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Amenability of Restricted Semigroup Algebras

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0. Introduction

In 1972, B.E. Johnson proved that for every discrete group G, $\ell^1(G)$ is amenable as a Banach algebra if and only if G is amenable as a group [9]. When S is a commutative semigroup, $\ell^1(S)$ is amenable if and only if S is a finite semilattice of abelian (and hence amenable) groups [7]. When S is a cancellative semigroup with identity, it is amenable if and only if S is an amenable group [6]. In 1978, J. Duncan and I. Namioka showed that if S is an arbitrary inverse semigroup with finite set of idempotents $E(S)$, then $\ell^1(S)$ is amenable if and only if each maximal group of S is amenable [4]. Also, they showed that $\ell^1(S)$ fails to be amenable if $E(S)$ is infinite, for E-unitary semigroups. In 1990, J. Duncan and A.L.T. Paterson completed the story for inverse semigroup by showing that the above result holds without the restriction of S being E-unitary [5]. Recently, G.K. Dales, A.T.-M. Lau and D. Strauss have shown that for an arbitrary semigroup S, $\ell^1(S)$ is amenable if and only if S is 'built up from amenable groups' [3, Theorem 10.12]. They used the methods of [4].

We apply some of the above results to the semigroup algebra $\ell^1(S_r)$, where S_r is the restricted 0-semigroup associated to an inverse semigroup S [1], to prove similar results about the restricted semigroup algebra $\ell^1_r(S)$, introduced by the second author and A.R. Medghalchi in [1]. We show that the restricted

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semigroup S_r is a 0-direct union of Brandt semigroups (Lemma 2.1). This is a key concept for us to show that $\ell^1(S_r)$ is an amenable Banach algebra if and only if these Brandt semigroups are amenable and $E(S)$ is finite (Theorem 2.4). We also prove that amenability of the Banach algebras $\ell^1_r(S)$, $\ell^1(S_r)$, and $\ell^1(S)$ are equivalent (Theorems 2.6, 2.8 and Corollary 2.9). We calculate S_r for some inverse semigroups S and discuss the amenability of the corresponding algebras.

If a Banach algebra is amenable then it has a bounded approximate identity. J. Duncan and I. Namioka characterized those inverse semigroups S for which $\ell^1(S)$ has a bounded approximate identity by introducing the condition (D_k) [4]. We show that $\ell^1_r(S)$ and $\ell^1(S_r)$ have bounded approximate identity if and only if $E(S)$ is a finite set. In general we know that $\ell^1_r(S)$ always has a (not necessarily bounded) approximate identity [1].

1. Preliminaries

An inverse semigroup S is a discrete semigroup such that for each $s \in S$ there is a unique element s^* with

$$ss^*s = s, s^*ss^* = s^*.$$

The map $s \longmapsto s^*$ is an involution on S [11], [10]. The set $E(S)$ of idempotents of S consists of elements of the form ss^*, where $s \in S$. It is easy to see that $E(S)$ is a commutative subsemigroup of S [11], [10]. There is a natural order $\leq$ on $E(S)$ defined by $e \leq f$ if and only if $ef = e$. We refer the interested reader to [10] for more details.

For an arbitrary inverse semigroup S, the restricted product of elements x and y of S is xy if $x^*x = yy^*$ and undefined, otherwise. The set S with this restricted product forms a discrete groupoid [10, 3.1.4]. If we adjoin a zero element 0 to this groupoid and put $0^* = 0$, we get an inverse semigroup S_r [10, 3.3.3] with multiplication rule

$$x \bullet y = \begin{cases} xy & x^*x = yy^* \\ 0 & otherwise \end{cases} \qquad (x, y \in S \cup \{0\})$$

which is called the ***restricted semigroup*** of S in [1].

We remind that a subgroup G of S is a subsemigroup of S such that G is a group. Also, a subset I of S is an (a two-sided) ideal , if $sa \in I$ and $as \in I$, for each $a \in I$ and $s \in S$. For every element $b \in S$, there is the smallest ideal containing b called the ***principal ideal containing*** b. For an inverse semigroup S, the principal ideal containing b is SbS [10]. One of the Green's relations J, is defined in terms of principal ideals. For $s, t \in S$

$$sJt \Leftrightarrow SsS = StS.$$

The J-class containing t is denoted by J_t. We refer the readers to [2], [8], and [10] for more details.

A nonzero idempotent in a semigroup with zero is said to be ***primitive***, if it is minimal relative to the natural partial order on the set of all nonzero idempotents. An inverse semigroup S with zero is called ***primitive***, if every nonzero idempotent is primitive. A semigroup S without zero is called ***simple*** if it has no proper ideals. A semigroup S with zero is called ***0-simple*** if

(i) $\{0\}$ and S are its only ideals,

(ii) $S^2 = SS \neq \{0\}$.

A semigroup is called ***completely 0-simple*** if it is 0-simple and primitive. A ***Brandt*** semigroup is a completely 0-simple inverse semigroup. References [8], [2], [10], and [11] contain more details about these concepts.

An ideal I of a semigroup S induces a relation ρ_I on S by

$$(s,t) \in \rho_I \Leftrightarrow \ \ \textit{either } s = t \textit{ or } \ s,t \in I.$$

This is a congruence on S. The quotient $\frac{S}{\rho_I}$ is called the ***Rees*** quotient semigroup and is usually denoted by $\frac{S}{I}$. Note that $\frac{S}{I}$ is S when the ideal I is collapsed to zero.

The minimum ideal of S (if it exists) is denoted by $K(S)$. A principal series of ideals for S is a chain

$$S = I_1 \supsetneqq I_2 \supsetneqq \ldots \supsetneqq I_m = K(S)$$

where $I_1, I_2, ..., I_m$ are ideals in S and there is no ideal of S strictly between I_j and I_{j+1}, for each $j \in \mathbb{N}_{m-1}$.

The Banach algebra $\ell^1(S)$ of all complex valued functions f on S satisfying

$$\|f\|_1 = (\sum_{x \in S} |f(x)|) < \infty$$

is called the ***semigroup algebra*** of S. This is a Banach algebra with convolution product

$$(f * g)(x) = \sum_{st=x} f(s)g(t) \quad (f, g \in \ell^1(S)).$$

When S is a $*$-semigroup, we put $\widetilde{f}(x) = \overline{f(x^*)}$, for each $f \in \ell^1(S)$. Following [1], we define

$$(f \bullet g)(x) = \sum_{x^*x = yy^*} f(xy)g(y^*) \quad (x \in S).$$

Under the usual ℓ^1-norm, $(\ell^1(S), \bullet, \tilde{\ })$ is a Banach $*$-algebra [1]. We denote this Banach algebra by $\ell^1_r(S)$ and call it the ***restricted semigroup algebra*** of S [1].

Example 1.1. *Let S be a meet semilattice, that is a semigroup such that, for every $a \in S$, $a^2 = a$. This means that $a^* = a$. Thus we can define the restricted semigroup S_r with the restricted product*

$$x \bullet y = \begin{cases} x & x = y \\ 0 & \textit{otherwise} \end{cases} \quad (x, y \in S \cup \{0\}).$$

In this case, S_r is also a meet semilattice.

Example 1.2. *Rees inverse semigroups are Brandt semigroups* [8, 5.1.8]. *If S is a Brandt semigroup then $S \cong M^0(G, I, n)$ for some group G with identity e, some $n \in \mathbb{N}$, and a matrix $I = (\delta_{ij})$ of order n defined by*

$$\delta_{ij} = \begin{cases} e & i = j \\ 0 & otherwise \end{cases} \quad (0 < i, j < n + 1).$$

We use the notation of [3]. *An arbitrary element of S is $(x)_{ij}$ for $x \in G$ (with the $(i, j)^{th}$ component x, and 0 elsewhere). In this semigroup $((x)_{ij})^* = (x^{-1})_{ji}$. Therefore*

$$((x)_{ij})^*(x)_{ij} = (x^{-1})_{ji}(x)_{ij} = (x^{-1}ex)_{jj} = (e)_{jj}$$

and also

$$(y)_{kl}((y)_{kl})^* = (y)_{kl}(y^{-1})_{lk} = (yey^{-1})_{kk} = (e)_{kk}$$

for $x, y \in G$. Thus we have

$$(x)_{ij} \bullet (y)_{kl} = \begin{cases} (x)_{ij}(y)_{kl} & j = k \\ 0 & otherwise \end{cases} \quad (x, y \in G)$$

that is

$$(x)_{ij} \bullet (y)_{kl} = \begin{cases} (xy)_{il} & j = k \\ 0 & otherwise \end{cases} \quad (x, y \in G).$$

This restricted product is exactly the ordinary product of S. Therefore $S = S_r$ for a Brandt semigroup S.

Example 1.3. *Let S be a Clifford semigroup. Then S is isomorphic to a strong semilattice of groups* [10, 5.2.12], *say $S = \cup_{e \in E(S)} G_e$. If $e, f \in E(S)$, $x \in G_e$ and $y \in G_f$ then $x^{-1}x = xx^{-1} = e$ and $y^{-1}y = yy^{-1} = f$. Therefore $x \bullet y = y \bullet x = 0$, unless $e = f$. For $x, y \in G_e$ we have $x \bullet y = xy$ and $y \bullet x = yx$. For every non-zero element $s \in G_e \subset S_r$,*

$$s^{-1} \bullet s = s^{-1}s = e = ss^{-1} = s \bullet s^{-1}.$$

Also,

$$0^{-1} \bullet 0 = 0^{-1}0 = 0 = 00^{-1} = 0 \bullet 0^{-1}.$$

Thus S_r is a Clifford semigroup [10, 5.2.12]. *Note that S_r is not 0-simple in general, because every $G_i \cup \{0\}$ is a proper ideal of S_r.*

Remark 1.4. *For every nontrivial inverse semigroup S, the restricted semigroup S_r is not an E-unitary semigroup, because $0 \in S_r \cap E(S_r)$.*

2. Amenability of the algebras $\ell^1(S_r)$ and $\ell^1_r(S)$

In this section we discuss the amenability properties of some function algebras on semigroups. We study the relation between amenability of the semigroup algebra $\ell^1(S_r)$ of the restricted semigroup S_r of an inverse semigroup S, the restricted semigroup algebra $\ell^1_r(S)$, and the semigroup algebra $\ell^1(S)$. For more details about amenability of semigroups and their function algebras, see [9], [4], [3] and [12]. For every inverse semigroup S, the restricted semigroup S_r is an inverse semigroup and $E(S_r) = E(S) \cup \{0\}$. For each elements

e, f of $E(S_r)$, if $ef \neq 0$ then $e = e^*e = ff^* = f$, hence every element of $E(S_r)$ is minimal relative to the canonical partial order. Thus S_r is a primitive semigroup.

By [8, theorem 3.3.4] or [10, theorem 3.3.5], we have the following lemma.

Lemma 2.1. *For every inverse semigroup S, the restricted semigroup S_r is 0-direct union of Brandt semigroups; i.e. $S_r = \cup_{i\in I} S_i$ such that every S_i is a Brandt semigroup and $S_i \cap S_j = S_iS_j = \{0\}$, if $i \neq j$.*

For each $i \in I$ there exist an element $e \in E(S)$ such that $S_i = J_e \cup \{0\}$. Because every S_i is a completely 0-simple inverse semigroup (Brandt), the only ideals of S_i are $\{0\}$ and S_i. Also, we have $(S_i)^2 = S_iS_i \neq \{0\}$.

If $E(S)$ (and so $E(S_r)$) is a finite set, then the set I in the above lemma is finite and we have $|I| \leq |E(S_r)|$. Suppose that $I = \{1, 2, ..., n\}$. Each S_i, for $i \in I$, and every finite union of these semigroups are ideals of S_r.

Lemma 2.2. *Let S be an inverse semigroup with $E(S)$ finite. The nontrivial ideals of the restricted semigroup S_r are exactly*

$$(S_1), (S_2), ..., (S_n), (S_1 \cup S_2), ..., (S_1 \cup S_2 \cup S_3), ..., (S_1 \cup ... \cup S_n).$$

proof. If K is a nontrivial ideal of S_r, then for some $i \in I$, $K \cap S_i \neq \emptyset$. The set $K \cap S_i$ is an ideal of S_i. Hence, it is $\{0\}$ or S_i, because S_i is 0-simple. This says that K is a finite union of semigroups S_i. ■

We need the next lemma for studying the amenability of $\ell^1(S_r)$.

Lemma 2.3. *For every inverse semigroup S with $E(S)$ finite, the restricted semigroup S_r has a principal series.*

proof. Consider the chain

$$S_r = (S_1 \cup ... \cup S_n) \supsetneqq (S_1 \cup ... \cup S_{n-1}) \supsetneqq ... \supsetneqq (S_1 \cup S_2) \supsetneqq (S_1) \supsetneqq \{0\} \supsetneqq \emptyset.$$

Like in the above lemma, there is no proper ideal of S_r strictly between two consecutive ideals of this chain. Let $2 \leqq i \leqq n$ be arbitrary. The Rees quotient semigroup

$$\frac{S_1 \cup S_2 \cup ... \cup S_{i-1} \cup S_i}{S_1 \cup S_2 \cup ... \cup S_{i-1}} = \{S_1 \cup S_2 \cup ... \cup S_{i-1}\} \cup (S_i - \{0\})$$

is just S_i with $(S_1 \cup S_2 \cup ... \cup S_{i-1})$ collapsed to zero. Hence, this Rees quotient semigroup is a Brandt semigroup. On the other hand, $\frac{S_1}{\{0\}} = \{\{0\}\} \cup (S_1 - \{0\})$ can be identified with S_1 and this Rees quotient semigroup is a Brandt semigroup, too. It is obvious that each two ideals of the above chain are distinct, which completes the proof. ■

We know that if $\ell^1(S)$ is amenable then $E(S)$ is finite [5]. The next result is one of the main theorems of this section, in which we find a necessary and sufficient condition for amenability of $\ell^1(S_r)$, where S is an inverse semigroup with finitely many idempotents. This follows from the above lemma and [3, theorem 10.12].

Theorem 2.4. *Let S be an inverse semigroup with finitely many idempotents. We know that $S_r = \cup_{i=1}^{n} S_i$ such that each S_i is a Brandt semigroup with the corresponding group G_i. Then $\ell^1(S_r)$ is amenable if and only if every G_i is an amenable group.*

Next we study the relation between the amenability of Banach algebras $\ell^1(S)$, $\ell^1_r(S)$ and $\ell^1(S_r)$, for an arbitrary inverse semigroup S.

Lemma 2.5. *If S is an inverse semigroup, then $\mathbb{C}\delta_0$ is a closed ideal of $\ell^1(S_r)$.*

proof. Let $f \in \ell^1(S_r)$ and $c\delta_0 \in \mathbb{C}\delta_0$ are arbitrary elements. For each nonzero element $x \in S_r$ we have

$$(f \bullet c\delta_0)(x) = \sum_{x^*x=yy^*} f(xy)c\delta_0(y^*) = 0$$

unless $y^* = 0$. If $y^* = 0$ then $y = 0$ and $x = 0$. Hence, for every nonzero element $x \in S_r$, $(f \bullet c\delta_0)(x) = 0$. For $x = 0$

$$(f \bullet c\delta_0)(0) = \sum_{0^*0=yy^*} f(0y)c\delta_0(y^*) = \sum_{y=0} f(0)c\delta_0(y^*)$$

Therefore $f \bullet c\delta_0 \in \mathbb{C}\delta_0$. On the other hand, for an arbitrary nonzero element $x \in S_r$, if $x^*x = yy^*$ then $xy \in S$ and $xy \neq 0$. Thus

$$((c\delta_0) \bullet f)(x) = \sum_{x^*x=yy^*} c\delta_0(xy)f(y^*) = 0.$$

Also

$$((c\delta_0) \bullet f)(0) = \sum_{0^*0=yy^*} c\delta_0(0y)f(y^*) = cf(0) \in \mathbb{C}.$$

Hence $c\delta_0 \bullet f \in \mathbb{C}\delta_0$. Therefore $\mathbb{C}\delta_0$ is an ideal of $\ell^1(S_r)$.

Now it remains to show that if $\{c_n\delta_0\} \subset \mathbb{C}\delta_0$ converges to $f \in \ell^1(S_r)$, then $f \in \mathbb{C}\delta_0$. Let $x \in S_r$ and $x \neq 0$. For each n, $c_n\delta_0(x) = 0$ and $c_n\delta_0(0) = c_n$. Since $\lim_{n\to\infty} \|f - c_n\delta_0\| = 0$, we have

$$0 = \lim_{n\to\infty} \sum_{x\in S_r} |f(x) - c_n(x)\delta_0(x)| = \lim_{n\to\infty} (|f(0) - c_n| + \sum_{0\neq x\in S} |f(x)|).$$

Hence $f(x) = 0$, for each $0 \neq x \in S$. It is obvious that $f(0) \in \mathbb{C}$. Therefore $f \in \mathbb{C}\delta_0$ and the proof is complete. ■

Theorem 2.6. *Let S be an inverse semigroup. The restricted semigroup algebra $\ell^1_r(S)$ is amenable if and only if $\ell^1(S_r)$ is amenable.*

proof. By [1, theorem 3.2], we know that $\ell^1_r(S) \cong \frac{\ell^1(S_r)}{\mathbb{C}\delta_0}$. But $\mathbb{C}\delta_0$ is an closed ideal of $\ell^1(S_r)$, hence, by [12, corollary 2.3.2] and [12, theorem 2.3.10], it follows that $\ell^1_r(S)$ is an amenable Banach algebra if and only if $\ell^1(S_r)$ is amenable as a Banach algebra. ■

Lemma 2.7. *Let S be an inverse semigroup. Every nonzero subgroup of S is a subgroup of S_r and every nonzero subgroup of S_r is a subgroup of S.*

proof. Let G be a nonzero subgroup of S. If $0 \in G$, then $00^{-1} = 0 \neq e$, which is a contradiction (here e is the identity of G). Hence G is a subset of S_r which $0 \notin G$. If $a, b \in G$ are two arbitrary elements of G, then $a^{-1}a = e = bb^{-1}$. Hence $a \bullet b = ab \neq 0$ and $a \bullet b \in G$. Thus $a \bullet a^{-1} = aa^{-1} = e$, $a^{-1} \bullet a = a^{-1}a = e$, $a \bullet e = ae = a$, and $e \bullet a = ea = a$. Therefore, G is a subgroup of S_r.

Conversely, let G_r be an arbitrary nonzero subgroup of S_r and e_r be the identity element of G_r. The zero element of S_r could not be in G_r. We conclude that $G_r \subset S$, as sets. Suppose that $a, b \in G_r$ are arbitrary. Then $a \bullet b \in G_r$ and $a \bullet b \neq 0$. Thus $ab = a \bullet b$, therefore G_r is a subgroup of S. ■

Theorem 2.8. *Let S be an inverse semigroup. The semigroup algebra $\ell^1(S)$ is amenable if and only if $\ell^1(S_r)$ is amenable.*

proof. By the previous lemma, every maximal nonzero subgroup of S is a maximal nonzero subgroup of S_r and vice versa. Also, $\{0\}$ is an amenable maximal subgroup of S_r and possibility of S. On the other hand, $E(S)$ is a finite set if and only if $E(S_r)$ is finite. Now, the result follows from [4, theorem 8] and [5, corollary 1]. ■

Corollary 2.9. *For an inverse semigroup S, $\ell^1(S)$ is amenable if and only if $\ell^1_r(S)$ is amenable.*

In particular, if $\ell^1_r(S)$ is an amenable Banach algebra, then S is an amenable semigroup, but the converse is false (see also [4, lemma 3 and theorem 10]). It is trivial that S_r is always 0-amenable [4, remark (2) on Page 311]. Theorems 2.6 and 2.8 show that, to study the amenability of $\ell^1(S)$, we need to calculate the semigroups S_i and their corresponding groups G_i, where $S_r = \cup_{i=1}^n S_i$, and investigate the amenability of the groups G_i. For more information about these groups, refer to [8].

We finish this section by presenting some examples.

Example 2.10. *Let $S \cong M^0(G, I, n)$ be a Brandt semigroup. In example 1.2, we proved that $S = S_r$. Therefore $\ell^1(S_r) = \ell^1(S)$ and we have*

$$\ell^1_r(S) = \frac{\ell^1(S)}{\mathbb{C}\delta_0} \cong M^0(\ell^1(G), I, n).$$

For more details, see [3, Page 62] *and* [1, theorem 3.2]. *Thus the Banach algebra $\ell^1_r(S)$, and so $\ell^1(S)$, is amenable if and only if G is an amenable group. We note that the index set I is finite, then the set $E(S)$ is finite.*

Example 2.11. *For every finite meet semilattice S, we have*

$$S_r = \cup_{i=1}^n S_i$$

where $S_i = \{0, a_i\}$ *with* $a_i \in S$. *Thus* S_i *is a Brandt semigroup with corresponding group* $G_i = \{a_i\}$. *The groups* G_i *are trivially amenable, therefore the Banach algebras* $\ell^1(S_r)$, $\ell^1_r(S)$ *and* $\ell^1(S)$ *are amenable.*

Example 2.12. *For each Clifford semigroup* S *with finitely many idempotents,* S_r *is a Clifford semigroup (see example 1.3). If* $S = \cup_{i=1}^n G_i$ *and* e_i *is the identity of* G_i, *the restricted semigroup* S_r *is equal to* $S \cup \{0\}$ *with the restricted product* $\bullet$. *Put* $S_i = G_i \cup \{0\}$ *for* $i = 1, 2, ..., n$. *For each* i, S_i *is a Brandt semigroup with corresponding group* G_i *and* $S_r = \cup_{i=1}^n S_i$. *Also we have* $S_i \cap S_j = S_i S_j = \{0\}$, *for every* $i \neq j$. *Thus* S_r *is* 0*-direct union of these Brandt semigroups* S_i. *Hence the Banach algebras* $\ell^1(S_r)$, $\ell^1_r(S)$ *and* $\ell^1(S)$ *are amenable if and only if each* G_i *is an amenable group.*

3. Bounded approximate identities for $\ell^1_r(S)$ and $\ell^1(S_r)$

A necessary condition for a Banach algebras to be amenable is that it possesses a bounded right approximate identity. In this section, we characterize those inverse semigroups S for which the restricted semigroup algebra $\ell^1_r(S)$ has a bounded right approximate identity. we do the same for the semigroup algebra $\ell^1(S_r)$.

Note that in an inverse semigroup S, for $s, t \in S$, $\delta_s * \delta_t = \delta_{st}$, but this relation fails for the dot product $\bullet$.

Lemma 3.1. *For an inverse semigroup* S *we have*

$$\delta_s \bullet \delta_t = \begin{cases} \delta_{st} & s^*s = tt^* \\ 0 & otherwise \end{cases} \quad (s, t \in S)$$

proof. Let $x \in S$ be an arbitrary element. Then

$$\delta_s \bullet \delta_t(x) = \sum_{x^*x = yy^*} \delta_s(xy)\delta_t(y^*) = 0$$

unless $x^*x = yy^*$, $s = xy$ and $t = y^*$. If these equalities hold, then $x = xx^*x = (xy)y^* = (s)y^* = st$ or $x = st$. Therefore $\delta_s \bullet \delta_t(x) = 0$, unless $x = st$. Now

$$\delta_s \bullet \delta_t(st) = \sum_{t^*s^*st = yy^*} \delta_s(sty)\delta_t(y^*) = 0$$

unless $y^* = t$, $sty = s$ and $t^*s^*st = yy^*$. These equalities imply that $t^*s^*st = t^*t$ and so $t(t^*s^*st) = t$ or $s^*st = t$. From this and $sty = s$ (or $stt^* = s$), we have $s^*s = (tt^*s^*)(stt^*) = s^*stt^*tt^* = (s^*st)t^* = tt^*$. Therefore

$$\delta_s \bullet \delta_t(st) = \begin{cases} 1 & s^*s = tt^* \\ 0 & otherwise \end{cases} \quad (s, t \in S).$$

These imply that

$$\delta_s \bullet \delta_t(x) = \begin{cases} 1 & x = st \ and \ s^*s = tt^* \\ 0 & otherwise \end{cases} \quad (s, t \in S)$$

which completes the proof. ■

Lemma 3.2. *Let S be an inverse semigroup. Then $\ell^1_r(S)$ admits a bounded right approximate identity with bound M if and only if given $\epsilon > 0$ and any number of elements $s_1, s_2, ..., s_n \in S$ there exists $\alpha \in \ell^1_r(S)$ such that*

$$\|\alpha\|_1 \leq M, \quad \|\delta_{s_i} - \delta_{s_i} \bullet \alpha\|_1 < \epsilon \quad (i = 1, 2, ..., n).$$

proof. One side is clear. We shall prove the other side. Suppose that $f = \sum_{s\in S} f(s)\delta_s \in \ell^1_r(S)$ and $\epsilon > 0$ are arbitrary. Since $\|f\|_1 = \sum_{s\in S} |f(s)| < \infty$, there are at most countably many $s \in S$, say $s_1, s_2, ...,$ for which $f(s) \neq 0$. Hence, for $\epsilon > 0$, there is $N \geq 1$ such that $\sum_{i=N+1}^{\infty} f(s_i) < \frac{\epsilon}{2(1+M)}$. By assumption, there exists $\alpha \in \ell^1_r(S)$ such that

$$\|\alpha\|_1 \leq M, \quad \|\delta_{s_i} - \delta_{s_i} \bullet \alpha\|_1 < \frac{\epsilon}{2N\|f\|_1} \quad (i = 1, 2, ..., N).$$

It is clear that for any $s \in S$ we have

$$\|\delta_s - \delta_s \bullet \alpha\|_1 \leq \|\delta_s\|_1 + \|\delta_s \bullet \alpha\|_1 \leq 1 + M.$$

Thus

$$\begin{aligned}
\|f - f \bullet \alpha\|_1 &= \|\sum_{i=1}^{\infty} f(s_i)\delta_{s_i} - \sum_{i=1}^{\infty} f(s_i)(\delta_{s_i} \bullet \alpha)\|_1 \\
&= \|\sum_{i=1}^{\infty} f(s_i)(\delta_{s_i} - \delta_{s_i} \bullet \alpha)\|_1 \\
&\leq \sum_{i=1}^{N} |f(s_i)|\, \|\delta_{s_i} - \delta_{s_i} \bullet \alpha\|_1 + \sum_{i=N+1}^{\infty} |f(s_i)|\, \|\delta_{s_i} - \delta_{s_i} \bullet \alpha\|_1 \\
&\leq \frac{\epsilon}{2N\|f\|_1} \sum_{i=1}^{N} |f(s_i)| + (1+M) \sum_{i=N+1}^{\infty} |f(s_i)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \blacksquare
\end{aligned}$$

Lemma 3.3. *$\ell^1_r(S)$ admits a bounded right approximate identity with bound M if and only if $\ell^1_r(E(S))$ has the same property.*

proof. Consider the linear map $T : \ell^1_r(S) \to \ell^1_r(E(S))$, defined by $\delta_s \mapsto \delta_{ss^*}$, for $s \in S$. If $f = (\sum_{s\in S} f(s)\delta_s) \in \ell^1_r(S)$ then

$$T(f) = T(\sum_{s\in S} f(s)\delta_s) = \sum_{s\in S} f(s)T(\delta_s) = \sum_{s\in S} f(s)\delta_{ss^*},$$

and so T is a norm decreasing linear map. By lemma 3.2, we have

$$T(\delta_e \bullet \delta_s) = \begin{cases} \delta_e \bullet T(\delta_s) & e = ss^* \\ 0 & otherwise \end{cases} \quad (e \in E(S), s \in S).$$

Therefore for $\alpha = (\sum_{s\in S} \alpha(s)\delta_s) \in \ell^1_r(S)$ and $e \in E(S)$,

$$T(\delta_e \bullet \alpha) = \sum_{s\in S} \alpha(s)T(\delta_e \bullet \delta_s) = \sum_{ss^*=e} \alpha(s)(\delta_e \bullet T(\delta_s)) = \sum_{ss^*=e} \alpha(s)\delta_e$$

and also

$$\delta_e \bullet T(\alpha) = \delta_e \bullet (\sum_{s\in S} \alpha(s)\delta_{ss^*}) = \sum_{ss^*=e} \alpha(s)\delta_e.$$

Therefore $T(\delta_e \bullet \alpha) = \delta_e \bullet T(\alpha)$. Now suppose that $\ell^1_r(S)$ admits a bounded right approximate identity with bound M. Let $\epsilon > 0$ and $e_1, ..., e_n \in E(S)$. There exist $\alpha \in \ell^1_r(S)$ such that

$$\|\alpha\|_1 \leq M, \quad \|\delta_{e_i} - \delta_{e_i} \bullet \alpha\|_1 < \epsilon \ \ (i = 1, ..., n).$$

Hence $T(\alpha) \in \ell^1_r(E(S))$, $\|T(\alpha)\|_1 \leq M$ and for $i = 1, ..., n$ we have

$$\|\delta_{e_i} - \delta_{e_i} \bullet T(\alpha)\|_1 = \|T(\delta_{e_i}) - T(\delta_{e_i} \bullet \alpha)\|_1 \leq \|\delta_{e_i} - \delta_{e_i} \bullet \alpha\|_1 < \epsilon.$$

Conversely suppose that $\ell^1_r(E(S))$ admits a bounded right approximate identity with bound M. Given $\epsilon > 0$ and $s_1, ..., s_n \in S$ there exists $\alpha \in \ell^1_r(E(S))$ such that

$$\|\alpha\|_1 \leq M, \quad \|\delta_{s_i^* s_i} - \delta_{s_i^* s_i} \bullet \alpha\|_1 < \epsilon \ \ (i = 1, ..., n).$$

Thus for $i = 1, ..., n$ we have

$$\begin{aligned}\|\delta_{s_i} - \delta_{s_i} \bullet \alpha\|_1 &= \|\delta_{s_i s_i^* s_i} - \delta_{s_i s_i^* s_i} \bullet \alpha\|_1 \\ &= \|\delta_{s_i} \bullet \delta_{s_i^* s_i} - (\delta_{s_i} \bullet \delta_{s_i^* s_i}) \bullet \alpha\|_1 \\ &= \|\delta_{s_i} \bullet (\delta_{s_i^* s_i} - \delta_{s_i^* s_i} \bullet \alpha)\|_1 \leq \|\delta_{s_i}\|_1 \|\delta_{s_i^* s_i} - \delta_{s_i^* s_i} \bullet \alpha\|_1 < \epsilon. \blacksquare\end{aligned}$$

One can prove a similar result for bounded left approximate identity. Because $E(S)$ is a commutative semigroup, $\ell^1_r(E(S))$ has a bounded right approximate identity if and only if it has a bounded left approximate identity.

Lemma 3.4. *$\ell^1_r(E(S))$ admits a bounded right approximate identity with upper bound M if and only if S has finitely many idempotents, with $|E(S)| \leq M$.*

proof. When $E(S)$ is finite, $\ell^1_r(S)$, and therefore $\ell^1_r(E(S))$, has a bounded approximate identity with upper bound $|E(S)|$ [1, Proposition 3.2].

Conversely suppose that $\ell^1_r(E(S))$ admits a bounded right approximate identity with upper bound M. Let k be an positive integer with $k \geq M$. Given $e_1, ..., e_{k+1} \in E(S)$ there exists $\alpha \in \ell^1_r(E(S))$ such that

$$\|\alpha\|_1 \leq M, \quad \|\delta_{e_i} - \delta_{e_i} \bullet \alpha\|_1 < \frac{1}{k+1} \ \ (i = 1, ..., k+1).$$

Let $\alpha = \sum_{u_r \in E(S)} \lambda_r u_r$. For $i = 1, ..., k+1$ put $u_{r_i} = e_i$. Then

$$1 - |\lambda_{r_i}| \leq |1 - \lambda_{r_i}| = \|\delta_{e_i} - \delta_{e_i} \bullet \alpha\|_1 < \frac{1}{k+1}$$

and so $\frac{k}{k+1} < |\lambda_{r_i}|$. If $e_1, ..., e_{k+1}$ are distinct elements of $E(S)$, then

$$M \geq \sum |\lambda_r| \geq \sum_{i=1}^{k+1} |\lambda_{r_i}| > (k+1)\frac{k}{k+1} = k$$

which contradicts our choice of k. Therefore $e_i = e_j$ for some i, j with $1 \leq i < j \leq k+1$. Thus $|E(S)| \leq k$. $\blacksquare$

Theorem 3.5. *For any inverse semigroup S, $\ell^1_r(S)$ admits a bonded approximate identity if and only if S has finitely many idempotents.*

proof. By lemmas 3.3 and 3.4, $\ell^1_r(S)$ admits a bonded right approximate identity if and only if S has finitely many idempotents. The statement for bonded left approximate identity is proved similarly. ■

Theorem 3.6. *For any inverse semigroup S, $\ell^1(S_r)$ has a bounded approximate identity if and only if S has finitely many idempotents.*

proof. If $E(S_r)$ has finitely many idempotents, say $E(S_r) = \{0, e_1, e_2, ..., e_k\}$, then $E(S_r)$ satisfies condition (D_k) of Duncan and Namioka [4, section 4], since each subset of $E(S_r)$ with $k+1$ members has at least two equal members or includes 0.

Conversely suppose that $E(S_r)$ is infinite. For $e, f \in E(S_r)$ we have

$$e \bullet f = \begin{cases} e & e = e^*e = ff^* = f \\ 0 & otherwise \end{cases}$$

and so for any positive integer k, there is a set of $k+1$ nonzero idempotents, which fails to satisfy condition (D_k). Therefore $\ell^1(S_r)$ does not admit a bounded approximate identity. ■

Corollary 3.7. *If S has infinitely many idempotents, then the algebras $\ell^1(S)$, $\ell^1_r(S)$ and $\ell^1(S_r)$ are not amenable.*

Proposition 3.8. *Let S be an inverse semigroup. If $\ell^1_r(S)$ (or $\ell^1(S_r)$) has a bounded approximate identity with upper bound M then $\ell^1(S)$ has a bounded approximate identity with upper bound $2^k - 1$, where $k = |E(S)|$.*

proof. If $\ell^1_r(S)$ (or $\ell^1(S_r)$) has a bounded approximate identity with upper bound M then $k = |E(S)|$ is finite and so $E(S)$ satisfies condition (D_k) [4]. Now the result follows from the proof of [4, Lemma 15]. ■

Example 3.9. *Consider $S = (\mathbb{N}, \wedge)$, where $m \wedge n = \max(m, n)$ and $n^* = n$, for $m, n \in \mathbb{N}$. Then $\ell^1(S)$ has a bounded approximate identity, but $E(S) = S$ is not finite. This shows that the converse of proposition 3.8 does not hold.*

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