Amenability via random walks

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Abstract

We use random walks to show that the Basilica group is amenable, answering an open question of Grigorchuk and Żuk. Our results separate the class of amenable groups from the closure of subexponentially growing groups under the operations of group extension and direct limits; these classes are separated even within the realm of finitely presented groups.

1 Introduction

The concept of amenability, introduced by von Neumann (1929), has been central to many areas of mathematics. Kesten (1959) showed that a countable group is amenable if and only if the spectral radius equals 1; in particular, if the random walk escapes at a sublinear rate. Although this connection has been deeply exploited to study the properties of random walks, it appears that it has not yet been used to prove the amenability of groups.

A group is amenable if it admits a finitely additive invariant probability measure. The simplest examples of amenable groups (AG) are

(i) finite and Abelian groups and, more generally,

(ii) groups of subexponential growth.

Amenability is preserved by taking subgroups, quotients, extensions, and direct limits. The classes of elementary amenable (EG), and subexponentially amenable (SG, see Grigorchuk (1998), and Ceccherini et al. (1999), §14) groups are the closure of (i), (ii) under

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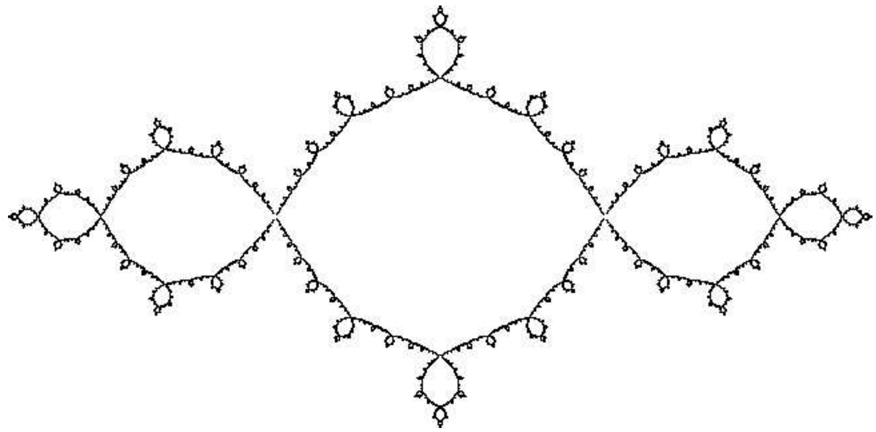

Figure 1: The Basilica, or the action of G on ∂T

these operations, respectively. We have

$$\mathsf{EG} \subseteq \mathsf{SG} \subseteq \mathsf{AG},$$

and the question arises whether these inclusions are strict: Day (1957) asked this about $\mathsf{EG} \subseteq \mathsf{AG}$ (see also Grigorchuk (1998)). Chou (1980) showed that there are no elementary amenable groups of intermediate growth. Thus Grigorchuk's group separates the class EG and SG, answering Day's question.

In this paper, we show by example that the inclusion $\mathsf{SG} \subset \mathsf{AG}$ is also strict.

The **Basilica group** G we are considering is the iterated monodromy group of the polynomial $z^2 - 1$. It was first studied by Grigorchuk and Żuk (2002a), who showed that G does not belong to the class SG. The main goal of this note is to show, using rate of escape for random walks, that G is amenable. This answers a question of the above authors.

Let T be the rooted binary tree with vertex set V consisting of all finite binary sequences, and edge set $E = \{(v, vi) : v \in V, i \in \{0,1\}\}$. Let $\varepsilon \in \mathrm{Aut}(T)$ send iv to $((i+1) \bmod 2)v$. For $g, h \in$Aut(T) (with the notation $g : v \mapsto v^g$) let (g,h) denote the element of $\mathrm{Aut}(T)$ sending $0v \mapsto 0v^g$ and $1v \mapsto 1v^h$. The Basilica group G is generated by the following two recursively defined elements of $\mathrm{Aut}(T)$:

$$a = (1, b), \quad b = (1, a)\varepsilon.$$

Then G is the iterated monodromy group of the polynomial $z^2 - 1$; the scaling limit of the Schreier graphs of its action on level n of T is the *Basilica*, i.e. the Julia set of this polynomial (see the survey Bartholdi, Grigorchuk, and Nekrashevych (2003) and Figure 1). Let Z_n be a symmetric random walk on G with step distribution supported on a, b and their inverses. We will study the speed of the random walk Z_n to show our main theorem.

Theorem 1 *The return probability $P(Z_{2n}=1)$ has slower than exponential decay.*

By Kesten's theorem, this implies that the group G is amenable, and $\mathsf{SG} \neq \mathsf{AG}$.

In the rest of the paper, we extend this result in two directions. In Section 4, Theorem 12 we give a finitely presented example $\tilde{G}$ separating AG and SG, showing that these classes are distinct even in this realm. Grigorchuk (1998) showed that $\mathsf{EG} \neq \mathsf{AG}$ (more precisely, $\mathsf{EG} \neq \mathsf{SG}$) for finitely presented groups.

In Section 3 Corollary 10, we give a quantitative upper bound of order $n^{5/6}$ on the rate of escape. For the heat kernel, we have the following quantitative lower bound.

Theorem 2 *There exists $c > 0$ so that for all n we have $\mathbf{P}(Z_{2n}=1) \geq e^{-cn^{2/3}}$.*

Motivated by their question about amenability, Grigorchuk and Żuk (2002b) study spectral properties of G. Amenability of G has been claimed in the preprint Bartholdi (2002), whose proof appears to contain serious gaps and is considered altogether incomplete. The present paper uses the same starting point as Bartholdi (2002), but follows a different path; we get specific heat kernel bounds for a less general family of groups.

2 A fractal distance

For $g \in G$ and $v \in T$ let $g[v] \in \mathrm{Aut}(T)$ denote the action of g on the descendant subtree of v, and let $g(v) \in C_2$ denote the action on the two children of v. Let S be a finite binary subtree of T containing the root (i.e. each vertex in S has zero or two descendants). Let $|\cdot|$ denote shortest-word distance in G with the above generators. Let ∂S denote its set of leaves, and let

$$\begin{aligned} \nu_S(g) &= \sum_{v\in\partial S}(|g[v]|+1)-1, \\ \nu(g) &= \min_S \nu_S(g). \end{aligned}$$

The quantity ν has the alternative recursive definition; for $g = (g_1, g_2)\varepsilon_*$, let

$$\nu(g) = \min(|g|, 1+\nu(g_1)+\nu(g_2)).$$

Lemma 3 *The function ν is a norm on G. Moreover, ν-balls have exponential growth.*

Proof. First note that since multiplying g by a increases $|g_1|+|g_2|$ by at most 1, we get $|g| \geq |g_1|+|g_2|$. This implies that if $\nu(g)=|g|$ then $\nu(g) \geq \nu(g_1)+\nu(g_2)$. So in general, we have

$$\nu(g_1)+\nu(g_2) \leq \nu(g) \leq \nu(g_1)+\nu(g_2)+1. \tag{1}$$

We now check that ν satisfies the triangle inequality; this is clear if $\nu(g) = |g|$, $\nu(h) = |h|$, otherwise we may assume that $\nu(g) = \nu(g_1) + \nu(g_2) - 1$. Then we get

$$\nu(gh) \leq 1 + \nu((gh)_1) + \nu((gh)_2) \leq 1 + \nu(g_1) + \nu(g_2) + \nu(h_1) + \nu(h_2) \leq \nu(g) + \nu(h),$$

where the first inequality holds by induction (some care is needed to show that the induction can be started).

We now claim that the balls $B_n = \{g : \nu(g) \leq n\}$ grow at most exponentially, more precisely, we have

$$\#B_n \leq 40^n \quad \text{for all } n. \tag{2}$$

Indeed, there are at most 4^n such subtrees S with at most n edges. Given the subtree S, the element $g \in B_n$ is defined by its action $g(v) \in C_2$ at the vertices of S that are not leaves (at most 2^n possibilities), as well as the words $g[v]$ at the vertices v that are leaves (these can be described with n symbols from the alphabet a, a^{-1}, b, b^{-1} and comma). Thus we have $\#B_n \leq (4 \cdot 2 \cdot 5)^n$.

For the other direction, note that ν-balls contain the word-distance balls of the same radius and G has exponential growth (see Grigorchuk and Żuk (2002a)). □

3 Self-similarity of random walks on G

Fix $r > 0$, and consider the random walk Z_n on the free group F_2 where each step is chosen from (a, a^{-1}, b, b^{-1}) according to weights $(1, 1, r, r)$, respectively. This walk projects to a subgroup of $F_2 \wr C_2$ via the substitution $\pi : a \mapsto (1, b)$ and $b \mapsto (1, a)\varepsilon$. Let $(Y_n, X_n)\varepsilon_n$ be the projection of Z_n.

For the sequel, we consider this definition from another point of view. Consider the subgroup $H_0 \subset F_2 \wr C_2$ of elements of the form $(g, 1)$. The right cosets of H_0 have representatives of the form $(1, g)\kappa$, with $g \in F_2, \kappa \in C_2$.

Given the walk Z_n, we may define X_n and ε_n by saying that $(1, X_n)\varepsilon_n$ is the representative of the coset $H_0\pi(Z_n)$, which we abbreviate $H_0 Z_n$. Similarly, $(Y_n, 1)\varepsilon_n$ can be defined as the representative of $H_1 Z_n$ where $H_1 \subset F_2 \wr C_2$ is the subgroup of elements of the form $(1, g)$.

Define the stopping times

$$\begin{aligned}
\sigma(0) &= 0, \\
\sigma(m+1) &= \min\{n > \sigma(m) : \varepsilon_n = 1,\, X_n \neq X_{\sigma(m)}\}, \quad m \geq 0, \\
\tau(0) &= \min\{n > 0 : \varepsilon_n = \varepsilon\}, \\
\tau(m+1) &= \min\{n > \tau(m) : \varepsilon_n = \varepsilon,\, Y_n \neq Y_{\tau(m)}\}, \quad m \geq 0.
\end{aligned}$$

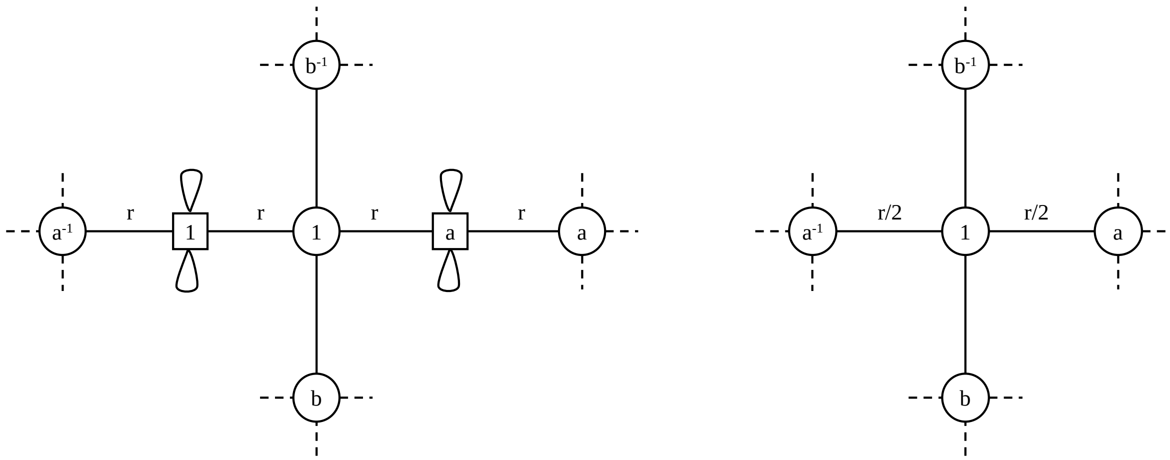

Figure 2: The random walks (X_n, ε_n) and $X_{\sigma(m)}$

Lemma 4 *$X_{\sigma(m)}$, $Y_{\tau(m)}$ are simple random walks on F_2 with step distribution given by the weights $(r/2, r/2, 1, 1)$ on (a, a^{-1}, b, b^{-1}), respectively.*

Proof. First note that the process (X_n, ε_n) is a reversible random walk on a weighted graph. More precisely, it is a walk on the weighted Schreier graph of the right action of $(F_2, \{a, a^{-1}, b, b^{-1}\})$ on the right cosets of H_0. See the left of Figure 2; there the value of ε_n is represented by a circle (1) or square (ε).

The Markov property of (X_n, ε_n) follows since F_2 acts on these cosets; reversibility follows from the reversibility of the original walk.

When we only look at this walk at the times $\sigma(n)$, the resulting process is still a Markov chain, where the transition probabilities are given by the hitting distribution on the 4 circles that are neighbors or separated by a single square.

The hitting distribution is given by effective conductances, and using the series law we get the picture on the right hand side of Figure 2. This process is a symmetric random walk with weights as claimed.

The proof for Y is identical. Because $\tau(0)$ has a different definition, the process $Y_{\tau(m)}$ does not start at 0, rather at a or a^{-1}. Note that the processes $X_{\sigma(m)}$ and $Y_{\tau(m)}$ are not independent. $\square$

Lemma 5 *With probability 1, we have* $\lim m/\sigma(m) = (2+r)/(4+4r) =: f(r)$*, and the same holds for τ.*

PROOF. The increments of the process $\sigma(m)$ are the time the random walk in Figure 2 spends between hitting two different circles. These increments are independent and identically distributed. Let $t_\circ$, $t_\square$ denote the expected times starting from a circle or a neighboring square to hit a different circle. Conditioning on the first step of the walk gives the equations

$$\begin{aligned} t_\circ &= 1 + r/(r+1)t_\square \\ t_\square &= 1 + r/(2(r+1))t_\circ + 1/(r+1)t_\square \end{aligned}$$

And the solution is $t_\circ = 4(1+r)/(2+r)$. The claim now follows from the strong law of large numbers. □

If $\overline{Z}_n$ denotes the image of Z_n in G, then by construction we have $\overline{Z}_n = (\overline{Y}_n, \overline{X}_n)\varepsilon_n$. In the rest of this section we will simply (ab)use the notation Z_n, X_n, Y_n for the images in G of the corresponding random words.

Proposition 6 *We have* $\lim \nu(Z_n)/n = 0 \quad$ *a.s.*.

PROOF. By Kingman's subadditive ergodic theorem (see Kallenberg (2002), Theorem 10.22), the random limit

$$s(r) = \lim \nu(Z_n)/n$$

exists and equals a constant with probability 1. By (1) we also have

$$\nu(Z_n) \leq \nu(X_n) + \nu(Y_n) + 1$$

and therefore

$$\begin{aligned} s(r) &\leq \limsup \nu(X_n)/n + \limsup \nu(Y_n)/n \\ &= \limsup \nu(X_{\sigma(m)})/\sigma(m) + \limsup \nu(Y_{\tau(m)})/\tau(m) \\ &= \left(\lim \nu(X_{\sigma(m)})/m\right)(\lim m/\sigma(m)) + \left(\lim \nu(Y_{\tau(m)})/m\right)(\lim m/\tau(m)) \\ &= 2s(2/r)f(r). \end{aligned}$$

In the last equality we used Lemmas 4 and 5. Iterating this inequality we get

$$s(r) \leq 4s(r)f(r)f(2/r) = 4s(r)/8$$

and since s is a finite constant we get $s = 0$ with probability 1. □

We get Theorem 1 as a simple corollary.

Corollary 7 *The return probability $P(Z_{2n} = 1)$ has slower than exponential decay.*

PROOF. Let $\varepsilon > 0$, n even. Recall that the most likely value of Z_n is 1. Thus

$$\mathbf{P}(\nu(Z_n) \le \varepsilon n) = \sum_{v \in B_{\varepsilon n}} \mathbf{P}(Z_n = v) \le (\#B_{\varepsilon n})\, \mathbf{P}(Z_n = 1) \tag{3}$$

Since balls grow at most exponentially (2), we get

$$\mathbf{P}(Z_n = 1) \ge \mathbf{P}(\nu(Z_n) \le \varepsilon n) 40^{-\varepsilon n}.$$

Since $\nu(Z_n)/n$ converges to 0 in probability, the first factor on the right converges to 1. Thus the return probability can be bounded below by an exponential with an arbitrary slow rate, as required. □

A more technical version of Proposition 6 gives a better bound on the ν-rate of escape.

Proposition 8 *There exists $c > 0$ depending on r so that for all $n \ge 1$*

$$u_r(n) := \mathbf{E} \max_{i \le n} \nu(Z_i) \le c n^{2/3}.$$

PROOF. Let $L(n)$ be the largest so that $\sigma(L) \le n$, and let $M(n)$ be the largest so that $\tau(M) \le n$. Following the argument of Proposition 6, we get

$$\begin{aligned} u_r(n) &\le \mathbf{E} \max_{i \le n} \nu(X_i) + \mathbf{E} \max_{i \le n} \nu(Y_i) + 1 \\ &= \mathbf{E} \max_{i \le L(n)} \nu(X_{\sigma(i)}) + \mathbf{E} \max_{i \le M(n)} \nu(Y_{\tau(i)}) + 1 \\ &\le 2 + 2u_{2/r}(f(r)n + k_n) + 2n\mathbf{P}(L(n) > f(r)n + k_n) \end{aligned} \tag{4}$$

where we can choose the constants $k_n = n^\alpha$ for some $\alpha > 1/2$ but less than $2/3$. The stopping time $\sigma(1)$ has an exponential tail i.e. for all $n > 0$

$$\mathbf{P}(\sigma(1) > n) \le c_1 e^{-c_2 n},$$

since starting from any position the walk has a nonzero chance to stop within two steps. By standard large deviation arguments, we get that last term of (4) can be bounded above by $c_3 n e^{-c_4 n^{2\alpha - 1}} < c_5$. Thus for $n \ge 1$ we get

$$u_r(n) \le u_{2/r}(f(r)n + k_n) + c_6 \le u_{2/r}(f(r)n) + (c_6 + 1)n^\alpha.$$

Applying this to $u_{2/r}$ as well and using the fact that $f(r)f(2/r) = 1/8$, we easily get

$$u_r(8n) \le 4u_r(n) + c_7 n^\alpha$$

with $u_r(1) \le 1$. Iteration at values of n that are powers of 8 gives that for such values

$$u_r(n) \le c_8 n^{2/3}.$$

Since $u_r(n)$ is monotone in n, the claim follows. $\square$

Corollary 9 *There exists $c > 0$ depending on r so that for all $n \ge 0$ we have*

$$\mathbf{P}(Z_{2n} = 1) \ge e^{-cn^{2/3}}.$$

PROOF. By Markov's inequality, we have

$$\mathbf{P}(\nu(Z_n) \le 2cn^{2/3}) \ge 1/2$$

and therefore by (3) for even n we have

$$\mathbf{P}(Z_n = 1) \ge 1/2 \cdot 40^{2cn^{2/3}}$$

as required. $\square$

Let $M_n = \max(|X_1|, \dots, |X_n|)$. We have the following bound on the rate of escape.

Corollary 10 *There exists $c > 0$ so that for all $a, n \ge 1$ we have $\mathbf{P}(M_n > an^{5/6}) < c/a$.*

PROOF. Let $K_n = \max(\nu(X_1), \dots, \nu(X_n))$. We have

$$\begin{aligned} \mathbf{P}(M_n > an^{5/6}) &= \mathbf{P}(M_n > an^{5/6}, K_n > ac_1 n^{2/3}) \\ &+ \mathbf{P}(M_n > an^{5/6}, K_n \le ac_1 n^{2/3}). \end{aligned} \tag{5}$$

The first term is at most $\mathbf{P}(K_n > ac_1 n^{2/3}) < c_2/(c_1 a)$ by Proposition 8 and Markov's inequality. The second term is bounded above by the sum of $\mathbf{P}(X_m = g)$ over all $m \le n$ and all g with $|g| > an^{5/6}$ and $\nu(g) \le ac_1 n^{2/3}$. By the Varopoulos-Carne bounds (see Carne (1985)

) the first constraint on g implies

$$\mathbf{P}(X_m = g) \le e^{-(an^{5/6})^2/(2n)},$$

and since ν-balls grow exponentially (2) the second term of (5) is bounded above by

$$ne^{-(an^{5/6})^2/(2n) + c_3 c_1 a n^{2/3}} = ne^{(c_3 c_1 a - a^2/2) n^{2/3}},$$

which is at most c_4/a for an appropriate choice of c_1. $\square$

4 A finitely presented example and generalizations

Our first goal is to show that an HNN-extension of G gives a finitely presented example separating AG and SG. The following lemma is needed.

Lemma 11 *G has the following presentation:*

$$G = \langle a, b \,|\, \sigma^n[a, a^b] \quad \forall n \in \mathbb{N}\rangle,$$

where σ is the substitution $b \mapsto a, a \mapsto b^2$.

Proof. By Proposition 9 of Grigorchuk and Żuk (2002a) we have

$$G = \langle a, b \,|\, \sigma^n[a, a^{b^{2m+1}}] \quad \forall n, m \in \mathbb{N}\rangle.$$

For odd i, we have $a^{b^i} \equiv [a^{-1}, b^{-2}]^b a^{b^{i-2}}$ using the relation $[b^{2a}, b^2] = \sigma([a^b, a])$; therefore $[a^{b^i}, a]$ follows from $[a^{b^{i-2}}, a]$ and $[[a^{-1}, b^{-2}]^b, a]$, which itself is a consequence of $[a^b, a]$. So the relations $\sigma^n([a, a^{b^{2m+1}}])$ may be eliminated for all $m > 1$ as long as $\sigma^n([a, a^b])$ and $\sigma^{n+1}([a, a^b])$ are kept. □

Theorem 12 *G embeds in the finitely presented group*

$$\tilde{G} = \langle a, t \,|\, a^{t^2} = a^2,\ [[[a, t^{-1}], a], a] = 1\rangle.$$

Furthermore, $\tilde{G}$ is also amenable, and does not belong to the class SG.

This implies that the classes SG and AG are distinct, even in the realm of finitely presented groups.

Proof. Let $\tilde{G}$ be the HNN extension of G along the endomorphism σ identifying G and $\sigma(G)$: it is given by the presentation

$$\tilde{G} = \langle a, b, t |\, a^t = \sigma(a),\ b^t = \sigma(b), \text{ relations in } G\rangle.$$

A simpler presentation follows by eliminating the generator b.

Consider the kernel H of the map $a \mapsto 1, t \mapsto t$ from $\tilde{G}$ to $\langle t\rangle$. Since the HNN extension is "ascending", we have $H = \bigcup_{n\in\mathbb{Z}} G^{t^n}$, an ascending union. Therefore H is amenable, and since $\tilde{G}$ is an extension of H by $\mathbb{Z}$, it is also amenable.

Finally, if $\tilde{G}$ were in SG, then G would also be in SG, since it is the subgroup of $\tilde{G}$ generated by a and $a^{t^{-1}}$. However, Proposition 13 of Grigorchuk and Żuk (2002a) shows that G is not in SG. □

Generalizations. In what setting does the proof for amenability work? Let G be a group acting spherically transitively on a b-ary rooted tree ($b \geq 2$), and suppose that it is defined recursively by the set S of generators $g_i = (g_{i,1}, \dots, g_{i,b})\sigma_i$, where each $g_{i,v}$ is one of the g_j. Consider the Schreier graph of the action of G on T_1, that is level 1 of the tree; we label level 1 of the tree by the integers $1, \dots b$. Furthermore, we label each directed edge (v, g_i) by $g_{i,v}$.

Fix a vertex at level 1, without loss of generality the vertex 1. Consider the set of cycles in the Schreier graph that go from 1 to 1 and may traverse edges either forwards or backwards; such a cycle is called "irreducible" if it only visits 1 at its endpoints. The label of a cycle is the product of the labels along its edges (taking inverses if we traverse an edge backwards).

If $\#S \geq 2$ then there are infinitely many irreducible cycles. A necessary condition (1) for our proof to work is that the set of labels of irreducible cycles is finite and agrees with the set S of generators together with the identity.

Given a probability distribution μ on the set of generators, we get a distribution on the set of irreducible loops by considering the path of a random walk on G up to the first positive time τ that it fixes 1 and has a cycle whose label is not 1. Call the distribution of this label μ'. The transformation $\mu \mapsto \mu'$ is a continuous map from a convex set to itself, so it has a fixed point.

A further necessary condition is that at least one fixed point is in the interior of the convex set, i.e. assigns positive weight to each generator. For this, it is sufficient that condition (1) does not hold for any proper subset of S.

Now let μ_0 be a such a fixed point, and let $\alpha = \log b / \log \mathbf{E}\tau$ for the corresponding random time τ. If $\alpha > 1/2$, then the argument above gives a heat kernel lower bound of e^{-cn^α}. The argument above cannot give an exponent below 1/2 as the rate of escape cannot be slower than $n^{1/2}$. In the proof, the large deviation bounds for σ break down at $\alpha = 1/2$.

Example. Consider the group acting on the binary tree generated by $a_i = (1, a_{i+1})$ for $i < k$, and $a_k = (1, a_1)\varepsilon$. The distribution $\mu = (m_1, \dots, m_k)$ on the generators (and symmetrically on their inverses) is then sent to $T\mu' = (m_k/2, m_1, \dots, m_{k-1})/(1 - m_k/2)$. A fixed point is given by $(1, 2^{1/k}, \dots, 2^{(k-1)/k})$ normalized to be a probability distribution. A simple computation gives $\mathbf{E}\tau = 2^{1+1/k}$, and we get the heat kernel lower bound e^{-cn^α} with $\alpha = k/(k+1)$.

In this example, it is not important to consider a fixed point. Since $T^k = 1$, one may iterate the decomposition process k times starting from an arbitrary μ. Then one is lead to consider 2^k processes having the same distribution as the original walk, each with time running slower by a constant factor. After massive cancellations, one finds that the constant $\prod \mathbf{E}\tau_i$ does not depend on μ, and equals 2^{k+1}. This gives the same heat kernel bound as above.

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