PACIFIC JOURNAL OF MATHEMATICS
Vol. 143, No. 2, 1990

AMENABILITY OF DISCRETE CONVOLUTION ALGEBRAS, THE COMMUTATIVE CASE

NIELS GRØNBÆK

A Banach algebra $\mathfrak{A}$ is called amenable if all bounded derivations into dual Banach $\mathfrak{A}$-modules are inner. Let S be a semigroup and let $l^1(S)$ be the corresponding discrete convolution algebra. This paper is on the theme: "On the hypothesis that $l^1(S)$ is amenable, what conclusions can be drawn about the (algebraic) structure of S?" We give a complete characterization of commutative semigroups carrying amenable semigroup algebras. If S is commutative, then $l^1(S)$ is amenable if and only if S is a finite semilattice of groups, that is, there is a finite semilattice Y and disjoint commutative groups G_α $(\alpha \in Y)$ such that $S = \bigcup_{\alpha\in Y} G_\alpha$ and $G_\alpha G_\beta \subseteq G_{\alpha\beta}$ $(\alpha, \beta \in Y)$.

The theme above has previously been studied in [3] and [4]. In both papers it is apparent that the condition of amenability imposes strong algebraic constraints on the semigroup. In [3] a rather complete description of inverse semigroups carrying amenable semigroup algebras is given. Of particular interest for this paper is that a semilattice carries an amenable semigroup algebra if and only if it is finite [3, Theorem 10]. In [4] it is proved that, if a one-sided cancellative semigroup carries an amenable semigroup algebra, then it is a group. The result of this paper, that for a commutative semigroup S, the semigroup algebra $l^1(S)$ is amenable if and only if S is a finite lattice of groups, is proved by looking at the gross structure of S by means of the "principle of maximal homomorphic image of a given type". Using the fact that homomorphic images of S carry amenable semigroup algebras when S does, we establish the necessity of the characterization by showing that each archimedean component of S is a group. This is obtained by applying the results from [3] and [4], mentioned above, to the maximal semilattice, the maximal cancellative, and the maximal separative homomorphic images of S. The sufficiency of the characterization is easily verified. Alternatively, it follows from [3, Theorem 8].

1. Preliminaries. We shall need some elementary semigroup theory. We prefer to keep our exposition self-contained, so although most of what follows can be found in standard texts on the subject, we shall, with a few exceptions, give proofs in some detail. For a further discussion the reader is referred to [**1**]. Throughout S will denote a commutative semigroup, with the binary operation written multiplicatively.

1.1. DEFINITIONS. Consider the following conditions on S:
(A) Each element of S is an idempotent.
(B) For all $s, t \in S$ there is $n \in \mathbf{N}$ such that

$$s^n \in tS \quad \text{and} \quad t^n \in sS.$$

(C) $s^2 = t^2 = st \Rightarrow s = t \quad (s, t \in S)$.
If S satisfies (A) we call S a *semilattice.*
If S satisfies (B) we call S *archimedean.*
If S satisfies (C) we call S *separative.*
An *ideal* in S is a subset I such that $SI \subseteqq I$. A *prime ideal* in S is an ideal, whose complement is a subsemigroup of S.
A *congruence* on S is an equivalence relation which is compatible with the semigroup operation.
A congruence $\sim$ on S will be called separative (cancellative, archimedean, etc.) if the semigroup $S/\sim$ is separative (cancellative, archimedean, etc.).

1.2. DEFINITION. (Principle of maximal homomorphic image of a given type). Let $\mathfrak{C}$ be a class of congruences on S, closed under intersections. Put $\rho_0 = \bigcap\{\rho | \rho \in \mathfrak{C}\}$. Then S/ρ_0 is the *maximal "type class $\mathfrak{C}$" homomorphic image of* S.
See also [**1**, p. 18] and [7, §1].

EXAMPLE. Let $\rho_0 = \bigcap\{\rho | s^2 \rho s \quad (s \in S)\}$. Then S/ρ_0 is the maximal semilattice homomorphic image of S.

1.3. DEFINITION. Let $s \in S$ and choose $m \in \mathbf{N}$ smallest possible so that $s^m = s^{m+r}$ for some $r \in \mathbf{N}$. Then $\operatorname{order}(s) = m$ and the smallest possible r is called $\operatorname{period}(s)$. If no such $m \in \mathbf{N}$ can be found we put $\operatorname{order}(s) = \infty$.

1.4. DEFINITION. Let S be a semigroup and suppose that there is a semilattice Y and disjoint subsemigroups $S_\alpha \quad (\alpha \in Y)$ of S such that $S = \bigcup_{\alpha \in Y} S_\alpha$ and $S_\alpha S_\beta \subseteqq S_{\alpha\beta} \quad (\alpha, \beta \in Y)$. Then S is called a *semilattice of the subsemigroups* $S_\alpha \quad (\alpha \in Y)$.

The following lemma is the main structure theorem for commutative semigroups.

1.5. LEMMA. *Let S be a commutative semigroup and let Y be the maximal semilattice homomorphic image of S. Then there are disjoint archimedean subsemigroups S_α $(\alpha \in Y)$ of S such that S is a semilattice of the semigroups S_α $(\alpha \in Y)$. This decomposition of S into archimedean subsemigroups is unique up to isomorphism of Y, and S is separative if and only if each archimedean component S_α is cancellative.*

Proof. See [**1**, §4.3].

1.6. LEMMA. *On S define the relations:*

$$s\, c\, t \Leftrightarrow \exists u \in S \; su = tu$$

and

$$s\, \sigma\, t \Leftrightarrow \exists n_0 \in \mathbf{N} \forall n \geq n_0 \; s^n = t^n.$$

Then c and σ are congruences and S/c is the maximal cancellative homomorphic image of S and S/σ is the maximal separative homomorphic image of S.

Proof. It is clear that both relations are congruences. Now suppose ρ is a cancellative congruence; that is, $su\,\rho\, tu \Rightarrow s\,\rho\, t$ $(s, t, u \in S)$. Then clearly $s\,c\,t \Rightarrow s\,\rho\,t$ $(s, t \in S)$ so that $c \subseteqq \rho$. Since c is cancellative we are done with the statements about c.

Now suppose that $s^2\,\sigma\, t^2\,\sigma\, st$; that is, there is $n_0 \in \mathbf{N}$ so that $s^{2n} = t^{2n} = s^n t^n$ for $n \geq n_0$. Then $s^{4n_0+1}t = ss^{2n_0} \cdot t^{2n_0} \cdot t = s^{2n_0+1}t^{2n_0+1} = s^{4n_0+2}$ so that for $n \geq 8n_0 + 2$ we have $s^n = t^n$. Hence $s\,\sigma\,t$, proving that σ is separative. Let ρ be a separative congruence. If $s\,\sigma\,t$, then there is $k \in \mathbf{N}$ so that $st^k = t^{k+1}$. In particular $st^k\,\rho\, t^{k+1}$. This gives

$$(st^{k-1})^2 = st^{k-2}st^k\,\rho\, st^{k-2}t^{k+1} = st^{k-1}t^k\,\rho\, t^{k+1}t^{k-1} = (t^k)^2.$$

With $x = st^{k-1}$ and $y = t^k$ we have $x^2\,\rho\, y^2\,\rho\, xy$ so that $x\,\rho\, y$, that is, $st^{k-1}\,\rho\, t^k$. Repeating as necessary, we get $st\,\rho\, t^2\,\rho\, s^2$, where the second relation follows from symmetry. Thus $s\,\rho\,t$, proving that $\sigma \subseteqq \rho$. □

1.7. LEMMA. $s^2\,\sigma\, s \Leftrightarrow$ order$(s) < \infty$ *and* period$(s) = 1$. *If e, f are idempotents in S, then $e\,\sigma\, f \Leftrightarrow e = f$.*

Proof. Suppose $s^2\,\sigma\, s$. Then there is $n_0 \in \mathbf{N}$ so that $s^{2n} = s^n$ for $n \geq n_0$. If r is the period of s we have $2n \equiv n \pmod r$ for $n \geq n_0$ so that $r = 1$. The rest is obvious. □

1.8. LEMMA. *S/σ is a group if and only if S is archimedean with unique idempotent.*

Proof. First suppose that S/σ is a group. From Lemma 1.7 it follows that S has a unique idempotent. Let $s, t \in S$. Since S/σ is a group there are u, $v \in S$ so that $su\,\sigma\, t$ and $tv\,\sigma\, s$. By definition of σ, s divides a power of t and t divides a power of s, that is, S is archimedean. Conversely, let $s \in S$ and let e denote the unique idempotent in S. Since S is archimedean there are t, $u \in S$ so that $st = e$ and $ue = s^{n_0}$ for some n_0. We have $(es)^{n_0+p} = e^{n_0+p}s^{n_0}s^p = e^{n_0+p}ues^p = ues^p = s^{n_0+p}$ $(p \in \mathbf{N})$ so that $es\,\sigma\, s$. Clearly $st\,\sigma\, e$, so S/σ is a group. □

2. The main theorem. For the remainder of this paper we shall assume that S is a commutative semigroup such that $l^1(S)$ is amenable. We shall make frequent use of the fact that, if T is a homomorphic image of S, then $l^1(T)$ is amenable, and if I is an ideal in S which is generated by an idempotent, then $l^1(I)$, being a closed $l^1(S)$-ideal which is unital as a Banach algebra, is amenable [**6**, Proposition 5.1]. Thus, if $S = \bigcup_{\alpha \in Y} S_\alpha$ is the decomposition of S into its archimedean components, then the semilattice Y is finite, since $l^1(Y)$ is amenable ([**3**, Theorem 10]). We give Y the usual semilattice ordering $\alpha \le \beta \Leftrightarrow \alpha\beta = \alpha$ $(\alpha, \beta \in Y)$. Since Y is finite, Y has a minimal element, namely the product of all elements in Y.

It is convenient to start with the case where S is separative; that is, we are assuming that each archimedean component is cancellative.

2.1. LEMMA. *Let S and Y be as above and let α_0 be the minimal element of Y. Then S_{α_0} is a group.*

Proof. By [**4**, Theorem 2.3] S/c is a group. Let $s \in S_{\alpha_0}$ Then there is $t \in S$ so that for all $u \in S$ $stu\,c\,u$, that is, for all $u \in S$ there is $v \in S$ so that $stuv = uv$. Since α_0 is minimal, $st \in S_{\alpha_0}$ and $uv \in S_{\alpha_0}$, so, using the cancellation law in S_{α_0}, we see that st is a neutral element in S_{α_0}. Consequently $l^1(S_{\alpha_0})$ can be identified canonically with an ideal generated by an idempotent in $l^1(S)$. It follows that $l^1(S_{\alpha_0})$ is amenable and therefore S_{α_0} is a group, again by [**4**, Theorem 2.3]. □

2.2. LEMMA. *Let $l^1(S)$ be amenable and suppose that S is separative. Then S is a finite semilattice of groups.*

Proof. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be the decomposition of S into its archimedean components. Let $\beta \in Y$, and define $T = \bigcup_{\alpha \geq \beta} S_\alpha$. Then T is a subsemigroup of S and $S \setminus T$ is a (prime) ideal in S. Hence the canonical Banach space direct sum $l^1(S) = l^1(T) \oplus l^1(S \setminus T)$ is a semidirect product, so that $l^1(T)$ is amenable. Since β is minimal in $\{\alpha \in Y | \alpha \geq \beta\}$, Lemma 2.1 implies that S_β is a group. But β was arbitrary in Y. □

We now turn to the general case.

2.3. LEMMA. *Suppose $l^1(S)$ is amenable. Then S is a finite semilattice of its archimedean components, $S = \bigcup_{\alpha \in Y} S_\alpha$. Each S_α has a unique idempotent e_α, and $e_\alpha S_\alpha$ is a group, isomorphic to the maximal separative homomorphic image of S_α.*

Proof. By Lemma 2.2 S/σ is a finite semilattice of groups, $S/\sigma = \bigcup_{\alpha \in Y} G_\alpha$. Let S_α be the preimage of G_α by the canonical map $S \to S/\sigma$. With slight abuse of notation we have $S_\alpha/\sigma = G_\alpha$, so that S_α is archimedean with unique idempotent, e_α say, by Lemma 1.8. It follows that $S = \bigcup_{\alpha \in Y} S_\alpha$ is the decomposition of S into its archimedean components. Now let $s \in S_\alpha$. Since G_α is a group, there is $t \in S_\alpha$ so that $st \,\sigma\, e_\alpha$, i.e. $(st)^n = e_\alpha$ for some $n \in \mathbf{N}$. Hence $e_\alpha s^{n-1} t^n$ is an inverse to $e_\alpha s$. Clearly the canonical map from $e_\alpha S_\alpha$ to G_α is surjective. Assume that $e_\alpha s \,\sigma\, e_\alpha$ for some $s \in S_\alpha$. Since $e_\alpha S_\alpha$ is a group it follows from Lemma 1.7 that $e_\alpha s = e_\alpha$, proving injectivity of the canonical map. □

We shall finish the proof of the main theorem by proving that $e_\alpha S_\alpha = S_\alpha$ for each $\alpha \in Y$. This is done by exploiting that $l^1(S)$, being amenable, has a bounded approximate identity. First we need a definition.

2.4. DEFINITION. Let $s \in S$. Then we define

$$[ss^{-1}] = \{u \in S | us = s\}.$$

Since $l^1(S)$ has a bounded approximate identity $[ss^{-1}] \neq \emptyset$ for all $s \in S$ [**4**, Theorem 1.1].

2.5. LEMMA. *Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be the decomposition of S into its archimedean components, as in Lemma 2.3, and let $s \in S_\alpha$. If $[ss^{-1}] \cap S_\alpha \neq \emptyset$, then $s \in e_\alpha S_\alpha$. If α is maximal in Y, then S_α is a group.*

Proof. Let $u \in [ss^{-1}] \cap S_\alpha$. Then $us\,\sigma\, e_\alpha s$. Since S_α/σ is a group we have $u\,\sigma\, e_\alpha$, i.e. $u^n = e_\alpha$ for some $n \in \mathbf{N}$. Hence $s = u^n s = e_\alpha s$. In general, if $s \in S_\alpha$ and $u \in [ss^{-1}] \cap S_\beta$, then $s = us \in S_\alpha \cap S_{\beta\alpha}$, so $\beta \geq \alpha$. Thus, when α is maximal in Y we have that $[ss^{-1}] \subseteqq S_\alpha$ for all $s \in S_\alpha$. It follows that $e_\alpha S_\alpha = S_\alpha$, so that S_α is a group by Lemma 2.3. □

2.6. LEMMA. *Let $s = \bigcup_{\alpha \in Y} S_\alpha$ be as in Lemma 2.3. Then $[ss^{-1}] \cap \{e_\alpha | \alpha \in Y\} \neq \varnothing$ for all $s \in S$. In particular $l^1(S)$ is unital.*

Proof. First note that, if $u \in [ss^{-1}]$, then $[uu^{-1}] \subseteqq [ss^{-1}]$. Let $s \in S$ and let S_{α_0} be the archimedean component of s. Put $u_0 = s$ and choose successively $u_k \in [u_{k-1}u_{k-1}^{-1}]$. Let S_{α_k} be the archimedean component of u_k. As noted in the proof of Lemma 2.5 we have $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq \cdots$. Since $\operatorname{card} Y < \infty$, we eventually have $S_{\alpha_k} = S_{\alpha_{k+1}}$, whence $[u_k u_k^{-1}] \cap S_{\alpha_k} \neq \varnothing$, so that $e_{\alpha_k} \in [u_k u_k^{-1}]$ by Lemma 2.5. As observed in the beginning of the proof $e_{\alpha_k} \in [ss^{-1}]$. From [**5**, Theorem 7.5] it follows that $l^1(S)$ has a unit. □

We are now able to prove:

2.7. THEOREM. *Let S be a commutative semigroup. Then $l^1(S)$ is amenable if and only if S is a finite semilattice of commutative groups.*

Proof. The sufficiency has been noted in the introduction. Hence we assume that $l^1(S)$ is amenable. Let $s = \bigcup_{\alpha \in Y} S_\alpha$ be the decomposition as in Lemma 2.3. By Lemma 2.5 the theorem is true if $\operatorname{card} Y = 1$. We proceed by induction on $n = \operatorname{card} Y$. Assume that $n \geq 2$ and that the theorem is true for semigroups which are semilattices of archimedean semigroups with cardinality of the semilattice strictly less than n. Let α_0 be the minimal element in Y. Let $\beta \in Y \setminus \{\alpha_0\}$, and define $T_\beta = \bigcup_{\alpha \geq \beta} S_\alpha$. As in the proof of Lemma 2.2, we see that $l^1(T_\beta)$ is amenable. Thus, by the induction hypothesis, we have that S_α is a group for $\alpha \in Y \setminus \{\alpha_0\}$. We finish the induction step by proving that $S_{\alpha_0} = e_{\alpha_0} S_{\alpha_0}$. To this end, define a congruence $\sim$ on S by

$$s \sim t \Leftrightarrow Ss = St \qquad (s, t \in S).$$

Note that, if $s \sim t$, then $s \in St$, since $[ss^{-1}] \neq \varnothing$. Using that S_α is a group for $\alpha \neq \alpha_0$, we see that $S/{\sim} \cong \bigcup_{\alpha \neq \alpha_0} \{e_\alpha\} \cup S_{\alpha_0}/{\sim}$. Hence $l^1(S_{\alpha_0}/{\sim})$ is (isomorphic to) a closed ideal of finite codimension in the amenable Banach algebra $l^1(S/\sim)$, and therefore $l^1(S_{\alpha_0}/\sim)$ is itself amenable [2, Theorem 4.1]. From Lemma 2.5 we get that $S_{\alpha_0}/\sim$ is a group. In particular we have for all $s \in S_{g\alpha_0}$ that $s \sim e_{\alpha_0}s$, so, by the note above, $S_{\alpha_0} \subseteqq e_{\alpha_0}S_{\alpha_0}$. The induction step is hereby completed. □

Acknowledgment. I wish to acknowledge a stimulating correspondence with Professor J. Duncan on the subject. I wish to thank Dr. K. B. Laursen for a careful reading of the manuscript.

References

[1] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys no. 7, Amer. Math. Soc., Providence, Rhode Island 1967.

[2] P. C. Curtis and R. J. Loy, *The structure of amenable Banach algebras*; (to appear in J. London Math. Soc.).

[3] J. Duncan and I. Namioka, *Amenability of inverse semigroups and their semigroup algebras*, Proc. Royal Soc. Edinburgh, **80A** (1978), 309–321.

[4] N. Grønbæk, *Amenability of weighted discrete convolution algebras on cancellative semigroups*, Proc. Royal Soc. Edinburgh, **110A** (1988), 351–360.

[5] E. Hewitt and H. S. Zuckerman, *The l_1-algebra of a commutative semigroup*, Trans. Amer. Math. Soc., **83** (1956), 70–97.

[6] B. E. Johnson, *Cohomology in Banach algebras*; Mem. Amer. Math. Soc., **127** (1972).

[7] T. Tamura, *The study of closets and free contents related to semilattice decomposition of semigroups*; in Semigroups, Proceedings of a symposium on semigroups held at Wayne State University, Detroit, Michigan, (1968); Academic Press, New York and London.

Received August 29, 1988.

KØBENHAVNS UNIVERSITETS MATEMATISKE INSTITUT
UNIVERSITETSPARKEN 5
2100 KØBENHAVN Ø, DENMARK