable to groups Γ which, like $\mathbf{F}_n$, are equipped with a "length" function

$$|\cdot|\colon \Gamma \to \mathbf{R}_+$$

satisfying $|e| = 0$ and the triangular inequality $|ts| \leq |t| + |s|$.

5.3. Definition. *A partial representation σ of Γ is said to be semi-saturated if, whenever $t, r \in \Gamma$ are such that $|tr| = |t| + |r|$, one has $\sigma(tr) = \sigma(t)\sigma(r)$.*

Of course, the representation S provided by (5.2) is a semi-saturated representation.

Whenever we are speaking of a partial representation σ of a group Γ, we will denote by $\mathrm{e}^\sigma(t)$, or simply by $\mathrm{e}(t)$, the range projections $\mathrm{e}(t) = \sigma(t)\sigma(t^{-1})$, of each $\sigma(t)$.

5.4. Proposition. *A partial representation σ of the group Γ is semi-saturated if and only if, whenever $|tr| = |t| + |r|$, one has that $\mathrm{e}(tr) \le \mathrm{e}(t)$ in the usual order of projections.*

Proof. By using the C^*-algebra relation $\|a\|^2 = \|aa^*\|$ and the axioms for partial representations, one checks that, for any $t, r \in \Gamma$,

$$\|\sigma(t)\sigma(r) - \sigma(tr)\|^2 = \|\sigma(tr)\sigma(r^{-1})\sigma(t^{-1}) - \mathrm{e}(tr)\|^2.$$

Now,

$$\begin{aligned}\sigma(tr)\sigma(r^{-1})\sigma(t^{-1}) &= \sigma(tr)\sigma(r^{-1})\sigma(t^{-1})\sigma(t)\sigma(t^{-1}) = \\ &= \sigma(tr)\sigma(r^{-1}t^{-1})\sigma(t)\sigma(t^{-1}) = \mathrm{e}(tr)\mathrm{e}(t),\end{aligned}$$

from which the proof can easily be concluded. □

Given a partial representation S of $\mathbf{F}_n$, such as the one provided by (5.2), the theory developed in [**13**] can be applied. Specifically we would like to recall that there is a one-to-one correspondence between partial representations of a group and representations of the partial group C^*-algebra, and so we get a representation of $C^*_p(\mathbf{F}_n)$.

To be more specific, let us denote by u_t the canonical partial isometry in $C^*_p(\mathbf{F}_n)$ associated to each $t \in \mathbf{F}_n$, as in the proof of [**13**, 6.5], and put $\mathrm{e}_t = u_t u_t^*$. Then the partial representation of $\mathbf{F}_n$ constructed above gives rise, via [**13**, 6.5], to a representation

$$\pi: C^*_p(\mathbf{F}_n) \to \mathcal{L}(H)$$

such that $\pi(u_{g_i}) = S_i$. This representation is somewhat special, since the Cuntz–Krieger relations imply that

(CK1') $\pi(e_{g_i} e_{g_j}) = 0$, for $i \ne j$,
(CK2') $\sum_{i=1}^n \pi(e_{g_i}) = 1$,
(CK3') $\pi(e_{g_i^{-1}}) = \sum_{j=1}^n a_{i,j}\pi(e_{g_j})$, for $i = 1, \ldots, n$.

Also, by the fact that S is semi-saturated, we have

(SS) $\pi(\mathrm{e}_{tr}) \le \pi(\mathrm{e}_t)$, whenever $|tr| = |t| + |r|$.

In other words, let J be the ideal, within $C^*_p(\mathbf{F}_n)$, generated by the elements

(CK1") $e_{g_i} e_{g_j}$, for $i \ne j$,
(CK2") $1 - \sum_{i=1}^n e_{g_i}$,
(CK3") $e_{g_i^{-1}} - \sum_{j=1}^n a_{i,j} e_{g_j}$ for $i = 1, \ldots, n$,
(SS') $\mathrm{e}_{tr}\mathrm{e}_t - \mathrm{e}_{tr}$, for $|tr| = |t| + |r|$.

Then we see that π vanishes on J. The exact logical relationship between representations of the Cuntz–Krieger relations and representations of $C^*_p(\mathbf{F}_n)$ boils down to the following:

5.5. Theorem. *The Cuntz–Krieger algebra $\mathcal{O}_A$ is canonically isomorphic to $C^*_p(\mathbf{F}_n)/J$.*

Proof. By (5.2) and (5.4) we see that there is a one to one correspondence between representations of the Cuntz–Krieger relations and partial representations of $\mathbf{F}_n$, satisfying (CK1'–3')+(SS). These, in turn, correspond to representations of $C^*_p(\mathbf{F}_n)$ vanishing on J. In other words, both $\mathcal{O}_A$ and $C^*_p(\mathbf{F}_n)/J$ have a universal property regarding representations of the Cuntz–Krieger relations and hence they must be isomorphic. □

Recall from [**13**] that $C^*_p(\mathbf{F}_n)$ is defined to be the partial crossed product of the abelian C^*-subalgebra A generated by all the projections e_t, by a certain partial action of $\mathbf{F}_n$. This is saying that $C^*_p(\mathbf{F}_n)$ is the full cross-sectional algebra of the corresponding semi-direct product bundle [**11**]. As such, it is a topologically graded C^*-algebra. The grading space corresponding to the unit group element coincides with the subalgebra A, mentioned above. Incidentally, this is where the generators of the ideal J belong, and so (3.11) applies and tell us that the quotient algebra, that is $\mathcal{O}_A$, is a topologically graded algebra. This grading turns out to be equivalent to the co-action of $\mathbf{F}_n$ on $\mathcal{O}_A$ studied by Quigg and Raeburn in [**20**].

By inspecting the grading in $C^*_p(\mathbf{F}_n)/J$ from the point of view of (3.11), it is easy to see that the grading of $\mathcal{O}_A$, thus obtained, is given by the subspaces

$$\mathcal{O}_A^t = \overline{\text{span}}\{\text{e}(r_1)\text{e}(r_2)\ldots\text{e}(r_k)\sigma(t)\colon k\in\mathbf{N},\ r_1,r_2,\ldots,r_k\in\mathbf{F}_n\},$$

where σ is the partial representation of $\mathbf{F}_n$ corresponding to any given faithful representation of $\mathcal{O}_A$. The Fell bundle corresponding to this grading will henceforth be referred to as the Cuntz–Krieger bundle.

6. Amenability of the Cuntz–Krieger bundle. The main goal of this section is to prove that the Cuntz–Krieger bundle satisfies the approximation property. In fact we will obtain a somewhat more general result, as we shall see in a moment. The quest for wider generality will have the added advantage of allowing the presentation to become pretty much independent of the previous section, although, of course, the main motivation for what we are about to do lies there.

With that in mind, let Γ be a discrete group (soon to be supposed equal to $\mathbf{F}_n$) and let

$$\sigma\colon\Gamma\to\mathcal{L}(H)$$

be any partial representation of Γ on H and define B^σ_t to be the closed linear subspace of $\mathcal{L}(H)$ given by

$$B^\sigma_t = \overline{\text{span}}\{\text{e}(r_1)\text{e}(r_2)\ldots\text{e}(r_k)\sigma(t)\colon k\in\mathbf{N},\ r_1,r_2,\ldots,r_k\in\Gamma\}.$$

By using that

$$\sigma(t)e(r) = \sigma(t)\sigma(r)\sigma(r^{-1}) = \sigma(tr)\sigma(r^{-1}) =$$
$$= \sigma(tr)\sigma(r^{-1}t^{-1})\sigma(tr)\sigma(r^{-1}) = \sigma(tr)\sigma(r^{-1}t^{-1})\sigma(t) = \text{e}(tr)\sigma(t),$$

and that

$$\sigma(t)\sigma(r) = \sigma(t)\sigma(t^{-1})\sigma(t)\sigma(r) = \sigma(t)\sigma(t^{-1})\sigma(tr) = \mathrm{e}(t)\sigma(tr),$$

one sees that $B^\sigma_{t_1} B^\sigma_{t_2} \subseteq B^\sigma_{t_1 t_2}$ and that $B^{\sigma *}_t = B^\sigma_{t^{-1}}$ for any $t, t_1, t_2 \in \Gamma$.

Regardless of whether or not the B^σ_t form an independent family of linear subspaces of $\mathcal{L}(H)$, we may equip the disjoint union of the B^σ_t with the Fell bundle structure over Γ, induced by the C^*-algebra operations of $\mathcal{B}(H)$.

6.1. Definition. *Given a partial representation of a discrete group Γ, we denote by $\mathcal{B}^\sigma$ the Fell bundle*

$$\mathcal{B}^\sigma = \big(B^\sigma_t\big)_{t\in\Gamma}$$

canonically associated to σ, as described above.

Let us suppose, for the time being, that Γ possesses a length function, as in our discussion before (5.3).

6.2. Definition. *A Fell bundle $\mathcal{B}$, over Γ is said to be semi-saturated if, whenever $t, r \in \Gamma$ are such that $|tr| = |t| + |r|$, one has that B_{tr} coincides with the closed linear span of the set of products $\{b_t c_r : b_t \in B_t, c_r \in B_r\}$ (compare* [**9**, 4.8] *and* [**14**, VIII.2.8]*).*

6.3. Proposition. *If σ is a semi-saturated partial representation of Γ, then $\mathcal{B}^\sigma$ is a semi-saturated Fell bundle.*

Proof. If $|tt'| = |t| + |t'|$ and if $r_1, r_2, \ldots, r_k, r'_1, r'_2, \ldots, r'_{k'} \in \Gamma$ we have

$$\mathrm{e}(r_1)\mathrm{e}(r_2)\ldots\mathrm{e}(r_k)\sigma(t) \quad \mathrm{e}(r'_1)\mathrm{e}(r'_2)\ldots\mathrm{e}(r'_{k'})\sigma(t') =$$
$$= \mathrm{e}(r_1)\mathrm{e}(r_2)\ldots\mathrm{e}(r_k)\mathrm{e}(tr'_1)\mathrm{e}(tr'_2)\ldots\mathrm{e}(tr'_{k'})\sigma(tt')$$

which implies that $\mathcal{B}^\sigma$ is semi-saturated. □

In what follows we will assume that σ is a semi-saturated partial representation of $\mathbf{F}_n$, satisfying

$$\sum_{i=1}^{n} \mathrm{e}(g_i) = 1.$$

Compare with (CK2). We should point out that we will make no further assumptions on σ and hence we will be facing a Fell bundle that is substantially more general than the Cuntz–Krieger bundle.

Observe that each $\mathrm{e}(g_i)$ is a projection and, because projections adding up to one must always be orthogonal to each other, we have for $i \neq j$ that

$$0 = \sigma(g_i)^*\mathrm{e}(g_i)\mathrm{e}(g_j)\sigma(g_j) = \sigma(g_i)^*\sigma(g_i)\sigma(g_i)^*\sigma(g_j)\sigma(g_j)^*\sigma(g_j) = \sigma(g_i)^*\sigma(g_j),$$

telling us that the ranges of $\sigma(g_i)$ and $\sigma(g_j)$ must be orthogonal to each other. What this really says, is that (CK1) follows from (CK0) and (CK2).

Now, since we are assuming that σ is semi-saturated, this implies, as in the case of the Cuntz–Krieger partial representation, that a necessary condition for $\sigma(t) \neq 0$ is that $t = \alpha\beta^{-1}$ for some $\alpha, \beta \in P$.

6.4. Lemma. *If k is a natural number then $\sum_{\alpha\in P_k} \mathrm{e}(\alpha) = 1$.*

Proof. This is obvious for $k = 0$. If $k \geq 1$ note that P_k is the disjoint union of $g_i P_{k-1}$, for $i = 1, \ldots, n$. Thus, by induction we have

$$\sum_{\alpha\in P_k} \sigma(\alpha)\sigma(\alpha)^* = \sum_{i=1}^{n} \sum_{\beta\in P_{k-1}} \sigma(g_i \cdot \beta)\sigma(g_i \cdot \beta)^* =$$

$$= \sum_{i=1}^{n} \sigma(g_i)\big(\sum_{\beta\in P_{k-1}} \sigma(\beta)\sigma(\beta)^*\big)\sigma(g_i)^* = \sum_{i=1}^{n} \sigma(g_i)\sigma(g_i)^* = 1. \qquad \square$$

A very important conclusion to be drawn by this is:

6.5. Lemma. *For each t in $\mathbf{F}_n$ and for each natural number k we have*

$$\sigma(t) = \sum_{\alpha\in P_k} \mathrm{e}(t\alpha)\sigma(t)\mathrm{e}(\alpha).$$

Proof. The right hand side equals

$$\sum_{\alpha\in P_k} \sigma(t\alpha)\sigma(\alpha^{-1}t^{-1})\sigma(t)\sigma(\alpha)\sigma(\alpha^{-1}) =$$

$$= \sum_{\alpha\in P_k} \sigma(t\alpha)\sigma(\alpha^{-1})\sigma(\alpha)\sigma(\alpha^{-1}) = \sum_{\alpha\in P_k} \sigma(t)\sigma(\alpha)\sigma(\alpha^{-1}) = \sigma(t). \qquad \square$$

6.6. Theorem. *For every semi-saturated partial representation σ of $\mathbf{F}_n$, such that*

$$\sum_{i=1}^{n} \mathrm{e}(g_i) = 1,$$

the bundle $\mathcal{B}^\sigma$ satisfies the approximation property and hence is amenable.

Proof. Let, for each $m \in \mathbf{N}$, the function $a_m\colon \mathbf{F}_n \to B_e^\sigma$ be defined by

$$a_m(t) = \begin{cases} m^{-\frac{1}{2}}\mathrm{e}(t) & \text{if } t \in P \text{ and } |t| \leq m, \\ 0 & \text{in all other cases,} \end{cases}$$

Observe that

$$\sum_{t\in\mathbf{F}_n} a(t)^* a(t) = \sum_{k=0}^{m} \sum_{\alpha\in P_k} m^{-1}\mathrm{e}(\alpha) = \sum_{k=0}^{m} m^{-1} = 1.$$

We will show this sequence satisfies the hypothesis of (4.4). In order to prove it, let us begin by showing that

$$\lim_{m\to\infty} \sum_{r\in\mathbf{F}_n} a_m(tr)^*\sigma(t)a_m(r) = \sigma(t), \quad t \in \mathbf{F}_n.$$

Since $\sigma(t) = 0$, unless $t = \alpha\beta^{-1}$ for some $\alpha, \beta \in P$, as observed shortly before (6.4), let us assume that this is the case. We may also assume that $|t| = |\alpha| + |\beta|$.

Observe that the sum above is, in fact, a finite sum, since $a_m(r) = 0$ except for $r \in P$ with $|r| \le m$. What is not so obvious is the behavior of $a_m(tr)$, for these r's. Making this point clear will take us some work.

We claim that, for any γ in P one has

i) if $|\gamma| < |\beta|$ then $a_m(t\gamma) = 0$,

ii) if $|\beta| \le |\gamma| \le m - |t|$ then $a_m(t\gamma) = m^{-\frac{1}{2}}\mathrm{e}(t\gamma)$,

To see this, note that, in the first case, $t\gamma$, which is given by $t\gamma = \alpha \cdot \beta^{-1}\gamma$, cannot belong to P and hence $a_m(t\gamma) = 0$ by definition.

In the second case, write $\gamma = \gamma_1 \cdot \gamma_2$ with $\gamma_1, \gamma_2 \in P$ and $|\gamma_1| = |\beta|$. If it so happens that $\gamma_1 \ne \beta$ then $t\gamma = \alpha \cdot \beta^{-1}\gamma_1 \cdot \gamma_2 \notin PP^{-1}$ which implies that $\sigma(t\gamma) = 0$ and hence also that $\mathrm{e}(t\gamma) = 0$. So, no matter which clause in the definition of $a_m(t\gamma)$ is used, we will have $a_m(t\gamma) = m^{-\frac{1}{2}}\mathrm{e}(t\gamma)$.

However, if $\gamma_1 = \beta$, we have that $t\gamma = \alpha\beta^{-1}\gamma_1\gamma_2 = \alpha\gamma_2 \in P$, and $|t\gamma| \le |t| + |\gamma| \le m$. So, by definition, $a_m(t\gamma) = m^{-\frac{1}{2}}\mathrm{e}(t\gamma)$, proving our claim.

This clarifies the role of $a_m(t\gamma)$ for most cases, but the case $|\gamma| > m - |t|$ is still missing. Regarding this we claim that

iii) If $k > |\beta|$ then $\|\sum_{\gamma \in P_k} \mathrm{e}(t\gamma)\| \le 1$.

For $\gamma \in P_k$, write $\gamma = \gamma_1\gamma_2$ with $|\gamma_1| = |\beta|$, repeating, somewhat, the proof of case (ii) above. If $\gamma_1 \ne \beta$, we saw that $\mathrm{e}(t\gamma) = 0$. So, $\mathrm{e}(t\gamma)$ can only be nonzero if $\gamma = \beta\eta$ with $\eta \in P_{k-|\beta|}$. For γ such as that we have $t\gamma = \alpha\eta$ and therefore

$$\sum_{\gamma \in P_k} \mathrm{e}(t\gamma) = \sum_{\eta \in P_{k-|\beta|}} \mathrm{e}(\alpha\eta) \le \sum_{\zeta \in P_{k-|\beta|+|\alpha|}} \mathrm{e}(\zeta) = 1,$$

the last equality following from (6.4). This proves (iii).

Now, back to the task of computing the limit referred to at the beginning of the present proof, observe that, since $a_m(r) = 0$ unless $r \in P$ and $|r| \le m$, we have,

$$\sum_{r \in \mathbf{F}_n} a_m(tr)^*\sigma(t)a_m(r) = \sum_{k=0}^{m} \sum_{\gamma \in P_k} a_m(t\gamma)^*\sigma(t)a_m(\gamma).$$

Let us now carefully analyze each sum $\sum_{\gamma \in P_k} a_m(t\gamma)^*\sigma(t)a_m(\gamma)$, in terms of the possible values of k. Of course, if $k < |\beta|$ this sum will vanish by (i). Next, if $|\beta| \le k \le m - |t|$, then by (ii) and (6.5) we have

$$\sum_{\gamma \in P_k} a_m(t\gamma)^*\sigma(t)a_m(\gamma) = m^{-1} \sum_{\gamma \in P_k} \mathrm{e}(t\gamma)^*\sigma(t)\mathrm{e}(\gamma) = m^{-1}\sigma(t).$$

Finally, assume that $m - |t| < k \le m$. Since we are only interested in the limit as $m \to \infty$, we may assume that m is as big as necessary to force $k > |\beta|$, and hence use the conclusion of (iii). In the computation below, we will also use the inequality

$$\|\sum_i x_i^* y_i\|^2 \le \|\sum_i x_i^* x_i\|\|\sum_i y_i^* y_i\|,$$

which holds in any C^*-algebra, and can be proved by considering matrices over the algebra.

We have, for $k > |\beta|$,

$$\|\sum_{\gamma\in P_k} a_m(t\gamma)^*\sigma(t)a_m(\gamma)\|^2 \le$$

$$\le \|\sum_{\gamma\in P_k} a_m(t\gamma)^* a_m(t\gamma)\|\|\sum_{\gamma\in P_k} a_m(\gamma)^*\sigma(t)^*\sigma(t)a_m(\gamma)\| \le$$

$$\le \|\sigma(t)\|^2\|\sum_{\gamma\in P_k} m^{-1}\mathrm{e}(t\gamma)\|\|\sum_{\gamma\in P_k} m^{-1}\mathrm{e}(\gamma)\| \le m^{-2}.$$

This said we have that

$$\sum_{k=0}^{m}\sum_{\gamma\in P_k} a_m(t\gamma)^*\sigma(t)a_m(\gamma) =$$

$$= \sum_{k=|\beta|}^{m-|t|}\sum_{\gamma\in P_k} a_m(t\gamma)^*\sigma(t)a_m(\gamma) + \sum_{k=m-|t|+1}^{m}\sum_{\gamma\in P_k} a_m(t\gamma)^*\sigma(t)a_m(\gamma).$$

The first summand after the last equal sign equals

$$\sum_{k=|\beta|}^{m-|t|} m^{-1}\sigma(t) = \frac{m-|t|-|\beta|+1}{m}\sigma(t),$$

and converges to $\sigma(t)$ as $m \to \infty$.

The second one has norm no bigger than

$$\sum_{k=m-|t|+1}^{m}\|\sum_{\gamma\in P_k} a_m(t\gamma)^*\sigma(t)a_m(\gamma)\| \le \sum_{k=m-|t|+1}^{m} m^{-1} = \frac{|t|}{m}$$

and converges to zero as $m \to \infty$.

This completes the task proposed at the beginning of the proof. Now, observe that each B_t^σ is spanned by the elements of the form

$$y = \mathrm{e}(r_1)\mathrm{e}(r_2)\ldots\mathrm{e}(r_k)\sigma(t),$$

with $k \in \mathbf{N}$ and $r_1, r_2, \ldots, r_k \in \mathbf{F}_n$. So, in view of the commutativity of B_e^σ [**13**, 2.4] we have that

$$a_m(tr)^* y a_m(r) = \mathrm{e}(r_1)\mathrm{e}(r_2)\ldots\mathrm{e}(r_k) a_m(tr)^* \sigma(t) a_m(r),$$

and we can use the conclusion above to affirm that

$$\lim_{m\to\infty} \sum_{r\in\Gamma} a_m(tr)^* y a_m(r) = y$$

for any y as above. Finally, since the operators

$$b_t \in B_t \mapsto \sum_{r\in\Gamma} a_m(tr)^* b_t a_m(r)$$

are uniformly bounded with m, and since the set of y's as above span a dense subset of B_t^σ, we conclude that

$$\lim_{m\to\infty} \sum_{r\in\Gamma} a_m(tr)^* x a_m(r) = x$$

for all $x \in B_t^\sigma$, as required. □

6.7. Corollary. *The Cuntz–Krieger bundle satisfies the approximation property and hence is amenable.*

Proof. Follows immediately from our last result. □

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