T. DE FERNEX AND A. LANTERI
KODAI MATH. J.
22 (1999), 83–98

AMPLE VECTOR BUNDLES AND DEL PEZZO MANIFOLDS

TOMMASO DE FERNEX AND ANTONIO LANTERI

Abstract

Let $\mathcal{E}$ be an ample vector bundle of rank r on a smooth complex projective manifold X of dimension $n \geq r+3$. Pairs $(X, \mathcal{E})$ as above are investigated under the assumption that $\mathcal{E}$ has a regular section vanishing along a Fano manifold Z of index $\dim Z - 1$ and Picard number $\rho(Z) \geq 2$.

Introduction

Let X be a complex projective manifold of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank $r \leq n-2$ on X having a regular section, i.e. there exists a section $s \in \Gamma(\mathcal{E})$ whose zero locus $Z := (s)_0$ is a smooth subvariety of the expected dimension $n-r$. Triplets $(X, \mathcal{E}, Z)$ as above have been investigated in several papers ([LM1], [LM2], [LM3], [dF], [LM4]) under the assumption that Z is some special variety. In particular the case when Z is a Fano manifold of index $\dim Z - 1$ and Picard number $\rho(Z) = 1$ was discussed in [LM1, (2.4)]. In this paper we focus on the case $\rho(Z) > 1$, assuming that $\dim Z \geq 3$. Actually as the results in [LPS] show, the same study when Z is a surface is far from being complete even in the case of divisors, i.e. when $r = 1$. We recall that extending several classification results known in the setting of ample divisors is the main motivation for investigating triplets $(X, \mathcal{E}, Z)$ as above [LM1].

To relate our Z to the title note that Fano manifolds of index $\dim Z - 1$ coincide with del Pezzo manifolds, with the only exception given by the pair $(\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}^3}(2))$. So, having assumed that $\dim Z \geq 3$, according to the classification of del Pezzo manifolds, [F, Chapter I, §8], Z is one of the following:

$$\text{(0.1)} \qquad \boldsymbol{P}^2 \times \boldsymbol{P}^2, \quad \boldsymbol{P}(T_{\boldsymbol{P}^2}), \quad \boldsymbol{B}_q(\boldsymbol{P}^3), \quad \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1,$$

where $\boldsymbol{B}_q(\boldsymbol{P}^3)$ stands for $\boldsymbol{P}^3$ blown-up at a point q. Note that this threefold has

---

1991 *Mathematics Subject Classification*. Primary 14F05, 14J60; secondary 14J40.

Key words and phrases. ample vector bundle; del Pezzo manifold; extremal ray.

Work partially supported by the M.U.R.S.T. of the Italian Government, in the framework of the National Research Project "Geometria Algebrica, Algebra Commutativa e Aspetti Computazionali"

Both authors are members of the G.N.S.A.G.A. of the Italian C.N.R.

Received June 8, 1998.

a $\boldsymbol{P}^1$-bundle structure over $\boldsymbol{P}^2$, a section of which is the exceptional divisor of the blowing-up. So in all cases $Z = \boldsymbol{P}_S(\mathscr{B})$, where $S = \boldsymbol{P}^2$ and $\mathscr{B}$ is any twist of $\mathcal{O}_{\boldsymbol{P}^2}^{\oplus 3}$, $T_{\boldsymbol{P}^2}$, $\mathcal{O}_{\boldsymbol{P}^2}(1) \oplus \mathcal{O}_{\boldsymbol{P}^2}$ in the first three cases, while $S = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ with $\mathscr{B}$ any twist of the trivial bundle of rank 2 in the last.

The main result of this paper is that triplets $(X, \mathscr{E}, Z)$ as above with Z as in (0.1) satisfy a very strong restriction. In fact the $\boldsymbol{P}$-bundle structure of Z described above extends to a $\boldsymbol{P}$-bundle structure of X. More precisely we have

THEOREM 1. *Let X be a smooth complex projective n-fold and let $\mathscr{E}$ be an ample vector bundle on X of rank r, $2 \le r \le n-3$, having a regular section with zero locus Z. If Z is a Fano manifold of index* $\dim Z - 1$ *with $\rho(Z) \ge 2$ then $X = \boldsymbol{P}_S(\mathscr{F})$ where $\mathscr{F}$ is an ample vector bundle of rank $n-1$ over a surface*

$$S \cong \begin{cases} \boldsymbol{P}^1 \times \boldsymbol{P}^1, & \text{if } Z = \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1 \\ \boldsymbol{P}^2, & \text{otherwise.} \end{cases}$$

Moreover $\mathscr{E} = H \otimes f^*\mathscr{G}$, where H is the tautological line bundle of $\mathscr{F}$ on X, $f : X \to S$ is the bundle projection and $\mathscr{G}$, a vector bundle of rank r on S, is the dual of the kernel of the vector bundle surjection $\mathscr{F} \to \mathscr{B}$ corresponding to the fibrewise inclusion of $Z = \boldsymbol{P}_S(\mathscr{B})$ into X.

The proof relies on a thorough analysis of the Mori cone of X in comparison with that of Z. Due to the ampleness of $\mathscr{E}$ the cone of Z can be seen as a subcone of X in the same real vector space $\boldsymbol{R}^{\rho(Z)}$. Moreover the special structure of Z allows us to prove that a negative extremal ray of Z is also a negative extremal ray for X. The corresponding contraction is the morphism f. To see that it makes X a $\boldsymbol{P}$-bundle over S we construct a suitable ample vector bundle of rank $n-2$ on X, whose adjoint bundle is not nef, to which a result of Maeda [M] applies.

Now, let $h \in \operatorname{Pic}(Z)$ be the ample line bundle such that $-K_Z = (\dim Z - 1)h$. Then, due to the ampleness of $\mathscr{E}$ there exists a unique line bundle $\mathscr{H} \in \operatorname{Pic}(X)$ such that $\mathscr{H}_Z = h$. When $\mathscr{H}$ is ample all possible triplets $(X, \mathscr{E}, \mathscr{H})$ occurring for $\dim Z \ge 2$ have been classified in [LM4, Theorem 4]; in particular it turns out that for $\dim Z \ge 3$ it cannot be $\rho(Z) > 1$. In other words in our setting $\mathscr{H}$ cannot be ample. On the other hand, when $\rho(Z) = 2$ and X is Fano, X has a second negative extremal ray, whose contraction morphism has been analyzed in [SzW]. This allows us to obtain a nice application of Theorem 1, improving the above conclusion as follows.

THEOREM 2. *Let X be a smooth complex projective n-fold and let $\mathscr{E}$ be an ample vector bundle on X of rank r, $2 \le r \le n-3$, having a regular section whose zero locus Z is a Fano manifold of index* $\dim Z - 1$ *with $\rho(Z) = 2$. Let $-K_Z = (\dim Z - 1)h$, with $h \in \operatorname{Pic}(Z)$ ample, and let $\mathscr{H} \in \operatorname{Pic}(X)$ be the line bundle extending h to X. Then $\mathscr{H}$ is not nef.*

As to Theorem 1, in fact we cannot assert that our result is effective. Actually it is well known that products like $\boldsymbol{P}^2 \times \boldsymbol{P}^2$ and $\boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$ cannot occur as ample divisors [S, Proposition IV]. The same is known for $\boldsymbol{B}_q(\boldsymbol{P}^3)$ [F, p. 66]. This leads us to suspect that these cases should not really occur also in our setting.

The paper is organized as follows. In Section 1 we collect some background material and prove a key lemma for comparing the cones of Z and X. The proof of Theorem 1 takes Sections 2 and 3. In Section 4 we prove Theorem 2.

1. Background material

(1.1) We only consider complex projective varieties. A smooth projective variety is frequently called a manifold, sometimes an n-fold to emphasize its complex dimension n. We use standard notation in algebraic geometry, but following current abuses we do not distinguish between vector bundles and the corresponding locally free sheaves. Moreover we adopt the additive notation for the tensor product of line bundles, reserving the multiplicative one for intersections in the Chow rings. We denote by $\mathscr{E}_Z$ the pull-back of a vector bundle $\mathscr{E}$ on a manifold X via an embedding $Z \subset X$.

(1.2) Let X be a manifold, and set

$$N^1(X) := (\mathrm{Pic}(X)/\equiv) \otimes \boldsymbol{R}, \quad N_1(X) := (Z_1(X)/\equiv) \otimes \boldsymbol{R},$$

where $\equiv$ denotes the numerical equivalence. These $\mathbf{R}$-vector spaces are in duality through the intersection of 1-cycles and divisors. Their dimension $\rho(X)$ is the Picard number of X. In $N_1(X)$ the numerical classes of effettive 1-cycles span a convex cone whose closure is denoted by $\overline{NE}(X)$. The corresponding dual cone in $N^1(X)$, which is spanned by the numerical classes of nef divisors, is the closure of the cone $\mathrm{Amp}(X)$ spanned by the classes of ample divisors, due to the Nakai-Moishezon-Kleiman ampleness criterion. If D is a divisor of X, we set

$$\overline{NE}_{D \gtreqless 0}(X) := \{[C] \in \overline{NE}(X) \mid C \cdot D \gtreqless 0\}.$$

According to [K, Chapter II, Definition 4.9], if V is a closed convex cone in $\boldsymbol{R}^n$, a subcone $W \subset V$ is called *extremal* if it is so in the sense of convexity. A polyhedral extremal subcone is called an *extremal face*. A one dimensional subcone is called a *ray*. A ray R of $\overline{NE}(X)$ is said to be *negative* if it belongs to $\overline{NE}_{K_X<0}(X)$. A divisor L is a *good supporting divisor* of an extremal face W of $\overline{NE}(X)$ if L is nef and $\overline{NE}_{L=0}(X) = W$.

(1.2.1) THE CONE THEOREM [M], [KMM]. *Let X be a manifold. Then there exist countably many negative extremal rays R_ι of $\overline{NE}(X)$ such that*

$$\overline{NE}(X) = \overline{NE}_{K_X \geq 0}(X) + \sum_\iota R_\iota.$$

Such rays R_ι are locally discrete in the open halfspace $\overline{NE}_{K_X<0}(X)$, and each of *them contains the numerical class of a rational curve* C_ι *such that*

$$0 < -K_X \cdot C_\iota \leq \dim X + 1.$$

(1.2.2) THE CONTRACTION THEOREM [KMM]. *Let X be a manifold and let R be a negative extremal ray. Let L be a good supporting divisor of R. Then* $|mL|$ *is base point free for* $m \gg 1$. *Thus, through Stein factorization, it defines a morphism* $f_R : X \to Y$ *over a normal variety Y, with connected fibres, such that* $-K_X$ *is* f_R*-ample and, if C is any irreducible curve on X, then* $f_R(C)$ *is a point if and only if* $[C] \in R$. *Moreover* $\rho(Y) = \rho(X) - 1$.

(1.3) Let R be a negative extremal ray of an n-fold X. We recall that the *length* of R is defined as

$$l(R) := \min\{-K_X \cdot C \mid C \text{ is a rational curve}, [C] \in R\}.$$

By the Cone Theorem (1.2.1) this number satisfies the inequality:

$$0 < l(R) \leq n + 1 \tag{1.3.1}$$

Now let $\mathscr{E}$ be an ample vector bundle on X, and consider the set $\Omega(X, \mathscr{E})$ of those negative extremal rays R of $\overline{NE}(X)$ having negative intersection with the adjoint bundle $K_X + \det \mathscr{E}$. If such set is not empty, for any $R \in \Omega(X, \mathscr{E})$ and C a rational curve in R such that $-K_X \cdot C = l(R)$ we define the positive number

$$\Lambda(X, \mathscr{E}, R) := -(K_X + \det \mathscr{E}) \cdot C.$$

(1.3.2) PROPOSITION. *Let* $\mathscr{E}$ *be an ample vector bundle on X and assume that* $\Omega(X, \mathscr{E})$ *is not empty. Then for every* $R \in \Omega(X, \mathscr{E})$

$$l(R) \geq \Lambda(X, \mathscr{E}, R) + \operatorname{rk} \mathscr{E}.$$

We refer for this to [W1, Theorem (3.3)]. Note that though stated for $\mathscr{E}$ ample and spanned, its proof does not involve the spannedness assumption, simply relying on the first part of [W1, Lemma (3.2)], where $\mathscr{E}$ is simply supposed to be ample.

(1.3.3) THEOREM [W2, Theorem (3.3)]. *Let R be a negative extremal ray on a manifold X,* $f = f_R : X \to Y$ *its contraction,* $E := \operatorname{Exc}(f)$ *the locus of the points of the curves belonging to R, F the general fibre of* $f|_E$. *Then*

$$\dim E + \dim F \geq \dim X - 1 + l(R).$$

(1.4) LEMMA. *Let Z be a smooth irreducible subvariety of a manifold X, and assume that the inclusion induces an isomorphism* $N_1(Z) \cong N_1(X)$. *Let R be a negative extremal ray in* $\overline{NE}(X)$ *and let* $f^X : X \to Y$ *be the corresponding contraction. Then the restriction* $f^X|_Z$ *of* f^X *to Z is not a finite morphism if and only if R is an extremal ray in* $\overline{NE}(Z)$ *as well. In this case, if R is also negative in* $\overline{NE}(Z)$, *denote by* f^Z *the relative contraction of Z; then* $f^X|_Z$ *factors through* f^Z *and a finite morphism.*

Proof. The fact the $f^X|_Z$ is not a finite morphism if and only if $R \subset \overline{NE}(Z)$ is a consequence of the Contraction Theorem (1.2.2); the extremality of R in $\overline{NE}(Z)$ follows from its extremality in $\overline{NE}(X)$ and the inclusion of the cones $\overline{NE}(Z) \subset \overline{NE}(X)$.

Now suppose that R is a negative extremal ray both in $\overline{NE}(X)$ and $\overline{NE}(Z)$. f^X is defined starting from a good suporting divisor L of R as extremal ray of $\overline{NE}(X)$; by inclusion of the cones of the curves, $l := L_Z$ is also a good supporting divisor of R as extremal ray of $\overline{NE}(Z)$. Let $\phi_{|mL|} : X \to Y_0$ and $\phi_{|ml|} : Z \to W_0$. Note that $\phi_{|mL|}|_Z$ is $\phi_{|ml|}$ followed by a projection π from some linear subspace of $\boldsymbol{P}(H^0(ml))$. Let

$$X \xrightarrow{f^X} Y \xrightarrow{u} Y_0, \quad Z \xrightarrow{f^Z} W \xrightarrow{v} W_0,$$

be the corresponding Stein factorizations, where u and v are finite morphisms. Then

$$(u \circ f^X)|_Z = \pi \circ v \circ f^Z.$$

Call h this morphism. Let G be a positive dimensional connected component of a fibre of h. Then every irreducible curve $C \subset G$ is contracted by f^X; hence $[C] \in R$, but since R is also an extremal ray of Z we see that $f^Z(C)$, hence $f^Z(G)$, is a point. It thus follows that π is finite. Moreover $f^X|_Z$ factors through f^Z and a morphism v' factoring $\pi \circ v$. But v' has to be finite, so being $\pi \circ v$. □

2. The $\boldsymbol{P}^{n-2}$-bundle structure of X

(2.1) Let $(X, \mathscr{E}, Z)$ be as in the assumption of Theorem 1. Then Z is one of the four manifolds appearing in (0.1). Since in all cases $\dim Z \geq 3$, the Lefschetz-Sommese theorem [LM3, Theorem (1.1)] tells us that the inclusion $i : Z \to X$ induces isomorphisms both on the second cohomology groups and on the Picard groups:

(2.1.1) $$i^* : \mathrm{Pic}(X) \xrightarrow{\cong} \mathrm{Pic}(Z).$$

Recall that, on the Picard groups, numerical and homological equivalence coincide (e.g. [H, Proposition 3.1]). Then, taking quotients with respect to numerical equivalence, (2.1.1) still induces an isomorphism between the numerical equivalence class groups. So, by tensoring with $\boldsymbol{R}$ and using duality, we get isomorphisms:

(2.1.2) $$N^1(X) \cong N^1(Z), \quad N_1(X) \cong N_1(Z),$$

and under these identifications we have the obvious inclusions

$$\overline{NE}(X) \supset \overline{NE}(Z), \quad \mathrm{Amp(X)} \subset \mathrm{Amp(Z)}.$$

Note that for any $[C] \in \overline{NE}(Z)$ we have $K_X \cdot C = (K_Z - \det \mathscr{E}_Z) \cdot C < 0$, which means that $\overline{NE}_{K_X<0}(X)$ is not empty. Hence by the Cone Theorem (1.2.1) there exists at least a negative extremal ray R of $\overline{NE}(X)$. Moreover $\Omega(X,\mathscr{E})$ it is not empty because $(K_X+\det\mathscr{E})_Z \cong K_Z$ is not nef.

(2.1.3) From now on we consider $R\in\Omega(X,\mathscr{E})$ and $[C]\in R$ such that $-K_X\cdot C=l(R)$. We denote by $f=f_R^X : X\to Y$ the contraction of R, by $E:=\mathrm{Exc}(f)$ the locus of the points of the curves belonging to R and by F the general fibre of $f|_E$. Note that it can be $E=X$.

We are going to investigate the contraction f_R of X according to the four cases in (0.1).

(2.1.4) Claim. *R has length $n-1$, and f is a morphism of fiber type onto a surface Y whose restriction $f|_Z$ to Z factors through the contraction of an extremal ray of $\overline{NE}(Z)$ and a finite morphism. Moreover there exists an ample vector bundle $\mathscr{E}'$ of rank $n-2$ on X such that $K_X+\det\mathscr{E}'$ is not nef.*

We will prove the Claim by cases.

(2.2) Case $Z\cong \boldsymbol{P}^2\times\boldsymbol{P}^2$. In this case $N_1(X)\cong N_1(Z)\cong\boldsymbol{R}^2$, hence $\overline{NE}(X)$ and $\overline{NE}(Z)$ are 2-dimensional cones. This implies that both admit exactly two extremal rays. The extremal rays of $\overline{NE}(Z)$ are spanned by the numerical classes of two lines C_1 and C_2 belonging respectively to fibres of the two projections of Z onto the factors. We can think of these two curves as the generators of $N_1(Z)$, hence of $N_1(X)$, through the identification (2.1.2). The line bundles corresponding to such curves by duality are $h_1:=\mathcal{O}_Z(1,0)$ and $h_2:=\mathcal{O}_Z(0,1)$. We denote by H_i $(i=1,2)$ the line bundle on X corresponding to h_i through the isomorphism (2.1.1).

Let $R=\boldsymbol{R}_{\geq 0}[C]$ and $f=f_R$ be as in (2.1.3). We can write $[C]=\sum_i\lambda_i[C_i]$ where

$$\lambda_i=h_i\cdot\sum_j\lambda_j[C_j]=H_i\cdot C\in\boldsymbol{Z}.$$

Hence

$$0<\Lambda(X,\mathscr{E},R)=-(K_X+\det\mathscr{E})\cdot C=-K_Z\cdot\sum_i\lambda_iC_i=3\sum_i\lambda_i,$$

which implies $\Lambda(X,\mathscr{E},R)\geq 3$. So $l(R)\geq n-1$ by (1.3.2). Now by Theorem (1.3.3) we get

$$\dim E+\dim F\geq n-1+l(R)\geq 2n-2. \tag{2.2.1}$$

(2.2.2) Suppose that f is birational, i.e. $\dim E<n$. Then (2.2.1) implies that $\dim E=\dim F=n-1$. So E is a divisor and $f(E)$ is a point. Note that E cannot contain Z, since otherwise f would map Z to a point while Lemma (1.4) tells us that $f|_Z$ cannot be the contraction of the whole cone $\overline{NE}(Z)$. Then, restricting the divisor E to Z, we see that $E_Z=E\cap Z$ is an effective divisor of Z which is nontrivial in view of the isomorphism (2.1.1). Therefore $f|_Z$ is a

birational morphism of Z on $f(Z)$ which contracts the divisor E_Z. But this cannot happen because $\boldsymbol{P}^2 \times \boldsymbol{P}^2$ does not admit any divisorial contraction, and so this case does not occur.

(2.2.3) Therefore f is of fiber type: $\dim E = n$, $\dim F \geq l(R) - 1 \geq n - 2$ and $\dim Y = n - \dim F \leq 2$. Note that $f|_Z$ is not finite: by appying Lemma (1.4) we get that R is also a (negative) extremal ray of $\overline{NE}(Z)$, say for example $R = \boldsymbol{R}_{\geq 0}[C_1]$, and $f|_Z : Z \to Y$ factors through one of the two projections of Z on $\boldsymbol{P}^2$ and a finite morphism. We deduce in particular that $\dim Y = 2$ and $l(R) = n - 1$.

(2.2.4) Note that $H_2 = \mathcal{O}_X(0,1)$ is a good supporting divisor of R, and in particular that $[H_2] \in \overline{\mathrm{Amp}}(X)$. Since $\overline{\mathrm{Amp}}(X) \subset \overline{\mathrm{Amp}}(Z)$, we deduce that $[(1/m)H_1 + H_2] \in \mathrm{Amp}(X)$ for a sufficient large $m \in \boldsymbol{N}$. In other words there is an ample line bundle of the form $H := H_1 + mH_2$. Let $\mathscr{E}' := \mathscr{E} \oplus H \oplus H$. Then $\mathscr{E}'$ is an ample vector bundle of rank $n - 2$ whose adjoint bundle $K_X + \det \mathscr{E}'$ is not nef, since

$$(K_X + \det \mathscr{E}') \cdot C_1 = (K_Z + 2H_Z) \cdot C_1 = -1.$$

(2.3) Case $Z \cong \boldsymbol{P}(T_{\boldsymbol{P}^2})$. In this case $N_1(X) \cong N_1(Z) \cong \boldsymbol{R}^2$ and the two cones $\overline{NE}(X)$ and $\overline{NE}(Z)$ are 2-dimensional. As is well known,

$$Z = \{(x, y) \in \boldsymbol{P}^2 \times \boldsymbol{P}^2 \mid x_0 y_0 + x_1 y_1 + x_2 y_2 = 0\} \in |\mathcal{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^2}(1,1)|$$

and $K_Z = -2h$, where $h := \mathcal{O}_{\boldsymbol{P}^2 \times \boldsymbol{P}^2}(1,1)_Z$. Note that

$$Z_x : Z \cap (\{x\} \times \boldsymbol{P}^2) \cong \{y \in \boldsymbol{P}^2 \mid x_0 y_0 + x_1 y_1 + x_2 y_2 = 0\} \cong \boldsymbol{P}^1$$

for any $x \in \boldsymbol{P}^2$ and, symmetrically, $Z_y = Z \cap (\boldsymbol{P}^2 \times \{y\}) \cong \boldsymbol{P}^1$ for any $y \in \boldsymbol{P}^2$. Hence Z has two structures of $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^2$ induced by the two projections of $\boldsymbol{P}^2 \times \boldsymbol{P}^2$. The extremal rays of $\overline{NE}(Z)$ are spanned by the numerical classes of the corresponding fibres C_1 and C_2, and $K_Z \cdot C_\iota = -2$.

Arguing as in the case (2.2), we find that if R is as in (2.1.3) then $l(R) \geq n - 1$. Then proceeding as in (2.2.2)–(2.2.4) we get that $f = f_R$ is of fibre type, R is an extremal ray of $\overline{NE}(Z)$, e.g. $R = \boldsymbol{R}_{\geq 0}[C_1]$, and $f(X)$ is a surface dominated by $\boldsymbol{P}^2$. In the same way as we did in (2.2.4), we construct an ample line bundle H on X inducing on Z a tautological line bundle for the $\boldsymbol{P}^1$-bundle at hand and then $\mathscr{E}' := \mathscr{E} \oplus H$ is an ample vector bundle on X, whose adjoint bundle is not nef.

(2.4) Case $Z \cong \boldsymbol{B}_q(\boldsymbol{P}^3) \cong \boldsymbol{P}_{\boldsymbol{P}^2}(\mathcal{O}_{\boldsymbol{P}}(1) \oplus \mathcal{O}_{\boldsymbol{P}})$. Here $N_1(X) \cong N_1(Z) \cong \boldsymbol{R}^2$, and so $\overline{NE}(X)$ and $\overline{NE}(Z)$ are 2-dimensional cones. In this case the two extremal rays of $\overline{NE}(Z)$ are spanned by $[C_1]$ and $[C_2]$, where C_1 is the strict transform of a line of $\boldsymbol{P}^3$ passing through q through the blow-up $\sigma : Z \cong \boldsymbol{B}_q(\boldsymbol{P}^3) \to \boldsymbol{P}^3$, and C_2 is a line on the exceptional locus A of σ. By duality the two extremal rays of $\mathrm{Amp}(Z)$ are spanned by $[h_1]$ and $[h_2]$, h_i being determined by the conditions

$h_i \cdot C_j = \delta_{ij}$. So

$$h_1 = \sigma^* \mathcal{O}_{\boldsymbol{P}^3}(1), \quad h_2 = \sigma^* \mathcal{O}_{\boldsymbol{P}^3}(1) - A.$$

Let $H_i \in \operatorname{Pic}(X)$ be such that $(H_i)_Z = h_i$, $i = 1, 2$. Thinking of Z as the $\boldsymbol{P}^1$-bundle $\pi : \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2}(1) \oplus \mathcal{O}_{\boldsymbol{P}^2}) \to \boldsymbol{P}^2$, C_1 can be seen as a fibre of π; hence $(K_Z)_{C_1} = K_{C_1}$, and so

$$K_Z \cdot C_1 = \deg K_{\boldsymbol{P}^1} = -2.$$

On the other hand, thinking of Z as the blow-up of $\boldsymbol{P}^3$, since $A \cong \boldsymbol{P}^2$ is the exceptional locus, we have $\mathcal{O}_A(A) \cong \mathcal{O}_{\boldsymbol{P}^2}(-1)$ and $K_A \cong \mathcal{O}_{\boldsymbol{P}^2}(-3)$; so, by adjunction, $(K_Z)_A = K_A - \mathcal{O}_A(A) \cong \mathcal{O}_{\boldsymbol{P}^2}(-2)$. Therefore

$$K_Z \cdot C_2 = (K_Z)_A \cdot C_2 = -2.$$

Let $R = \boldsymbol{R}_{\geq 0}[C]$ and let $f : X \to Y$ be as in (2.1.3). As in (2.2) we have $[C] = \sum_i \lambda_i [C_i]$ with $\lambda_i \in \boldsymbol{Z}$, and

$$0 < \Lambda(X, \mathscr{E}, R) = -(K_X + \det \mathscr{E}) \cdot C = -K_Z \cdot \sum_i \lambda_i C_i = 2 \sum_i \lambda_i,$$

which implies $\Lambda(X, \mathscr{E}, R) \geq 2$. So $l(R) \geq n - 1$ by (1.3.2), and again

$$\dim E + \dim F \geq n - 1 + l(R) \geq 2n - 2. \tag{2.4.1}$$

(2.4.2) Birational Case. In this case $\dim E < n$. Then $\dim E = \dim F = n - 1$ by (2.4.1). So E is a divisor and $f(E)$ is a point. As we have seen in (2.2.2), E cannot contain Z, hence $E_Z = E \cap Z$ is a nontrivial (effective) divisor on Z and $f|_Z$ is the divisorial contraction of such divisor. So necessarily $E_Z = A$ and $f|_Z = \sigma$. This means in particular that the ray R is spanned by $[C_2]$.

In the same way as we did in (2.2.4), we get that $H := mH_1 + H_2$ is an ample line bundle on X for a large $m \in \boldsymbol{N}$. Let $\mathscr{E}' := \mathscr{E} \oplus H$. Then $\mathscr{E}'$ is an ample vector bundle on X of rank $n - 2$ with non nef adjoint bundle, since

$$(K_X + \det \mathscr{E}') \cdot C_2 = (K_Z + H_Z) \cdot C_2 = -1.$$

By applying [M] or [Z, Theorem 1.1], we conclude that

$$(E, \mathcal{O}_E(E), \mathscr{E}_E) \cong (\boldsymbol{P}^{n-1}, \mathcal{O}_{\boldsymbol{P}^{n-1}}(-1), \mathcal{O}_{\boldsymbol{P}^{n-1}}(1)^{\oplus(n-3)})$$

and $f : X \to Y$ is the blow-down of E to a smooth point of Y. In particular Y is smooth. Then [LM2, Lemma (5.1)] applies and so $\mathscr{E} = f^* \tilde{\mathscr{E}} \otimes \mathcal{O}_X(-E)$, where $\tilde{\mathscr{E}}$ is an ample vector bundle on Y with a regular section vanishing along $f(Z) = \sigma(Z) \cong \boldsymbol{P}^3$. This in turn implies that $(Y, \tilde{\mathscr{E}}) \cong (\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(1)^{\oplus(n-3)})$ by [LM1, Theorem A]. So we would get that $X \cong \boldsymbol{B}_q(\boldsymbol{P}^n)$ is $\boldsymbol{P}^n$ blown-up at a point q and $\mathscr{E} = f^* \mathcal{O}_{\boldsymbol{P}^n}(1)^{\oplus(n-3)} \otimes \mathcal{O}_X(-E)$. However note that this is a contradiction because each summand of $\mathscr{E}$, $f^* \mathcal{O}_{\boldsymbol{P}^n}(1) \otimes \mathcal{O}_X(-E)$, is not ample, having intersection 0 with the proper transform of every line of $\boldsymbol{P}^n$ passing through q.

(2.4.3) FIBERING CASE. Now assume that f is of fiber type, i.e. $\dim E = n$. Since Y has at least dimension two, as we have seen before, we deduce by (2.4.1) that $\dim Y = 2$, $\dim F = n-2$ and $l(R) = n-1$. By applying Lemma (1.4) we have that R is spanned by $[C_1]$ and $\boldsymbol{P}^2$ dominates Y. By arguing as in the previous cases we find an ample line bundle H on X inducing $\mathcal{O}_{\boldsymbol{P}^1}(1)$ on the fibres of Z. Then $\mathscr{E}' = \mathscr{E} \oplus H$ is an ample vector bundle on X whose adjoint bundle is not nef.

(2.5) CASE $Z \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$. In this case $N_1(X) \cong N_1(Z) \cong \boldsymbol{R}^3$, and the cones of the effective curves are 3-dimensional. The extremal rays of $\overline{NE}(Z)$ are spanned by the numerical classes of three lines C_1, C_2 and C_3 which are respectively fibres of the three different projections of Z on $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. As in (2.2), we look at their classes as the generators of $N_1(Z)$, hence of $N_1(X)$ through the identification (2.1.2). Let $h_i \in \mathrm{Pic}(Z)$ be the corresponding dual line bundles and $H_i \in \mathrm{Pic}(X)$ such that $(H_i)_Z = h_i$, $i = 1, 2, 3$.

Let $R = \boldsymbol{R}_{\geq 0}[C]$ and f be as in (2.1.3). Exactly as in (2.2) we can write $[C] = \sum_i \lambda_i [C_i]$ with $\lambda_i \in \boldsymbol{Z}$, and we have

$$0 < \Lambda(X, \mathscr{E}, R) = -(K_X + \det \mathscr{E}) \cdot C = -K_Z \cdot \sum_i \lambda_i C_i = 2 \sum_i \lambda_i,$$

which implies $\Lambda(X, \mathscr{E}, R) \geq 2$. So $l(R) \geq n-1$ by (1.3.2).

(2.5.1) By applying (1.3.3) and arguing as in (2.2.2), (2.2.3), we get that R is one of the three extremal rays of $\overline{NE}(Z)$, say $\boldsymbol{R}_{\geq 0}[C_1]$, and $f|_Z : Z \to Y$ factors through a projection of Z onto $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ and a finite morphism. Hence Y is a surface dominated by $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ and $l(R) = n-1$.

(2.5.2) Let $R = \boldsymbol{R}_{\geq 0}[C_1]$. By the Cone Theorem R is an isolated extremal ray on the cone $\overline{NE}(X)$. Then there are two distinct 2-dimensional extremal faces V_1 and V_2 of $\overline{NE}(X)$ of which R is the common edge. Let L_1 and L_2 be the corresponding good supporting divisors. Note in particular that $L_i \cdot C_1 = 0$, hence, by duality, their numerical classes are necessarily contained in the cone W spanned by $[H_2]$ and $[H_3]$.

Let W' be the cone spanned by $[L_1]$ and $[L_2]$: this is a 2-dimensional subcone of W, since L_1 and L_2 are not numerically equivalent, and any divisor H with numerical class $[H]$ in the interior of W' is a good supporting divisor of R. W' is an extremal 2-dimensional face of $\overline{\mathrm{Amp}}(X)$; more precisely $W' = \overline{\mathrm{Amp}}(X) \cap W$. Then we can find a $\boldsymbol{Q}$-divisor $H' = a_2 H_2 + a_3 H_3$ in W such that $[\varepsilon H_1 + H'] \in \mathrm{Amp}(X)$ for $\varepsilon > 0$ small. This implies that $H := H_1 + m_2 H_2 + m_3 H_3$ is an ample line bundle if $m_i = m a_i$ $(i = 2, 3)$, where m is a sufficiently large common multiple of the denominators of a_2 and a_3. Now let $\mathscr{E}' := \mathscr{E} \oplus H$. Then $\mathscr{E}'$ is an ample vector bundle of rank $n-2$ and its adjoint bundle $K_X + \det \mathscr{E}'$ is not nef, since

$$(K_X + \det \mathscr{E}') \cdot C_1 = (K_Z + H) \cdot C_1 = -1.$$

This concludes the proof of the Claim (2.1.4).

(2.6) Due to (2.1.4) we have the following commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\iota} & X \\ {\scriptstyle f_R^Z}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{v} & Y, \end{array}$$

where f_R^Z denotes the contraction of R as ray of $\overline{NE}(Z)$, v is a finite morphism and

$$S \cong \begin{cases} \boldsymbol{P}^1 \times \boldsymbol{P}^1, & \text{if } Z = \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1 \\ \boldsymbol{P}^2, & \text{otherwise.} \end{cases}$$

Applying [M] to the pair $(X, \mathscr{E}')$ and splitting $\mathscr{E}'_F$ as $\mathscr{E}_F \oplus H_F^{\oplus(\dim Z-2)}$ for every fibre F of f, we get that $f : X \to Y$ is a $\boldsymbol{P}^{n-2}$-bundle over a smooth surface, and $\mathscr{E}_F \cong \mathscr{O}_{\boldsymbol{P}^{n-2}}(1)^{\oplus r}$ for every fibre F. Finally, if $S \cong \boldsymbol{P}^2$, the smoothness of Y combined with the existence of a surjective morphism $v : \boldsymbol{P}^2 \to Y$ implies that $Y \cong \boldsymbol{P}^2$. In the same way, if $S \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ we get $Y \cong \boldsymbol{P}^2$ or $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. But the equality of Picard numbers $\rho(Y) = \rho(X) - 1 = \rho(Z) - 1 = \rho(S)$ rules out the first possibility. Therefore $Y \cong S$.

3. Concluding the proof of Theorem 1

As shown in Section 2, in each one of the four cases (0.1), X is a $\boldsymbol{P}^{n-2}$-bundle over a smooth surface S, where

$$S \cong \begin{cases} \boldsymbol{P}^1 \times \boldsymbol{P}^1, & \text{if } Z = \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1 \\ \boldsymbol{P}^2, & \text{otherwise} \end{cases}$$

and $\mathscr{E}_F \cong \mathscr{O}_{\boldsymbol{P}^{n-2}}(1)^{\oplus r}$. Moreover in each case we constructed an ample line bundle $H \in \operatorname{Pic}(X)$ such that $H_F \cong \mathscr{O}_{\boldsymbol{P}^{n-2}}(1)$. Thus (X, H) is a scroll over S. Let $\mathscr{F} = f_*H$, where $f : X \to S$ is the bundle projection; then $\mathscr{F}$ is an ample vector bundle of rank $n-1$ on S, $X = \boldsymbol{P}_S(\mathscr{F})$ and $H = H(\mathscr{F})$ is the tautological line bundle of $\mathscr{F}$. Moreover, since $\mathscr{E} \otimes H^{-1}$ restricts trivially to every fibre F we conclude that there is a vector bundle $\mathscr{G} := f_*(\mathscr{E} \otimes H^{-1})$, of rank r, on S such that $\mathscr{E} = H \otimes f^*\mathscr{G}$.

Write $Z = \boldsymbol{P}_S(\mathscr{B})$, where $\mathscr{F} \to \mathscr{B}$ is the vector bundle surjection corresponding to the fibrewise inclusion of Z in X. Let $\mathscr{A}$ denote the kernel; then we have an exact sequence on S:

$$0 \to \mathscr{A} \to \mathscr{F} \to \mathscr{B} \to 0.$$

According to [LPS, Lemma 0.8], the normal bundle of Z inside X is $N_{Z/X} = (H \otimes f^*\mathscr{A}^*)_Z$. On the other hand, $N_{Z/X} = \mathscr{E}_Z$, and then the expression of $\mathscr{E}$ given above combined with the injectivity of f^* identifies $\mathscr{A}$ as the dual $\mathscr{G}^*$ of $\mathscr{G}$. In other words, for any vector bundle $\mathscr{F}$ defining the scroll structure of X, $\mathscr{F}$ and the corresponding vector bundle $\mathscr{G} = f_*(\mathscr{E} \otimes H(\mathscr{F})^{-1})$ are connected by the exact sequence

(3.0.1) $$0 \to \mathcal{G}^* \to \mathcal{F} \to \mathcal{B} \to 0.$$

This concludes the proof of Theorem 1.

In particular, taking the first Chern classes, (3.0.1) gives the relation

(3.0.2) $$c_1(\mathcal{G}) = c_1(\mathcal{B}) - c_1(\mathcal{F}).$$

4. The case $\rho = 2$

Let $(X, \mathcal{E}, Z)$ be as in Theorem 1. Condition $\rho(Z) = 2$ is equivalent to assuming that $S = \boldsymbol{P}^2$. Recall that, according to Section 2, $N^1(X)$ is generated by the classes of the line bundles $H_2 = f^*\mathcal{O}_{\boldsymbol{P}^2}(1)$ and H_1, which is the line bundle extending either $\mathcal{O}_Z(1,0)$, the tautological line bundle of $T_{\boldsymbol{P}^2}(-1)$ on Z, or the line bundle $\sigma^*\mathcal{O}_{\boldsymbol{P}^3}(1)$, according to whether

i) $Z = \boldsymbol{P}^2 \times \boldsymbol{P}^2$,

ii) $Z = \boldsymbol{P}(T_{\boldsymbol{P}^2})$ or

iii) $Z = \boldsymbol{B}_q(\boldsymbol{P}^3)$.

Note that in all three cases we have

$$-K_Z = (\dim Z - 1)(H_1 + H_2)_Z.$$

So, the line bundle $\mathcal{H} := H_1 + H_2$ extends $(\dim Z - 1)^{-1}(-K_Z)$ to X. Note that

(4.1) $\mathcal{H}$ cannot be ample.

This follows from [LM4, Theorem 4], as we observed in the Introduction. Alternately (4.1) can be directly seen as a consequence of Theorem 1. Actually, since $S = \boldsymbol{P}^2$, up to twist with a line bundle, we can replace the vector bundle $\mathcal{F}$ in Section 3, with a new vector bundle $\mathcal{F}_0$ whose tautological line bundle $H_0 := H(\mathcal{F}_0)$ is nef non-ample. Then, according to the constructions made in Section 2, we have

(4.1.1) $$H_0 \equiv H_1 + mH_2,$$

where m is a nonnegative integer. Recall that the boundary of the ample cone $\mathrm{Amp}(X)$ consists of the rays generated by the classes of H_0 and H_2. Therefore $\mathcal{H}$ is ample if and only if $m = 0$. Note however that $m = 0$ would imply $\mathrm{Amp}(X) = \mathrm{Amp}(Z)$, hence $NE(X) = NE(Z)$. Let $R' = \boldsymbol{R}_{\geq 0}[C_2]$. Then R' would be a negative extremal ray of X. Moreover $\Lambda(X, \mathcal{E}, R') > 0$ and then the same arguments as in Section 2 show that $l(R') = l(R) = n - 1$. Therefore

$$-K_X = (-K_X \cdot C_1)H_1 + (-K_X \cdot C_2)H_2 = (n-1)\mathcal{H}.$$

Thus $(X, \mathcal{H})$ would be a del Pezzo manifold. But the classification [F, p. 72, Theorem 8.11] shows that this is impossible since $\rho(X) \geq 2$ and $n \geq 5$. This gives

(4.2) Proposition. *$m \geq 1$ in (4.1.1) and equality holds if and only if $\mathcal{H}$ is nef.*

Proof. The argument above shows that $m \geq 1$. If $m = 1$ then $\mathscr{H} = H_0$, which is nef by construction. Conversely let $\mathscr{H}$ be nef; then, since it is not ample, its class lies on the boundary of $\mathrm{Amp}(X)$, i.e. generates an extremal ray. Since $\mathscr{H}$ is not a multiple of H_2, this is the same extremal ray as that generated by H_0. But then comparing the expression of $\mathscr{H}$ with (4.1.1) we see that $m = 1$. □

(4.3) Assume that $\mathscr{H}$ is nef. By adjunction

$$-K_X = \det \mathscr{E} = (\dim Z - 1)\mathscr{H}.$$

Thus $-K_X$ is ample, being the sum of an ample and a nef line bundle, i.e. X is a Fano manifold. Since $S = \boldsymbol{P}^2$ and X is Fano, [SzW] gives a list of possibilities for X. In most cases X appears as the blow-up of some projective variety. All these cases cannot occur in our setting since X does not admit divisorial contractions of negative extremal rays, as we proved in Section 2 (see (2.2.2) for case i), (2.3) of case ii), (2.4.2) for case iii)). The surviving cases in [SzW] are the following:

(a) $\mathscr{F}_0 = \mathscr{O}_{\boldsymbol{P}^2}^{\oplus(n-1)}$ (hence $X = \boldsymbol{P}^2 \times \boldsymbol{P}^{n-2}$),

(b) $\mathscr{F}_0 = T_{\boldsymbol{P}^2}(-1) \oplus \mathscr{O}_{\boldsymbol{P}^2}^{\oplus(n-3)}$,

(c) $\mathscr{F}_0$ fits into an exact sequence $0 \to \mathscr{O}_{\boldsymbol{P}^2}(-2) \to \mathscr{O}_{\boldsymbol{P}^2}^{\oplus n} \to \mathscr{F}_0 \to 0$. Moreover $\mathscr{F}_0$ has a trivial summand $\mathscr{O}_{\boldsymbol{P}^2}$ as soon as its rank is ≥ 6.

Note that in all these cases the bundle $\mathscr{F}_0$ in normalized in the sense of [SzW, p. 92].

(4.4) Lemma. *Let $\mathscr{G}_0 = f_*(\mathscr{E} \otimes H_0^{-1})$. If there exists a section $S'(\cong \boldsymbol{P}^2)$ of $f : X \to \boldsymbol{P}^2$ such that $(H_0)_{S'}$ is trivial, then $\mathscr{G}_0$ is ample. Moreover if $\mathscr{H}$ is nef then $\mathscr{G}_0(1)$ is always ample.*

Proof. Let S' be such a section. Since $\mathscr{E} = H_0 \otimes f^*\mathscr{G}_0$ is ample, so is its restriction to S':

$$\mathscr{E}_{S'} = (H_0)_{S'} \otimes (f^*\mathscr{G}_0)_{S'} = f|_{S'}^*\mathscr{G}_0 \cong \mathscr{G}_0,$$

which shows that $\mathscr{G}_0$ is ample.

Since

$$\mathscr{E} = H_0 \otimes f^*\mathscr{O}_{\boldsymbol{P}^2}(-1) \otimes (f^*\mathscr{G}_0(1)) = (H_0 - H_2) \otimes (f^*\mathscr{G}_0(1)),$$

in the same way we see that $\mathscr{G}_0(1)$ is ample provided that f has a section S' to which $H_0 - H_2$ restricts trivially. In fact we can always find such a section. To do this take a section of $f|_Z : Z \to \boldsymbol{P}^2$ on which h_1 has trivial restriction. Then $(H_1)_{S'}$ is trivial; on the other hand $H_0 - H_2 \equiv H_1$ by Proposition (4.2). □

Now we are ready to prove Theorem 2.

(4.5) *Proof of Theorem* 2. Let $(X, \mathscr{E}, Z)$ be as in Theorem 1, with $\rho(Z) = 2$ and, by contradiction, suppose that $\mathscr{H}$ is nef. We will show that all cases (a), (b), (c) appearing in (4.3) lead to contradict the exacteness of the sequence

$$0 \to \mathscr{G}_0^* \to \mathscr{F}_0 \to \mathscr{B}_0 \to 0, \tag{4.5.1}$$

obtained by an appropriate twist from (3.0.1). Note that

$$\mathscr{B}_0 = \mathcal{O}_{\boldsymbol{P}^2}(1)^{\oplus 3}, \quad T_{\boldsymbol{P}^2}, \quad \text{or } \mathcal{O}_{\boldsymbol{P}^2}(2) \oplus \mathcal{O}_{\boldsymbol{P}^2}(1),$$

according to whether Z is as in i), ii), or iii) respectively. It is also convenient to note that

$$h^0(\mathscr{B}_0) = 9, 8, 9, \tag{4.5.2}$$

according to the three cases above.

Assume that X is as in case (a). Then the exact sequence (4.5.1) becomes

$$0 \to \mathscr{G}_0^* \to \mathcal{O}_{\boldsymbol{P}^2}^{\oplus(n-1)} \to \mathscr{B}_0 \to 0. \tag{4.5.3}$$

Thus, since $\mathscr{F}_0$ is trivial, (3.0.2) gives $c_1(\mathscr{G}_0) = 3$ in all three cases i), ii), iii). On the other hand, let $S' := \boldsymbol{P}^2 \times \{p\} \subset X$: this is a section of $f : X \to \boldsymbol{P}^2$ and $(H_0)_{S'} = (\mathcal{O}_X(0,1))_{S'} = \mathcal{O}_{S'}$. Hence $\mathscr{G}_0$ is ample by Lemma (4.4). We thus get

$$n - \dim Z = \operatorname{rk} \mathscr{E} = \operatorname{rk} \mathscr{G}_0 \le c_1(\mathscr{G}_0) = 3.$$

So either $n = 3 + \dim Z$ and $\mathscr{G}_0$ is a uniform bundle of rank 3 of splitting type $(1,1,1)$, or $n = 2 + \dim Z$ and $\mathscr{G}_0$ is a uniform bundle of rank 2 of splitting type $(2,1)$. According to [OSS, Theorem 3.2.1, p. 51 and p. 59], this gives the following three possibilities:

(1) $\mathscr{G}_0 = \mathcal{O}_{\boldsymbol{P}^2}(1)^{\oplus 3}$,
(2) $\mathscr{G}_0 = \mathcal{O}_{\boldsymbol{P}^2}(2) \oplus \mathcal{O}_{\boldsymbol{P}^2}(1)$,
(3) $\mathscr{G}_0 = T_{\boldsymbol{P}^2}$.

Taking into account cases i), ii), and iii), this leads to nine possibilities for the exact sequence (4.5.3). We rule out all of them. In case (1), the exact cohomology sequence induced by (4.5.1) gives

$$0 = H^0(\mathcal{O}_{\boldsymbol{P}^2}(-1)^{\oplus 3}) \to H^0(\mathcal{O}_{\boldsymbol{P}^2}^{\oplus(n-1)}) \to H^0(\mathscr{B}_0) \to H^1(\mathcal{O}_{\boldsymbol{P}^2}(-1)^{\oplus 3}) = 0,$$

the last term being zero by the Kodaira vanishing theorem. Thus

$$n - 1 = h^0(\mathcal{O}_{\boldsymbol{P}^2}^{\oplus(n-1)}) = h^0(\mathscr{B}_0), \tag{4.5.4}$$

which contradicts (4.5.2), since $n = \dim Z + \operatorname{rk} \mathscr{G}_0 \le 7$. In case (2) we get the same contradiction since by the Kodaira vanishing theorem the cohomology sequence induced by (4.5.3) still gives (4.5.4). Finally consider case (3). In this case (4.5.3) becomes

$$0 \to \Omega^1_{\boldsymbol{P}^2} \to \mathcal{O}_{\boldsymbol{P}^2}^{\oplus(n-1)} \to \mathscr{B}_0 \to 0. \tag{4.5.5}$$

Its extension class lives in $H^1(\Omega^1_{\boldsymbol{P}^2} \otimes \mathscr{B}_0^*)$. If Z is as in i) or iii), then Bott formula [OSS, p. 8] immediately shows that this H^1 is trivial. So (4.5.5) would split, which clearly gives a contradiction, since the vector bundle in the middle is decomposable, while the summand $\Omega^1_{\boldsymbol{P}^2}$ is not. Finally suppose that Z is as in ii). Then (4.5.5) becomes

$$0 \to \Omega^1_{\boldsymbol{P}^2} \to \mathcal{O}_{\boldsymbol{P}^2}^{\oplus 4} \to T_{\boldsymbol{P}^2} \to 0.$$

Taking cohomology we thus get

$$h^1(\Omega^1_{\boldsymbol{P}^2}) = h^0(T_{\boldsymbol{P}^2}) - h^0(\mathcal{O}_{\boldsymbol{P}^2}^{\oplus 4}) = 8 - 4,$$

which gives a contradiction, since $h^1(\Omega^1_{\boldsymbol{P}^2}) = 1$ [OSS, p. 8]. This concludes the proof in case (a).

Now assume that X is as in (b). Since $\mathcal{F}_0$ has a trivial summand, there is a section S' of $f : X \to \boldsymbol{P}^2$ satisfying the hypotesis of Lemma (4.4), hence $\mathcal{G}_0$ is ample. Note that $c_1(\mathcal{G}_0) = 3 - 1 = 2$ by (3.0.2). Then $\mathcal{G}_0$ must have rank 2 and splitting type (1,1), so it is uniform, which in turn implies that $\mathcal{G}_0 = \mathcal{O}_{\boldsymbol{P}^2}(1)^{\oplus 2}$ by [OSS, Theorem 3.2.1, p. 51]. Thus the exact sequence (4.5.1) becomes

$$0 \to \mathcal{O}_{\boldsymbol{P}^2}(-1)^{\oplus 2} \to \mathcal{F}_0 \to \mathcal{B}_0 \to 0.$$

Taking cohomology and noting that $H^q(\mathcal{O}_{\boldsymbol{P}^2}(-1)^{\oplus 2}) = 0$ for $q = 1, 2$, we get $h^0(\mathcal{F}_0) = h^0(\mathcal{B}_0)$; but this contradicts (4.5.2), since

$$h^0(\mathcal{F}_0) = h^0(T_{\boldsymbol{P}^2}(-1) \oplus \mathcal{O}_{\boldsymbol{P}^2}^{\oplus(n-3)}) = 3 + n - 3 = n = \operatorname{rk}\mathcal{G}_0 + \dim Z = 6, 5, 5,$$

according to cases i), ii), iii) respectively. In conclusion, case (b) does not occur as well.

It remains to examine case (c). In this case $c_1(\mathcal{G}_0) = 3 - 2 = 1$ by (3.0.2). If $\operatorname{rk}\mathcal{F}_0 \geq 6$, then $\mathcal{F}_0$ has a trivial summand; hence there is a section S' of $f : X \to \boldsymbol{P}^2$ satisfying the hypotesis of Lemma (4.4). But this implies that $\mathcal{G}_0$ is an ample vector bundle, with $\operatorname{rk}\mathcal{G}_0 \geq 2$ and $c_1(\mathcal{G}_0) = 1$, which is impossible. So $\operatorname{rk}\mathcal{F}_0 = n - 1 \leq 5$ and $\mathcal{G}_0$ is not ample. We have the following possibilities:

(1) Z as in i), $n = 6$, $\operatorname{rk}\mathcal{F}_0 = 5$, $\operatorname{rk}\mathcal{G}_0 = 2$,

(2) Z as in ii) or iii), $n = 5$, $\operatorname{rk}\mathcal{F}_0 = 4$, $\operatorname{rk}\mathcal{G}_0 = 2$, or

(3) Z as in ii) or iii), $n = 6$, $\operatorname{rk}\mathcal{F}_0 = 5$, $\operatorname{rk}\mathcal{G}_0 = 3$.

Note that by the second part of Lemma (4.4) any splitting type of $\mathcal{G}_0$ must have nonnegative indexes; hence $\mathcal{G}_0$ has splitting type either (1.0) or (1,0,0) according to the rank. In particular $\mathcal{G}_0$ is uniform, and applying [OSS, Theorem 2.2.2, p. 211 and 3.4, p. 70] we get:

$$\mathcal{G}_0 = \mathcal{O}_{\boldsymbol{P}^2}(1) \oplus \mathcal{O}_{\boldsymbol{P}^2} \quad \text{or} \quad T_{\boldsymbol{P}^2}(-1) \quad \text{in cases (1) and (2)},$$

$$\mathcal{G}_0 = \mathcal{O}_{\boldsymbol{P}^2}(1) \oplus \mathcal{O}_{\boldsymbol{P}^2}^{\oplus 2} \quad \text{or} \quad T_{\boldsymbol{P}^2}(-1) \oplus \mathcal{O}_{\boldsymbol{P}^2} \quad \text{in case (3)}.$$

In any case $H^1(\mathcal{G}_0^*) = 0$. Then the cohomology sequence induced by (4.5.1) shows that there is a surjection $H^0(\mathcal{F}_0) \to H^0(\mathcal{B}_0) \to 0$. Now, comparing the dimensions, we get in all cases

$$h^0(\mathcal{B}_0) \leq h^0(\mathcal{F}_0) = \operatorname{rk}\mathcal{F}_0 + 5 - c_2(\mathcal{F}_0) \leq 6,$$

by [SzW, (2.10)] recalling that $c_2(\mathscr{F}_0) = 4$. But this contradicts (4.5.2). This concludes the proof. □

References

[dF] T. de Fernex, Ample vector bundles with sections vanishing along conic fibrations over curves, Collect. Math., **49** (1998), 67–79.

[F] T. Fujita, Classification Theories of Polarized Varieties, London Math. Soc. Lecture Notes Ser., **155**, Cambridge Univ. Press, 1990.

[H] R. Hartshorne, Equivalence relations on algebraic cycles and subvarieties of small codimension, Algebraic Geometry, Arcata 1974, Proc. Sympos. Pure Math., **29**, Amer. Math. Soc., 1975, 129–164.

[KMM] Y Kawamata, K. Matsuda and K. Matsuki, Introduction to the minimal model problem, Adv. Stud. Pure Math., **10** (1987), 283–360.

[K] J. Kollár, Rational Curves on Algebraic Varieties, Springer Verlag, Berlin-Heidelberg-New York, 1996.

[LM1] A. Lanteri and H. Maeda, Ample vector bundles with sections vanishing on projective spaces or quadrics, Internat. J. Math., **6** (1995), 587–600.

[LM2] A. Lanteri and H. Maeda, Geometrically ruled surfaces as zero loci of ample vector bundles, Forum Math., **9** (1997), 1–15.

[LM3] A. Lanteri and H. Maeda, Ample vector bundle characterizations of projective bundles and quadric fibrations over curves, Higher Dimensional Complex Varieties, Proc. Internat. Conf. Trento, June 1994 (M. Andreatta, Th. Peternell, eds.), de Gruyter, 1996, 247–259.

[LM4] A. Lanteri and H. Maeda, Special varieties in adjunction theory and ample vector bundles, preprint.

[LPS] A. Lanteri, M. Palleschi and A. J. Sommese, Del Pezzo surfaces as hyperplane sections, J. Math. Soc. Japan, **49** (1997), 501–529.

[M] H. Maeda, Nefness of adjoint bundles for ample vector bundles, Matematiche (Catania), **50** (1995), 73–82.

[Mo] S. Mori, Threefolds whose canonical bundles are not numerically effective, Ann. of Math., **116** (1982), 133–176.

[OSS] Ch. Okonek, M. Schneider and H. Spindler, Vector Bundles on Complex Projective Spaces, Birkhäuser, Boston, 1980.

[S] A. J. Sommese, On manifolds that cannot be ample divisors, Math. Ann., **221** (1976), 55–72.

[SzW] M. Szurek and J. A. Wiśniewski, On Fano manifolds, which are $\boldsymbol{P}^k$-bundles over $\boldsymbol{P}^2$, Nagoya Math. J., **120** (1990), 89–101.

[W1] J. A. Wiśniewski, Length of extremal rays and generalized adjunction, Math. Z., **200** (1989), 409–427

[W2] J. A. Wiśniewski, On contractions of extremal rays of Fano manifolds, J. Reine Angew. Math., **417** (1991), 141–157.

[Z] Q. Zhang, A theorem on the adjoint system for vector bundles, Manuscripta Math., **70** (1991), 189–201.

Dipartimento di Matematica
Università
Via Dodecaneso 35
I-16146 Genova
Italy
E-mail: defernex@dima.unige.it

Dipartimento di Matematica "F. Enriques"
Università
Via C. Saldini 50
I–20133 Milano
Italy
E-mail: lanteri@vmimat.mat.unimi.it