

AMPLE VECTOR BUNDLES ON A RATIONAL SURFACE (HIGHER RANK)

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Introduction

In the previous paper [1], we showed that the set of simple vector bundles of rank 2 on a rational surface with fixed Chern classes is bounded and we gave a sufficient condition for an H -stable vector bundle of rank 2 on a rational surface to be ample. In this paper, we shall extend the results of [1] to the case of higher rank.

Let k be an algebraically closed field of arbitrary characteristic. Throughout this paper, the ground field k will be fixed.

In §1, we shall prove the following;

THEOREM 1. *Let X be the projective plane \mathbf{P}^2 or the rational ruled surface Σ_n . For a divisor C_1 on X and integers $C_2, r (\geq 2)$, put $\mathcal{F} = \{E; \text{ simple vector bundle of rank } r \text{ on } X \text{ with } C_i(E) = C_i \text{ for } i = 1, 2\}$, then \mathcal{F} is bounded.*

For a vector bundle E of rank r on a non-singular projective surface, define an integer $\Delta(E)$ to be $(r-1)C_1(E)^2 - 2rC_2(E)$. It is easy to see that $-\Delta(E)$ is the second Chern class of $\text{End}(E)$. Hence if L is a line bundle, then $\Delta(E \otimes L) = \Delta(E)$. Let H be a hyperplane of \mathbf{P}^2 . For a vector bundle E of rank r on \mathbf{P}^2 , there exists uniquely a line bundle L on \mathbf{P}^2 such that $C_1(E \otimes L) = aH$ with $-r+1 \leq a \leq 0$. Put $a(E) = a$. In §2, we shall prove the following;

THEOREM 2. *Let E be an H -stable vector bundle of rank r on \mathbf{P}^2 . If $(C_1(E), H) \geq -\frac{1}{2}\Delta(E) + (a+2r)(2-a-r)/2$ then E is ample where $a = a(E)$.*

Let $\Sigma_n = \mathbf{P}(O_{\mathbf{P}^1}(-n) \oplus O_{\mathbf{P}^1})$ be a rational ruled surface and let M be a minimal section of Σ_n and N be a fibre of Σ_n . The divisor class group

of Σ_n is generated by the classes of M and N . For a couple of integers (α, β) , we denote $\alpha(M + nN) + \beta N$ by $H_{\alpha, \beta}$. $H_{\alpha, \beta}$ is ample if and only if $\alpha > 0, \beta > 0$. For a vector bundle E of rank r on Σ_n , there exists uniquely a line bundle L on Σ_n such that $C_1(E \otimes L) = aM + bN$ with $-r + 1 \leq a, b \leq 0$. Put $a(E) = a$ and $b(E) = b$. In §3, we shall prove the following;

THEOREM 3. *Let E be an $H_{\alpha, \beta}$ -stable vector bundle of rank r on Σ_n ($\alpha > 0, \beta > 0$). If $(C_1(E), N) \geq -\frac{1}{2}\Delta(E) + c(a, b, r, n) + a$ and $(C_1(E), M) \geq -\frac{1}{2}\Delta(E) + c(a, b, r, n) - an + b$ then E is ample where $a = a(E), b = b(E)$ and $c(a, b, r, n) = \frac{1}{2}an(a + r) - r(a + b + ab + r - 2)$.*

In §4, we shall show that Theorem 2 is best possible in some cases. If E is an H -stable vector bundle of rank r on P^2 with $C_1(E) = \pm H$, then $C_2(E) \geq r - 1$ (Lemma 4.1). Conversely for any couple of integers (r, n) such that $n \geq r - 1 \geq 1$, there is an H -stable vector bundle E of rank r on P^2 with $C_1(E) = H$ and $C_2(E) = n$ such that $E(t)$ is ample if and only if $E(t)$ satisfies the condition of Theorem 2 and $E^*(t)$ is ample if and only if $E^*(t)$ satisfies the condition of Theorem 2 (Theorem 4).

§1. Simple vector bundles

Let S be a non-singular projective variety defined over k and E be a vector bundle (i.e. a locally free sheaf of finite rank) on S .

DEFINITION. E is called simple if any global endomorphism of E is constant i.e. $H^0(S, \text{End}(E)) = k$.

DEFINITION. A set \mathcal{F} of vector bundles on S is bounded if there are an algebraic k -scheme T and a vector bundle V on $T \times S$ such that each E in \mathcal{F} is isomorphic to $V_t = V|_{t \times S}$ for some closed point t in T .

Let X be the projective plane P^2 or a rational ruled surface $\Sigma_n = P(O_{P^1}(-n) \oplus O_{P^1})$ ($n \geq 0$). Let M be a minimal section of Σ_n and N be a fibre of Σ_n . By the same symbol H , we denote a hyperplane of P^2 when $X = P^2$, $H_{1,1} = (M + nN) + N$ when $X = \Sigma_n$. H is a very ample divisor on X and a general member of the complete linear system $|H|$ is isomorphic to the projective line P^1 . If K_X is the canonical divisor on X , then $K_X \sim -3H$ when $X = P^2$, $K_X \sim -2M - (n + 2)N$ when $X = \Sigma_n$. For a divisor D on X and a coherent sheaf E on X , we denote $E \otimes O_X(D)$ by $E(D)$, $E \otimes O_X(mH)$ by $E(m)$ and the dual sheaf $\text{Hom}_{O_X}(E, O_X)$ of E

by E^* . The aim of this section is;

THEOREM 1. *Let X be P^2 or Σ_n . For a divisor C_1 on X and integers $C_2, r (\geq 2)$, put $\mathcal{F} = \{E; \text{ simple vector bundle of rank } r \text{ on } X \text{ with } C_i(E) = C_i \text{ for } i = 1, 2\}$ then \mathcal{F} is bounded.*

Proof. For an integer d , let \mathcal{F}_d be the subset of \mathcal{F} which consists of E in \mathcal{F} such that $H^0(X, E(d)) = (0)$ and $H^0(X, E(d+1)) \neq (0)$, then $\mathcal{F} = \cup \mathcal{F}_d$. We separate the proof into two steps;

- (a) For almost all d , \mathcal{F}_d is empty,
- (b) \mathcal{F}_d is bounded for all d .

If (a) and (b) are proved then \mathcal{F} is considered as a finite union of bounded families and so \mathcal{F} is bounded. Before proving (a) and (b), we introduce one more notation. For E in \mathcal{F} , let P be the numerical polynomial defined by $P(m) = \chi(X, E(m)) = \sum (-1)^i h^i(X, E(m))$ where $h^i(X, E(m)) = \dim_k H^i(X, E(m))$. Since H is ample and X is a surface, P is of degree two and $P(m) \rightarrow \infty$ if $m \rightarrow \pm\infty$. P is independent from a choice of E in \mathcal{F} .

(a) We shall prove that if \mathcal{F}_d is not empty then $P(d) \leq 0$. Hence such d 's are finite. Assume that \mathcal{F}_d is not empty. Let E be an element of \mathcal{F}_d , then $H^0(X, E(d)) = (0)$ and $H^0(X, E(d+1)) \neq (0)$. We want to prove that $H^2(X, E(d)) = (0)$. If this is proved, then $P(d) = -h^1(X, E(d)) \leq 0$. The dual of $H^2(X, E(d))$ is isomorphic to $H^0(X, E(d)^* \otimes O_X(K_X))$ (Serre duality) and $E(d)^* \otimes O_X(K_X) \cong E(d+1)^* \otimes O_X(K_X + H)$. Since $-K_X - H$ is linearly equivalent to an effective divisor, $H^0(X, E(d)^* \otimes O_X(K_X)) \subset H^0(X, E(d+1)^*)$. Therefore it suffices to prove that $H^0(X, E(d+1)^*) = (0)$. This follows from;

LEMMA 1.1. ((4) Proposition 1.) *Let E' be a vector bundle on a non-singular variety S defined over k . If $H^0(S, E') \neq (0)$, $H^0(S, E'^*) \neq (0)$ and E' is not a line bundle then E' is not simple.*

Since E is simple and $H^0(X, E(d+1)) \neq (0)$, $H^0(X, E(d+1)^*) = (0)$ by Lemma 1.1.

(b) By a theorem of Kleiman ((2) Theorem 1.13), it is sufficient to show that there are integers m_1, m_2 such that for any E in $\mathcal{F}_d(d)$, i) $h^0(X, E) \leq m_1$ ii) $h^0(\ell, E|_\ell) \leq m_2$ for a general member ℓ in $|H|$ where $\mathcal{F}_d(d) = \{E(d); E \text{ in } \mathcal{F}_d\}$. By the definition of \mathcal{F}_d , $m_1 = 0$ satisfies i). We now show ii). For a general member ℓ in $|H|$ and E in $\mathcal{F}_d(d)$,

there is a long exact sequence of cohomologies;

$$\dots \rightarrow H^0(X, E) \rightarrow H^0(\ell, E|_\ell) \rightarrow H^1(X, E(-1)) \rightarrow \dots .$$

Since $H^0(X, E) = (0)$,

$$h^0(\ell, E|_\ell) \leq h^1(X, E(-1)) . \quad (1)$$

If $X = \mathbf{P}^2$ then $h^2(\mathbf{P}^2, E(-1)) = h^0(\mathbf{P}^2, E(-1)^* \otimes O_{\mathbf{P}^2}(K_{\mathbf{P}^2})) = h^0(\mathbf{P}^2, E(1)^* \otimes O_{\mathbf{P}^2}(-1))$. Since $h^0(\mathbf{P}^2, E(1)) \neq 0$ and E is simple, $h^0(\mathbf{P}^2, E(1)^*) = 0$ by Lemma 1.1. Hence $h_2(\mathbf{P}^2, E(-1)) = 0$ and also $h^0(\mathbf{P}^2, E(-1)) = 0$. Therefore $h^1(\mathbf{P}^2, E(-1)) = -P(d-1)$. This and (1) show that $m_2 = -P(d-1)$ satisfies ii) when $X = \mathbf{P}^2$. Now assume $X = \Sigma_n$. Put $F = E(1)^*$ and consider the following long exact sequence of cohomologies;

$$\dots \rightarrow H^0(\Sigma_n, F) \rightarrow H^0(N, F|_N) \rightarrow H^1(\Sigma_n, F(-N)) \rightarrow \dots .$$

Since $H^0(\Sigma_n, F) = (0)$, we have $h^0(N, F|_N) \leq h^1(\Sigma_n, F(-N))$. On the other hand, $h^2(\Sigma_n, F(-N)) = h^0(\Sigma_n, F(-N)^* \otimes O_{\Sigma_n}(K_{\Sigma_n})) = h^0(\Sigma_n, E \otimes O_{\Sigma_n}(-M)) = 0$ and $h^0(\Sigma_n, F(-N)) = 0$, therefore $h^0(N, F|_N) \leq -\chi(\Sigma_n, F(-N))$. Note that $\chi(\Sigma_n, F(-N))$ is dependent only on \mathcal{F} and d . Since N is a fibre of Σ_n , $F(mN)|_N \cong F|_N$ for any integer m . Now consider the following long exact sequences of cohomologies;

$$0 \rightarrow H^0(\Sigma_n, F((m-1)N)) \rightarrow H^0(\Sigma_n, F(mN)) \rightarrow H^0(N, F(mN)|_N) \rightarrow \dots$$

for $m = 0, \dots, n$, then we have;

$$\begin{aligned} h^0(\Sigma_n, F(nN)) &\leq h^0(\Sigma_n, F((n-1)N)) + h^0(N, F(nN)|_N) \\ &= h^0(\Sigma_n, F((n-1)N)) + h^0(N, F|_N) \\ &\dots \\ &\leq nh^0(N, F|_N) \leq -n\chi(\Sigma_n, F(-N)) . \end{aligned} \quad (2)$$

Since $h^0(\Sigma_n, E(-1)) = 0$ and $h^2(\Sigma_n, E(-1)) = h^0(\Sigma_n, E(-1)^* \otimes O_{\Sigma_n}(K_{\Sigma_n})) = h^0(\Sigma_n, E(1)^* \otimes O_{\Sigma_n}(nN)) = h^0(\Sigma_n, F(nN))$, $h^1(\Sigma_n, E(-1)) = -P(d-1) + h^0(\Sigma_n, F(nN))$. Therefore, (1) and (2) show that $m_2 = -P(d-1) - n\chi(\Sigma_n, F(-N))$ satisfies ii) when $X = \Sigma_n$.

§2. H -stable vector bundles on \mathbf{P}^2

Let E be a vector bundle on a non-singular projective surface S defined over k and H be an ample divisor on S .

DEFINITION. E is H -stable if for every non-zero coherent subsheaf

F of E of rank $< r(E)$, $(C_1(F), H)/r(F) < (C_1(E), H)/r(E)$ where $r(F)$ is the rank of F .

We refer to [5] for basic properties of H -stable vector bundles. For a vector bundle E on S , put $\Delta(E) = (r-1)C_1(E)^2 - 2rC_2(E)$. This integer is equal to $-C_2(\text{End } E)$. If E is a vector bundle of rank r on \mathbf{P}^2 then there exists uniquely a line bundle L on \mathbf{P}^2 such that $C_1(E \otimes L) = aH$ with $-r+1 \leq a \leq 0$, where H is a hyperplane of \mathbf{P}^2 . Put $a(E) = a$. The aim of this section is;

THEOREM 2. *Let E be an H -stable vector bundle of rank r on \mathbf{P}^2 . If $(C_1(E), H) \geq -\frac{1}{2}\Delta(E) + (a+2r)(2-a-r)/2$ then E is ample where $a = a(E)$.*

In order to prove Theorem 2, we need the following lemma.

LEMMA 2.1. *Let E be an H -stable vector bundle of rank r on \mathbf{P}^2 such that $C_1(E) = aH$ with $a = a(E)$ then;*

- (1) $h^0(\mathbf{P}^2, E) = 0$,
- (2) $h^2(\mathbf{P}^2, E(m)) = 0$ for any $m \geq 0$,
- (3) $h^1(\mathbf{P}^2, E(m)) \leq h^1(\mathbf{P}^2, E(m-1))$ for any $m \geq 1$,
- (4) If $h^1(\mathbf{P}^2, E(m)) = h^1(\mathbf{P}^2, E(m-1))$ for some $m \geq 1$,

then $E(m)$ is generated by its global sections.

Proof. (1) If $h^0(\mathbf{P}^2, E) \neq 0$ then E contains $O_{\mathbf{P}^2}$ as a subsheaf but $(C_1(E), H) = a \leq 0$. Since E is H -stable, this cannot occur. (2) Since E^* is also H -stable and $(C_1(E(m)^* \otimes O_{\mathbf{P}^2}(-3)), H) = -a - r(m+3) \leq 0$ for any $m \geq 0$, $h^2(\mathbf{P}^2, E(m)) = 0$ for any $m \geq 0$ by the Serre duality. (3) Let F_m be the smallest subsheaf of $E(m)$ such that $H^0(\mathbf{P}^2, F_m) = H^0(\mathbf{P}^2, E(m))$ and $E(m)/F_m$ is torsion free. Note that $H^0(\mathbf{P}^2, F_m(-1)) = H^0(\mathbf{P}^2, E(m-1))$. Let ℓ be a general member of $|H|$ such that $F_m|_\ell$ is locally free on ℓ and $0 \rightarrow F_m(-1) \rightarrow F_m \rightarrow F_m|_\ell \rightarrow 0$ is exact. Since F_m is generically generated by its global sections and $\ell \cong \mathbf{P}^1$, $F_m|_\ell$ is generated by its global sections and $h^1(\ell, F_m|_\ell) = 0$ for a suitable choice of ℓ . Considering the following long exact sequence of cohomologies;

$$\begin{aligned} \dots \rightarrow H^1(\mathbf{P}^2, F_m(-1)) &\rightarrow H^1(\mathbf{P}^2, F_m) \rightarrow H^1(\ell, F_m|_\ell) \\ &\rightarrow H^2(\mathbf{P}^2, F_m(-1)) \rightarrow H^2(\mathbf{P}^2, F_m) \rightarrow 0 \end{aligned}$$

we have $h^1(\mathbf{P}^2, F_m) \leq h^1(\mathbf{P}^2, F_m(-1))$ and $h^2(\mathbf{P}^2, F_m) = h^2(\mathbf{P}^2, F_m(-1))$. Hence we have;

$$\begin{aligned}
& h^1(\mathbf{P}^2, E(m)) - h^1(\mathbf{P}^2, E(m-1)) \\
&= h^0(\mathbf{P}^2, E(m)) - h^0(\mathbf{P}^2, E(m-1)) - (\chi(\mathbf{P}^2, E(m)) - \chi(\mathbf{P}^2, E(m-1))) \\
&= h^0(\mathbf{P}^2, F_m) - h^0(\mathbf{P}^2, F_m(-1)) - (r + (C_1(E(m)), H)) \\
&= h^1(\mathbf{P}^2, F_m) - h^1(\mathbf{P}^2, F_m(-1)) + (\chi(\mathbf{P}^2, F_m) \\
&\quad - \chi(\mathbf{P}^2, F_m(-1))) - (r + (C_1(E(m)), H)) \\
&\leq (r' + (C_1(F_m), H)) - (r + (C_1(E(m)), H))
\end{aligned}$$

where $r' = \text{rank of } F_m$. Since E is H -stable and $(C_1(E(m)), H) = a + rm > 0$, $(C_1(F_m), H) \leq (C_1(E(m)), H)$ therefore $h^1(\mathbf{P}^2, E(m)) \leq h^1(\mathbf{P}^2, E(m-1))$.

(4) If $h^1(\mathbf{P}^2, E(m)) = h^1(\mathbf{P}^2, E(m-1))$ then $F_m = E(m)$ by the above inequality. Hence for a general member ℓ in $|H|$, $E(m)|_\ell$ is generated by its global sections and $h^1(\ell, E(m)|_\ell) = 0$. Consider the following long exact sequence of cohomologies;

$$\begin{aligned}
\cdots \rightarrow H^0(\mathbf{P}^2, E(m)) \rightarrow H^0(\ell, E(m)|_\ell) \\
\rightarrow H^1(\mathbf{P}^2, E(m-1)) \rightarrow H^1(\mathbf{P}^2, E(m)) \rightarrow H^1(\ell, E(m)|_\ell) \rightarrow \cdots
\end{aligned}$$

Since $h^1(\ell, E(m)|_\ell) = 0$ and $h^1(\mathbf{P}^2, E(m)) = h^1(\mathbf{P}^2, E(m-1))$, $H^0(\mathbf{P}^2, E(m)) \rightarrow H^0(\ell, E(m)|_\ell)$ is surjective. Hence for any closed point x in ℓ , $H^0(\mathbf{P}^2, E(m)) \rightarrow E(m) \otimes k(x)$ is surjective. On the other hand for any closed point y in $X - \ell$, take a member ℓ' in $|H|$ such that ℓ' contains y and take x in $\ell \cap \ell'$. Now consider the following commutative diagram;

$$\begin{array}{ccc}
H^0(\mathbf{P}^2, E(m)) & \longrightarrow & E(m) \otimes k(x) \\
& \searrow & \nearrow \\
& & H^0(\ell', E(m)|_{\ell'})
\end{array}$$

Since $H^0(\mathbf{P}^2, E(m)) \rightarrow E(m) \otimes k(x)$ is surjective, $H^0(\ell', E(m)|_{\ell'}) \rightarrow E(m) \otimes k(x)$ is surjective therefore $E(m)|_{\ell'}$ is generated by its global sections and $h^1(\ell', E(m)|_{\ell'}) = 0$. As the above argument for $E(m)|_\ell$, we have that $H^0(\mathbf{P}^2, E(m)) \rightarrow E(m) \otimes k(y)$ is surjective. Hence $E(m)$ is generated by its global sections by Nakayama's lemma.

COROLLARY 2.2. *Let E be as in Lemma 2.1 then $E(-\chi(\mathbf{P}^2, E) + 2)$ is ample.*

Proof. $h^1(\mathbf{P}^2, E) = -\chi(\mathbf{P}^2, E)$ by Lemma 2.1 (1) and (2). Put $c = -\chi(\mathbf{P}^2, E)$, then by Lemma 2.1 (3) we have;

$$c = h^1(\mathbf{P}^2, E) \geq h^1(\mathbf{P}^2, E(1)) \geq \cdots \geq h^1(\mathbf{P}^2, E(c)) \geq h^1(\mathbf{P}^2, E(c+1)) \geq 0.$$

Hence there must be an integer m ($1 \leq m \leq c+1$) such that $h^1(\mathbf{P}^2, E(m)) = h^1(\mathbf{P}^2, E(m-1))$. Hence $E(m)$ is generated by its global sections by Lemma 2.1 (4) therefore $E(-\chi(\mathbf{P}^2, E) + 2)$ is ample.

Proof of Theorem 2. Let E be as in Theorem 2, then there is a line bundle L on \mathbf{P}^2 such that for $E' = E \otimes L$, $C_1(E') = aH$. It is easily calculated that $(C_1(E'(-\chi(\mathbf{P}^2, E') + 2)), H) = -\frac{1}{2}\Delta(E) + (a+2r)(2-a-r)/2$. For $E'' = E'(-\chi(\mathbf{P}^2, E') + 2)$, there is a line bundle L' on \mathbf{P}^2 such that $E = E'' \otimes L'$. By the condition of Theorem 2, we have $(C_1(E'' \otimes L'), H) \geq (C_1(E''), H)$. Hence $(C_1(L'), H) \geq 0$. This is equivalent to that L' is generated by its global sections. Since E'' is ample by Corollary 2.2, $E = E'' \otimes L'$ is ample.

§3. $H_{\alpha, \beta}$ -stable vector bundles on Σ_n

Let $\Sigma_n = \mathbf{P}(O_{\mathbf{P}^1}(-n) \oplus O_{\mathbf{P}^1})$ ($n \geq 1$) be a rational ruled surface and let M be a minimal section of Σ_n and N be a fibre of Σ_n . The divisor class group of Σ_n is generated by the classes of M and N . For a couple of integers (α, β) , we denote $\alpha(M + nN) + \beta N$ by $H_{\alpha, \beta}$. The intersection numbers $(H_{\alpha, \beta}, N)$ and $(H_{\alpha, \beta}, M)$ are α and β respectively. $H_{\alpha, \beta}$ is ample if and only if $\alpha > 0$, $\beta > 0$ and the complete linear system $|H_{\alpha, \beta}|$ is base point free if and only if $\alpha \geq 0$, $\beta \geq 0$ ((1) Lemma (3.1)). For a vector bundle E of rank r on Σ_n , there exists uniquely a line bundle L on Σ_n such that $C_1(E \otimes L) = aM + bN$ with $-r+1 \leq a, b \leq 0$. Put $a(E) = a$ and $b(E) = b$. The aim of this section is;

THEOREM 3. *Let E be an $H_{\alpha, \beta}$ -stable vector bundle of rank r on Σ_n ($\alpha > 0, \beta > 0$). If $(C_1(E), N) \geq -\frac{1}{2}\Delta(E) + c(a, b, r, n) + a$ and $(C_1(E), M) \geq -\frac{1}{2}\Delta(E) + c(a, b, r, n) - an + b$ then E is ample where $a = a(E)$, $b = b(E)$ and $c(a, b, r, n) = \frac{1}{2}an(a+r) - r(a+b+ab+r-2)$.*

In order to prove Theorem 3, we need some lemmas.

LEMMA 3.1. *Let E be an $H_{\alpha, \beta}$ -stable vector bundle of rank r on Σ_n with $C_1(E) = aM + bN$ such that $a = a(E)$, $b = b(E)$, then;*

- (1) $h^0(\Sigma_n, E) = 0$
- (2) $h^2(\Sigma_n, E(D)) = 0$ for any effective divisor D on Σ_n .

Proof. The proof is similar to that of Lemma 2.1 (1), (2).

LEMMA 3.2. *Let E be an $H_{\alpha,\beta}$ -stable vector bundle of rank r on Σ_n with $C_1(E) = aM + bN$ such that $a \geq a(E)$, $b \geq b(E)$ and let F be the smallest subsheaf of $E(H_{1,1})$ such that $H^0(\Sigma_n, F) = H^0(\Sigma_n, E(H_{1,1}))$ and $E(H_{1,1})/F$ is torsion free, then;*

(1) *if $r' = \text{rank of } F < r$ then $h^1(\Sigma_n, E(H_{0,1})) < h^1(\Sigma_n, E)$ or $h^1(\Sigma_n, E(H_{1,0})) < h^1(\Sigma_n, E)$,*

(2) *if $r' = r$ (i.e. $E(H_{1,1})$ is generically generated by its global sections) then $h^1(\Sigma_n, E(H_{1,1})) \leq h^1(\Sigma_n, E)$ and if $h^1(\Sigma_n, E(H_{1,1})) = h^1(\Sigma_n, E)$ then $E(H_{1,1})$ is generated by its global sections.*

Proof. (1) Put $C_1(E(H_{1,1})) = uM + vN$ and $C_1(F) = u'M + v'N$, then by the stability of E we have;

$$\frac{\beta u' + \alpha v'}{r'} < \frac{\beta u + \alpha v}{r}.$$

Since $\alpha > 0$, $\beta > 0$, $u > 0$, $v > 0$ and $r' < r$, we have $u' < u$ or $v' < v$. We want to prove that (i) if $u' < u$ then $h^1(\Sigma_n, E(H_{0,1})) < h^1(\Sigma_n, E)$ (ii) if $v' < v$ then $h^1(\Sigma_n, E(H_{1,0})) < h^1(\Sigma_n, E)$.

(i) Assume $u' < u$. Let ℓ be a general member of $|H_{0,1}|$ such that $F|_\ell$ is locally free and $0 \rightarrow F(-H_{1,1}) \rightarrow F(-H_{1,0}) \rightarrow F(-H_{1,0})|_\ell \rightarrow 0$ is exact. Since ℓ is a fibre of Σ_n , ℓ is isomorphic to the projective line and since F is generically generated by its global sections, $F|_\ell$ is generated by its global sections for a suitable choice of ℓ . The intersection number $(-H_{1,0}, \ell)$ is -1 so we have $h^1(\ell, F(-H_{1,0})|_\ell) = 0$ for a suitable choice of ℓ . Considering the following long exact sequence of cohomologies;

$$\begin{aligned} \dots &\rightarrow H^1(\Sigma_n, F(-H_{1,1})) \rightarrow H^1(\Sigma_n, F(-H_{1,0})) \rightarrow H^1(\ell, F(-H_{1,0})|_\ell) \\ &\rightarrow H^2(\Sigma_n, F(-H_{1,1})) \rightarrow H^2(\Sigma_n, F(-H_{1,0})) \rightarrow 0 \end{aligned}$$

we have $h^1(\Sigma_n, F(-H_{1,0})) \leq h^1(\Sigma_n, F(-H_{1,1}))$ and $h^2(\Sigma_n, F(-H_{1,0})) = h^2(\Sigma_n, F(-H_{1,1}))$. Note that $h^0(\Sigma_n, E(H_{0,1})) = h^0(\Sigma_n, F(-H_{1,0}))$ and $h^0(\Sigma_n, E) = h^0(\Sigma_n, F(-H_{1,1}))$, hence we have;

$$\begin{aligned} &h^1(\Sigma_n, E(H_{0,1})) - h^1(\Sigma_n, E) \\ &= h^0(\Sigma_n, E(H_{0,1})) - h^0(\Sigma_n, E) - (\chi(\Sigma_n, E(H_{0,1})) - \chi(\Sigma_n, E)) \\ &= h^0(\Sigma_n, F(-H_{1,0})) - h^0(\Sigma_n, F(-H_{1,1})) - (r + (C_1(E(H_{0,1})), H_{0,1})) \\ &= h^1(\Sigma_n, F(-H_{1,0})) - h^1(\Sigma_n, F(-H_{1,1})) + (\chi(\Sigma_n, F(-H_{1,0})) \\ &\quad - \chi(\Sigma_n, F(-H_{1,1})) - u) \\ &\leq u' - u < 0. \end{aligned}$$

(ii) Assume $v' < v$. A general member ℓ of $|H_{1,0}|$ is a section of Σ_n so ℓ is isomorphic to the projective line and $(-H_{0,1}, \ell) = -1$. Hence $h^1(\Sigma_n, E(H_{1,0})) < h^1(\Sigma_n, E)$ is similarly obtained as above.

(2) The proof is similar to that of Lemma 2.1 (3), (4).

COROLLARY 3.3. *Let E be as in Lemma 3.1, then $E((-\chi(\Sigma_n, E) + 2)H_{1,1})$ is ample.*

Proof. $h^1(\Sigma_n, E) = -\chi(\Sigma_n, E)$ by Lemma 3.1. Put $c = -\chi(\Sigma_n, E)$. By Lemma 3.2 (1), there are integers $p \geq 0, q \geq 0$ such that for $E' = E(H_{p,q})$, $h^1(\Sigma_n, E') \leq c - (p + q)$ and $E'(H_{1,1})$ is generically generated by its global sections. Put $c' = h^1(\Sigma_n, E')$ then by Lemma 3.2 (2) we have;

$$\begin{aligned} c' &= h^1(\Sigma_n, E') \geq h^1(\Sigma_n, E'(H_{1,1})) \geq \dots \\ &\geq h^1(\Sigma_n, E'(c'H_{1,1})) \geq h^1(\Sigma_n, E'((c' + 1)H_{1,1})) \geq 0. \end{aligned}$$

Hence there must be an integer m ($1 \leq m \leq c' + 1$) such that $h^1(\Sigma_n, E'((m - 1)H_{1,1})) = h^1(\Sigma_n, E'(mH_{1,1}))$. Hence by Lemma 3.2 (2), $E'(mH_{1,1})$ is generated by its global sections, therefore $E'((c' + 2)H_{1,1})$ is ample. On the other hand $E((c + 2)H_{1,1}) = E'((c' + 2)H_{1,1}) \otimes O_{\Sigma_n}(H_{q,p} + (c - (p + q) - c')H_{1,1})$ and $c - (p + q) - c' \geq 0$, so $E((c + 2)H_{1,1})$ is ample.

Proof of Theorem 3. Let E be as in Theorem 3, then there is a line bundle L on Σ_n such that for $E' = E \otimes L$, $C_1(E') = aM + bN$. It is easily calculated that for $E'' = E'((-\chi(\Sigma_n, E') + 2)H_{1,1})$, $(c_1(E''), N) = -\frac{1}{2}\Delta(E) + c(a, b, r, n) + a$ and $(C_1(E''), M) = -\frac{1}{2}\Delta(E) + c(a, b, r, n) - an + b$. There are integers p, q such that $E = E''(H_{p,q})$. By the condition of Theorem 3, we have $(C_1(E''(H_{p,q})), N) \geq (C_1(E''), N)$ and $(C_1(E''(H_{p,q})), M) \geq (C_1(E''), M)$. Hence $(H_{p,q}, N) = p \geq 0$ and $(H_{p,q}, M) = q \geq 0$. This is equivalent to that $O_{\Sigma_n}(H_{p,q})$ is generated by its global sections. Since E'' is ample by Corollary 3.5, $E = E''(H_{p,q})$ is ample.

§4. Examples of H -stable vector bundles on P^2

In this section we shall show that Theorem 2 is best possible when $a = -r + 1$ or -1 . Let H be a hyperplane of P^2 . We begin with a simple lemma.

LEMMA 4.1. *Let E be an H -stable vector bundle of rank r on P^2 . If $C_1(E) = H$ or $-H$ then $C_2(E) \geq r - 1$.*

Proof. Since $C_1(E^*) = -C_1(E)$ and $C_2(E^*) = C_2(E)$, we may assume

$C_1(E) = -H$. By Lemma 2.1 (1), (2), $h^0(\mathbf{P}^2, E) = h^2(\mathbf{P}^2, E) = 0$. Hence $-h^1(\mathbf{P}^2, E) = \chi(\mathbf{P}^2, E) = r + (C_1(E), 3H)/2 + (C_1(E)^2 - 2C_2(E))/2 = r - 1 - C_2(E)$ by the Riemann-Roch theorem. Therefore $C_2(E) \geq r - 1$.

The following lemma is due to Maruyama ((3) Theorem 4.6).

LEMMA 4.2. *Let ℓ be a line on \mathbf{P}^2 and $n \geq 1$ be an integer, then there is an H -stable vector bundle of rank 2 on \mathbf{P}^2 such that $C_1(E) = H$, $C_2(E) = n$ and $E|_\ell \cong O_\ell(-n+1) \oplus O_\ell(n)$ where $O_\ell(n)$ is the line bundle on ℓ with $\deg(O_\ell(n)) = n$.*

LEMMA 4.3. *Let E be an H -stable vector bundle of rank r on \mathbf{P}^2 with $C_1(E) = H$. If there is a short exact sequence of vector bundles;*

$$0 \rightarrow O_{\mathbf{P}^2} \rightarrow E' \rightarrow E \rightarrow 0 \quad (*)$$

and this is not split then E' is H -stable.

Proof. Let F be a non-trivial subsheaf of E' such that the rank of $F < r + 1$ and E'/F is torsion free. Since $C_1(E') = H$, it is sufficient to show that $(C_1(F), H) \leq 0$. Put $L = F \cap O_{\mathbf{P}^2}$ and F' be the image of F in E , then there is a short exact sequence $0 \rightarrow L \rightarrow F \rightarrow F' \rightarrow 0$. Since $O_{\mathbf{P}^2}$ and E are H -stable, $(C_1(L), H) \leq 0$ and $(C_1(F'), H) \leq 1$ hence $(C_1(F), H) \leq 1$. Therefore it is sufficient to show that $(C_1(F), H) \neq 1$. If it were happened then $(C_1(L), H) = 0$ and $(C_1(F'), H) = 1$. This is possible if and only if $L = (0)$ and $\dim \text{supp}(E/F') \leq 0$, by the H -stability of E . Since (*) is not split, $E/F' \neq (0)$. There is a short exact sequence $0 \rightarrow O_{\mathbf{P}^2} \rightarrow E'/F' \rightarrow E/F' \rightarrow 0$. But $H^0(\mathbf{P}^2, (E'/F')(m)) \neq (0)$ and $H^1(\mathbf{P}^2, O_{\mathbf{P}^2}(m)) = (0)$ for all m and since E'/F' is torsion free, $H^0(\mathbf{P}^2, (E'/F')(m)) = (0)$ for $m \ll 0$. This is a contradiction.

The aim of this section is the following theorem which shows that the converse of Lemma 4.1 and that Theorem 2 is best possible when $a = -r + 1$ or -1 .

THEOREM 4. *Put $A = \{(r, n); n \geq r - 1 \geq 1\}$. Let ℓ be a line on \mathbf{P}^2 . Then there is a set $S = \{E_{(r, n)}\}_{(r, n) \in A}$ of vector bundles on \mathbf{P}^2 which satisfies the following conditions;*

- (1) *S consists of H -stable vector bundles,*
- (2) *the rank of $E_{(r, n)}$ is r , $C_1(E_{(r, n)}) = H$ and $C_2(E_{(r, n)}) = n$ for all $(r, n) \in A$,*
- (3) *there is a short exact sequence $0 \rightarrow O_{\mathbf{P}^2} \rightarrow E_{(r, n)} \rightarrow E_{(r-1, n)} \rightarrow 0$ and*

this is not split,

$$(4) \quad h^1(\mathbf{P}^2, E_{(r,n)}^*) = n - r + 1,$$

$$(5) \quad E_{(r,n)}|_{\ell} \cong O_{\ell}(-n + 1) \oplus O_{\ell}(n - r + 2) \oplus \sum_{i=1}^{r-2} O_{\ell}(1) \text{ where } \sum_{i=1}^{r-2} O_{\ell}(1) = O_{\ell}(1) \oplus \dots \oplus O_{\ell}(1) \text{ (} r - 2 \text{ times),}$$

$$(6) \quad H^1(\mathbf{P}^2, E_{(r,n)}^*) \cong H^1(\ell, E_{(r,n)}^*|_{\ell}) \text{ canonically,}$$

(7) $E_{(r,n)}(t)$ is ample if and only if $E_{(r,n)}(t)$ satisfies the condition of Theorem 2,

(8) $E_{(r,n)}^*(t)$ is ample if and only if $E_{(r,n)}^*(t)$ satisfies the condition of Theorem 2.

Proof. The above conditions are not independent each other. In fact;

- (i) (1), (2) and (3) for $E_{(r-1,n)} \Leftrightarrow$ (1) for $E_{(r,n)}$ by Lemma 4.3,
- (ii) (2) and (3) for $E_{(r-1,n)} \Leftrightarrow$ (2) for $E_{(r,n)}$,
- (iii) (1) and (2) \Leftrightarrow (4) by the Riemann-Roch theorem and Lemma 2.1 (1), (2),
- (iv) (1), (2), (4) and (5) \Leftrightarrow (6),
- (v) (1), (2) and (5) \Leftrightarrow (7),
- (vi) (1), (2) and (5) \Leftrightarrow (8).

(v) and (vi) are easily checked by considering $E_{(r,n)}(t)|_{\ell}$ and $E_{(r,n)}^*(t)|_{\ell}$ respectively. We now show (iv). Consider the following long exact sequence of cohomologies;

$$\dots \rightarrow H^1(\mathbf{P}^2, E_{(r,n)}^*) \rightarrow H^1(\ell, E_{(r,n)}^*|_{\ell}) \rightarrow H^2(\mathbf{P}^2, E_{(r,n)}^*(-1)) \rightarrow \dots$$

Since $(C_1(E_{(r,n)}(-2)), H) < 0$ by (2), $H^2(\mathbf{P}^2, E_{(r,n)}^*(-1)) = (0)$ by (1). Moreover $h^1(\mathbf{P}^2, E_{(r,n)}^*) = n - r + 1$ by (4) and $h^1(\ell, E_{(r,n)}^*|_{\ell}) = n - r + 1$ by (5) hence we have $H^1(\mathbf{P}^2, E_{(r,n)}^*) \cong H^1(\ell, E_{(r,n)}^*|_{\ell})$ canonically.

By Lemma 4.2, for any $n \geq 1$, there is a vector bundle $E_{(2,n)}$ such that $E_{(2,n)}$ satisfies (1), (2) and (5). Lastly we constant $E_{(r,n)}$ which satisfies (3) and (5) by (5) and (6) for $E_{(r-1,n)}$. There is a short exact sequence;

$$0 \rightarrow O_{\ell} \rightarrow O_{\ell}(-n + 1) \oplus O_{\ell}(n - r + 2) \oplus \sum_{i=1}^{r-1} O_{\ell}(1) \rightarrow E_{(r-1,n)}|_{\ell} \rightarrow 0 \quad (\ast)$$

of vector bundles on ℓ by (5) for $E_{(r-1,n)}$. (\ast) has an obstruction in $H^1(\ell, E_{(r-1,n)}^*|_{\ell})$ hence there is a short exact sequence $0 \rightarrow O_{\mathbf{P}^2} \rightarrow E_{(r,n)} \rightarrow E_{(r-1,n)} \rightarrow 0$ such that its restriction to ℓ is isomorphic to (\ast) by (6) for $E_{(r-1,n)}$. This short exact sequence is not split and $E_{(r,n)}$ satisfies (5) by

(*) . All these together we have constructed $S = \{E_{(r,n)}\}_{(r,n) \in A}$ which satisfies (1)–(8).

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