AN 8-PERIODIC EXACT SEQUENCE OF WITT GROUPS OF AZUMAYA ALGEBRAS WITH INVOLUTION

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Abstract. Given an Azumaya algebra with involution (A,σ) over a commutative ring R and some auxiliary data, we construct an 8-periodic chain complex involving the Witt groups of (A,σ) and other algebras with involution, and prove it is exact when R is semilocal. When R is a field, this recovers an 8-periodic exact sequence of Witt groups of Grenier-Boley and Mahmoudi, which in turn generalizes exact sequences of Parimala–Sridharan–Suresh and Lewis. We apply this result in several ways: We establish the Grothendieck–Serre conjecture on principal homogeneous bundles and the local purity conjecture for certain outer forms of $\mathbf{GL}_n$ and $\mathbf{Sp}_{2n}$, provided some assumptions on R. We show that a 1-hermitian form over a quadratic étale or quaternion Azumaya algebra over a semilocal ring R is isotropic if and only if its trace (a quadratic form over R) is isotropic, generalizing a result of Jacobson. We also apply it to characterize the kernel of the restriction map $W(R)\to W(S)$ when R is a (non-semilocal) 2-dimensional regular domain and S is a quadratic étale R-algebra, generalizing a theorem of Pfister. In the process, we establish many fundamental results concerning Azumaya algebras with involution and hermitian forms over them.

Introduction

Central simple algebras with involution over fields, in the sense of [40, §2], play a major role in the study of classical algebraic groups. Indeed, all forms of $\mathbf{GL}_n$, $\mathbf{O}_n$ and $\mathbf{Sp}_{2n}$ arise as the algebraic groups of unitary elements in a central simple algebra with involution.

When the base field is replaced with a (commutative) ring R (always with $2\in R^\times$), the role of central simple algebras with involution is played by Azumaya algebras with involution. These are the locally free R-algebras with R-involution (A,σ) which specialize to a central simple algebra with involution at the residue field of every prime $\mathfrak{p}\in\operatorname{Spec}R$.

The Witt group of ε-hermitian forms over (A,σ), denoted $W_\varepsilon(A,\sigma)$, is an important invariant of (A,σ), capturing fine arithmetic properties. For example, when A is a field F and $\sigma=\mathrm{id}_F$, the affirmation of the quadratic form version of Milnor's conjecture by Orlov, Vishik and Voevodsky [50] shows that the cohomology groups $\mathrm{H}^i_{\text{ét}}(F,\boldsymbol{\mu}_{2,F})$ can be recovered from $W(F):=W_1(F,\mathrm{id}_F)$; this was recently generalized to the case where A is a semilocal commutative ring by Jacobson [35].

In this paper, we introduce an 8-periodic chain complex involving the Witt group of (A,σ) — an Azumaya algebra with involution over R — and prove it is exact when R is semilocal. Several applications of the exactness are then presented.

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The Main Result. Let R be a commutative ring with $2 \in R^\times$, let (A, σ) be an Azumaya R-algebra with involution (σ is applied exponentially) and let $\varepsilon \in A$ be a central element such that $\varepsilon^\sigma \varepsilon = 1$. Let $\lambda, \mu \in A^\times$ be elements satisfying $\lambda^\sigma = -\lambda$, $\mu^\sigma = -\mu$, $\lambda\mu = -\mu\lambda$ and $\lambda^2 \in R$. Then the centralizer of λ in A, denoted B, is Azumaya over its center and $\tau_1 := \sigma|_B$ and $\tau_2 := \mathrm{Int}(\mu^{-1}) \circ \tau_1$ are involutions of B. We construct an octagon, i.e. an 8-periodic chain complex, of Witt groups:

$$
\begin{array}{ccccccc}
W_\varepsilon(A,\sigma) & \xrightarrow{\pi_1^{(\varepsilon)}} & W_\varepsilon(B,\tau_1) & \xrightarrow{\rho_1^{(\varepsilon)}} & W_{-\varepsilon}(A,\sigma) & \xrightarrow{\pi_2^{(-\varepsilon)}} & W_\varepsilon(B,\tau_2) \\
\uparrow{\scriptstyle \rho_2^{(-\varepsilon)}} & & & & & & \downarrow{\scriptstyle \rho_2^{(\varepsilon)}} \\
W_{-\varepsilon}(B,\tau_2) & \xleftarrow[\pi_2^{(\varepsilon)}]{} & W_\varepsilon(A,\sigma) & \xleftarrow[\rho_1^{(-\varepsilon)}]{} & W_{-\varepsilon}(B,\tau_1) & \xleftarrow[\pi_1^{(-\varepsilon)}]{} & W_{-\varepsilon}(A,\sigma)
\end{array}
\tag{0.1}
$$

Its maps are induced by functors between the relevant categories of hermitian forms; their definition, which depends on λ and μ, is given in 3A below.

The main result of this paper (Theorem 3.4) asserts that the octagon (0.1) is exact when R is semilocal.

When R is a field, (0.1) is isomorphic to an octagon of Witt groups introduced by Grenier-Boley and Mahmoudi [30, §6], who also proved it is exact.[1] Many special cases of the latter result were known previously. For example, Parimala, Sridharan and Suresh [10, Appendix] established the exactness of the top row of (0.1) when R is a field. Furthermore, the 5-term and 7-term exact sequences of Witt groups that Lewis [43] associates to quadratic field extensions and quaternion division algebras, respectively, can be recovered from (0.1) when R is a field. Predating Lewis, Baeza [5, Korollar 2.9] and Mandelberg [46, Proposition 2.1] established the exactness of Lewis' 5-term sequence at two places when R is semilocal; we extend these works in 8A, showing that both of Lewis' sequences remain exact when base ring is semilocal.

When R is a general, the octagon (0.1) seem related to the octagon of L-groups considered by Ranicki in [57, Remark 22.22]. Other octagons involving Witt groups of central simple algebras with involution appear in [44] and [45].

In the process of proving the exactness of (0.1) when R is semilocal, we give necessary and sufficient conditions for a hermitian space to be in the image of the functors $\pi_1^{(\pm\varepsilon)}, \pi_2^{(\pm\varepsilon)}, \rho_1^{(\pm\varepsilon)}, \rho_2^{(\pm\varepsilon)}$ (the exactness of the octagon answers this only up to Witt equivalence), see Theorem 7.1. These conditions, which seem novel even when R is a field, involve the Brauer classes of A and B and the discriminant of the hermitian space at hand; they are needed for some of the applications. For example, given a unimodular $(-\varepsilon)$-hermitian space (P, f) over (A, σ), we show that there exists a unimodular ε-hermitian space (Q, g) over (B, τ_1) such that $\rho_1^{(\varepsilon)}(Q, g) \cong (P, f)$ if and only if $\pi_2(P, f)$ is hyperbolic and at least one of the following hold: (1) (τ_2, ε) is not orthogonal (see 1D), (2) the Brauer class of B is nontrivial, (3) (τ_2, ε) is orthogonal, $n := \frac{\mathrm{rk}_R P}{\deg A}$ is even and the discriminant of f (see 2H) equals $\lambda^n (R^\times)^2$. We further we show that any *anistropic* hermitian space whose Witt class lives in the kernel of some map in (0.1) is the image of a hermitian space under the functor corresponding to the preceding map in (0.1), see Corollary 7.2.

While proving that the octagon (0.1) is exact when R is a field takes only several pages, showing the exactness when R is semilocal is significantly more involved; the proof occupies most of this paper, and is surveyed in 3C. One reason why the

[1] The term $W_{-\varepsilon}(B, \tau_1)$ on the bottom row of (0.1) and the maps adjacent to it differ from their counterparts in [30, p. 980]. However, the octagons become the same once identifying the term $W_{-\varepsilon}(B, \tau_1)$ on the bottom row of (0.1) with the corresponding term $W_\varepsilon(B, \tau_1)$ in *op. cit.* via λ-*conjugation* ("scaling by λ") in the sense of 2G below.

field case is simpler is the fact that when R is a field, every Witt class contains a representative with no isotropic vectors, which allows for a short clean proof; see Remark 3.7. In contrast, the proof when R is semilocal relies on two ingredients: careful analysis of the image of the functors $\pi_*^{(\pm\varepsilon)}$ and $\rho_*^{(\pm\varepsilon)}$ when R is a field, and lifting of information from the residue fields of R to R itself, usually using results from [25]. The complexity of the former ingredient manifests in the length of Theorem 7.1, which describes the images of $\pi_*^{(\pm\varepsilon)}$ and $\rho_*^{(\pm\varepsilon)}$ when R is semilocal.

We do not know if (3.1) remains exact when R is not assumed to be semilocal. However, Theorem 8.13 below (see also the proof of Corollary 8.3) can be regarded as a positive partial result when R is a regular 2-dimensional domain. We further note that if the Witt group $W_\varepsilon(A,\sigma)$ is replaced by the Zariski sheaf associated to the presheaf $U \mapsto W_\varepsilon(A_U,\sigma_U)$ on $\operatorname{Spec} R$, then (3.1) becomes an exact sequence of sheaves. Indeed, it is exact at the stalks.

The octagon (0.1) also seems to be related with Bott periodicity. Clarifying this connection seems an interesting problem, which may lead to new insights.

Applications. A celebrated application of the well-known exactness of (0.1) when R is a field is Bayer-Fluckiger and Parimala's proof of Serre's Conjecture II for classical groups [10].

Knowing that (0.1) is exact when R is semilocal allows for a new set of applications. Here, we use it to establish some open cases of the Grothendieck–Serre conjecture and the local purity conjecture, show that the trace of a 1-hermitian form over a quadratic étale or quaternion Azumaya algebra is isotropic if and only if the original form is isotropic, and characterize the kernel of the restriction map $W(R) \to W(S)$ when R is an arbitrary 2-dimensional regular domain and S is a quadratic étale R-algebra. Further applications appear in [13].

In more detail, given a regular local ring R with fraction field K, Grothendieck [31, Remark 3, pp. 26-27], [33, Remark 1.11.a] and Serre [67, p. 31] conjectured that for every reductive (connected) group R-scheme $\mathbf{G}$, the kernel of the restriction map

$$\mathrm{H}^1_{\text{ét}}(R,\mathbf{G}) \to \mathrm{H}^1_{\text{ét}}(K,\mathbf{G})$$

is trivial. Under the same assumptions, the local purity conjecture predicts that

$$\operatorname{im}\left(\mathrm{H}^1_{\text{ét}}(R,\mathbf{G}) \to \mathrm{H}^1_{\text{ét}}(K,\mathbf{G})\right) = \bigcap_{\mathfrak{p}\in R^{(1)}} \left(\mathrm{H}^1_{\text{ét}}(R_{\mathfrak{p}},\mathbf{G}) \to \mathrm{H}^1_{\text{ét}}(K,\mathbf{G})\right),$$

where $R^{(1)}$ is the set of height-1 primes of R; we then say that purity holds for $\mathbf{G}$. In fact, both conjectures are believed to hold under the milder assumption that R is a regular *semilocal* domain, which we assume through the following paragraphs.

The Grothendieck–Serre conjecture was addressed by numerous authors and is known to hold in many cases. Most notably, Nisnevich [48] proved the conjecture when R is a discrete valuation ring, Guo [34] established the case where R is a semilocal Dedekind domain, and Fedorov–Panin [21] and Panin [53] proved the conjecture when R contains a field. Many positive results for particular groups $\mathbf{G}$ are known as well, see [51, §5] for a survey.

The local purity conjecture is also known to hold in many cases: Colloit-Thélène and Sansuc showed that it holds for all reductive group schemes when $\dim R \le 2$, even without assuming that R is semilocal, see [16, Corollary 6.14]. When R is a regular local ring containing a field k of characteristic 0, purity was established for $\mathbf{O}_n$, $\mathbf{SO}_n$, $\mathbf{PGL}_n$, $\mathbf{SL}_1(A)$ (A is a central simple k-algebra), $\mathbf{SL}_n/\boldsymbol{\mu}_d$ ($d \mid n$) and $\mathbf{Spin}_n$ in [52], and for groups of type G_2 in [15]. In fact, for $\mathbf{O}_n$, it is enough to assume that k is any field of characteristic not 2, see Scully [66, p. 12] and also Panin–Pimenov [54, Corollary 3.1].

To relate the octagon (0.1) to the Grothendieck–Serre conjecture, let (A,σ) be a degree-n Azumaya R-algebra with involution and let $\mathbf{U}(A,\sigma)$ denote the group R-scheme of unitary elements in (A,σ). Then $\mathbf{U}(A,\sigma)$ is a form of $\mathbf{GL}_n$, $\mathbf{O}_n$ or $\mathbf{Sp}_n$, depending on whether σ is unitary, orthogonal or symplectic, respectively. We show (Proposition 8.7) that if the restriction map

$$W_1(A,\sigma)\to W_1(A_K,\sigma_K) \tag{0.2}$$

is injective, then the Grothendieck–Serre conjecture holds for the neutral component of $\mathbf{U}(A,\sigma)$ (and, more generally, for the neutral component of the isometry group scheme of any unimodular 1-hermitian form over (A,σ)); this is well-known when $A=R$ [17, Proposition 1.2].

In accordance with the Grothendieck–Serre conjecture, it is conjectured that (0.2) is injective when R is regular semilocal. Provided $2\in R^\times$, this has been established by Balmer–Walter [9, Corollary 10.4] (see also Pardon [55]) and Balmer–Preeti [8, p. 3] when $\dim R\leq 4$ and $A=R$, and when R is local and contains a field by Gille [29, Theorem 7.7]. We use the former result and the exactness of (0.1) to establish the injectivity of $W_1(A,\sigma)\to W_1(A_K,\sigma_K)$ in when $\dim R\leq 4$ and one of the following hold:

(1) σ is unitary and $\operatorname{ind} A=1$;
(2) σ is symplectic and $\operatorname{ind} A\leq 2$;

see Theorem 8.9. As a result, the Grothendieck–Serre conjectures holds for $\mathbf{U}(A,\sigma)$ if (1) or (2) holds and $\dim R\leq 4$ (Corollary 8.10).

By similar means, we use (0.1) together with results of Gille [29, Theorem 7.7] and Scully [66, Theorem 5.1] to show that purity holds for $\mathbf{U}(A,\sigma)$ in cases (1) and (2), provided that R is regular local and contains a field of characteristic not 2 (Theorem 8.12). Here, the exactness of (0.1) is not sufficient, and we have to use the finer information provided by Theorem 7.1 about the image of $\pi_*^{(\pm\varepsilon)}$, $\rho_*^{(\pm\varepsilon)}$.

Suppose next that R is any semilocal ring and let (A,σ) be a quadratic étale R-algebra with its standard involution, or a degree-2 Azumaya R-algebra with its (unique) symplectic involution. Write Tr for the trace map from A to R. If (P,f) is a unimodular 1-hermitian space over (A,σ), then $(P,\mathrm{Tr}\circ f)$ is a unimodular 1-hermitian space over (R,id_R). We show that $\mathrm{Tr}\circ f$ is isotropic if and only if f is isotropic (Theorem 8.5). When R is a field, this goes back to Jacobson [36] (see also [65, Theorems 10.1.1, 10.1.7]). The quick proof in the case R is a field does not apply over rings, see Remark 8.6, and we instead appeal to our Theorem 7.1.

Finally, assume that R is any regular domain, possibly non-semilocal, let S be a quadratic étale R-algebra and let θ be its standard involution (see 1C). When S is a field, a famous theorem of Pfister [65, Theorem I.5.2] states that the kernel of the restriction map $W(R)\to W(S)$ is generated by the diagonal quadratic form $\langle 1,-\alpha\rangle$, where $S=R[\sqrt{\alpha}]$. Using our main result and Colloit-Thélène and Sansuc's purity result [16, Corollary 6.14], we generalize Pfister's theorem to the case where $\dim R\leq 2$, showing that the sequence

$$W_1(S,\theta)\xrightarrow{[g]\mapsto[\mathrm{Tr}_{S/R}\circ g]} W(R)\xrightarrow{[f]\mapsto[f_S]} W(S)$$

is exact in the middle (Theorem 8.13). This result also applies in the generality of quadratic étale coverings of regular integral 2-dimensional schemes.

Additional Results. The first two sections of this paper are concerned with generalizing many fundamental results about central simple algebras with involution and hermitian forms over them to the context of Azumaya algebras with involution over semilocal rings. For example, letting (A,σ) denote an Azumaya algebra with involution over a semilocal ring R with $2 \in R^\times$, and letting (P, f) be a unimodular ε-hermitian space over (A, σ), it is shown that:

- A contains a *full* idempotent $e \in A$ with $\deg eAe = \operatorname{ind} A$ (Theorem 1.25).
- If σ is orthogonal or unitary, then the idempotent e can be chosen to satisfy $e^\sigma = e$ (Theorem 1.30).
- (P, f) cancels from orthogonal sums (Theorem 2.2); this is essentially due to Reiter [59] and Keller [37].
- If the Witt class of (P, f) is 0, then (P, f) is hyperbolic. If (P', f') is Witt equivalent to (P, f) and $P \cong P'$, then $(P, f) \cong (P', f')$ (Theorem 2.8).
- If (σ, ε) is orthogonal or unitary, then (P, f) is diagonalizable whenever P is a free A-module (Proposition 2.13).
- When $\mathrm{Z}(A)$ is connected, the isometry group of (P, f) acts transitively on the set of Lagrangians of (P, f), provided it is nonempty (Lemma 2.22).

We note that the first result is false when R is not semilocal, see [2]. The second result is particularly convenient when hermitian Morita theory is needed.

Outline. Sections 1 and 2 are preliminary and recall Azumaya algebras with involution and hermitian forms, respectively. In Section 3, we construct the octagon (0.1), prove it is a chain complex, and survey the proof of its exactness when R is semilocal. The proof itself is carried in Sections 4–6 and is concluded in Section 7. Finally, the applications to the Grothendieck–Serre conjecture, the local purity conjecture and the generalizations of Jacobson and Pfister's theorems are given in Section 8.

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Notation and Conventions

Throughout this paper, a ring means a commutative (unital) ring. Algebras are unital and associative, but not necessarily commutative. *We assume that* 2 *is invertible in all rings and algebras.*

Unless otherwise indicated, R denotes a ring. Unadorned tensors and Hom-sets are always taken over R. An R-ring means a commutative R-algebra. Given $\mathfrak{p} \in \operatorname{Spec} R$, we let $k(\mathfrak{p})$ denote the fraction field of $R/\mathfrak{p}$.

Let S be an R-ring. Given (right) R-modules M, N and $f \in \operatorname{Hom}(M, N)$, we write $M_S := M \otimes S$ and $f_S := f \otimes \mathrm{id}_S \in \operatorname{Hom}_S(M_S, N_S)$. When $S = k(\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Spec} R$, we write $M(\mathfrak{p}) = M_{k(\mathfrak{p})}$ and $f(\mathfrak{p}) = f_{k(\mathfrak{p})}$, and let $m(\mathfrak{p})$ denote the image of $m \in M$ in $M(\mathfrak{p})$. When $S = R_\mathfrak{p}$, we write $M_\mathfrak{p} = M_{R_\mathfrak{p}}$ and $f_\mathfrak{p} = f_{R_\mathfrak{p}}$.

Let M be a finite (i.e. finitely generated) projective R-module. The R-rank of M, denoted $\operatorname{rk}_R M$, is the function $\operatorname{Spec} R \to \mathbb{Z}_{\geq 0}$ sending $\mathfrak{p}$ to $\dim_{k(\mathfrak{p})} M(\mathfrak{p})$; it is locally constant relative to the Zariski topology [27, Theorem 2.3.5]. Thus, when R is connected, we shall freely regard $\operatorname{rk}_R M$ as an integer.

Statements and operations involving locally constant functions from $\operatorname{Spec} R$ to $\mathbb{Z}$ should be interpreted point-wise. For example, the sum of two such functions is taken point-wise, and relations such as "$<$" should be understood as holding after evaluation at every $\mathfrak{p} \in \operatorname{Spec} R$.

We will need to compare integer-valued functions defined on spectra of different rings. To that end, given a ring homomorphism $\iota : R \to S$ and $f : \operatorname{Spec} R \to \mathbb{Z}$, define $\iota f : \operatorname{Spec} S \to \mathbb{Z}$ by $(\iota f)(\mathfrak{q}) = f(\iota^{-1}(\mathfrak{q}))$. For example, $\iota \operatorname{rk}_R M = \operatorname{rk}_S M_S$. In addition, if S is finite projective over R and N is a finite projective S-module of rank that is constant along the fibers of $\operatorname{Spec} S \to \operatorname{Spec} R$, then

$$\operatorname{rk}_S N \cdot \iota \operatorname{rk}_R S = \iota \operatorname{rk}_R N. \tag{0.3}$$

Given an R-algebra A, the units, the center, the Jacobson radical and the opposite algebra of A are denoted $A^\times$, $\mathrm{Z}(A)$, $\operatorname{Jac} A$ and A^{op}, respectively. We write $\mathrm{Z}_A(X)$ for the centralizer of a subset $X \subseteq A$ in A. The category of finite projective right A-modules is denoted $\mathcal{P}(A)$. If $a \in A^\times$, then $\operatorname{Int}(a)$ denotes the inner automorphism $x \mapsto axa^{-1} : A \to A$. Given an R-ring S, and $P, Q \in \mathcal{P}(A)$, the natural map $\operatorname{Hom}_A(P, Q) \otimes S \to \operatorname{Hom}_{A_S}(P_S, Q_S)$ is an isomorphism [27, Theorem 1.3.26], and we shall freely identify these S-modules.

In situations when an abelian group M can be regarded as a module over multiple R-algebras, we shall sometimes write M_A to denote "M, viewed as a right A-module". In particular, A_A denotes "A, viewed as a right module over itself". Similar notation will be applied to left modules, but with the subscript written on the left, e.g., ${}_AA$.

If $e \in A$ is an idempotent, we shall freely identify $\operatorname{End}_A(eA)$ with eAe, where eAe acts on eA via multiplication on the left. We say that e is *full* if $AeA = A$, or equivalently, if eA is a progenerator [42, §18B]. (A right A-module M is called a progenerator if M is finite projective and A_A is isomorphic to a summand of M^n for some $n \in \mathbb{N}$.) The idempotent e is called *primitive* if $e \neq 0$ and eAe contains no idempotents except 0 and e.

An R-algebra with involution means a pair (A, σ) consisting of an R-algebra A and an *R-linear* involution $\sigma : A \to A$. Involutions are applied exponentially to elements of A, i.e., a^σ stands for $\sigma(a)$. Given $\varepsilon \in \mathrm{Z}(A)$ with $\varepsilon^\sigma \varepsilon = 1$, we let $\mathcal{S}_\varepsilon(A, \sigma) = \{a \in A : a = \varepsilon a^\sigma\}$.

1. Azumaya Algebras With Involution

We recall the definition and some properties of Azumaya algebras with involution, giving particular attention to the case where the base ring R is semilocal. When R is a field, all the material can be found in [40, Chapter I].

1A. Separable Projective Algebras.

Recall that an R-algebra A is called separable if A is projective when endowed with the right $A^{\mathrm{op}} \otimes A$-module structure determined by $a \cdot (x^{\mathrm{op}} \otimes y) = xay$, or equivalently, if the right $A^{\mathrm{op}} \otimes A$-module homomorphism $x^{\mathrm{op}} \otimes y \mapsto xy : A^{\mathrm{op}} \otimes A \to A$ admits an $A^{\mathrm{op}} \otimes A$-linear section.

By definition, the *Azumaya* R-algebras are the central separable R-algebras, and the *finite étale* R-algebras are the finite projective commutative separable R-algebras. There are many other equivalent definitions, see [27] and [39, III.§5], for instance.

In the sequel, we shall often consider R-algebras A such that A is Azumaya over $\mathrm{Z}(A)$ and $\mathrm{Z}(A)$ is finite étale over R. The following proposition lists a few convenient equivalent characterizations of such algebras, which we call *separable projective* after condition (SP2).

Proposition 1.1. *Let A be an R-algebra. The following conditions are equivalent.*

(SP1) A is Azumaya over $\mathrm{Z}(A)$ and $\mathrm{Z}(A)$ is finite étale over R.
(SP2) A is projective as an R-module and separable as an R-algebra.

(SP3) *A is finite projective as an R-module and, for all $\mathfrak{m} \in \operatorname{Max} R$, the $k(\mathfrak{m})$-algebra $A(\mathfrak{m})$ is semisimple and its center is a product of separable field extensions of $k(\mathfrak{m})$.*

Proof. (SP1) $\Longrightarrow$ (SP2) follows from [20, Theorem II.3.4(iii), Theorem II.3.8]. (SP2) $\Longrightarrow$ (SP1) follows from [20, Theorem II.3.8, Lemma II.3.1]. (SP2) $\Longrightarrow$ (SP3) follows from [20, Proposition II.2.1, Corollary II.2.4] and the fact that (SP2) continues to hold after base-change. (SP3) $\Longrightarrow$ (SP2) follows from [20, Theorem II.7.1, Corollary II.2.4]. □

We collect several facts about separable projective algebras.

Lemma 1.2 ([64, Proposition 2.14])**.** *Let A be a separable R-algebra and let M be a right A-module. If M is projective over R, then M is projective over A. The converse holds when A is projective over R.*

Lemma 1.3. *Let A be a separable projective R-algebra and let $S \subseteq \mathrm{Z}(A)$ be an R-subalgebra such that S is separable over R or an R-summand of $\mathrm{Z}(A)$. Then A is separable projective over S and S is separable projective over R.*

Proof. Suppose first that S is separable over R. That A is separable over S follows from [20, Proposition II.1.12]. Since A is projective over R and S is separable over R, the algebra A is projective as an S-module by Lemma 1.2. It is faithful over S since S is a subring of A. Now, by [20, Corollary II.4.2], S is a summand of A, so S is projective over R.

If S is summand of $Z := \mathrm{Z}(A)$, then $S \in \mathcal{P}(R)$. Thus, for every $\mathfrak{p} \in \operatorname{Spec} R$, the map $S(\mathfrak{p}) \to Z(\mathfrak{p})$ is injective. Since $Z(\mathfrak{p})$ is a finite product of separable field extensions of $k(\mathfrak{p})$, the same holds for $S(\mathfrak{p})$, and we conclude that S is also separable. Proceed as in the previous paragraph. □

Lemma 1.4. *Let A be a separable projective R-algebra, let $B \subseteq A$ be a separable projective R-subalgebra and let S be an R-ring. Then the natural map $\mathrm{Z}_A(B) \otimes S \to \mathrm{Z}_{A_S}(B_S)$ is an isomorphism. In particular, $\mathrm{Z}(A) \otimes S = \mathrm{Z}(A_S)$.*

Proof. Write $C = B \otimes A^{\mathrm{op}}$ and view A as a left C-module by setting $(b \otimes a^{\mathrm{op}}) \cdot x = bxa$ ($a, x \in A$, $b \in B$). Since C is separable over R and $A \in \mathcal{P}(R)$, Lemma 1.2 implies that A is projective as a C-module. Thus, the natural map $\operatorname{End}_C(A) \otimes S \to \operatorname{End}_{C_S}(A_S)$ is an isomorphism. However, $\operatorname{End}_C(A) \cong \mathrm{Z}_A(B)$ via $\varphi \mapsto \varphi(1)$, and likewise $\operatorname{End}_{C_S}(A_S) \cong \mathrm{Z}_{A_S}(B_S)$. The resulting isomorphism $\mathrm{Z}_B(A) \otimes S \to \operatorname{End}_C(A) \otimes S \to \operatorname{End}_{C_S}(A_S) \to \mathrm{Z}_{A_S}(B_S)$ is the natural map $\mathrm{Z}_A(B) \otimes S \to \mathrm{Z}_{A_S}(B_S)$ and the proposition follows. □

Lemma 1.5. *Let A be a separable projective R-algebra. Then $\operatorname{Jac} A = \operatorname{Jac} R \cdot A = \bigcap_{\mathfrak{m} \in \operatorname{Max} R} \mathfrak{m} A$.*

Proof. Since A is finite over R, we have $\operatorname{Jac} R \cdot A \subseteq \operatorname{Jac} A$ [39, Corollary II.4.2.4]. In addition, for all $\mathfrak{m} \in \operatorname{Max} R$, the ring $A/\mathfrak{m}A = A(\mathfrak{m})$ is semisimple by (SP3), hence $\operatorname{Jac} A \subseteq \bigcap_{\mathfrak{m} \in \operatorname{Max} R} \mathfrak{m}A$. It remains to show that $\bigcap_{\mathfrak{m} \in \operatorname{Max} R} \mathfrak{m}A \subseteq \operatorname{Jac} R \cdot A$.

Consider the exact sequence of R-modules $0 \to \operatorname{Jac} R \to R \to \prod_{\mathfrak{m} \in \operatorname{Max} R} R/\mathfrak{m}$. Since A is a flat over R, tensoring with A gives an exact sequence $0 \to \operatorname{Jac} R \otimes A \to R \otimes A \to (\prod_{\mathfrak{m} \in \operatorname{Max} R} R/\mathfrak{m}) \otimes A$. Furthermore, since A is finitely presented, the natural map $(\prod_{\mathfrak{m} \in \operatorname{Max} R} R/\mathfrak{m}) \otimes A \to \prod_{\mathfrak{m} \in \operatorname{Max} R} (R/\mathfrak{m}) \otimes A \cong \prod_{\mathfrak{m} \in \operatorname{Max} R} A/\mathfrak{m}A$ is an isomorphism [42, Proposition 4.44]. Thus, $0 \to \operatorname{Jac} R \otimes A \to A \to \prod_{\mathfrak{m} \in \operatorname{Max} R} A/\mathfrak{m}A$ is exact, and the exactness at A means that $\bigcap_{\mathfrak{m} \in \operatorname{Max} R} \mathfrak{m}A = \operatorname{Jac} R \cdot A$. □

We also record the following general lemmas.

Lemma 1.6. *Let A be a finite R-algebra and let $a \in A$. If $a(\mathfrak{m}) \in A(\mathfrak{m})^{\times}$ for all $\mathfrak{m} \in \operatorname{Max} R$, then $a \in A^{\times}$.*

Proof. Consider the map $\phi : A \to A$ given by $\phi(x) = ax$. Then $\phi(\mathfrak{m}) : A(\mathfrak{m}) \to A(\mathfrak{m})$ is bijective for all $\mathfrak{m} \in \operatorname{Max} R$. As A is a finite R-module, ϕ is surjective ([68, Tag 05GE] with $f = 1$), so $aA = \operatorname{im}\phi = A$. Likewise, $Aa = A$, so $a \in A^\times$. □

Lemma 1.7. *Let A be a finite projective R-algebra, let $P \in \mathcal{P}(A)$ and let U and V be summands of P. Suppose that $P(\mathfrak{m}) = U(\mathfrak{m}) \oplus V(\mathfrak{m})$ for all $\mathfrak{m} \in \operatorname{Max} R$. Then $P = U \oplus V$.*

Proof. We need to show that $\psi : U \times V \to P$ given by $\psi(u, v) = u + v$ is an isomorphism. By assumption, $\operatorname{im}\psi + P\mathfrak{m} = P$ for all $\mathfrak{m} \in \operatorname{Max} R$. By Nakayama's Lemma $\operatorname{ann}_R(P/\operatorname{im}\psi)$ is not contained in any maximal ideal, so it must be R. Thus, $P/\operatorname{im}\psi = 0$ and ψ is onto. Let $K = \ker\psi$. Since P is projective, ψ splits and $U \times V \cong P \times K$, hence $\operatorname{rk}_R U + \operatorname{rk}_R V = \operatorname{rk}_R P + \operatorname{rk}_R P$. Since $P(\mathfrak{m}) = U(\mathfrak{m}) \oplus V(\mathfrak{m})$ for all $\mathfrak{m} \in \operatorname{Max} R$, we have $\operatorname{rk}_R U + \operatorname{rk}_R V = \operatorname{rk}_R P$, so $\operatorname{rk}_R K = 0$ and $\ker\psi = K = 0$. □

1B. **Azumaya Algebras.** We refer the reader to [27, 7.§3], [39, III.§5.3] or [64, Chapter 3] for the definition of the *Brauer group* of R. We denote it as $\operatorname{Br} R$ and write its binary operation additively. The *Brauer class* of an Azumaya R-algebra A is denoted $[A]$.

As usual, the *degree* of an Azumaya R-algebra A is $\deg A := \sqrt{\operatorname{rk}_R A}$, and its *index* is $\operatorname{ind} A := \gcd\{\deg A' \mid A' \in [A]\}$. Recall that both $\deg A$ and $\operatorname{ind} A$ are functions from $\operatorname{Spec} R$ to $\mathbb{N}$, and that the "gcd" in the definition of $\operatorname{ind} A$ is evaluated point-wise. Since A is a finite projective R-module, $\deg A$ is locally constant relative to the Zariski topology, and with a little more work, one sees that the same holds for $\operatorname{ind} A$. When R is connected, both $\deg A$ and $\operatorname{ind} A$ are constant and may be regarded as elements of $\mathbb{N}$.

We alert the reader that in general, there may be no $A' \in [A]$ with $\deg A' = \operatorname{ind} A$; see [2]. However, this is true when R is semilocal, by Theorem 1.25 below.

Theorem 1.8 (Saltman [63])**.** *Let A be an Azumaya R-algebra of degree dividing $n \in \mathbb{N}$. Then $n \cdot [A] = 0$ in $\operatorname{Br} R$.*

Given $P \in \mathcal{P}(A)$, the reduced rank $\operatorname{rk}_R P$ is (point-wise) divisible by $\deg A$; indeed, by [40, pp. 5–6], $\deg A(\mathfrak{p}) \mid \dim_{k(\mathfrak{p})} P(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$. It is therefore convenient to introduce the *reduced A-rank* of P, defined by

$$\operatorname{rrk}_A P := \operatorname{rk}_R P / \deg A.$$

This agrees with the *reduced dimension* defined in *op. cit.* when R is a field. For example, $\operatorname{rrk}_A(A_A) = \deg A$. If $\iota : R \to S$ is a ring homomorphism, then $\operatorname{rrk}_{A_S} P_S = \iota \operatorname{rrk}_A P$. In particular, $\deg A_S = \iota \deg A$.

Remark 1.9. If A is an R-algebra which is Azumaya over its center $\mathrm{Z}(A)$, then we regard $\deg A$, $\operatorname{ind} A$ and $\operatorname{rrk}_A P$ ($P \in \mathcal{P}(A)$) as functions from $\operatorname{Spec} \mathrm{Z}(A)$ to $\mathbb{Z}$. Note also that $[A]$ is a member of $\operatorname{Br} \mathrm{Z}(A)$, rather than $\operatorname{Br} R$.

We record a number of properties of the reduced rank which will be used many times in the sequel.

Proposition 1.10. *Let A be an Azumaya R-algebra and let $P \in \mathcal{P}(A)$. Then $\operatorname{rrk}_A P > 0$ if and only if P is a progenertor.*

Proof. Since $\operatorname{rrk}_A(A_A) = \deg A > 0$, if P is a progenerator, then $\operatorname{rrk}_A P > 0$.

To see the converse, let $T = \sum_\phi \operatorname{im}\phi$ where ϕ ranges over $\operatorname{Hom}_A(P, A)$. It is enough to prove that $T = A$, see [27, pp. 7–8]. Fix $\mathfrak{m} \in \operatorname{Max} R$. Since $(\operatorname{rrk}_A P)(\mathfrak{m}) > 0$, the $A(\mathfrak{m})$-module $P(\mathfrak{m})$ is nonzero. Since $A(\mathfrak{m})$ is simple artinian, there exists $n \in \mathbb{N}$ and a surjection $\varphi : P(\mathfrak{m})^n \to A(\mathfrak{m})$. Since P is projective, there exists

$\hat{\varphi} \in \mathrm{Hom}_A(P^n, A)$ such that $\varphi = \hat{\varphi}(\mathfrak{m})$, hence $\mathrm{im}(\hat{\varphi}) + \mathfrak{m}A = A$. Since $\mathrm{im}(\hat{\varphi}) \subseteq T$, this means that $T + \mathfrak{m}A = A$, or rather, $(A/T)\mathfrak{m} = A/T$. By Nakayama's Lemma $\mathrm{ann}_R(A/T)$ is not contained in $\mathfrak{m}$. As this holds for all $\mathfrak{m} \in \mathrm{Max}\, R$, we must have $A/T = 0$, so $T = A$. □

Proposition 1.11. *Let A be an Azumaya R-algebra and suppose that $P \in \mathcal{P}(A)$ satisfies $\mathrm{rrk}_A\, P > 0$. Then:*

(i) $B := \mathrm{End}_A(P)$ *is an Azumaya R-algebra,* $\deg B = \mathrm{rrk}_A\, P$ *and* $[B] = [A]$.
(ii) For all $Q \in \mathcal{P}(B)$, we have $\mathrm{rrk}_B\, Q = \mathrm{rrk}_A(Q \otimes_B P)$.
(iii) For every $B' \in [A]$, there exists $P' \in \mathcal{P}(A)$ with $B' \cong \mathrm{End}_A(P')$ and $\mathrm{rrk}_A\, P' = \deg B' > 0$.

Proof. (i) By Proposition 1.10, P_A is a progenerator, and in particular faithful. Thus, A^{op} embeds as an R-subalgebra of $\mathrm{End}_R(P)$ via $a^{\mathrm{op}} \mapsto [x \mapsto xa]$, and $B = \mathrm{Z}_{A^{\mathrm{op}}}(\mathrm{End}_R(P))$. Since both A^{op} and $\mathrm{End}_R(P)$ are Azumaya R-algebras, B is Azumaya over R and $A^{\mathrm{op}} \otimes B \cong \mathrm{End}_R(P)$ [20, Theorem II.4.3]. This implies that $\deg A^{\mathrm{op}} \cdot \deg B = \deg \mathrm{End}_R(P) = \mathrm{rk}_R\, P$. It follows that $\deg B = \mathrm{rrk}_A\, P$ and $[A^{\mathrm{op}}] + [B] = [\mathrm{End}_R(P)] = 0$, so $[A] = [B]$.

(ii) By definition of the reduced rank, it enough to check the statement after specializing to $k(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}\, R$. (Recall that $\psi \mapsto \psi \otimes \mathrm{id}_{k(\mathfrak{p})} : B(\mathfrak{p}) = \mathrm{End}_A(P) \otimes k(\mathfrak{p}) \to \mathrm{End}_{A(\mathfrak{p})}(P(\mathfrak{p}))$ is an isomorphism because $P \in \mathcal{P}(A)$.) Now that R is a field, we may regard the reduced rank as an integer and further specialize to the algebraic closure, as it would not affect the reduced rank. When R is algebraically closed, we may assume that $A = \mathrm{M}_n(R)$, $P = \mathrm{M}_{m\times n}(R)$, $B = \mathrm{M}_m(R)$, $Q = \mathrm{M}_{t\times m}(R)$, and checking that $\mathrm{rrk}_B\, Q = t = \mathrm{rrk}_A(Q \otimes_A P)$ is routine.

(iii) By a Theorem of Bass, see [23, Theorem 9.2], exists is an progenerator $P' \in \mathcal{P}(A)$ such that $B' \cong \mathrm{End}_A(P')$. The claim now follows from Proposition 1.10 and (i). □

Corollary 1.12. *Let A be an Azumaya R-algebra and let $e \in A$ be an idempotent. Then e is full (i.e. $AeA = A$) if and only if* $\mathrm{rrk}_A\, eA > 0$. *In this case, eAe is an Azumaya R-algebra,* $\deg eAe = \mathrm{rrk}_A\, eA$, $[A] = [eAe]$ *and for all $P \in \mathcal{P}(A)$, we have* $\mathrm{rrk}_A\, P = \mathrm{rrk}_{eAe}\, Pe$.

Proof. Recall that e is full if and only if eA_A is a progenerator, and this is equivalent to $\mathrm{rrk}_A\, eA > 0$ by Proposition 1.10. Since $eAe = \mathrm{End}_A(eA)$, the first three assertions follow from Proposition 1.11(i). For the last assertion, note that by Morita theory, $Ae \in \mathcal{P}(eAe)$ is a progenerator and $A = \mathrm{End}_{eAe}(Ae)$ [42, Corollary 18.21]. Applying Proposition 1.11(ii) with eAe, A, Ae in place of A, B, P, we see that $\mathrm{rrk}_{eAe}\, Pe = \mathrm{rrk}_{eAe}(P \otimes_A Ae) = \mathrm{rrk}_A\, P$. □

Corollary 1.13. *Let A be an Azumaya R-algebra and let $P \in \mathcal{P}(A)$. Then* $\mathrm{ind}\, A \mid \mathrm{rrk}_A\, P$.

Proof. Since $\mathrm{ind}\, A \mid \deg A = \mathrm{rrk}_A\, A$, we may replace P with $P \oplus A$ and assume that $\mathrm{rrk}_A\, P > 0$. By Proposition 1.11(i), $\mathrm{rrk}_A\, P = \deg \mathrm{End}_A(P)$ and $\mathrm{End}_A(P) \in [A]$, so $\mathrm{rrk}_A\, P$ is divisible by $\mathrm{ind}\, A$. □

Proposition 1.14. *Let A be an Azumaya R-algebra, let S be a finite étale R-subalgebra of A and let $\iota : R \to S$ be the inclusion map. Then $B := \mathrm{Z}_A(S)$ is Azumaya over S and $[B] = [A \otimes S]$ in* $\mathrm{Br}\, S$. *Furthermore, A is projective as a right S-module, and if* $\mathrm{rk}_S\, A_A$ *is constant along the fibers of* $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$, *then:*

(i) $\deg B \cdot \iota\, \mathrm{rk}_R\, S = \iota \deg A$, *and* $\mathrm{rrk}_B\, P = \iota\, \mathrm{rrk}_A\, P$ *for all $P \in \mathcal{P}(A)$, and*
(ii) $\iota\, \mathrm{rrk}_A(Q \otimes_B A) = \iota\, \mathrm{rk}_R\, S \cdot \mathrm{rrk}_B\, Q$ *for all $Q \in \mathcal{P}(A)$ such that* $\mathrm{rrk}_B\, Q$ *is constant along the fibers of* $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$.

Proof. That B is Azumaya over S and $[B] = [A \otimes S]$ is well-known, see [64, Theorem 3.10], for instance. That A is projective as a right S-module follows from Lemma 1.2.

By Lemma 1.4, it is enough to prove (i) and (ii) after base changing to every residue field of R, so assume R is a field. In fact, we may further base-change to an algebraic closure of R and assume that R is algebraically closed. In this case, $S \cong R^t$ for $t = \mathrm{rk}_R S$ and $A = \mathrm{M}_n(R)$ for $n = \deg A$.

Let $e_{ij} \in \mathrm{M}_n(R)$ denote the $n \times n$ matrix with 1 at the (i, j)-entry and zeroes elsewhere. Let $f_1, \dots, f_t$ denote the primitive idempotents of S. Since $\mathrm{rrk}_S A_S$ is constant, $\dim_R Af_i$ is independent of i, so ${}_AAf_1 \cong \dots \cong {}_AAf_t$. This means that each f_i is an idempotent of rank $s := \frac{n}{t}$ in $A = \mathrm{M}_n(R)$. Since all such idempotents are conjugate, we may choose the identification of A with $\mathrm{M}_n(R)$ such that $f_i = \sum_{j=(i-1)s+1}^{is} e_{jj}$. Thus,

$$B = \begin{bmatrix} \mathrm{M}_s(R) & & \\ & \ddots & \\ & & \mathrm{M}_s(R) \end{bmatrix} \subseteq \mathrm{M}_n(R).$$

Furthermore, every right A-module is isomorphic to $\mathrm{M}_{m\times n}(R)$ for some $m \geq 0$ and any right B-module with constant S-rank is isomorphic to $\mathrm{M}_{\ell\times s}(R) \times \dots \times \mathrm{M}_{\ell\times s}(R)$ (t times) for some $\ell \geq 0$. Now, verifying (i) and (ii) is straightforward. □

The requirement that $\mathrm{rk}_S A_A$ is contstant along the fibers of $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$ is guaranteed when $\mathrm{rk}_R S = \deg A$.

Corollary 1.15. *Let A be an Azumaya R-algebra and let S be a finite étale subalgebra of A such that $\mathrm{rk}_R S = \deg A$. Then $\mathrm{rk}_S A_A = \iota \deg A$ ($\iota : R \to S$ is the inclusion), $S = \mathrm{Z}_A(S)$ and $[A_S] = 0$.*

Proof. We only need to show that $\mathrm{rk}_S A_A = \iota \deg A$. The remaining assertions then follow from Proposition 1.14.

The algebra A is an (A, S)-bimodule and hence a right module over $A^{\mathrm{op}} \otimes S$. By Lemma 1.2, $A_{A^{\mathrm{op}}\otimes S}$ is projective, so $\deg A_S \mid \mathrm{rk}_S A_A$. Furthermore, $\mathrm{rk}_S A_A > 0$ because A is faithful as a right S-module. Let $\mathfrak{p} \in \mathrm{Max}\, R$ and write $S(\mathfrak{p}) = \prod_{i=1}^t K_i$, where K_i is a $k(\mathfrak{p})$-field. We need to show that $\dim_{K_i}(A \otimes_S K_i) = \deg A(\mathfrak{p})$. Write $n = \deg A(\mathfrak{p})$. Then $n^2 = \dim_{k(\mathfrak{p})} A(\mathfrak{p}) = \sum_i [K_i : k(\mathfrak{p})] \dim_{K_i}(A \otimes_S K_i)$ and $\sum_i [K_i : k(\mathfrak{p})] = \dim_{k(\mathfrak{p})} S(\mathfrak{p}) = n$. Since $\dim_{K_i}(A \otimes_S K_i)$ is positive and divisible by n, we must have $\dim_{K_i}(A \otimes_S K_i) = n$ for all i, as required. □

1C. **Quadratic Étale Algebras.** Finite étale R-algebras of R-rank 2 are also called *quadratic étale* algebras. Every such algebra S admits a unique R-involution $\theta : S \to S$ such that $R = \{s \in S : s^\theta = s\}$; it is given by $x^\theta = \mathrm{Tr}_{S/R}(x) - x$ and satisfies $\mathrm{Nr}_{S/R}(x) = x^\theta x$.[2] See [39, Proposition I.1.3.4] for its uniqueness. Following [39, I.§1.3], we call θ the *standard R-involution* of S.

For example, the R-algebra $R \times R$ is quadratic étale and its standard involution is the exchange involution $(x, y) \mapsto (y, x)$. Furthermore, by our standing assumption that $2 \in R^\times$, the R-algebra $R[x \,|\, x^2 = a]$ is quadratic étale whenever $a \in R^\times$ (use (SP3) above), and its standard involution is determined by $x^\theta = -x$.

Lemma 1.16. *Let S be a quadratic étale R-algebra. If R is connected and S is not connected, then $S \cong R \times R$ as R-algebras.*

[2] If A is a finite projective R-algebra of rank $n \in \mathbb{N}$, then the *trace* and *norm* maps $\mathrm{Tr}_{A/R}, \mathrm{Nr}_{A/R} : A \to R$ take $a \in A$ to $-c_1(a)$ and $(-1)^n c_n(a)$, respectively, where $X^n + c_1(a)X^{n-1} + \dots + c_n(a)X^0$ is the characteristic polynomial of $[x \mapsto ax] \in \mathrm{End}_R(A)$ in the sense of [27, Example 5.3.3].

Proof. Let θ denote the standard R-involution of S, and let $e \in S$ be a nontrivial idempotent. Then $e^\theta e$ is a non-invertible idempotent of R, hence $e^\theta e = 0$ (because R is connected). This means that $e + e^\theta$ is also an idempotent in R, and it is nonzero because $e(e + e^\theta) = e \neq 0$. Since R is connected, $e + e^\theta = 1$. It is now routine to check that $r \mapsto er : R \to eS$ and $s \mapsto s + s^\theta : eS \to R$ are mutually inverse. Since the former map is an R-algebra homomorphism, we see that $R \cong eS$ as R-algebras, and similarly $R \cong e^\theta S = (1-e)S$. The lemma follows because $S \cong eS \times (1-e)S$. □

Lemma 1.17. *Let S be a quadratic étale R-algebra and let $\sigma : S \to S$ be an R-involution. Then there exists a factorization $R = R_1 \times R_2$ such that $\sigma_{R_1} : S_{R_1} \to S_{R_1}$ is the standard R_1-involution of S_{R_1} and $\sigma_{R_2} : S_{R_2} \to S_{R_2}$ is the identity. In particular, if R is connected, then σ is either the standard R-involution of S or id_S.*

Proof. This is a restatement of [39, Proposition III.4.1.2]. □

Lemma 1.18. *Let S be a quadratic étale R-algebra. Then $S_S \cong S \times S$ as S-algebras.*

Proof. This follows from the discussion in [39, III.§4.1]. □

Lemma 1.19. *Suppose that R is semilocal and let S be a quadratic étale R-algebra with standard involution θ. Then there exists $\lambda \in S$ such that $\lambda^2 \in R^\times$, $\lambda^\theta = -\lambda$ and $\{1, \lambda\}$ is an R-basis of S.*

Proof. Since $2 \in R^\times$, we have $S = \mathcal{S}_1(S,\theta) \oplus \mathcal{S}_{-1}(S,\theta) = R \oplus \mathcal{S}_{-1}(S,\theta)$, and so $\mathcal{S}_{-1}(S,\theta)$ is a rank-1 projective R-module. Since R is semilocal, $\mathcal{S}_{-1}(S,\theta)$ is free. Let λ be a generator of $\mathcal{S}_{-1}(S,\theta)$. Then $\lambda^2 = -\lambda \cdot \lambda^\theta = -\operatorname{Nr}_{S/R}(\lambda) \in R$ and $\{1, \lambda\}$ is an R-basis of S. As a result, $S \cong R[x \,|\, x^2 - a]$, where $a = \lambda^2$. If $a \notin R^\times$, then there exists $\mathfrak{m} \in \operatorname{Max} R$ with $a \in \mathfrak{m}$, and it follows that $S(\mathfrak{m}) \cong k(\mathfrak{m})[x \,|\, x^2 = 0]$ is not étale over $k(\mathfrak{m})$. Thus, we must have $\lambda^2 = a \in R^\times$. □

1D. **Azumaya Algebras With Involution.** Recall our standing assumption that $2 \in R^\times$. An *Azumaya algebra with involution*[3] over R is an R-algebra with involution (A, σ) such that A is separable projective over R and the homomorphism $r \mapsto r \cdot 1_A : R \to A$ identifies R with the σ-fixed elements of $\mathrm{Z}(A)$. *Note that A is not necessarily Azumaya as an R-algebra.* Rather, A is Azumaya over $\mathrm{Z}(A)$, so that $\deg A$ is a function from $\operatorname{Spec} \mathrm{Z}(A)$ to $\mathbb{Z}$ and $[A] \in \operatorname{Br} \mathrm{Z}(A)$, cf. Remark 1.9.

If (A, σ) is an Azumaya R-algebra with involution and S is an R-ring, then (A_S, σ_S) is an Azumaya S-algebra with involution. Indeed, $\mathrm{Z}(A_S) = \mathrm{Z}(A) \otimes S$ by Lemma 1.4, and the exact sequence $0 \to R \xrightarrow{r \mapsto r \cdot 1_A} \mathrm{Z}(A) \xrightarrow{a \mapsto a - a^\sigma} R \to 0$ is split at $\mathrm{Z}(A)$ because $\mathrm{Z}(A) = R1_A \oplus \mathcal{S}_{-1}(\mathrm{Z}(A), \sigma)$, so it remains exact after tensoring with S. Together, this means that $s \mapsto s \cdot 1_A : S \to \{a \in \mathrm{Z}(A_S) \,:\, a - a^{\sigma_S} = 0\}$ is an isomorphism, hence our claim.

Example 1.20. Let A be a separable projective R-algebra, let $\sigma : A \to A$ be an R-involution and let $R_1 := \{s \in \mathrm{Z}(A) \,:\, s^\sigma = s\}$. Then (A, σ) is an Azumaya R_1-algebra with involution. Indeed, R_1 is a R-summand of $\mathrm{Z}(A)$ because $2 \in R^\times$, so by Lemma 1.3, A is separable projective over R_1 and R_1 is finite étale over R.

When R is a field F, an Azumaya F-algebra with involution, (A, σ), is a central simple F-algebra with involution in the sense of [40, pp. 13, 20]. The center of A is then either F or a quadratic étale extension of F. In first case, A is a central simple F-algebra and σ can be either of *orthogonal* or *symplectic* type, see [40, §2.A]. When σ is symplectic, $\deg A$ must be even [40, Proposition 2.6]. In the case

[3] This should be understood as "Azumaya algebra-with-involution" rather than "Azumaya-algebra with involution".

$\mathrm{Z}(A) \neq F$, the center is either $F \times F$ or a quadratic separable field extension of F, and σ is said to be of *unitary* type, see [40, §2.B].

Returning to the case R is arbitrary, we turn to define the *type* of the involution σ. In fact, it will be convenient to define the type of a pair (σ, ε), where $\varepsilon \in \mathrm{Z}(A)$ satisfies $\varepsilon^\sigma \varepsilon = 1$, with the type of σ being the type of $(\sigma, 1)$.

To that end, suppose first that R is a field. We say that the type of (σ, ε) is unitary if σ is unitary, i.e., when $\mathrm{Z}(A) \neq R$. Suppose now that $\mathrm{Z}(A) = R$. Then $\varepsilon \in \{\pm 1\}$ and σ is either orthogonal or symplectic. We say that (σ, ε) is of orthogonal type if either σ is orthogonal and $\varepsilon = 1$, or σ is symplectic and $\varepsilon = -1$. In all other cases, (σ, ε) is said to be of symplectic type.

When R is arbitrary, the type of (σ, ε) is the function from $\operatorname{Spec} R$ to the set $\{\text{orthogonal}, \text{symplectic}, \text{unitary}\}$ assigning $\mathfrak{p}$ the type of $(\sigma(\mathfrak{p}), \varepsilon(\mathfrak{p}))$. The type of σ is the type of $(\sigma, 1)$; this agrees with the definition of [39, III.§8]. We also say that (σ, ε) is orthogonal (resp. symplectic, unitary) at $\mathfrak{p}$ if $(\sigma(\mathfrak{p}), \varepsilon(\mathfrak{p}))$ is orthogonal (resp. symplectic, unitary). The pair (σ, ε) is called orthogonal (resp. symplectic, unitary) if this holds at all primes $\mathfrak{p} \in \operatorname{Spec} R$. We remark that (σ, ε) is unitary if and only if σ (i.e. $(\sigma, 1)$) is unitary.

Recall that $\mathcal{S}_\varepsilon(A, \sigma) = \{a \in A : \varepsilon a^\sigma = a\}$.

Proposition 1.21. *Let (A, σ) be an Azumaya R-algebra with involution, let $\varepsilon \in \mathrm{Z}(A)$ be an element satisfying $\varepsilon^\sigma \varepsilon = 1$ and write $n = \deg A$.*

(i) (σ, ε) is orthogonal if and only if $\operatorname{rk}_R \mathcal{S}_\varepsilon(A, \sigma) = \frac{1}{2}n(n+1)$ and $\mathrm{Z}(A) = R$.
(ii) (σ, ε) is symplectic if and only if $\operatorname{rk}_R \mathcal{S}_\varepsilon(A, \sigma) = \frac{1}{2}n(n-1)$ and $\mathrm{Z}(A) = R$.
(iii) (σ, ε) is unitary if and only if $\operatorname{rk}_R \mathrm{Z}(A) = 2$. In this case, $\mathrm{Z}(A)$ is a quadratic étale R-algebra, $\sigma|_{\mathrm{Z}(A)}$ is its standard involution and $\operatorname{rk}_R \mathcal{S}_\varepsilon(A, \sigma) = n^2$.
(iv) There exists a factorization $R \cong R_o \times R_s \times R_u$ such that $(\sigma_{R_o}, \varepsilon \otimes 1_{R_o})$ is orthogonal, $(\sigma_{R_s}, \varepsilon \otimes 1_{R_s})$ is symplectic and $(\sigma_{R_u}, \varepsilon \otimes 1_{R_u})$ is unitary.
(v) If R is connected, then (σ, ε) is either orthogonal, symplectic or unitary.

Proof. Suppose first that R is a field. If $\mathrm{Z}(A) = R$, then $\varepsilon \in \{\pm 1\}$ and (i)–(iii) follow from [40, Proposition 2.6]. If $\mathrm{Z}(A) \neq R$, then by Hibert's Theorem 90, there exist $\delta \in \mathrm{Z}(A)$ such that $\delta^\sigma \delta^{-1} = \varepsilon^{-1}$. One readily checks that $\delta \cdot \mathcal{S}_1(A, \sigma) = \mathcal{S}_\varepsilon(A, \sigma)$, so $\dim_R \mathcal{S}_\varepsilon(A, \sigma) = \dim_R \mathcal{S}_1(A, \sigma)$, and the right hand side is n^2 by [40, Proposition 2.17]. It follows that (i)–(iii) hold in this case as well.

Parts (i)–(iii) for general R will follow from the field case if we show that the natural maps $\mathrm{Z}(A)(\mathfrak{p}) \to \mathrm{Z}(A(\mathfrak{p}))$ and $(\mathcal{S}_\varepsilon(A, \sigma))(\mathfrak{p}) \to \mathcal{S}_\varepsilon(A(\mathfrak{p}), \sigma(\mathfrak{p}))$ are isomorphisms for all $\mathfrak{p} \in \operatorname{Spec} R$. The former isomorphism is Lemma 1.4. To establish the second, note that the short exact sequence $\mathcal{S}_\varepsilon(A, \sigma) \to A \to \mathcal{S}_{-\varepsilon}(A, \sigma)$ in which the right arrow is given by $a \mapsto a - \varepsilon a^\sigma$ is split, because $2 \in R^\times$, and thus it remains exact after base-change along $R \to k(\mathfrak{p})$.

Now, part (iv) follows readily from the fact that $\operatorname{rk}_R \mathrm{Z}(A)$ and $\operatorname{rk}_R \mathcal{S}_\varepsilon(A, \sigma)$ are locally constant functions, and part (v) follows from (iv). □

Corollary 1.22. *Let (A, σ) be an Azumaya R-algebra with involution and let $\varepsilon \in \mathrm{Z}(A)$ be an element satisfying $\varepsilon^\sigma \varepsilon = 1$.*

(i) For every $\delta \in \mathrm{Z}(A)$ satisfying $\delta^\sigma \delta = 1$ and every $\mu \in \mathcal{S}_\delta(A, \sigma) \cap A^\times$, the pair $(A, \operatorname{Int}(\mu) \circ \sigma)$ is an Azumaya R-algebra with involution and the type of $(\operatorname{Int}(\mu) \circ \sigma, \delta\varepsilon)$ is the same as the type of (σ, ε).
(ii) For every idempotent $e \in A$ with $\operatorname{rrk}_A eA > 0$ and $e^\sigma = e$, the pair $(eAe, \sigma|_{eAe})$ is an Azumaya R-algebra with involution and the type of $(\sigma|_{eAe}, e\varepsilon)$ is the same as the type of (σ, ε).

Proof. (i) Checking that $(A, \operatorname{Int}(\mu) \circ \sigma)$ is an Azumaya R-algebra with involution is straightforward. It is routine to check that $x \mapsto \mu x : \mathcal{S}_\varepsilon(A, \sigma) \to \mathcal{S}_{\delta\varepsilon}(A, \operatorname{Int}(\mu) \circ \sigma)$

is an R-module isomorphism, hence $\operatorname{rk}_R \mathcal{S}_\varepsilon(A,\sigma) = \operatorname{rk}_R \mathcal{S}_{\delta\varepsilon}(A, \operatorname{Int}(\mu)\circ\sigma)$. By Proposition 1.21, this means that (σ,ε) and $(\operatorname{Int}(\mu)\circ\sigma, \delta\varepsilon)$ have the same type.

(ii) Write $\sigma_e := \sigma|_{eAe}$. By Corollary 1.12, eAe is Azumaya over $\mathrm{Z}(A)$. In particular, $a \mapsto ea$ defines an isomorphism $\mathrm{Z}(A) \to \mathrm{Z}(eAe)$. This isomorphism is compatible with σ, so $r \mapsto er : R \to eAe$ identifies R with the σ_e-fixed elements in $\mathrm{Z}(eAe)$. Thus, (eAe, σ_e) is an Azumaya R-algebra with involution.

Let $\mathfrak{p} \in \operatorname{Spec} R$. Since $\operatorname{rk}_R \mathrm{Z}(A) = \operatorname{rk}_R \mathrm{Z}(eAe)$, Proposition 1.21 implies that (σ,ε) is unitary at $\mathfrak{p}$ if and only if $(\sigma_e, e\varepsilon)$ is unitary at $\mathfrak{p}$. Furthermore, by [24, Proposition 2.12], (σ,ε) is orthogonal at $\mathfrak{p}$ if and only if $(\sigma_e, e\varepsilon)$ is orthogonal at $\mathfrak{p}$. Thus, (σ,ε) and $(\sigma_e, \varepsilon e)$ have the same type. □

For later reference, we record the following easy consequence of Lemma 1.5 and the Chinese Remainder Theorem.

Lemma 1.23. *Let (A,σ) be an Azumaya algebra with involution over a semilocal ring R. Write $\overline{A} = A/\operatorname{Jac} A$ and let $\overline{\sigma} : \overline{A} \to \overline{A}$ be the induced involution. Then $(\overline{A}, \overline{\sigma}) \cong \prod_{\mathfrak{m}\in \operatorname{Max} R} (A(\mathfrak{m}), \sigma(\mathfrak{m}))$ as R-algebras with involution, and each factor $(A(\mathfrak{m}), \sigma(\mathfrak{m}))$ is a central simple $k(\mathfrak{m})$-algebra with involution.*

1E. **Azumaya Algebras Over Semilocal Rings.** We now specialize to the case where R is semilocal and establish several results about Azumaya algebras and Azumaya algebras with involution.

Lemma 1.24. *Let A be an Azumaya algebra over a semilocal ring R and let $P, Q \in \mathcal{P}(A)$. Then $P \cong Q$ if and only if $\operatorname{rrk}_A P = \operatorname{rrk}_A Q$. Furthermore, P is isomorphic to a summand of Q if and only if $\operatorname{rrk}_A P \leq \operatorname{rrk}_A Q$.*

Proof. The "only if" part of both statements is clear.

We first prove the "if" part of the second statement. Since $\operatorname{rrk}_A P \leq \operatorname{rrk}_A Q$, we have $\dim_{k(\mathfrak{m})} P(\mathfrak{m}) \leq \dim_{k(\mathfrak{m})} Q(\mathfrak{m})$ for all $\mathfrak{m} \in \operatorname{Max} R$. Since $A(\mathfrak{m})$ is a central simple $k(\mathfrak{m})$-algebra, this means that $P(\mathfrak{m})$ is isomorphic to an $A(\mathfrak{m})$-summand of $Q(\mathfrak{m})$.

Write $S = R/\operatorname{Jac} R$. Since R is semilocal, we have $S = \prod_{\mathfrak{m}\in\operatorname{Max} R} k(\mathfrak{m})$, $A_S = \prod_{\mathfrak{m}\in\operatorname{Max} R} A(\mathfrak{m})$, $P_S = \prod_{\mathfrak{m}\in\operatorname{Max} R} P(\mathfrak{m})$ and $Q_S = \prod_{\mathfrak{m}\in\operatorname{Max} R} Q(\mathfrak{m})$; the products are all finite. By the previous paragraph there exists an A-module epimorphism $\varphi : Q_S \to P_S$. Since P is projective, φ lifts to an A-module homomorphism $\psi : Q \to P$. Since $\operatorname{im}\varphi = P_S$, we have $\operatorname{im}\psi + P\mathfrak{m} = P$ for all $\mathfrak{m} \in \operatorname{Max} R$. Thus, as in the proof of Lemma 1.7, $\operatorname{im}\psi = P$. Since P is projective, this means that P is isomorphic to a summand of Q.

To prove the "if" part of the first statement, argue as above and note that $\ker\psi = 0$, because $\operatorname{rrk}_A P = \operatorname{rrk}_A Q$. □

Theorem 1.25. *Let A be an Azumaya algebra over a semilocal ring R. Then there exists an idempotent $e \in A$ such that $eAe \in [A]$ and $\operatorname{rrk}_A eA = \deg eAe = \operatorname{ind} A$.*

Proof. We first claim that there exists $P \in \mathcal{P}(A)$ with $\operatorname{rrk}_A P = \operatorname{ind} A$. Write $R = \prod_{i=1}^t R_i$, where each R_i is connected. By working over each factor separately, we may assume that R is connected. As a result, $\operatorname{rrk}_A P$ is constant for all $P \in \mathcal{P}(A)$.

Since every $B \in [A]$ is isomorphic to $\operatorname{End}_A(P)$ for some $P \in \mathcal{P}(A)$ with $\operatorname{rrk}_A P > 0$ and $\deg B = \operatorname{rrk}_A P$ (Proposition 1.11(iii)), we have $\operatorname{ind} A = \gcd\{\operatorname{rrk}_A P \,|\, P \in \mathcal{P}(A), \operatorname{rrk}_A P > 0\}$. Thus, in order to establish the existence of $P \in \mathcal{P}(A)$ with $\operatorname{rrk}_A P = \operatorname{ind} A$, it is enough to show that for any $P, Q \in \mathcal{P}(A)$ with $\operatorname{rrk}_A Q \leq \operatorname{rrk}_A P$, there exists $S \in \mathcal{P}(A)$ with $\operatorname{rrk}_A S = \operatorname{rrk}_A P - \operatorname{rrk}_A Q$. This follows readily from Lemma 1.24.

Let $P \in \mathcal{P}(A)$ be a module with $\operatorname{rrk}_A P = \operatorname{ind} A$. By Lemma 1.24, P is isomorphic to a summand of A_A, because $\operatorname{rrk}_A P \leq \deg A = \operatorname{rrk}_A A_A$. Therefore, there exists an idempotent $e \in A$ such that $P \cong eA$. The theorem now follows from Corollary 1.12. □

We now turn to consider Azumaya R-algebras with involution.

Lemma 1.26. *Let (A,σ) be an Azumaya algebra with involution over a semilocal ring R and let $\varepsilon \in \mathrm{Z}(A)$ be an element with $\varepsilon^\sigma\varepsilon = 1$. If for every $\mathfrak{m} \in \operatorname{Max} R$, the type of (σ,ε) at $\mathfrak{m}$ is not symplectic, or $\deg A(\mathfrak{m})$ is even, then $\mathcal{S}_\varepsilon(A,\sigma) \cap A^\times \neq \emptyset$.*

Proof. Suppose first that R is a field and let $S = \mathrm{Z}(A)$. Then either S is a field, or $S = R \times R$. If S is a field, then the map $s \mapsto \varepsilon s^\sigma : S \to S$ is an involution and its nonzero fixed points are contained in $\mathcal{S}_\varepsilon(A,\sigma) \cap A^\times$. If there are no such points, then $s = -\varepsilon s^\sigma$ for all $s \in S$, which implies $\varepsilon = -1$ (take $s = 1$) and $\sigma|_S = \mathrm{id}_S$. In this case, $\mathcal{S}_\varepsilon(A,\sigma) \cap A^\times \neq \emptyset$ by [40, Corollary 2.8]. If $S = R \times R$, then $\sigma|_S$ is the exchange involution $(x,y) \mapsto (y,x)$ and $\varepsilon = (\alpha,\alpha^{-1})$ for some $\alpha \in R^\times$, so $(\alpha,1) \in \mathcal{S}_\varepsilon(A,\sigma) \cap A^\times$.

For general R, let $\mathfrak{m}_1,\dots,\mathfrak{m}_t$ denote the maximal ideals of R. By the previous paragraph, for each $i \in \{1,\dots,t\}$, there exists $a_i \in \mathcal{S}_\varepsilon(A(\mathfrak{m}_i),\sigma(\mathfrak{m}_i)) \cap A(\mathfrak{m}_i)^\times$. By the Chinese Remainder Theorem, there exists $a \in A$ with $a(\mathfrak{m}_i) = a_i$, for all i. Replacing a with $\frac{1}{2}(a + \varepsilon a^\sigma)$, we may assume that $a \in \mathcal{S}_\varepsilon(A,\sigma)$. By Lemma 1.6, $a \in A^\times$, so we are done. □

We finish with showing that idempotent e of Theorem 1.25 can sometimes be chosen to be invariant under a given involution of A.

Proposition 1.27. *Let (A,σ) be a central simple algebra with involution over a field F and let n be a natural number divisible by $\operatorname{ind} A$ and not exceeding $\deg A$. If σ is symplectic, we further require that n is even. Then there exists an idempotent $e \in A$ such that $e^\sigma = e$ and $\deg eAe = \operatorname{rrk}_A eA = n$*

Proof. If A contains no σ-invariant idempotents other than 0 and 1, then [22, Theorem 8.2] (for instance) implies that $e = 1$ is the required idempotent. Suppose now that $u \in A$ is a nontrivial σ-invariant idempotent and let $v = 1-u$. Since AuA is a nonzero two-sided ideal of A invariant under σ, and since (A,σ) is a simple ring with involution, $AuA = A$, and likewise $AvA = A$. Now, by Corollary 1.12, $\deg uAu + \deg vAv = \operatorname{rrk}_A(uA \oplus vA) = \operatorname{rrk}_A A = \deg A$, $\operatorname{ind} A = \operatorname{ind} uAu = \operatorname{ind} vAv$, by Corollary 1.22(ii), $\sigma|_{uAu}$ and $\sigma|_{vAv}$ have the same type as σ. Express n as $n_1 + n_2$ with $n_1 \le \deg uAu$, $n_2 \le \deg vAv$ and such that n_1, n_2 are divisible by $\operatorname{ind} A$, or $\operatorname{lcm}\{2, \operatorname{ind} A\}$ if σ is symplectic. Applying induction to $(uAu, \sigma|_{uAu})$ and $(vAv, \sigma|_{vAv})$, we get σ-invariant idempotents $e_1 \in uAu$, $e_2 \in vAv$ with $\deg e_iAe_i = n_i$ $(i = 1,2)$. Take $e = e_1 + e_2$. □

Lemma 1.28. *Let (A,σ) be an R-algebra with involution, let $\overline{A} = A/\operatorname{Jac} A$ and let $\overline{\sigma} : \overline{A} \to \overline{A}$ denote the induced involution. Let $\eta \in \overline{A}$ be a σ-invariant idempotent. If η is the image of an idempotent in A, then η is the image of a σ-invariant idempotent in A.*

Proof. Denote the image of $a \in A$ in $\overline{A}$ as $\overline{a}$. Let $e \in A$ be an idempotent with $\overline{e} = \eta$. Since $\eta = \eta^\sigma$, we have $eA + (1-e)^\sigma A + \operatorname{Jac} A = A$, so $eA + (1-e)^\sigma A = A$ by Nakayama's Lemma. On the other hand, if $a \in eA \cap (1-e)^\sigma A$, then $(1-e)a = e^\sigma a = 0$, hence $(1 - e + e^\sigma)a = 0$. Since $\overline{1 - e + e^\sigma} = \overline{1}$, we have $1 - e + e^\sigma \in A^\times$, so $a = 0$. Thus, $A = eA \oplus (1-e)^\sigma A$. Write $1 = e_1 + f_1$ with $e_1 \in eA$, $f_1 \in (1-e)^\sigma A$. It is well-known that e_1 and f_1 are idempotents satisfying $e_1A = eA$ and $f_1 = (1-e)^\sigma A$. Now, $e_1 - e_1^\sigma e_1 = (1-e_1)^\sigma e_1 = f_1^\sigma e_1 = ((1-e)^\sigma f_1)^\sigma ee_1 = f_1^\sigma(1-e)ee_1 = 0$, so $e_1 = e_1^\sigma e_1$. It follows that $e_1^\sigma = (e_1^\sigma e_1)^\sigma = e_1^\sigma e_1 = e_1^\sigma$. Finally, since $\overline{e_1} \in \eta\overline{A}$ and $\overline{1 - e_1} \in (1-\eta)\overline{A}$, we must have $\overline{e_1} = \eta$, because $\overline{A} = \eta\overline{A} \oplus (1-\eta)\overline{A}$ and $\overline{1} = \eta + (1-\eta)$. □

Lemma 1.29. *Let A be a semilocal R-algebra, let $\overline{A} := A/\operatorname{Jac} A$ and let $\eta \in \overline{A}$ be an idempotent. Then there exists an idempotent $e \in A$ with $\overline{e} := e + \operatorname{Jac} A = \eta$ if and only if there exists $P \in \mathcal{P}(A)$ such that $\overline{P} := P/P\operatorname{Jac} A \cong \eta\overline{A}$ as right A-modules.*

Proof. For the "only if" part, take $P = eA$. We turn to prove the "if" part.

Note that $P \to \overline{P} \cong \eta\overline{A}$ is a projective covering; denote this map by f. Consider the surjective homomorphism $g : A_A \to \eta\overline{A}$ given by $g(a) = \eta\overline{a}$. Since $f : P \to \eta\overline{A}$ is a projective covering, there exists a factorization $A_A = P_1 \oplus Q$ and an isomorphism $P \to P_1$ such that the composition $P \to P_1 \xrightarrow{g} \eta\overline{A}$ is f. In particular, $\overline{P_1} = \eta\overline{A}$. Choose an idempotent $e_1 \in A$ such that $P_1 = e_1A$. Then $\overline{e_1}\overline{A} = \eta\overline{A}$ and $\overline{(1-e_1)}\overline{A} \cong \overline{A}/\overline{e_1}\overline{A} = \overline{A}/\eta\overline{A} \cong (1-\eta)\overline{A}$. Now, by [41, Exercise 21.16], there exists $x \in \overline{A}^{\times}$ with $x\overline{e'}x^{-1} = \eta$. Choose $y \in A$ with $\overline{y} = x$ and take $e = ye_1y^{-1}$. □

Theorem 1.30. *Let (A, σ) be an Azumaya algebra with involution over a semilocal ring R. Write $S := \mathrm{Z}(A)$ and let $n \in \Gamma(\operatorname{Spec} S, \mathbb{N})$. Suppose that n is invariant under $\sigma|_S$ and satisfies $\operatorname{ind} A \mid n$ and $n \le \deg A$. If σ is symplectic at $\mathfrak{p} \in \operatorname{Spec} R$, we also require that $n(\mathfrak{p})$ is even. Then there exists an idempotent $e \in A$ such that $e^\sigma = e$ and $\deg eAe = \operatorname{rrk}_A eA = n$.*

We remark that $\operatorname{ind} A$ is a $\sigma|_S$-invariant function from $\operatorname{Spec} S$ to $\mathbb{N}$.

Proof. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_t$ denote the maximal ideals of R. We use Lemma 1.23 to identify $\overline{A} := A/\operatorname{Jac} A$ with $\prod_{i=1}^t A(\mathfrak{m}_i)$. Since $\operatorname{ind} A(\mathfrak{m}_i) \mid (\operatorname{ind} A)(\mathfrak{m}_i)$, we may apply Proposition 1.27 to $(A(\mathfrak{m}_i), \sigma(\mathfrak{m}_i))$ and $n(\mathfrak{m}_i)$ and get a σ-invariant idempotent $\eta_i \in A(\mathfrak{m}_i)$ with $\operatorname{rrk}_{A(\mathfrak{m}_i)} \eta_i A(\mathfrak{m}_i) = n(\mathfrak{m}_i)$. Let $\eta = (\eta_i)_{i=1}^t \in \overline{A}$. Then $\eta^{\overline{\sigma}} = \eta$.

By Theorem 1.25, there exists $P \in \mathcal{P}(A)$ such that $\operatorname{rrk}_A P = n$. Comparing reduced ranks, one sees that $\overline{P} = P/P\operatorname{Jac} A \cong P \otimes_A \overline{A}$ is isomorphic to $\eta\overline{A}$. Thus, by Lemmas 1.28 and 1.29, there exists a σ-invariant idempotent $e \in A$ projecting onto η. Since $\operatorname{rrk}_{A(\mathfrak{m}_i)} eA(\mathfrak{m}_i) = \operatorname{rrk}_{A(\mathfrak{m}_i)} \eta_i A(\mathfrak{m}_i) = n(\mathfrak{m}_i)$ for all $1 \le i \le t$, and since $\operatorname{rrk}_A eA$ is locally constant, we must have $\operatorname{rrk}_A eA = n$. □

2. Hermitian Forms

This section concerns with hermitian forms, mainly over Azumaya algebras with involution, and related objects. See [39, Chapter I] for an extensive discussion of hermitian forms in general.

Throughout this section, (A, σ) denotes an R-algebra with involution and ε is an element of $\mathrm{Z}(A)$ satisfying $\varepsilon^\sigma\varepsilon = 1$. Recall our standing assumption that $2 \in R^\times$.

2A. **Hermitian Forms.** We define ε-hermitian spaces over (A, σ) in the usual way, i.e., as pairs (P, f) where $P \in \mathcal{P}(A)$ and $f : P \times P \to A$ is a biadditive map satisfying $f(xa, yb) = a^\sigma f(x, y)b$ and $f(x, y) = \varepsilon f(y, x)^\sigma$ $(x, y \in P,\ a, b \in A)$. We also say that f is an ε-hermitian form on P.

Given ε-hermitian spaces (P, f), (P', f') over (A, σ), an isometry $(P, f) \to (P', f')$ is an A-module isomorphism $\varphi : P \to P'$ such that $f'(\varphi x, \varphi y) = f(x, y)$ $(x, y \in P)$. If such an isometry exists, we write $(P, f) \cong (P', f')$ or $f \cong f'$. The group of isometries from (P, f) into itself is denoted $U(f)$. Orthogonal sums of hermitian spaces or hermitian forms are defined in the usual way and are written using the symbol $\oplus$. The n-fold orthogonal sum $(P, f) \oplus \cdots \oplus (P, f)$ is denoted $n \cdot (P, f)$.

Example 2.1. Let $a_1, \dots, a_n \in \mathcal{S}_\varepsilon(A, \sigma)$. Then the map $f : A^n \times A^n \to A$ given by $f((x_i), (y_i)) = \sum_i x_i^\sigma a_i y_i$ is an ε-hermitian form over (A, σ). We call f a diagonal form and denote it by $\langle a_1, \dots, a_n \rangle_{(A,\sigma)}$. A hermitian form which is isomorphic to a diagonal form is called diagonalizable.

Given $P \in \mathcal{P}(A)$, let P^* denote $\mathrm{Hom}_A(P, A)$ endowed with the *right* A-module structure given by $(\phi a)x = a^\sigma(\phi x)$ ($\phi \in P^*$, $a \in A$, $x \in P$). If f is an ε-hermitian form on P, then the map $x \mapsto f(x, -) : P \to P^*$ is an A-module homomorphism. When it is an isomorphism, we say that (P, f), or f, is *unimodular*[4]. The category of unimodular ε-hermitian spaces over (A, σ) with isometries as its morphisms is denoted

$$\mathcal{H}^\varepsilon(A, \sigma).$$

We shall need the following versions of Witt's Cancellation Theorem and Witt's Extension Theorem. The cancellation is derived from cancellation results of Reiter [59, Theorem 6.2] and Keller [37, Theorem 3.4.2].

Theorem 2.2. *Suppose that (A, σ) is an Azumaya R-algebra with involution and R is semilocal, and let $(P_1, f_1), (P_2, f_2), (Q, g) \in \mathcal{H}^\varepsilon(A, \sigma)$. If $f_1 \oplus g \cong f_2 \oplus g$, then $f_1 \cong f_2$.*

Proof. Write $R = \prod_i R_i$ with each R_i a connected semilocal ring. Working over each factor separately, we may assume that R is connected. Under this assumption, we may further assume that $\mathrm{rk}_R P_1 > 0$, because otherwise $\mathrm{rk}_R P_1 = \mathrm{rk}_R P_2 = 0$, which implies $P_1 = P_2 = 0$ and $f_1 \cong f_2$. At this point, we claim that we may apply Keller's cancellation result [37, Theorem 3.4.2(iii)] and conclude that $f_1 \cong f_2$. Indeed, in order to apply Keller's theorem, we need to check that the number r defined in *op. cit.* is 0 for the hermitian space (P_1, f_1). By Lemma 1.23, this is equivalent to having $P_1(\mathfrak{m}) \neq 0$ for all $\mathfrak{m} \in \mathrm{Max}\, R$, and this holds by our assumption that $\mathrm{rk}_R P_1 > 0$. Alternatively, one can use Reiter's version of Witt's Extension Theorem [59, Theorem 6.2], which applies under similar conditions, to conclude the proof. □

Theorem 2.3. *Suppose that R is a henselian local ring and (A, σ) is a finite R-algebra with involution. Let $(P, f) \in \mathcal{H}^\varepsilon(A, \tau)$ and let U, V be summands of P. Then any isometry $f|_{U \times U} \to f|_{V \times V}$ extends to an isometry of f.*

Proof. By a theorem of Azumaya [4, Theorem 24], A is a *semiperfect* ring. The theorem therefore follows from [24, Corollary 4.9]. □

2B. **The Witt Group.** As usual, a *Lagrangian* of a unimodular hermitian space $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ is a summand L of P such that $L = L^\perp := \{x \in P : f(x, L) = 0\}$. If (P, f) admits a Lagrangian, it is called *metabolic*. A convenient way to verify that an A-submodule $L \leq P$ with $f(L, L) = 0$ is a Lagrangian is to exhibit another submodule $M \leq P$ such that $f(M, M) = 0$ and $L \oplus M = P$. If such L and M exist, (P, f) is called *hyperbolic*. In this case, the map $x \mapsto f(x, -) : M \to L^*$ is an isomorphism of A-modules, and the induced map $P = L \oplus M \to L \oplus L^*$ is an isometry from (P, f) to $(L \oplus L^*, \mathbb{h}^\varepsilon_L)$, where $\mathbb{h}^\varepsilon_L$ is the ε-hermitian form given by

$$\mathbb{h}^\varepsilon_L(x \oplus \phi, x' \oplus \phi') = \phi x' + \varepsilon(\phi' x)^\sigma$$

($x, x' \in P$, $\phi, \phi' \in P^*$). Since we assume that $2 \in A^\times$, any Lagrangian L admits a Lagrangian M with $L \oplus M = P$ [39, Proposition I.3.7.1], so metabolic spaces are hyperbolic. Therefore, we shall only consider hyperbolic spaces in the sequel.

Recall that the Witt group of ε-hermitian forms over (A, σ), denoted

$$W_\varepsilon(A, \sigma),$$

is the Grothendieck group of $\mathcal{H}^\varepsilon(A, \sigma)$, relative to orthogonal sum, divided by the subgroup spanned by the (representatives of) hyperbolic spaces. The class represented by (P, f) in $W_\varepsilon(A, \sigma)$ is denoted $[P, f]$ or $[f]$. Two forms f, f' representing the same element in $W_\varepsilon(A, \sigma)$ will be called Witt-equivalent; this happens if and

[4]Some texts use "regular" or "nondegenerate".

only if there exist hyperbolic forms h, h' such that $f \oplus h \cong f' \oplus h'$. Note that $-[f] = [-f]$ because $f \oplus (-f)$ is hyperbolic.

Example 2.4. We say that $\sigma : A \to A$ is an *exchange involution* if there exists an idempotent $\eta \in \mathrm{Z}(A)$ such that $\eta^\sigma = 1 - \eta$. For example, this is the case if (A, σ) is an Azumaya R-algebra with involution and $\mathrm{Z}(A) = R \times R$, because $\sigma|_{\mathrm{Z}(A)}$ is the involution $(r, s) \mapsto (s, r)$ (see Proposition 1.21). In this situation, there exists an R-algebra B such that $(A, \sigma) \cong (B \times B^{\mathrm{op}}, (x, y^{\mathrm{op}}) \mapsto (y, x^{\mathrm{op}}))$, hence the name "exchange involution". Indeed, take $B = \eta A$; the required isomorphism $A \to B \times B^{\mathrm{op}}$ is given by $a \mapsto (\eta a, (\eta a^\sigma)^{\mathrm{op}})$.

It easy to see that for any $P \in \mathcal{P}(A)$, we have $P = P\eta \oplus P\eta^\sigma$. Furthermore, if f is a unimodular ε-hermitian form on P, then $f(P\eta, P\eta) = f(P\eta^\sigma, P\eta^\sigma) = 0$ (because $\eta^\sigma\eta = 0$), so f is hyperbolic and $f \cong \mathbb{h}^\varepsilon_{P\eta}$. From this we see that every $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ is hyperbolic and is determined up to isomorphism by the isomorphism class of P. In particular, $W_\varepsilon(A, \sigma) = 0$.

Recall that an ε-hermitian space $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ is called *isotropic* if P admits a nonzero summand M such that $f(M, M) = 0$. When no such M exists, (P, f) is called *anisotropic*. We alert the reader that at this level of generality, the existence of $0 \neq x \in P$ such that $f(x, x) = 0$ does not imply that (P, f) is isotropic. However, when A is semisimple artinian, xA is a summand of P, so f is isotropic if and only if $f(x, x) = 0$ for some nonzero $x \in P$.

Proposition 2.5 ([39, Proposition I.3.7.9])**.** *Every $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ can be written as an orthogonal sum of an anisotropic space and a hyperbolic space. In particular, (P, f) is Witt equivalent to an anisotropic ε-hermitian space.*

We proceed with showing that if (A, σ) is an Azumaya R-algebra with involution and R is semilocal, then every hermitian form representing 0 in $W_\varepsilon(A, \sigma)$ is hyperbolic. Furthermore, two hermitian spaces in the same Witt class having the same reduced rank are isomorphic. These statements may already fail for $(A, \sigma) = (R, \mathrm{id}_R)$ if R is not semilocal; see [7, Example 1.2.6], for instance.

Lemma 2.6. *Suppose that (A, σ) is an Azumaya R-algebra with involution and let $P \in \mathcal{P}(A)$. Then $\mathrm{rk}_R P = \mathrm{rk}_R P^*$ and $\mathrm{rrk}_A P^* = \sigma \,\mathrm{rrk}_A P$, i.e., $(\mathrm{rrk}_A P^*)(\mathfrak{p}) = (\mathrm{rrk}_A P)(\mathfrak{p}^\sigma)$ for all $\mathfrak{p} \in \mathrm{Spec}\,\mathrm{Z}(A)$. In particular, if there exists a unimodular ε-hemritian form on P, then $\mathrm{rrk}_A P$ is σ-invariant.*

Proof. Write $S = \mathrm{Z}(A)$. It is enough to prove lemma after specializing to the residue fields of R, so assume R is a field.

If S is connected, then $\sigma \,\mathrm{rrk}_A P = \mathrm{rrk}_A P$ and A is a simple artinian ring. Length considerations force $P \cong P^*$, hence $\mathrm{rk}_R P = \mathrm{rk}_R P^*$ and $\mathrm{rrk}_A P^* = \mathrm{rrk}_A P = \sigma \,\mathrm{rrk}_A P$.

If S is not connected, then $S = R \times R$ and $\sigma|_R$ is the exchange involution. Thus, as in Example 2.4, there exists a central simple R-algebra B such that $(A, \sigma) \cong (B \times B^{\mathrm{op}}, \tau)$ where $(x, y^{\mathrm{op}})^\tau = (y, x^{\mathrm{op}})$. Identifying A with $B \times B^{\mathrm{op}}$, we can write $P = P_1 \times P_2$ where $P_1 \in \mathcal{P}(B)$ and $P_2 \in \mathcal{P}(B^{\mathrm{op}})$. Regarding P_1 and P_2 as A-modules, one readily checks that $(1_B, 0_B^{\mathrm{op}})$ annihilates P_1^* and $(0_B, 1_B^{\mathrm{op}})$ annihilates P_2^*, so $P_1^* \in \mathcal{P}(B^{\mathrm{op}})$ and $P_2^* \in \mathcal{P}(B)$. Since $* : \mathcal{P}(A) \to \mathcal{P}(A)$ is a duality and $\mathcal{P}(A)$ is abelian semisimple, P_1 and P_1^* have the same A-length, so $\mathrm{length}_B P_1 = \mathrm{length}_{B^{\mathrm{op}}} P_1^*$. Likewise, $\mathrm{length}_{B^{\mathrm{op}}} P_2 = \mathrm{length}_B P_2^*$. Since B and B^{op} are central simple R-algebras of equal degree, this means that $\mathrm{rk}_R P = \mathrm{rk}_R P^*$ and $\mathrm{rrk}_A P^* = \mathrm{rrk}_A(P_2^* \times P_1^*) = \sigma \,\mathrm{rrk}_A P$.

Finally, if P carries a unimodular hermitian form, then $P \cong P^*$, so $\mathrm{rrk}_A P = \mathrm{rrk}_A P^* = \sigma \,\mathrm{rrk}_A P$. □

Lemma 2.7. *Suppose that (A,σ) is an Azumaya algebra with involution over a semilocal ring R, and let $(P_1,f_1),(P_2,f_2)\in\mathcal{H}^\varepsilon(A,\sigma)$ be hyperbolic. If $\mathrm{rrk}_A P_1\leq \mathrm{rrk}_A P_2$, then there is $V\in\mathcal{P}(A)$ such that $f_1\oplus\mathbb{h}^\varepsilon_V\cong f_2$. In particular, if $\mathrm{rrk}_A P_1=\mathrm{rrk}_A P_2$, then $f_1\cong f_2$.*

Proof. Write $S=\mathrm{Z}(A)$. As in the proof of Theorem 2.2, we may assume that R is connected.

Suppose first that S is not connected. By Lemma 1.16, $S=R\times R$. Put $\eta=(1_R,0_R)\in S$. Then by Example 2.4, $f_1\cong\mathbb{h}^\varepsilon_{P_1\eta}$ and $f_2\cong\mathbb{h}^\varepsilon_{P_2\eta}$. The assumption $\mathrm{rrk}_A P_1\leq\mathrm{rrk}_A P_2$ means that $\mathrm{rrk}_A P_1\eta\leq\mathrm{rrk}_A P_2\eta$, so by Lemma 1.24, there is $V\in\mathcal{P}(A)$ such that $P_1\eta\oplus V\cong P_2\eta$. Then $f_1\oplus\mathbb{h}^\varepsilon_V\cong\mathbb{h}^\varepsilon_{P_1\eta}\oplus\mathbb{h}^\varepsilon_V\cong\mathbb{h}^\varepsilon_{P_2\eta}\cong f_2$.

Now assume that S is connected and write $f_1\cong\mathbb{h}^\varepsilon_{U_1}$ and $f_2\cong\mathbb{h}^\varepsilon_{U_2}$ with $U_1,U_2\in\mathcal{P}(A)$. Lemma 2.6 and the connectivity of S imply that $2\,\mathrm{rrk}_A U_1=\mathrm{rrk}_A P_1\leq\mathrm{rrk}_A P_2=2\,\mathrm{rrk}_A U_2$. By Lemma 1.24, there is $V\in\mathcal{P}(A)$ such that $U_1\oplus V\cong U_2$. Then $f_1\oplus\mathbb{h}^\varepsilon_V\cong f_2$. □

Theorem 2.8. *Suppose that (A,σ) is an Azumaya algebra with involution over a semilocal ring R, and let $(P,f),(P',f')\in\mathcal{H}^\varepsilon(A,\sigma)$.*

(i) If $\mathrm{rrk}_A P\leq\mathrm{rrk}_A P'$, then there exists $V\in\mathcal{P}(A)$ such that $f\oplus\mathbb{h}^\varepsilon_V\cong f'$.
(ii) If $[f]=0$, then f is hyperbolic.
(iii) If $[f]=[f']$ and $\mathrm{rrk}_A P=\mathrm{rrk}_A P'$, then $f\cong f'$.

Proof. (i) There are $U,W\in\mathcal{P}(A)$ such that $f\oplus\mathbb{h}^\varepsilon_U\cong f'\oplus\mathbb{h}^\varepsilon_W$. Then $\mathrm{rrk}_A(U\oplus U^*)-\mathrm{rrk}_A(W\oplus W^*)=\mathrm{rrk}_A P'-\mathrm{rrk}_A P\geq 0$, so by Lemma 2.7, there is $V\in\mathcal{P}(A)$ such that $\mathbb{h}^\varepsilon_U\cong\mathbb{h}^\varepsilon_W\oplus\mathbb{h}^\varepsilon_V$. (Caution: $U\cong W\oplus V$ is *a priori* not guaranteed.) Then $(f\oplus\mathbb{h}^\varepsilon_V)\oplus\mathbb{h}^\varepsilon_W\cong f\oplus\mathbb{h}^\varepsilon_U\cong f'\oplus\mathbb{h}^\varepsilon_W$. By Theorem 2.2, this means that $f\oplus\mathbb{h}^\varepsilon_V\cong f'$.

(ii) Apply (i) with (P,f) being the zero hermitian space.

(iii) By (i), $f\oplus\mathbb{h}^\varepsilon_V\cong f'$ for some $V\in\mathcal{P}(A)$, and $V=0$ because $\mathrm{rrk}_A V=\mathrm{rrk}_A P'-\mathrm{rrk}_A P=0$. □

We also record the following useful corollary to Lemma 2.6.

Corollary 2.9. *Suppose that (A,σ) is an Azumaya R-algebra with involution and let $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma)$.*

(i) If R is connected, then $\mathrm{rrk}_A P$ is constant.
(ii) If $S:=\mathrm{Z}(A)$ is connected and f is hyperbolic, then there exists $V\in\mathcal{P}(A)$ with $\mathrm{rrk}_A P=2\,\mathrm{rrk}_A V$.

Proof. (i) If $S=R$, then this is clear. If $S\neq R$, then S is a quadratic étale R-algebra and $\sigma|_S$ is the standard R-involution of S (see 1C). Thus, σ acts transitively on every fiber of $\operatorname{Spec}S\to\operatorname{Spec}R$. By Lemma 2.6, this means that $\mathrm{rrk}_A P$ is constant on the fibers of $\operatorname{Spec}S\to\operatorname{Spec}R$. Thus, by (0.3), we have $\iota\,\mathrm{rk}_R P=\iota\,\mathrm{rk}_R S\cdot\mathrm{rk}_S P=2\,\mathrm{rk}_S P$, where $\iota:R\to S$ is the inclusion. Since the left hand side is constant (R is connected), $\mathrm{rrk}_A P$ is also constant.

(ii) There exists $V\in\mathcal{P}(A)$ such that $P=V\oplus V^*$. By Lemma 2.6, $\mathrm{rrk}_A P=\mathrm{rrk}_A V+\sigma\,\mathrm{rrk}_A V$, and $\sigma\,\mathrm{rrk}_A V=\mathrm{rrk}_A V$ because S is connected. □

2C. **Base Change.** Let $R\to S$ be a ring homomorphism. Given an ε-hermitian space (P,f) over (A,σ), define its *base change* along $R\to S$ to be the ε-hermitian space (P_S,f_S) over (A_S,σ_S), where $P_S=P\otimes S$ and f_S is determined by $f_S(x\otimes s,y\otimes t)=f(x,y)\otimes st$ ($x,y\in P$, $s,t\in S$). It is well-known that if (P,f) is unimodular, resp. hyperbolic, then so is (P_S,f_S). When $S=k(\mathfrak{p})$ for $\mathfrak{p}\in\operatorname{Spec}R$, we shall write $f(\mathfrak{p})$ instead of $f_{k(\mathfrak{p})}$.

Let $\rho:(B,\tau)\to(A,\sigma)$ be a homomorphism of R-algebras with involution and let $\delta\in\mathrm{Z}(B)$ be an element such that $\delta^\tau\delta=1$ and $\varepsilon:=\rho(\delta)\in\mathrm{Z}(A)$. We view

A as a left B-module via ρ. For every δ-hermitian space (Q,g) over (B,τ), define $\rho(Q,g)$ to be $(Q\otimes_B A,\rho g)$, where $\rho g:(Q\otimes_B A)\times(Q\otimes_B A)\to A$ is the biadditive pairing determined by $\rho g(x\otimes a,x'\otimes a')=a^\sigma\cdot\rho(f(x,x'))\cdot a'$ $(x,x'\in P,\ a,a'\in A)$. It is routine to check that $\rho(Q,g)$ is an ε-hermitian space over (A,σ). Furthermore, it is unimodular, resp. hyperbolic, when (Q,g) is. The assignment ρ extends to a functor $\rho:\mathcal{H}^\delta(B,\tau)\to\mathcal{H}^\varepsilon(A,\sigma)$ by setting $\rho\varphi=\varphi\otimes_B \mathrm{id}_A$.[5]

2D. **Adjoint Involutions.** Let $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma)$. It is well-known that there exists a unique R-linear involution $\theta:\mathrm{End}_A(P)\to\mathrm{End}_A(P)$ satisfying $f(\varphi x,y)=f(x,\varphi^\theta y)$ for all $\varphi\in\mathrm{End}_A(P)$, see [39, I.§9.2]. It is called the adjoint involution of f. Notice that $U(f)$ coincides with the group $U(\mathrm{End}_A(P),\theta):=\{\varphi\in\mathrm{End}_A(P):\varphi^\theta\varphi=\varphi\varphi^\theta=1\}$.

If $R\to S$ is a ring homomorphism, then $P\in\mathcal{P}(A)$ implies that the natural map $\mathrm{End}_A(P)\otimes S\to\mathrm{End}_{A_S}(P_S)$ is an isomorphism. Under this isomorphism, θ_S is the adjoint involution of f_S.

Example 2.10. Let $\alpha,\beta\in\mathcal{S}_\varepsilon(A,\sigma)\cap A^\times$ and consider the diagonal binary ε-hermitian form $\langle\alpha,\beta\rangle_{(A,\sigma)}$ on A^2 (notation as in Example 2.1). Direct computation shows that, upon realizing $\mathrm{End}_A(A^2_A)$ as $\mathrm{M}_2(A)$, the adjoint involution of $\langle\alpha,\beta\rangle_{(A,\sigma)}$ is given by $\left[\begin{smallmatrix}x&y\\z&w\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}\alpha^{-1}x^\sigma\alpha&\alpha^{-1}z^\sigma\beta\\\beta^{-1}y^\sigma\alpha&\beta^{-1}w^\sigma\beta\end{smallmatrix}\right]$ $(x,y,z,w\in A)$. When $\alpha,\beta\in\mathrm{Z}(A)$, this simplifies into $\left[\begin{smallmatrix}x&y\\z&w\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}x^\sigma&\gamma z^\sigma\\\gamma^{-1}y^\sigma&w^\sigma\end{smallmatrix}\right]$, where $\gamma=\alpha^{-1}\beta$ lives in $\mathcal{S}_1(\mathrm{Z}(A),\sigma)$.

Proposition 2.11. *Suppose that (A,σ) is an Azumaya R-algebra with involution. Let $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma)$ and let $\theta:\mathrm{End}_A(P)\to\mathrm{End}_A(P)$ be the adjoint involution of f. If $\mathrm{rrk}_A P>0$, then $(\mathrm{End}_A(P),\theta)$ is an Azumaya R-algebra with involution and θ and (σ,ε) have the same type.*

Proof. Write $S=\mathrm{Z}(A)$. By Proposition 1.11(i), $\mathrm{End}_A(P)$ is Azumaya over S. It is easy to check that $\theta|_S=\sigma|_S$, hence $(\mathrm{End}_A(P),\theta)$ is an Azumaya R-algebra with involution. To see that the types of θ and (σ,ε) coincide, we need to check that they coincide at every $\mathfrak{p}\in\mathrm{Spec}\,R$, so we may assume R is a field. In this case, it is clear that θ is unitary if and only if (σ,ε) is unitary. For the orthogonal and symplectic cases, see [40, Theorem 4.2(1)]. □

The converse of Proposition 2.11, namely, that every involution of $\mathrm{End}_A(P)$ is adjoint to some hermitian form, holds when R is a field; see [40, Theorem 4.2]. In fact, it holds in general if one allows hermitian forms to take values in (A^{op},A)-progenerators; see [22] and [23, §3–4]. We shall need a special case of the latter observation.

Proposition 2.12. *Suppose that (A,σ) is an Azumaya R-algebra with involution, $S:=\mathrm{Z}(A)$ is semilocal and $A=\mathrm{End}_S(Q)$ for some $Q\in\mathcal{P}(S)$. Then there exists $\delta\in S$ with $\delta^\sigma\delta=1$ and a unimodular δ-hermitian form $g:Q\times Q\to S$ over $(S,\sigma|_S)$ such that σ is adjoint to g. One has $\delta=1$ when σ is orthogonal and $\delta=-1$ when σ is symplectic.*

Proof. By [62, Theorem 4.2] (or, alternatively, [23, Proposition 4.6]), there exist $\delta_1\in S$ with $\delta_1^\sigma\delta_1=1$, a rank-1 projective S-module L, a $\sigma|_S$-linear involutive automorphism $\tau:L\to L$, and a unimodular L-valued $\sigma|_S$-sesquilinear form $g:Q\times Q\to L$ satisfying $g(x,y)=\delta_1 g(y,x)^\tau$ and having σ as its adjoint involution. (Here, unimodularity means that $x\mapsto g(x,-):P\to\mathrm{Hom}_S(P,L)$ is bijective.) Since S is semilocal, $L\cong S_S$, so we may assume $L=S$. Put $\delta_2=(1_S)^\tau$. Then $\sigma\circ\tau\in\mathrm{End}_S(S)=S$ maps 1_S to δ_2^σ, and so it coincides with the S-endomorphism

[5]We do not write $\rho(Q,g)$ as (P_A,f_A) because we reserve the subscript notation for base change relative to the base ring R.

$s \mapsto \delta_2^\sigma s : S \to S$. As a result, $s^{\tau\sigma} = \delta_2^\sigma s$, or rather $s^\tau = \delta_2 s^\sigma$, for all $s \in S$. Taking $s = \delta_2$ and noting that $\delta_2^\tau = 1$ (because $1^\tau = \delta_2$), we get $\delta_2\delta_2^\sigma = 1$. Thus, $g : P \times P \to S$ is a unimodular $\delta_1\delta_2$-hermitian form over $(S, \sigma|_S)$ with adjoint involution σ. Write $\delta = \delta_1\delta_2$.

By Proposition 2.11, the type of σ is the same as the type of $(\sigma|_S, \delta)$. Thus, $\delta = 1$ if σ is orthogonal and $\delta = -1$ if σ is symplectic. □

The following proposition can be proved directly, but we use Theorem 1.30 together with adjoint involutions to give a short proof.

Proposition 2.13. *Suppose that (A, σ) is an Azumaya R-algebra with involution and R is semilocal. Let $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ and suppose that P is a free A-module. If (σ, ε) is symplectic at $\mathfrak{p} \in \operatorname{Spec} R$, we also assume that $2 \mid (\deg A)(\mathfrak{p})$. Then (P, f) is diagonalizable (see Example 2.1).*

Proof. If $P = 0$, there is nothing to prove, so assume $P \neq 0$. Since P is free, we have $\operatorname{rrk}_A P > 0$. Let $B = \operatorname{End}_A(P)$ and let $\theta : B \to B$ be the adjoint involution of f. By Proposition 1.11(i), $\operatorname{ind} B = \operatorname{ind} A \mid \deg A$ and $\deg A \leq \operatorname{rrk}_A P = \deg B$. Furthermore, if (σ, ε), equivalently θ, is symplectic at $\mathfrak{p} \in \operatorname{Spec} R$, then $2 \mid (\deg A)(\mathfrak{p})$. Thus, by Theorem 1.30, there exists an idempotent $e \in B$ such that $e^\sigma = e$ and $\operatorname{rrk}_B eB = \deg A$. Write $e' = 1_B - e$. It is easy to check that $(P, f) = (eP, f|_{eP\times eP}) \oplus (e'P, f|_{e'P\times e'P})$. Furthermore, by Proposition 1.11(ii), $\operatorname{rrk}_A eP = \operatorname{rrk}_A eB \otimes_B P = \operatorname{rrk}_B eB = \deg A$, so $eP \cong A_A$ (Lemma 1.24). Thus, $f|_{eP\times eP} \cong \langle a\rangle$ for some $a \in \mathcal{S}_\varepsilon(A, \sigma)$. Proceed by induction on $(e'P, f|_{e'P\times e'P})$. □

2E. **The Isometry Group Scheme.** Suppose that A is finite projective over R and let $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$. By [11, Appendix], the functor $S \mapsto U(f_S)$ from R-rings to groups is represented by a smooth affine group R-scheme, denoted $\mathbf{U}(f)$. Since $\mathbf{U}(f) \to \operatorname{Spec} R$ is smooth, by [32, Corollaire 15.6.5], $\mathbf{U}(f)$ admits a unique open subgroup, $\mathbf{U}^0(f)$, such that $\mathbf{U}^0(f) \to \operatorname{Spec} R$ is connected, i.e., the fiber $\mathbf{U}^0(f) \times_R k(\mathfrak{p})$ over $\operatorname{Spec} k(\mathfrak{p})$ is connected for all $\mathfrak{p} \in \operatorname{Spec} R$. Moreover, $\mathbf{U}^0(f) \to \operatorname{Spec} R$ is geometrically connected [68, Tag 04KV]. We call $\mathbf{U}^0(f)$ the neutral component of $\mathbf{U}(f)$ and write $U^0(f) = \mathbf{U}^0(f)(R)$.

Remark 2.14. In case the base ring R is not clear from the context, we shall write $\mathbf{U}_R(f)$, $\mathbf{U}^0_R(f)$, $U^0_R(f)$ instead of $\mathbf{U}(f)$, $\mathbf{U}^0(f)$, $U^0(f)$. However, somewhat conveniently, if A is projective over $R_1 := \mathcal{S}_1(\mathrm{Z}(A), \sigma)$, e.g., when A is separable projective over R (Example 1.20), then $U^0(f)$ is independent of the base ring R.

Indeed, by [27, Proposition 2.4.6(1)], R_1 is an R_1-summand of A, and therefore R_1 is finite projective over R. Note that $\mathbf{U}_R(f) = \mathcal{R}_{R_1/R}\mathbf{U}_{R_1}(f)$, where $\mathcal{R}_{R_1/R}$ is the Weil restriction; see [14, §7.6], for instance. By [14, Proposition 7.6.2(i)], the R_1/R-Weil restriction of an open immersion is an open immersion, so $\mathcal{R}_{R_1/R}\mathbf{U}^0_{R_1}(f)$ is open in $\mathbf{U}_R(f)$. In addition, since $\mathbf{U}^0_{R_1}(f) \to \operatorname{Spec} R_1$ is geometrically connected, affine and smooth, the fibers of $\mathcal{R}_{R_1/R}\mathbf{U}^0_{R_1}(f) \to \operatorname{Spec} R$ are connected ([19, Proposition A.5.9]). As a result, $\mathbf{U}^0_R(f)$ conincides with $\mathcal{R}_{R_1/R}\mathbf{U}^0_{R_1}(f)$ and $U^0_R(f) = (\mathcal{R}_{R_1/R}\mathbf{U}^0_{R_1})(R) = U^0_{R_1}(f)$. In particular, $U^0_R(f)$ is determined by (A, σ) and is independent of R.

Remark 2.15. Keeping the previous assumptions, write $E = \operatorname{End}_A(P)$ and let θ denote the adjoint involution of f. Then $\mathbf{U}(f)$ coincides with $\mathbf{U}(E, \theta)$, the group R-scheme representing the functor $S \mapsto U(E_S, \theta_S) := \{x \in E_S : x^\theta x = 1\}$. Indeed, for any R-ring S, we have $\mathbf{U}(f)(S) = U(f_S) = U(E_S, \theta_S)$ upon identifying $\operatorname{End}_{A_S}(P_S)$ with $\operatorname{End}_A(P)_S$. As a result, $\mathbf{U}^0(f)$ is the neutral component of $\mathbf{U}(E, \theta) \to \operatorname{Spec} R$, denoted $\mathbf{U}^0(E, \theta)$.

We describe $\mathbf{U}^0(f)$ explicitly when (A,σ) is an Azumaya R-algebra with involution and $\operatorname{rrk}_A P > 0$. Since we can factor R as $\prod_{i=1}^t R_i$ such that $\operatorname{rrk}_{A\otimes R_i} P_{R_i}$ is constant for all i, it is enough to consider the case where $\operatorname{rrk}_A P$ is constant. By Proposition 1.21(v), (σ,ε) is now either orthogonal, symplectic or unitary.

When (σ,ε) is symplectic or unitary, the fibers of $\mathbf{U}(f) \to \operatorname{Spec} R$ are well-known to be outer forms of $\mathbf{Sp}_{2n}$ or $\mathbf{GL}_n$, respectively; see [40, §23A]. Thus, they are connected and $\mathbf{U}^0(f) = \mathbf{U}(f)$.

Suppose now that (σ,ε) is orthogonal, let $E = \operatorname{End}_A(P)$ and let $\theta : E \to E$ be the adjoint involution of f. Then (E,θ) is an Azumaya R-algebra with an orthogonal involution (Proposition 2.11). Let $\boldsymbol{\mu}_{2,R} \to \operatorname{Spec} R$ denote the affine group R-scheme representing the functor $S \mapsto \mu_2(S) := \{s \in S : s^2 = 1\}$. For every R-ring S, the reduced norm map, $\operatorname{Nrd}_{E_S/S} : E_S \to S$ (see [27, p. 410]), is compatible with base change and restricts to a group homomorphism $U(f_S) = U(E_S,\theta_S) \to \mu_2(S)$.[6] Thus, it determines a morphism of affine group R-schemes

$$\operatorname{Nrd} : \mathbf{U}(f) \to \boldsymbol{\mu}_{2,R}.$$

The scheme-theoretic kernel of this morphism — call it $\mathbf{K}$ — is $\mathbf{U}^0(f)$. Indeed, $\mathbf{K}$ is open in $\mathbf{U}(f)$ because the trivial group R-scheme $\mathbf{1}$ is open in $\boldsymbol{\mu}_{2,R}$ (recall that $2 \in R^\times$), and the fiber of $\mathbf{K}$ over $\mathfrak{p} \in \operatorname{Spec} R$ is $\mathbf{U}(E(\mathfrak{p}),\theta(\mathfrak{p}))$, which is a form of $\mathbf{SO}_n$ for $n = \deg A(\mathfrak{p})$ [40, §23B], hence connected.

We conclude the previous discussion with:

Proposition 2.16. *Let (A,σ) be an Azumaya R-algebra with involution and let $(P,f) \in \mathcal{H}^\varepsilon(A,\sigma)$. Assume that $\operatorname{rrk}_A P > 0$. If (σ,ε) is symplectic or unitary, then $\mathbf{U}^0(f) = \mathbf{U}(f)$. If (σ,ε) is orthogonal, then $\mathbf{U}^0(f) = \ker(\operatorname{Nrd} : \mathbf{U}(f) \to \boldsymbol{\mu}_{2,R})$.*

The following lemma is convenient for verifying equalities in $\mu_2(R)$.

Lemma 2.17. *Let $\alpha,\beta \in \mu_2(R)$. Then $\alpha = \beta$ if and only if $\alpha(\mathfrak{m}) = \beta(\mathfrak{m})$ for all $\mathfrak{m} \in \operatorname{Max} R$.*

Proof. Write $\gamma = \alpha^{-1}\beta$. It is enough to prove that if $\gamma(\mathfrak{m}) = 1$ for all $\mathfrak{m} \in \operatorname{Max} R$, then $\gamma = 1$. Note that $\frac{1}{2}(1-\gamma)$ is an idempotent. If $\gamma(\mathfrak{m}) = 1$ for all $\mathfrak{m} \in \operatorname{Max} R$, then $1-\gamma \in \operatorname{Jac} R$, so the idempotent $\frac{1}{2}(1-\gamma)$ must be 0 and $\gamma = 1$. □

Following are two theorems that will play a major role in the sequel.

Theorem 2.18. *Suppose that (A,σ) is an Azumaya R-algebra with involution and R is semilocal, and let $(P,f) \in \mathcal{H}^\varepsilon(A,\sigma)$. Then the specialization map $U^0(f) \to \prod_{\mathfrak{m}\in\operatorname{Max} R} U^0(f(\mathfrak{m}))$ is surjective.*

Proof. Write $R = \prod_{i=1}^t R_i$ with each R_i connected. Working over each factor separately, we may assume R is connected. We may further assume that $\operatorname{rrk}_A P > 0$.

Let E and θ be as in Remark 2.15 and write $U^0(E,\theta) = \mathbf{U}^0(E,\theta)(R) = U^0(f)$. Then (E,θ) is Azumaya over R (Proposition 2.11), θ is either orthogonal, symplectic, or unitary (Propositions 1.21(v)), and the theorem is equivalent to $U^0(E,\theta) \to \prod_{\mathfrak{m}\in\operatorname{Max} R} U^0(E(\mathfrak{m}),\theta(\mathfrak{m}))$ being surjective. This holds by [25, Theorem 2] (and Proposition 2.16) when θ is orthogonal and by [25, Theorem 6] when θ is not orthogonal. □

[6] One can show that $\operatorname{Nrd}_{E/R}$ maps $U(E,\theta)$ to $\mu_2(R)$, and similarly after base-changing to S, as follows: By [39, III.§8.5] or [26, Theorems 5.17 & 5.37, Examples 7.3 & 7.4], there exists a faithfully flat R-ring R' such that $(E_{R'},\theta_{R'}) \cong (\mathrm{M}_n(R'),\mathrm{t})$, where t is the transpose involution. Now, for all $x \in U(\mathrm{M}_n(R'),\mathrm{t})$, we have $\operatorname{Nrd}(x)^2 = \det(x)^2 = \det(x^{\mathrm{t}}x) = 1$, so $\operatorname{Nrd}(x) \in \mu_2(R')$. As a result, $\operatorname{Nrd}_{E/R}$ maps $U(E,\theta)$ to $R \cap \mu_2(R') = \mu_2(R)$.

Remark 2.19. Under the assumptions of Theorem 2.18, the specialization map $U(f) \to \prod_{\mathfrak{m}\in \mathrm{Max}\, R} U(f(\mathfrak{m}))$ may fail to be surjective in general. For example, take R to be a connected semilocal ring with two maximal ideals, let $(A,\sigma) = (R, \mathrm{id}_R)$ and consider the 1-hermitian form $f(x,y) = xy$ on R.

Theorem 2.20. *Suppose that (A,σ) is an Azumaya R-algebra with involution, (σ,ε) is orthogonal and R is semilocal. Let $(P,f) \in \mathcal{H}^{\varepsilon}(A,\sigma)$ be hermitian space with $\mathrm{rrk}_A P > 0$. Then $\mathrm{Nrd} : U(f) \to \mu_2(R)$ is surjective if and only if $[A] = 0$.*

Proof. As in the proof of Theorem 2.18, we may assume that R is connected, and hence, $\mu_2(R) = \{\pm 1\}$. Now, the theorem follows by applying [25, Theorem 1] to the adjoint involution of f. Note that $\mathrm{End}_A(P)$ is Azumaya over R because $\mathrm{rrk}_A P > 0$ (Proposition 1.11(i)). □

We finish with the following well-known theorem.

Theorem 2.21. *Suppose that R is a field and A is finite dimensional over R, and and let $(P,f) \in \mathcal{H}^{\varepsilon}(A,\sigma)$. Then $\mathbf{U}^0(f) \to \mathrm{Spec}\, R$ is a rational variety.*

Proof (sketch). We already know that $\mathbf{U}^0(f) \to \mathrm{Spec}\, R$ is irreducible. Let θ be the adjoint involution of f and let $\mathbf{V}$ denote the affine R-variety representing the functor $S \mapsto \mathcal{S}_{-1}(\mathrm{End}_{A_S}(P_S), \theta_S)$; it is isomorphic to $\mathbb{A}^n_R$ for $n = \dim_R \mathcal{S}_{-1}(\mathrm{End}_A(P), \theta)$. A birational equivalence between $\mathbf{V}$ and $\mathbf{U}^0(f)$ is given by the Cayley transform, $y \mapsto (1+y)(1-y)^{-1} : \mathbf{V} \dashrightarrow \mathbf{U}^0(f)$, and its inverse, $x \mapsto -(1+x)^{-1}(1-x) : \mathbf{U}^0(f) \dashrightarrow \mathbf{V}$. □

2F. **More on Lagrangians.** Throughout this subsection, (A,σ) denotes an Azumaya R-algebra with involution. Given $(P,f) \in \mathcal{H}^{\varepsilon}(A,\sigma)$, let

$$\mathrm{Lag}(f) = \{L \subseteq P \,:\, L \text{ is a Lagrangian of } f \text{ and } \mathrm{rrk}_A L = \tfrac{1}{2}\mathrm{rrk}_A P\}.$$

Recall from 2B that if L is a Lagrangian of f, then $P = L \oplus L^*$, hence $\mathrm{rrk}_A P = \mathrm{rrk}_A L + \sigma\, \mathrm{rrk}_A L$ by Lemma 2.6. Thus, if $\sigma|_{\mathrm{Z}(A)} = \mathrm{id}_{\mathrm{Z}(A)}$, or if $\mathrm{Z}(A)$ is connected, then $\mathrm{Lag}(f)$ consists of all Lagrangians of f.

In this subsection, we collect several facts about the action of $U^0(f)$ on $\mathrm{Lag}(f)$. Some of the results will require the use of sheaves, and we refer the reader to [39, Chapter III.§2] for a scheme-free introduction, or [47] for an extensive treatment.

The map $S \mapsto \mathrm{Lag}(f_S)$ naturally extends to a functor, $\mathbf{Lag}(f)$, from R-rings to sets. It is routine to check that Lagrangians descend along along faithfully flat ring homomorphisms. That is, if $S \to T$ is a faithfully flat map of R-rings, $i_1, i_2 : T \to T \otimes_S T$ are the maps $t \mapsto t \otimes 1$ and $t \mapsto 1 \otimes t$, and $L \in \mathrm{Lag}(f_T)$ satsfies $\mathbf{Lag}(i_1)(L) = \mathbf{Lag}(i_2)(L)$, then there exists a unique $L_0 \in \mathrm{Lag}(f_S)$ with $(L_0) \otimes_S T = L$; consult [39, III.§§1–2]. Thus, $\mathbf{Lag}(f)$ is sheaf relative to the fppf topology on the category of affine R-schemes, denoted $(\mathcal{A}ff/R)_{\mathrm{fppf}}$.[7] (In fact, it can be shown that $\mathbf{Lag}(f)$ is represented by a non-affine R-scheme, but this fact will not be needed in this work.) The group $U(f)$ acts on $\mathrm{Lag}(f)$ in a way which is compatible with base change, thus giving rise to an action of $\mathbf{U}(f)$ on $\mathbf{Lag}(f)$.

For the next results, given $P, Q \in \mathcal{P}(A)$ and $f \in \mathrm{Hom}_A(P,Q)$, recall that the dual homomorphism $f^* \in \mathrm{Hom}_A(Q^*, P^*)$ is defined by $f^*\phi = \phi \circ f$ $(\phi \in Q^*)$.

Lemma 2.22. *Suppose that (A,σ) is an Azumaya R-algebra with involution and R is semilocal. Let $(P,f) \in \mathcal{H}^{\varepsilon}(A,\sigma)$. Then $U(f)$ acts transitively on $\mathrm{Lag}(f)$, provided it is nonempty.*

[7] With the appropriate definitions, this functor also extends to a sheaf on the site of all R-schemes with the fpqc topology.

Proof. Let $L_1, L_2 \in \mathrm{Lag}(f)$. As explained in 2B, we can find isometries $\varphi_i : \mathbb{h}^{\varepsilon}_{L_i} \to f$ $(i = 1, 2)$ such that φ_i restricts to the identity on L_i. Since $\mathrm{rrk}_A L_1 = \frac{1}{2}\mathrm{rrk}_A P = \mathrm{rrk}_A L_2$, there is an A-module isomorphism $\psi : L_1 \cong L_2$ (Lemma 1.24). Then $\hat{\psi} := \psi \oplus (\psi^*)^{-1} : \mathbb{h}^{\varepsilon}_{L_1} \to \mathbb{h}^{\varepsilon}_{L_2}$ is an isometry taking L_1 to L_2. Now, $\varphi_2\hat{\psi}\varphi_1^{-1}$ is an element of $U(f)$ taking L_1 to L_2. □

Proposition 2.23. *Suppose that (A, σ) is an Azumaya R-algebra with involution, and let $(P, f) \in \mathcal{H}^{\varepsilon}(A, \sigma)$. When viewed as sheaves on $(\mathcal{A}ff/R)_{\mathrm{Zar}}$ — the category of affine R-schemes with the Zariski topology — the group $\mathbf{U}(f)$ acts transitively on $\mathbf{Lag}(f)$, provided $\mathbf{Lag}(f) \neq \emptyset$.*

Proof. The statement means that for every R-ring S and $L, M \in \mathrm{Lag}(f_S)$, there exist $\alpha_1, \dots, \alpha_t \in S$ and $\varphi_i \in U(f_{S_{\alpha_i}})$ $(i = 1, \dots, t)$ such that $S = \sum_i \alpha_i S$ and $\varphi_i(L_{S_{\alpha_i}}) = M_{S_{\alpha_i}}$ for all i. Here, S_{α_i} denotes the localization of S with respect to $\{1, \alpha_i, \alpha_i^2, \dots\}$.

Fix some $\mathfrak{p} \in \mathrm{Spec}\, S$. By Lemma 2.22, there exists an isometry $\psi \in U(f_{S_{\mathfrak{p}}})$ with $\psi(L_{S_{\mathfrak{p}}}) = M_{S_{\mathfrak{p}}}$. It is easy to see that there exists $\alpha = \alpha^{(\mathfrak{p})} \in S \setminus \mathfrak{p}$ and $\varphi = \varphi^{(\mathfrak{p})} \in U(f_{S_\alpha})$ such that $\psi = \varphi_{S_{\mathfrak{p}}}$ and $\varphi(L_{S_\alpha}) = M_{S_\alpha}$. Now, since $\sum_{\mathfrak{p}} \alpha_{\mathfrak{p}} S = S$, there exist $\mathfrak{p}_1, \dots, \mathfrak{p}_t \in S$ such that $\sum_{i=1}^t \alpha^{(\mathfrak{p}_i)} S = S$. The elements $\alpha_i := \alpha^{(\mathfrak{p}_i)}$ and the isometries $\varphi_i = \varphi^{(\mathfrak{p}_i)}$ fulfill all the requirements. □

Given $P \in \mathcal{P}(A)$ and $b \in \mathrm{Hom}_A(P, P^*)$, write b^{t} for the element of $\mathrm{Hom}_A(P, P^*)$ determined by $(b^{\mathrm{t}}x)y = ((by)x)^{\sigma}$ $(x, y \in P)$. It is straightforward to check that $b^{\mathrm{tt}} = b$ and $(b \circ \psi)^{\mathrm{t}} = \psi^* \circ b^{\mathrm{t}}$ for all $\psi \in \mathrm{End}_A(P)$. We set $\mathcal{S}_{\varepsilon}(P) = \{b \in \mathrm{Hom}_A(P, P^*) : b = \varepsilon b^{\mathrm{t}}\}$.

Lemma 2.24. *Let $L \in \mathcal{P}(A)$ and let B denote the subgroup of $U(\mathbb{h}^{\varepsilon}_L)$ consisting of isometries φ satisfying $\varphi(0 \oplus L^*) = 0 \oplus L^*$. Then, writing elements of $\mathrm{End}_A(L \oplus L^*)$ as 2×2 matrices, we have*

$$B = \left\{ \begin{bmatrix} a & 0 \\ b & (a^*)^{-1} \end{bmatrix} : a \in \mathrm{Aut}_A(L),\, b \in \mathrm{Hom}_A(L, L^*),\, a^* \circ b \in \mathcal{S}_{-\varepsilon}(L) \right\}.$$

Proof. That elements of B live in $U(\mathbb{h}^{\varepsilon}_L)$ and preserve $0 \oplus L^*$ is routine. Conversely, every element $\varphi \in U(f)$ satisfying $\varphi(0 \oplus L^*) = 0 \oplus L^*$ can be written as $\left[\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right]$ with $a \in \mathrm{Aut}_A(L)$, $b \in \mathrm{Hom}_A(L, L^*)$, $c \in \mathrm{Aut}_A(L^*)$. Let $x, x' \in L$ and $\phi \in L^*$. Unfolding the equality $\mathbb{h}^{\varepsilon}_L(\left[\begin{smallmatrix} 0 \\ \phi \end{smallmatrix}\right], \left[\begin{smallmatrix} x' \\ 0 \end{smallmatrix}\right]) = \mathbb{h}^{\varepsilon}_L(\varphi\left[\begin{smallmatrix} 0 \\ \phi \end{smallmatrix}\right], \varphi\left[\begin{smallmatrix} x' \\ 0 \end{smallmatrix}\right])$ gives $\phi x' = (c\phi)(ax') = (a^*(c\phi))x'$, so $a^*c = \mathrm{id}_{L^*}$, or rather, $c = (a^*)^{-1}$. Unfolding $\mathbb{h}^{\varepsilon}_L(\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} x' \\ 0 \end{smallmatrix}\right]) = \mathbb{h}^{\varepsilon}_L(\varphi\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right], \varphi\left[\begin{smallmatrix} x' \\ 0 \end{smallmatrix}\right])$ gives $0 = (bx')(ax) + \varepsilon((bx)(ax'))^{\sigma} = (a^*(bx'))x + \varepsilon(b^{\mathrm{t}}(ax'))x$, so $a^* \circ b + \varepsilon b^{\mathrm{t}} \circ a = 0$, which means that $a^* \circ b \in \mathcal{S}_{-\varepsilon}(L)$. □

The following proposition provides information about the $\mathbf{U}^0(f)$-orbits in $\mathbf{Lag}(f)$ when (A, σ) is an Azumaya R-algebra with involution and (σ, ε) is orthogonal. It will feature a number of times in the sequel.

Proposition 2.25. *Suppose that (A, σ) is an Azumaya R-algebra with involution and (σ, ε) is orthogonal. Let $(P, f) \in \mathcal{H}^{\varepsilon}(A, \sigma)$, let $L \in \mathrm{Lag}(f)$ and suppose that $\mathrm{rrk}_A P > 0$. Then there exists a unique $\mathbf{U}(f)$-equivariant natural transformation of functors from R-rings to sets,*

$$\Phi_L = \Phi_L^{(f)} : \mathbf{Lag}(f) \to \boldsymbol{\mu}_{2,R},$$

such that $\Phi_L(L) = 1$; here, $\mathbf{U}(f)$ acts on $\boldsymbol{\mu}_{2,R}$ via $\mathrm{Nrd} : \mathbf{U}(f) \to \boldsymbol{\mu}_{2,R}$. The map Φ_L has the following additional properties:

(i) $\Phi_L(M)\Phi_M(K) = \Phi_L(K)$ and $\Phi_L(M) = \Phi_M(L)$ for all $L, M, K \in \mathrm{Lag}(f)$.

(ii) Given $(P', f') \in \mathcal{H}^{\varepsilon}(A, \sigma)$ and $L' \in \mathrm{Lag}(f')$, we have $\Phi_{L \oplus L'}(M \oplus M') = \Phi_L(M)\Phi_{L'}(M')$ for all $M \in \mathrm{Lag}(f)$, $M' \in \mathrm{Lag}(f')$.

Proof. We may assume without loss of generality that $(P, f) = (L \oplus L^*, \mathbb{h}_L^\varepsilon)$ and identify L with its copy in $P = L \oplus L^*$. A sheaf means a sheaf on the site $(\mathcal{A}ff/R)_{\mathrm{fppf}}$.

Given an R-ring S, let $B = B(L)$ be as in Lemma 2.24. Let $\mathbf{B}$ denote the subfunctor of $\mathbf{U}(f)$ determined by $\mathbf{B}(S) = B(L_S)$. It is routine to check that $\mathbf{B}$ is a group subsheaf of $\mathbf{U}(f)$. We let $\mathbf{U}(f)/\mathbf{B}$ denote the quotient sheaf (note that $(\mathbf{U}(f)/\mathbf{B})(S)$ is in general larger than $U(f_S)/B(L_S)$). By definition, $\mathbf{B}$ is the stabilizer of the global section $0 \oplus L^*$ of $\mathbf{Lag}(f)$ under the action of $\mathbf{U}(f)$. Thus, we have an induced morphism $\Psi : \mathbf{U}(f)/\mathbf{B} \to \mathbf{Lag}(f)$, which is an isomorphism by Proposition 2.23.

For every $a \in \mathrm{End}_A(L)$, we have $\mathrm{Nrd}(a^*) = \mathrm{Nrd}(a)$. Indeed, $a \mapsto a^* : \mathrm{End}_A(L) \to \mathrm{End}_A(L^*)^{\mathrm{op}}$ is an isomorphism of Azumaya R-algebras, and thus respects the reduced norm. This implies readily that $\mathbf{B} \subseteq \ker(\mathrm{Nrd} : \mathbf{U}(f) \to \boldsymbol{\mu}_{2,R})$. As a result, there is an induced $\mathbf{U}(f)$-equivariant map $\overline{\mathrm{Nrd}} : \mathbf{U}(f)/\mathbf{B} \to \boldsymbol{\mu}_{2,R}$. Let Φ_0 denote the composition $\overline{\mathrm{Nrd}} \circ \Psi^{-1} : \mathbf{Lag}(f) \to \boldsymbol{\mu}_{2,R}$. Then Φ_0 is $\mathbf{U}(f)$-equivariant. Writing $\xi = \Phi_0(L) \in \mu_2(R)$ and defining $\Phi_L = \xi \cdot \Phi_0$, we see that Φ_L is $\mathbf{U}(f)$-equivariant and satisfies $\Phi_L(L) = 1$.

Suppose that $\Phi' : \mathbf{Lag}(f) \to \boldsymbol{\mu}_{2,R}$ is another $\mathbf{U}(f)$-equivariant natural transformation satisfying $\Phi'(L) = 1$. Let R' be an R-ring and let $M \in \mathrm{Lag}(f_{R'})$. By Proposition 2.23, there exists a faithfully flat R'-algebra R'' and $\varphi \in U(f_{R''})$ such that $\varphi(L_{R''}) = M \otimes_{R'} R''$. Thus, $\Phi'(M \otimes_R R'') = \mathrm{Nrd}(\varphi)\Phi'(L) = \mathrm{Nrd}(\varphi)\Phi_L(L) = \Phi_L(M \otimes_R R'')$ in $\mu_2(R'')$. Since $R' \to R''$ is faithfully flat, this means that $\Phi'(M) = \Phi_L(M)$ in $\mu_2(R')$, and we have shown that $\Phi' = \Phi_L$.

We turn to prove (i) and (ii):

(i) We apply Proposition 2.23 to assert the existence of a faithfully flat R-algebra R' and $\varphi, \psi \in U(f_{R'})$ such that $\varphi(L_{R'}) = M_{R'}$ and $\psi(M_{R'}) = K_{R'}$. Note that $\Phi_L(M) = \mathrm{Nrd}(\varphi)\Phi_L(L) = \mathrm{Nrd}(\varphi)$ in $\mu_2(R')$, and similarly, $\Phi_M(K) = \mathrm{Nrd}(\psi)$, $\Phi_L(K) = \mathrm{Nrd}(\psi\varphi)$ and $\Phi_M(L) = \mathrm{Nrd}(\varphi)^{-1}$. The identities in (i) follow readily from these equalities and the fact that $\mu_2(R)$ is 2-torsion.

(ii) By Proposition 2.23, there exists a faithfully flat R-algebra S and $\varphi \in U(f_S)$, $\varphi' \in U(f'_S)$ such that $\varphi L = M$ and $\varphi' L' = M'$. Then $\Phi_{L\oplus L'}(M \oplus M') = \Phi_{L \oplus L'}((\varphi \oplus \varphi')(L \oplus L')) = \mathrm{Nrd}(\varphi \oplus \varphi') = \mathrm{Nrd}(\varphi)\,\mathrm{Nrd}(\varphi') = \Phi_L(M) \cdot \Phi_{L'}(M')$. □

2G. **Conjugation and Transfer.** We now recall two special instances of hermitian Morita equivalence that will be used repeatedly in the sequel. We address them simply as "μ-conjugation" and "e-transfer".

Recall that $\varepsilon \in \mathrm{Z}(A)$ satisfies $\varepsilon^\sigma \varepsilon = 1$. Let $\delta \in \mathrm{Z}(A)$ be another element satisfying $\delta^\sigma \delta = 1$ and let $\mu \in \mathcal{S}_\delta(A, \sigma) \cap A^\times$. One readily checks that $\mathrm{Int}(\mu) \circ \sigma$ is also an R-involution and $(\delta\varepsilon)^{\mathrm{Int}(\mu)\circ\sigma}(\delta\varepsilon) = 1$. Given $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$, define $\mu f : P \times P \to A$ by $(\mu f)(x, y) = \mu \cdot f(x, y)$. Then μf is an $\varepsilon\delta$-hermitian form over $(A, \mathrm{Int}(\mu) \circ \sigma)$ and

$$(P, f) \mapsto (P, \mu f) : \mathcal{H}^\varepsilon(A, \sigma) \to \mathcal{H}^{\delta\varepsilon}(A, \mathrm{Int}(\mu) \circ \sigma)$$

is an equivalence of categories; morphisms are mapped to themselves. We call this equivalence *μ-conjugation.* It has the following properties:

(c1) For every R-ring S, we have $\mu(f_S) = (\mu f)_S$.
(c2) $U(f) = U(\mu f)$. If A is finite projective over R, then $\mathbf{U}(f) = \mathbf{U}(\mu f)$, $U^0(f) = U^0(\mu f)$ and $\mathbf{U}^0(f) = \mathbf{U}^0(\mu f)$.
(c3) The forms f and μf have the same Lagrangians. In particular, f is hyperbolic if and only if μf is hyperbolic.

Suppose further that (A,σ) is an Azumaya R-algebra with involution. Then, by Corollary 1.22(i), $(A,\mathrm{Int}(\mu)\circ\sigma)$ is also an Azumaya R-algebra with involution, and the types of (σ,ε) and $(\mathrm{Int}(\mu)\circ\sigma,\delta\varepsilon)$ are equal. When (σ,ε) is orthogonal, we have:

(c4) $\mathrm{Lag}(f)=\mathrm{Lag}(\mu f)$ and $\mathbf{Lag}(f)=\mathbf{Lag}(\mu f)$.

(c5) For every $L\in\mathrm{Lag}(f)=\mathrm{Lag}(\mu f)$, the maps $\Phi_L^{(f)}:\mathbf{Lag}(f)\to\boldsymbol{\mu}_{2,R}$ and $\Phi_L^{(\mu f)}:\mathbf{Lag}(\mu f)\to\boldsymbol{\mu}_{2,R}$ of Proposition 2.25 coincide.

(Item (c5) follows from the uniqueness part in Proposition 2.25.)

Items (c1)–(c5) allow us to rephrase certain claims about ε-hermitian forms over (A,σ) as claims about $\delta\varepsilon$-hermitian forms over $(A,\mathrm{Int}(\mu)\circ\sigma)$. We shall address this process as μ-conjugation in the sequel.

Next, let $e\in A$ be an idempotent such that $e^\sigma=e$ and eA_A is a progenerator, or equivalently, $AeA=A$. When A is Azumaya over its center, this is also equivalent to having $\mathrm{rrk}_A\, eA>0$ (Proposition 1.10). By Morita theory, the functor $\mathcal{P}(A)\to\mathcal{P}(eAe)$ sending a module P to Pe and a morphism $\varphi:P\to Q$ to $\varphi_e:=\varphi|_{Pe}$ is an equivalence; see [42, Example 18.30].

Write $\sigma_e:=\sigma|_{eAe}$ and note that $(e\varepsilon)^{\sigma_e}(e\varepsilon)=1$. Given $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma)$, let $f_e=f|_{Pe\times Pe}$. It is well-known, see [24, Proposition 2.5, Remark 2.1] for instance, that

$$(P,f)\mapsto(Pe,f_e)\in\mathcal{H}^\varepsilon(A,\sigma)\to\mathcal{H}^{e\varepsilon}(eAe,\sigma_e)$$

defines an equivalence of categories; isometries φ are mapped to φ_e. We call this equivalence *e-transfer*. It has the following additional properties:

(t1) For every R-ring S, there is a natural isomorphism $(f_S)_e\cong(f_e)_S$.

(t2) The map $\varphi\mapsto\varphi_e$ defines an isomorphism $U(f)\to U(f_e)$. If A is finite projective over R, then it also defines isomorphisms $\mathbf{U}(f)\to\mathbf{U}(f_e)$, $U^0(f)\to U^0(f_e)$ and $\mathbf{U}^0(f)\to\mathbf{U}^0(f_e)$.

(t3) The map $L\mapsto Le$ defines a bijection between the Lagrangians of f and the Lagrangians of f_e. In particular, f is hyperbolic if and only if f_e is hyperbolic.

Suppose further that (A,σ) is an Azumaya R-algebra with involution. By Corollary 1.22(ii), (eAe,σ_e) is also an Azumaya R-algebra with involution and the types of (σ,ε) and $(\sigma_e,e\varepsilon)$ are the same. When (σ,ε) is orthogonal, we have:

(t4) The isomorphism $\varphi\mapsto\varphi_e:\mathbf{U}(f)\to\mathbf{U}(f_e)$ respects the reduced norm.

(t5) The map $L\mapsto Le$ defines isomorphisms $\mathrm{Lag}(f)\to\mathrm{Lag}(f_e)$ and $\mathbf{Lag}(f)\to\mathbf{Lag}(f_e)$; its inverse is $L'\mapsto L'A$.

(t6) The composition $\mathbf{Lag}(f)\xrightarrow{\sim}\mathbf{Lag}(f_e)\xrightarrow{\Phi_{Le}}\boldsymbol{\mu}_{2,R}$ coincides with Φ_L (see Proposition 2.25).

(Item (t4) follows from the fact that $\varphi\mapsto\varphi_e:\mathrm{End}_A(P)\to\mathrm{End}_{eAe}(Pe)$ is an isomorphism of Azumaya algebras and so preserves the reduced norm. Item (t5) follows from (t3) Corollary 1.12. Item (t6) follows from (t4) and the uniqueness part of Proposition 2.25.) Note also that e-transfer preserves reduced rank by Corollary 1.12.

Items (t1)–(t6) allow us to rephrase certain claims about ε-hermitian forms over (A,σ) as claims about $e\varepsilon$-hermitian forms over (eAe,σ_e). We shall address this process as e-transfer in the sequel.

As a first example of using conjugation and transfer, we prove the following result, which provides an alternative way to evaluate Φ_L.

Proposition 2.26. *With the notation of Proposition 2.25, let $L,M\in\mathrm{Lag}(f)$. For every $\mathfrak{p}\in\operatorname{Spec}R$, let $I_\mathfrak{p}$ denote the intersection of $L(\mathfrak{p})$ and $M(\mathfrak{p})$ in $P(\mathfrak{p})$. Then*

$$\Phi_L(M)(\mathfrak{p})=(-1)^{\mathrm{rrk}_{A(\mathfrak{p})}L(\mathfrak{p})-\mathrm{rrk}_{A(\mathfrak{p})}I_\mathfrak{p}}$$

in $\mu_2(k(\mathfrak{p}))$*. In particular, if* $P = L \oplus M$*, then* $\Phi_L(M) = (-1)^{\operatorname{rrk}_A L}$ *in* $\mu_2(R)$*.*

We alert the reader that $I_{\mathfrak{p}}$ is in general not the image of $L \cap M$ in $P(\mathfrak{p})$.

Proof. It is enough to prove the proposition when R is a field and $\mathfrak{p} = 0$ (the last assertion will follow by virtue of Lemma 2.17). Note further that base-changing from the field R to an algebraic closure does not affect the R-dimension of A, L, M and $I_0 = L \cap M$, and thus $\operatorname{rrk}_A M$, $\operatorname{rrk}_A L$ and $\operatorname{rrk}_A I_0$ remain unchanged. This allows us to further restrict to the case where R is an algebraically closed field. In particular, $[A] = 0$ in $\operatorname{Br} R$.

If σ is symplectic, then $\varepsilon = -1$ (because (σ, ε) is orthogonal) and $\deg A$ is even. By Lemma 1.26, there exists $\mu \in \mathcal{S}_{-1}(A, \sigma) \cap A^\times$. Applying μ-conjugation, we may replace σ, ε, f with $\operatorname{Int}(\mu) \circ \sigma$, $-\varepsilon$, μf and assume that σ is orthogonal and $\varepsilon = 1$.

Now, by Proposition 1.27, there exists an idempotent $e \in A$ with $e^\sigma = e$ and $\deg eAe = 1$. Applying e-transfer, we may replace A, σ, P, f, L, M with eAe, σ_e, Pe, f_e, Le, Me and assume that $A = R$ and $\sigma = \mathrm{id}_R$ henceforth.

Write $I = I_0 = L \cap M$ and fix R-subspaces $W \subseteq L$, $W' \subseteq M$ such that $L = I \oplus W$ and $M = I \oplus W'$. Let $N = (W \oplus W')^\perp$ and fix a basis $\{x_1, \dots, x_n\}$ to W. The kernel of $y \mapsto f(y, -) : M \to L^*$ is $M \cap L^\perp = M \cap L = I$, so $y \mapsto f(y, -) : W' \to L^*$ is injective. Since any element in the image of this map vanishes on I, it follows that $y \mapsto f(y, -) : W' \to W^*$ is also injective, and thus bijective by conisdering R-dimensions. This means that there exists a basis $\{y_1, \dots, y_n\}$ to W' satisfying $f(x_i, y_j) = \delta_{ij}$. Consequently, $f|_{W \oplus W'}$ is unimodular, and thus $P = N \oplus W \oplus W'$. Let $\varphi \in \operatorname{End}_R(P)$ denote the endomorphism exchanging x_i and y_i and fixing N. Then $\varphi \in U(f)$ and $\operatorname{Nrd}(\varphi) = (-1)^{\dim_R W} = (-1)^{\operatorname{rrk}_A L - \operatorname{rrk}_A I_0}$. Since $\varphi L = \varphi(W + I) = W' + I = M$, we have $\Phi_L(M) = \operatorname{Nrd}(\varphi)$, so we are done. □

2H. **The Discriminant.** Classically, the discriminant of a nondegenerate symmetric bilinear space (V, b) over a field F is the coset in $F^\times/(F^\times)^2$ represented by $(-1)^{\frac{1}{2}\dim V(\dim V - 1)}$ times the determinant of some Gram matrix of b, see [40, p. 80]. If F carries a nontrivial involution $\sigma : F \to F$ with a fixed subfield F_0, then the discriminant of a unimodular 1-hermitian space (V, h) over (F, σ) is defined similarly, but this time it is regarded as an element of $F_0^\times / \operatorname{Nr}_{F/F_0}(F^\times)$ [40, p. 114]. These definitions do not generalize naively to hermitian forms over R-algebras with involution (A, σ) because projective A-modules need not be free. However, in [40, §7, §8, §10], a discriminant invariant was defined for 1-hermitian forms over central simple algebras with an orthogonal or unitary involution. It agrees with the classical discriminant and is compatible with extending the base field. Moreover, it is invariant under conjugation and e-transfer (see 2G), because it is defined as an invariant of the adjoint involution of the hermitian space, which is unaffected by these operations.

Suppose henceforth that (A, σ) is an Azumaya R-algebra with involution. We will need a generalization of the discriminant defined in [40, §7, §8, §10] to ε-hermitian forms over (A, σ) when (σ, ε) is orthogonal or unitary. Unfortunately, such a definition seems missing in the literature, and introducing one is out of the scope of this work. We therefore give an ad hoc generalization of the definition in *op. cit.* to some specific R, A, σ that will be needed in this work, and prove that it has desired properties such as being invariant under conjugation and e-transfer. Specifically, we shall restrict to rings R which are connected semilocal and consider only the cases where (1) (σ, ε) is orthogonal, or (2) σ is unitary and $[A] = 0$ in $\operatorname{Br} \mathrm{Z}(A)$.

Suppose first that (σ, ε) is orthogonal and R is connected semilocal. Let $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ be a hermitian space such that $n := \operatorname{rrk}_A P$ is even and positive. Write

$E = \mathrm{End}_A(P)$ and let θ denote the adjoint involution of f. By Lemma 1.26, there exists $\varphi \in \mathcal{S}_{-1}(E,\theta)\cap E^\times$. Following [40, §7], we define the discriminant of f to be

$$\mathrm{disc}(f) = (-1)^{n/2}\,\mathrm{Nrd}(\varphi)\cdot (R^\times)^2 \in R^\times/(R^\times)^2.$$

This is well-defined by the following proposition. The discriminant of the zero form is defined to be the trivial class $(R^\times)^2$.

Proposition 2.27. *Under the previous assumptions:*

(i) $\mathrm{disc}(f)$ *is well-defined, i.e., it is independent of the choice of* φ.

(ii) Isomorphic forms have equal discriminants. The discriminant is unchanged under μ*-conjugation and* e*-transfer (see 2G).*

(iii) If $(P', f') \in \mathcal{H}^\varepsilon(A,\sigma)$ *and* $\mathrm{rrk}_A P'$ *is even, then* $\mathrm{disc}(f\oplus f') = \mathrm{disc}(f)\,\mathrm{disc}(f')$.

(iv) If $(A,\sigma) = (R, \mathrm{id}_R)$, $\varepsilon = 1$ *and* $f = \langle \alpha_1,\dots,\alpha_{2n}\rangle_{(R,\mathrm{id}_R)}$, *then* $\mathrm{disc}(f) \equiv (-1)^n \prod_i \alpha_i \bmod (R^\times)^2$.

(v) If $d := \deg A$ *is even,* $u \in \mathcal{S}_{-\varepsilon}(A,\sigma)\cap A^\times$ *and* $a_1,\dots,a_n \in \mathcal{S}_\varepsilon(A,\sigma)$, *then* $\mathrm{disc}(\langle a_1,\dots,a_n\rangle_{(A,\sigma)}) \equiv (-1)^{nd/2}\,\mathrm{Nrd}(u)^n \prod_{i=1}^n \mathrm{Nrd}(a_i) \bmod (R^\times)^2$.

Proof. (i) See [40, Proposition 7.1] for the case R is a field. The same proof works when R is general; for the definition of the *Pfaffian* over general rings and a proof that its square is the reduced characteristic polynomial, see [38, p. 3].

(ii) The definition of $\mathrm{disc}(f)$ depends only on the isomorphism class of (E,θ) and this remains unchanged if we replace f with an isomorphic form or perform μ-conjugation or e-transfer.

(iii) Write $n' = \mathrm{rrk}_A P'$, let θ' be the adjoint involution of f' and let $\varphi' \in \mathcal{S}_{-1}(\mathrm{End}_A(P'),\theta')$. One readily checks that

$$(f\oplus f')((\varphi\oplus\varphi')(x\oplus x'), y\oplus y') = -(f\oplus f')(x\oplus x', (\varphi\oplus\varphi')(y\oplus y'))$$

for all $x,y\in P$ and $x',y'\in P'$. Thus, the adjoint involution of $f\oplus f'$ takes $\varphi\oplus\varphi'$ to $-(\varphi\oplus\varphi')$, and, by definition, $\mathrm{disc}(f\oplus f') \equiv (-1)^{(n+n')/2}\,\mathrm{Nrd}(\varphi\oplus\varphi') \equiv (-1)^{n/2}\,\mathrm{Nrd}(\varphi)(-1)^{n'/2}\,\mathrm{Nrd}(\varphi') \equiv \mathrm{disc}(f)\,\mathrm{disc}(f')$ modulo $(R^\times)^2$.

(iv) The proof of [40, Proposition 7.3(3)] applies verbatim.

(v) By (iii), it is enough to prove the case $n=1$. Writing $a = a_1$, and identifying $\mathrm{End}(A_A)$ with A via $\varphi\mapsto\varphi(1_A)$, the adjoint involution of $\langle a\rangle$ is $\theta := \mathrm{Int}(a^{-1})\circ\sigma$. Thus, $ua \in \mathcal{S}_{-1}(A,\theta)\cap A^\times$ and $\mathrm{disc}\langle a\rangle \equiv (-1)^{d/2}\,\mathrm{Nrd}(ua)$ modulo $(R^\times)^2$. □

Given a quadratic étale R-algebra S with standard involution θ (see 1C), we define the norm form $n_{S/R} : S\times S\to R$ by $n_{S/R}(x,y) = \frac{1}{2}(x^\theta y + y^\theta x)$; it is a 1-hermitian form over (R,id_R). When R is semilocal, there is $\lambda\in S$ such that $\{1,\lambda\}$ is an R-basis of S, $\lambda^2\in R^\times$ and $\lambda^\theta = -\lambda$ (Lemma 1.19). Using this basis to identify S with R^2, one finds that $n_{S/R} \cong \langle 1, -\lambda^2\rangle_{(R,\mathrm{id})}$. In this case, we define

$$\mathrm{disc}(S/R) := \mathrm{disc}(n_{S/R}) = \lambda^2 (R^\times)^2.$$

Keeping our assumption that R is connected semilocal, we now proceed with defining a discriminant for ε-hermitian spaces over (A,σ) when σ is unitary and $[A]=0$ in $\mathrm{Br}\,\mathrm{Z}(A)$. Note that the reduced rank of any $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma)$ is constant by Corollary 2.9(i). Write $S = \mathrm{Z}(A)$ and let $\mathrm{Nr}_{S/R} : S\to R$ denote the norm map; it is given by $\mathrm{Nr}_{S/R}(x) = x^\sigma x$ because $\sigma|_S$ is the standard R-involution of S.

Suppose first that $\deg A = 1$ and let $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma)$. Then $A = S$ and P is free. Let $\{x_i\}_{i=1}^n$ be an S-basis of P and let $g = (f(x_i,x_j))_{i,j}$ denote the corresponding Gram matrix. Since g is (σ,ε)-hermitian, $\det g = \varepsilon^n(\det g)^\sigma$. When $n = \mathrm{rrk}_A P$ is even, this means that $(-\varepsilon)^{-n/2}\det g = ((-\varepsilon)^{-n/2}\det g)^\sigma$, so $(-\varepsilon)^{-n/2}\det g\in R^\times$. In this case, the discriminant of f is defined to be

$$\mathrm{disc}(f) = (-\varepsilon)^{-n/2}\det g\cdot \mathrm{Nr}_{S/R}(S^\times) \in R^\times/\,\mathrm{Nr}_{S/R}(S^\times).$$

It is easy to see that this is independent of the basis $\{x_i\}_{i=1}^n$. Moreover, isomorphic forms have the same discriminant.

We extend this to any A with $[A]=0$ as follows: Use Theorem 1.30 to choose an idempotent $e\in A$ with $e^\sigma=e$ and $\operatorname{rrk}_A eA=1$. Noting that $eAe\cong S$, we define

$$\operatorname{disc}(f):=\operatorname{disc}(f_e)\in R^\times/\operatorname{Nr}_{S/R}(S^\times)$$

for every $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma)$ with $\operatorname{rrk}_A P$ even. Here, $f_e: Pe\times Pe\to eAe$ is the e-transfer of f, see 2G. This is well-defined by the following proposition.

Proposition 2.28. *Under the previous assumptions:*

(i) $\operatorname{disc}(f)$ *is well-defined, i.e., it is independent of the choice of* e*.*

(ii) Isomorphic forms have equal discriminants. The discriminant is unchanged under μ*-conjugation and* e'*-transfer (see 2G).*

(iii) If $(P',f')\in\mathcal{H}^\varepsilon(A,\sigma)$ *and* $\operatorname{rrk}_A P'$ *is even, then* $\operatorname{disc}(f\oplus f')=\operatorname{disc}(f)\operatorname{disc}(f')$.

Proof. (i) Let $e'\in A$ be another idempotent with $e'^\sigma=e'$ and $\operatorname{rrk}_A e'A=1$. Then $eA\cong e'A$ (Lemma 1.24). Every A-module homomorphism $eA\to e'A$ is given by multiplication on the left with a unique element in $e'Ae$, so there exist $u\in e'Ae$ and $v\in eAe'$ such that $uv=e'$ and $vu=e$. We also see that $v^\sigma v$ is invertible in $e'Ae'=\operatorname{End}_A(e'A)$. Since $\deg e'Ae'=1$, we have $e'Ae'=Se'$. Write $v^\sigma v=\alpha e'$ with $\alpha\in S^\times$. Then $\alpha\in R^\times$ because $v^\sigma v$ is fixed under σ. Identifying eAe and $e'Ae'$ with $S=\mathrm{Z}(A)$, it is routine to check that $x\mapsto xv: Pe\to Pe'$ defines an isometry from αf_e to $f_{e'}$ (its inverse is $y\mapsto yu: Pe'\to Pe$). Since $\operatorname{rrk}_{eAe} Pe=\operatorname{rrk}_A P$ is even, $\operatorname{disc}(\alpha f_{e'})=\operatorname{disc}(f_{e'})$ and it follows that $\operatorname{disc}(f_e)=\operatorname{disc}(f_{e'})$.

(ii) If $(P,f)\cong(P',f')$, then $f_e\cong f'_e$, so $\operatorname{disc}(f)=\operatorname{disc}(f_e)=\operatorname{disc}(f'_e)=\operatorname{disc}(f')$.

Let $\mu\in\mathcal{S}_\delta(A,\sigma)$, where $\delta\in\mathrm{Z}(A)$ satisfies $\delta^\sigma\delta=1$, and write $\tau=\operatorname{Int}(\mu)\circ\sigma$. Then τ is also unitary, and so there exists an idempotent $e'\in A$ with $\operatorname{rrk}_A e'A=1$ and $e'^\tau=e'$. As in the proof of (i), choose $u\in e'Ae$, $v\in eAe'$ such that $uv=e'$ and $vu=e$. We have $\mu v^\sigma v, uu^\sigma\mu^{-1}\in e'Ae'$ because $e'\mu v^\sigma v=\mu\mu^{-1}e'\mu v^\sigma v=\mu e'^\sigma v^\sigma v=\mu(ve')^\sigma v=\mu v^\sigma v$ and similarly $uu^\sigma\mu^{-1}e'=uu^\sigma\mu^{-1}$. Furthermore, $\mu v^\sigma v\cdot uu^\sigma\mu^{-1}=\mu v^\sigma euu^\sigma\mu^{-1}=\mu(uev)^\sigma\mu^{-1}=\mu e'^\sigma\mu^{-1}=e'^\tau=e'$, hence $\mu v^\sigma v\in(e'Ae')^\times=e'S^\times$. Write $\mu v^\sigma v=\alpha e'$ with $\alpha\in S^\times$. As in the proof of (i), identifying eAe and $e'Ae'$ with S, we see that $x\mapsto xv: Pe\to Pe'$ is an isometry from αf_e to $(\mu f)_{e'}$. Thus,

$$\operatorname{disc}((\mu f)_{e'})=\alpha^{2n}\delta^{-n}\operatorname{disc}(f_e),$$

where $\operatorname{rrk}_A P=2n$. Straightforward computation shows that $\delta(\mu v^\sigma v)^\tau=\mu v^\sigma v$. Since $\tau|_S=\sigma|_S$, this means that $\delta\alpha^\sigma=\alpha$, or rather, $\alpha^2=\delta\operatorname{Nr}_{S/R}(\alpha)$. Thus, $\operatorname{disc}(\mu f)=\operatorname{disc}((\mu f)_{e'})=\alpha^{2n}\delta^{-n}\operatorname{disc}(f_e)=\operatorname{disc}(f_e)=\operatorname{disc}(f)$.

Next, let $e'\in A$ be an idempotent with $e'^\sigma=e'$ and $\operatorname{rrk}_A e'A>0$. Then, using Theorem 1.30, we can choose an idempotent $e\in e'Ae'$ with $e^\sigma=e$ and $\operatorname{rrk}_{e'Ae'} eAe'=1$. By Corollary 1.12, $\operatorname{rrk}_A eA=\operatorname{rrk}_{e'Ae'} eAe'=1$, so $\operatorname{disc}(f)=\operatorname{disc}(f_e)=\operatorname{disc}(f_{e'})$.

(iii) We may replace f and f' with f_e and f'_e and assume that $A=S$. The statement is now straightforward. □

We continue to assume that $[A]=0$ and (σ,ε) is unitary. Let $S=\mathrm{Z}(A)$ and $\theta=\sigma|_S$. Recall that with every $\alpha\in R^\times$, we can associate a crossed produced R-algebra

$$(S/R,\alpha).$$

Its underlying R-module is the free right S-module with basis $\{1,u\}$ and its multiplication is determined by the product in S and the rules $u^2=\alpha$ and $su=us^\theta$ for all $s\in S$. It is well-known that $(S/R,\alpha)$ is a quaternion (i.e. degree-2) Azumaya R-algebra. Moreover, the map

$$\alpha\mapsto[(S/R,\alpha)]$$

determines a group homomorphism from $R^\times / \operatorname{Nr}_{S/R}(S^\times)$ to $\operatorname{Br} R$; see [64, Theorem 7.1a] or [39, Lemma III.5.4.1, Corollary III.5.4.6].

Following [40, §10], given $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ of even reduced rank, we write

$$D(f) = (S/R, \operatorname{disc} f)$$

and define the *discriminant Brauer class* of f to be $[D(f)]$.[8] We remark that since R is semilocal, using the discriminant Brauer class instead of the discriminant causes no loss of information:

Proposition 2.29. *Assume R is semilocal, let S be a quadratic étale R-algebra and let $\alpha, \beta \in R^\times$. Then $[(S/R, \alpha)] = [(S/R, \beta)]$ if and only if $\alpha \equiv \beta \bmod \operatorname{Nr}_{S/R}(S^\times)$.*

Proof. We only need to check the "only if" part. Write $A = (S/R, \alpha)$, $B = (S/R, \beta)$ and let θ denote the standard R-involution of S. We have $A = S \oplus uS$ with $u^2 = \alpha$ and $B = S \oplus vS$ with $v^2 = \beta$. Since R is semilocal and $\deg A = \deg B$, there exists an R-algebra isomorphism $\iota : A \to B$, see [61, Corollary 3.3].

We claim that there is an isomorphism $\psi : A \to B$ which restricts to the identity on S. To see this, view A as a right A_S-module via $x \cdot (a \otimes s) = sxa$ and B as a right A_S-module via $y \cdot (a \otimes s) = sy \cdot \iota a$ ($x \in A$, $y \in B$, $a \in A$, $s \in S$). Since $\operatorname{rk}_S({}_A A) = 2 = \operatorname{rk}_S({}_B B)$, we have $\operatorname{rrk}_{A_S} A = \operatorname{rrk}_{A_S} B$. By Lemma 1.24, there exists an A_S-module isomorphism $\xi : A \to B$. It induces an isomorphism $\operatorname{Int}(\xi) : \operatorname{End}_A(A_A) \to \operatorname{End}_B(B_B)$. Identifying $\operatorname{End}_A(A_A)$ with A and $\operatorname{End}_B(B_B)$ with B via $\varphi \mapsto \varphi(1)$, we get an isomorphism $\psi : A \to B$. Now, for all $s \in S$, we have $\psi(s) = (\xi \circ [x \mapsto sx] \circ \xi^{-1})(1_B) = \xi(s \cdot \xi^{-1}(1_B)) = s \cdot \xi^{-1}\xi(1_B) = s$.

Let $E_A = \{a \in A : sa = as^\theta \text{ for all } s \in S\}$ and define E_B similarly. One readily checks that $E_A = uS$ and $E_B = vS$. Since ψ fixes S, we have $\psi(E_A) \subseteq E_B$. Thus, $\psi(u) = vs$ for some $s \in S^\times$. Now, $\alpha = u^2 = \psi(u^2) = (vs)^2 = \operatorname{Nr}_{S/R}(s) v^2 = \operatorname{Nr}_{S/R}(s)\beta$. □

3. An Octagon of Witt Groups

In this section, we introduce an 8-periodic chain complex — an octagon, for short — of Witt groups of Azumaya algebras with involution, generalizing a similar octagon defined by Grenier-Boley and Mahmoudi for central simple algebras with involution [30]. By the end of Section 7, we will show that this octagon is exact when the base ring R is semilocal.

3A. **The Octagon.** Recall that R denotes a ring with $2 \in R^\times$. Suppose we are given the following data:

- (G1) (A, σ) is an Azumaya R-algebra with involution (see 1D),
- (G2) $\varepsilon \in \mathrm{Z}(A)$ satisfies $\varepsilon^\sigma \varepsilon = 1$,
- (G3) $\lambda, \mu \in A^\times$ satisfy $\lambda^\sigma = -\lambda$, $\mu^\sigma = -\mu$, $\lambda\mu = -\mu\lambda$ and $\lambda^2 \in \mathrm{Z}(A)$.

Define the following:

- (N1) $S = \mathrm{Z}(A)$,
- (N2) B is the commutant of λ in A,
- (N3) $T = \mathrm{Z}(B)$,
- (N4) $\tau_1 := \sigma|_B$,
- (N5) $\tau_2 := \operatorname{Int}(\mu^{-1}) \circ \sigma|_B$, i.e. $x^{\tau_2} = \mu^{-1} x^\sigma \mu$

Note that $R \subseteq S \subseteq T \subseteq B \subseteq A$ and τ_1, τ_2 are R-linear involutions on B. Also, $\lambda^2 \in R$ because $(\lambda^2)^\sigma = (-\lambda)^2 = \lambda^2$ and $R = \mathcal{S}_1(\mathrm{Z}(A), \sigma)$.

Lemma 3.1. *In the previous notation, the following hold:*

[8] When R is a field, our definition of $D(f)$ does not agree with the definition given in [40, §10]. However, both definitions give the same Brauer class by [40, Corollary 10.35].

(i) T is a quadratic étale S-algebra, $\{1,\lambda\}$ is an S-basis of T and $\mathrm{rk}_T(A_A)$ is constant along the fibers of $\mathrm{Spec}\,T \to \mathrm{Spec}\,S$.

(ii) B is an Azumaya T-algebra, $[B]=[A\otimes_S T]$ in $\mathrm{Br}\,T$ and $\deg B = \frac{1}{2}\iota\deg A$, where $\iota : S\to T$ is the inclusion map.

(iii) $A = B\oplus \mu B$, $\mu B = B\mu$ and $\mu^2\in B$.

Proof. Write $T' = S[\lambda]$. Since $\lambda^2\in S$, we have $T' = S+\lambda S$. If $a\in S\cap\lambda S$, then a commutes and anti-commutes with μ, so $2a\mu = 0$ and $a = 0$. Since $\mathrm{ann}_S\,\lambda = 0$, this means that $T'\cong S[x]/(x^2-\lambda^2)$. Thus, T' is a quadratic étale S-algebra (see 1C).

We claim that $\mathrm{rk}_{T'} A_A$ is constant along the fibers of $\mathrm{Spec}\,T'\to\mathrm{Spec}\,S$. To see this, note that $\mu\lambda\mu^{-1} = -\lambda$, hence $\mathrm{Int}(\mu)|_{T'}$ coincides with the standard S-involution of T', call it θ. This involution acts transitively on every fiber of $\mathrm{Spec}\,T'\to\mathrm{Spec}\,S$, so it is enough to show that $\mathrm{rk}_{T'} A_A = \theta\,\mathrm{rk}_{T'} A_A$. However, this follows from the fact that $\mathrm{Int}(\mu) : A\to A$ defines θ-linear isomorphism from A, viewed as a right T'-module, to itself.

Now, by Proposition 1.14, $B = \mathrm{Z}_A(T')$ is an Azumaya T'-algebra, $[B] = [A\otimes_S T']$, and $2\deg B = \deg B\cdot \iota\,\mathrm{rk}_S T' = \iota\deg A$, where $\iota : S\to T'$ is the inclusion map. Since $T = \mathrm{Z}(B) = T'$, we have established (i) and (ii).

To prove (iii), let E denote the set of elements of A which anti-commute with λ. One readily checks that $\mu B\subseteq E$ and $\mu^{-1}E\subseteq B$, hence $E = \mu B$. Likewise, $E = B\mu$, so $\mu B = B\mu$. Furthermore, $B\cap\mu B = B\cap E$ consists of elements which commute and anti-commutes with λ, so $B\cap \mu B = 0$ (because $2\lambda\in A^\times$). Finally, every $a\in A$ can be written as $\frac{1}{2}(a+\lambda^{-1}a\lambda)+\frac{1}{2}(a-\lambda^{-1}a\lambda)$. It is easy to check, using $\lambda^2\in\mathrm{Z}(A)$, that $a+\lambda^{-1}a\lambda\in B$ and $a-\lambda^{-1}a\lambda\in E$, so $A = B+E = B+\mu B$. We conclude that $A = B\oplus\mu B$. Finally, $\mu^2\in B$ because $\mu^2\lambda = -\mu\lambda\mu = \lambda\mu^2$. □

Before proceeding, let us present some examples where (G1)–(G3) hold.

Example 3.2. (i) Let $\alpha,\beta\in R^\times$. Take A to be the quaternion Azumaya R-algebra $R\langle\lambda,\mu\,|\,\lambda\mu=-\mu\lambda, \lambda^2=\alpha, \mu^2=\beta\rangle$ and σ to be the R-involution of A determined by $\lambda^\sigma = -\lambda$ and $\mu^\sigma = -\mu$. Then A,σ,λ,μ and any $\varepsilon\in\mu_2(R)$ satisfy (G1)–(G3). In this case, $\{1,\lambda,\mu,\lambda\mu\}$ is an R-basis of A, and it is routine to check that $S = R$, $B = R+\lambda R = R[\lambda]$, τ_1 is the standard R-involution of $T = B$, and $\tau_2 = \mathrm{id}_B$.

(ii) Write $A_0,\sigma_0,\lambda_0,\mu_0$ for A,σ,λ,μ defined in (i) and let (A_1,σ_1) be another Azumaya R-algebra with involution. Then $(A,\sigma) := (A_0\otimes A_1, \sigma_0\otimes\sigma_1)$ is also Azumaya over R because $\mathrm{Z}(A) = \mathrm{Z}(A_0)\otimes\mathrm{Z}(A_1) = R\otimes\mathrm{Z}(A_1)$ (see [60, Propositio 5.3.10(ii)] for the first equality), which means that $R = \{a\in\mathrm{Z}(A) : a^\sigma = a\}$. Then $A,\sigma,\lambda := \lambda_0\otimes 1, \mu := \mu_0\otimes 1$ and any $\varepsilon\in\mu_2(R)$ satisfy (G1)–(G3). Writing $S_1 = \mathrm{Z}(A_1)$ and letting θ denote the standard R-involution of $R[\lambda_0]$, we have $S = R\otimes S_1$, $B = R[\lambda_0]\otimes A_1$, $T = R[\lambda_0]\otimes S_1$ (Lemma 1.4), $\tau_1 = \theta\otimes\sigma_1$, and $\tau_2 = \mathrm{id}_{R[\lambda_0]}\otimes\sigma_1$.

Using Lemma 3.1(iii), we can define the following maps:

(N6) $\pi_1,\pi_2 : A\to B$ are defined by $\pi_i(b_1+\mu b_2) = b_i$ ($b_1,b_2\in B$, $i\in\{1,2\}$).

(For a definition of π_1 not involving μ, see Lemma 4.2(i) below.) We now introduce four functors:

(N7) For $i=1,2$, let $\pi_i^{(\varepsilon)} : \mathcal{H}^\varepsilon(A,\sigma)\to\mathcal{H}^{(-1)^{i+1}\varepsilon}(B,\tau_i)$ be defined by $\pi_i^{(\varepsilon)}(P,f) = (P,\pi_i f)$, where $\pi_i f = \pi_i\circ f$; morphisms are mapped to themselves.

(N8) For $i=1,2$, let $\rho_i^{(\varepsilon)} : \mathcal{H}^\varepsilon(B,\tau_i)\to\mathcal{H}^{-\varepsilon}(A,\sigma)$ be defined by $\rho_i^{(\varepsilon)}(Q,g) = (Q\otimes_B A, \rho_i g)$, where $\rho_i g : (Q\otimes_B A)\times(Q\otimes_B A)\to A$ is determined by

$$(\rho_1 g)(x\otimes a, x'\otimes a') = a^\sigma\lambda g(x,x')a',$$
$$(\rho_2 g)(x\otimes a, x'\otimes a') = a^\sigma(\lambda\mu) g(x,x')a'$$

($x,x'\in Q$, $a,a'\in A$); for a morphism φ, set $\rho_i^{(\varepsilon)}\varphi = \varphi\otimes_B\mathrm{id}_A$.

When there is no risk of confusion, we will drop the superscript "(ε)". The functors $\pi_1, \pi_2, \rho_1, \rho_2$ are well-defined by the following lemma.

Lemma 3.3. *The assignments $\pi_1^{(\varepsilon)}$, $\pi_2^{(\varepsilon)}$, $\rho_1^{(\varepsilon)}$, $\rho_2^{(\varepsilon)}$ are functors. Moreover, they take hyperbolic hermitian forms to hyperbolic hermitian forms.*

Proof. Everything is straightforward except the fact that $\pi_1, \pi_2, \rho_1, \rho_2$ take unimodular hermitian forms to unimodular hermitian forms. We verify this fact case-by-case.

The inclusion $B \to A$ induces a homomorphisms of R-algebras with involution $(B, \tau_1) \to (A, \sigma)$, which we denote by ρ. Given $(Q, g) \in \mathcal{H}^\varepsilon(B, \tau_1)$, the hermitian space $\rho_1(Q, g)$ is just $\rho(Q, \lambda g)$, where λg is the λ-conjugation of g, see 2G, and ρ is base change in the sense of 2C. Since both λ-conjugation and base change preserve unimodularity, $\rho_1(Q, g)$ is unimodular.

Similarly, to see that $\rho_2(Q, g)$ is unimodular, let $\sigma_2 := \mathrm{Int}((\lambda\mu)^{-1}) \circ \sigma$ and note that the inclusion $B \to A$ also defines a morphism $\rho' : (B, \tau_2) \to (A, \sigma_2)$. It is straightforward to check that $\rho_2 g = (\lambda\mu)(\rho' g)$, so $\rho_2 g$ is unimodular.

We proceed with checking that $\pi_1 f$ is unimodular for all $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$. If $x \in P$ satisfies $\pi_1 f(x, P) = 0$, then $f(x, P)$ is a right ideal of A contained in $\ker \pi = B\mu$. Thus, $f(x, P) \subseteq B\mu \cap (B\mu)\mu^{-1} = 0$, and $x = 0$ because f is unimodular. Suppose now that $\phi \in \mathrm{Hom}_B(P, B)$. Define $\hat{\phi} : P \to A$ by $\hat{\phi}x = \phi x + \phi(x\mu)\mu^{-1}$. It is routine to check that $\hat{\phi} \in \mathrm{Hom}_A(P, A)$ and $\pi_1 \circ \hat{\phi} = \phi$. Since f is unimodular, there exists $x \in P$ with $\hat{\phi} = f(x, -)$, so $\phi = \pi_1 f(x, -)$.

That $\pi_2 f$ is unimodular is shown similarly; define $\hat{\phi}$ by $\hat{\phi}x = \mu \cdot \phi(x\mu) \cdot \mu^{-1} + \mu\phi x$. □

Lemma 3.3 implies that $\pi_1^{(\varepsilon)}$, $\pi_2^{(\varepsilon)}$, $\rho_1^{(\varepsilon)}$, $\rho_2^{(\varepsilon)}$ induce maps between the relevant Witt groups. These maps can be arranged in an octagon-shaped diagram:

$$
\begin{array}{ccccccc}
W_\varepsilon(A,\sigma) & \xrightarrow{\pi_1^{(\varepsilon)}} & W_\varepsilon(B,\tau_1) & \xrightarrow{\rho_1^{(\varepsilon)}} & W_{-\varepsilon}(A,\sigma) & \xrightarrow{\pi_2^{(-\varepsilon)}} & W_\varepsilon(B,\tau_2) \\
{\scriptstyle \rho_2^{(-\varepsilon)}}\uparrow & & & & & & \downarrow{\scriptstyle \rho_2^{(\varepsilon)}} \\
W_{-\varepsilon}(B,\tau_2) & \xleftarrow[\pi_2^{(\varepsilon)}]{} & W_\varepsilon(A,\sigma) & \xleftarrow[\rho_1^{(-\varepsilon)}]{} & W_{-\varepsilon}(B,\tau_1) & \xleftarrow[\pi_1^{(-\varepsilon)}]{} & W_{-\varepsilon}(A,\sigma)
\end{array}
\tag{3.1}
$$

We will see in Proposition 3.5 below that the octagon is a chain complex of abelian groups.

The octagon is known to be exact when R is a field [30]; see the Introduction for the history of this result. The purpose of this paper is to extend the exactness of the octagon to semilocal rings. Specifically, we prove:

Theorem 3.4. *Suppose that R is semilocal. Then the octagon* (3.1) *is exact.*

The proof will occupy the following four sections and be concluded in Section 7; its highlights are given in 3C. In the course of the proof, we will also determine the images of the functors $\pi_1, \pi_2, \rho_1, \rho_2$ when T is connected semilocal (the exactness of the octagon answers this only up to Witt equivalence), see Theorem 7.1. This finer version will be required for some of the applications.

The remainder of this section is dedicated to proving that the octagon is a complex, providing equivalent conditions for its exactness, and surveying how these conditions will be proved under the assumption that R is semilocal.

3B. **Equivalent Conditions for Exactness.** Keep the assumptions of 3A. In this subsection, we show that the exactness of the octagon (3.1) is equivalent to a certain list of conditions involving $R, A, \sigma, \varepsilon, \lambda, \mu$. The proof generally follow the same lines as the corresponding arguments given in [30, §3] and [10, Appendix], both addressing the case S is a field.

Given a right B-module Q and $a \in A$, we write $Q \otimes a$ for the subset $\{q \otimes a \mid q \in Q\}$ of $Q \otimes_B A$; it is a B-submodule of $Q \otimes_B A$ if $aB \subseteq Ba$. If M is a subset of a right A-module P, we write MA for the A-submodule generated by M.

We begin by showing that the octagon is a chain complex for any ring R.

Proposition 3.5. *In the notation of 3A,* (3.1) *is a chain complex of abelian groups.*

Proof. By symmetry, we only need to consider the top row of (3.1).

(3.1) *is a complex at* $W_\varepsilon(A, \sigma)$*.* Let $(Q, g) \in \mathcal{H}^{-\varepsilon}(B, \tau_2)$. Then $\pi_1\rho_2(Q, g) = (Q \otimes_B A, \pi_1\rho_2 g)$. Straightforward calculation shows that the B-sumodules $M_1 := Q \otimes 1$ and $M_2 := Q \otimes \mu$ satisfy $\pi_1\rho_2 g(M_1, M_1) = \pi_1\rho_2 g(M_2, M_2) = 0$ and $M_1 + M_2 = Q \otimes_B A$. Thus, $\pi_1\rho_2(Q, g)$ is hyperbolic.

(3.1) *is a complex at* $W_{-\varepsilon}(A, \sigma)$*.* The proof for $W_\varepsilon(A, \sigma)$ applies verbatim.

(3.1) *is a complex at* $W_\varepsilon(B, \tau_1)$*.* Let $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$. Then $\rho_1\pi_1(P, f) = (P \otimes_B A, \rho_1\pi_1 f)$. Define

$$L_1 = \{x\mu \otimes 1 + x \otimes \mu \mid x \in P\},$$
$$L_2 = \{x\mu \otimes 1 - x \otimes \mu \mid x \in P\}.$$

It is easy to check that L_1 and L_2 are B-submodules of $P \otimes_A B$ and that $L_1 + L_2 = P \otimes_A B$ (recall that $2 \in R^\times$). We claim that $\rho_1\pi_1 f(L_1, L_1) = \rho_1\pi_1 f(L_2, L_2) = 0$. Indeed, let $x, y \in P$ and write $f(x, y) = \alpha + \mu\beta$ with $\alpha, \beta \in B$. Then

$$\begin{aligned}
&\rho_1\pi_1 f(x\mu \otimes 1 + x \otimes \mu, y\mu \otimes 1 + y \otimes \mu)\\
&\qquad = \rho_1\pi_1 f(x\mu \otimes 1, y\mu \otimes 1) + \rho_1\pi_1 f(x\mu \otimes 1, y \otimes \mu) + \rho_1\pi_1 f(x \otimes \mu, y\mu \otimes 1)\\
&\qquad\quad + \rho_1\pi_1 f(x \otimes \mu, y \otimes \mu)\\
&\qquad = \lambda\pi_1(\mu^\sigma(\alpha + \mu\beta)\mu) + \lambda\pi_1(\mu^\sigma(\alpha + \mu\beta))\mu + \mu^\sigma\lambda\pi_1((\alpha + \mu\beta)\mu)\\
&\qquad\quad + \mu^\sigma\lambda\pi_1(\alpha + \mu\beta)\mu\\
&\qquad = \lambda\mu^\sigma\alpha\mu + \lambda\mu^\sigma\mu\beta\mu + \mu^\sigma\lambda\mu\beta\mu + \mu^\sigma\lambda\alpha\mu\\
&\qquad = -\lambda\mu\alpha\mu - \lambda\mu^2\beta\mu + \lambda\mu^2\beta\mu + \lambda\mu\alpha\mu = 0,
\end{aligned}$$

hence $\rho_1\pi_1 f(L_1, L_1) = 0$. Likewise, $\rho_1\pi_1 f(L_2, L_2) = 0$, so $\rho_1\pi_1 f$ is hyperbolic.

(3.1) *is a complex at* $W_\varepsilon(B, \tau_2)$*.* This is similar to the proof of the case $W_\varepsilon(B, \tau_1)$; define L_1 and L_2 in the same manner. □

We now give equivalent conditions for the exactness of the octagon (3.1).

Theorem 3.6. *With the notation of 3A, consider the following conditions:*

- *(E1) For every $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ such that $[\pi_1(P, f)] = 0$ in $W_\varepsilon(B, \tau_1)$, there exists (P', f') in the Witt class of (P, f) and a Lagrangian M of $\pi_1(P', f')$ with $M \cdot A = P'$.*
- *(E2) For every $(Q, g) \in \mathcal{H}^\varepsilon(B, \tau_1)$ such that $[\rho_1(Q, g)] = 0$ in $W_{-\varepsilon}(A, \sigma)$, there exists (Q', g') in the Witt class of (Q, g) and a Lagrangian L of $\rho_1(Q', g')$ with $L \oplus (Q' \otimes 1) = Q' \otimes_B A$ as B-modules.*
- *(E3) For every $(P, f) \in \mathcal{H}^{-\varepsilon}(A, \sigma)$ such that $[\pi_2(P, f)] = 0$ in $W_\varepsilon(B, \tau_2)$, there exists (P', f') in the Witt class of (P, f) and a Lagrangian M of $\pi_2(P', f')$ with $M \cdot A = P'$.*
- *(E4) For every $(Q, g) \in \mathcal{H}^\varepsilon(B, \tau_2)$ such that $[\rho_2(Q, g)] = 0$ in $W_{-\varepsilon}(A, \sigma)$, there exists (Q', g') in the Witt class of (Q, g) and a Lagrangian L of $\rho_2(Q', g')$ with $L \oplus (Q' \otimes 1) = Q' \otimes_B A$ as B-modules.*

Then the exactness of (3.1) *at the terms* $W_\varepsilon(A,\sigma)$, $W_\varepsilon(B,\tau_1)$, $W_{-\varepsilon}(A,\sigma)$, $W_\varepsilon(B,\tau_2)$ *on the top row is equivalent to the conditions (E1), (E2), (E3), (E4), respectively.*

Remark 3.7. Conditions (E1)–(E4) are not difficult to verify when R is a field. We illustrate this for (E2) and (E1).

In the context of (E2), using Proposition 2.5, choose (Q',g') to be anisotropic. Since B is semisimple artinian, this means that $g'(x,x)\neq 0$ whenever $x\neq 0$. Now, if L is a Lagrangian of $\rho_1 g'$, then every $x\in L\cap(Q'\otimes 1)$ satisfies $g'(x,x)=0$, hence $L\cap(Q'\otimes 1)=0$. On the other hand, by Lemma 2.6, we have $2\dim_R L=\dim_R L+\dim_R L^*=\dim_R(Q'\otimes_B A)$, and since $Q'\otimes_B A=(Q'\otimes 1)\oplus(Q'\otimes\mu)$, we have $\dim_R(Q'\otimes 1)=\frac{1}{2}\dim_R(Q'\otimes_B A)$, so R-dimension considerations force $L\oplus(Q'\otimes 1)=Q'\otimes_B A$.

Similarly, in the context of (E1), we may choose (P',f') so that $f'(x,x)\neq 0$ whenever $x\neq 0$. If M is a Lagrangian of $\pi_1 f'$, then every $x\in M\cap M\mu$ satisfies $\pi_1 f(x,x)=\pi_1 f(x\mu^{-1},x)=0$, which means that $f(x,x)=0$. Thus, $M\cap M\mu=0$. Since $P\cong M\oplus M^*$ (the dual is taken relative to B), $\mathrm{rk}_{T_0}P=\mathrm{rk}_{T_0}M+\mathrm{rk}_{T_0}M^*$, where $T_0=\mathcal{S}_1(T,\tau)$. By Lemma 2.6, $2\,\mathrm{rk}_{T_0}M=\mathrm{rk}_{T_0}P$, so $2\dim_R M=\dim_R P$ and R-dimension considerations force $M+M\mu=P$. In particular, $MA=P$.

This argument relies critically on the fact that R is a field, and thus cannot be naively generalized to more general rings.

Before proving Theorem 3.6, we first prove the following lemma.

Lemma 3.8. *With notation as in 3A, let* $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma)$ *and let* M *be a Lagrangian of* $\pi_1 f$*, resp.* $\pi_2 f$*. Then* $MA=P$ *if and only if* $M\oplus M\mu=P$*.*

Proof. It is clear that $M\oplus M\mu=P$ implies $MA=P$, so we prove the converse. As both M and $M\mu$ are R-summands of P, the lemma will follow from Lemma 1.7 if we show that $M(\mathfrak{m})\oplus M\mu(\mathfrak{m})=P(\mathfrak{m})$ for all $\mathfrak{m}\in\operatorname{Max}R$. We may therefore assume that R is a field (the setting of 3A is preserved under base-change by Lemma 1.4). Since $MA=P$ and $A=B+B\mu$, we have $M+M\mu=P$. We observed in Remark 3.7 that $2\dim_R M=\dim_R P$, so this means that $M\oplus M\mu=P$. □

Proof of Theorem 3.6. We showed that the octagon is a chain complex in Proposition 3.5. Moreover, the proof of that proposition shows that if $(Q,g)=\pi_1(P,f)$ for $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma)$, then $\rho_1(Q,f)$ admits a Lagrangian L — L_1 or L_2 in the notation of that proof — with $L\oplus(Q\otimes 1)=Q\otimes_B A$. Thus, condition (E2) follows from the exactness of the octagon at $W_\varepsilon(B,\tau_1)$, and, in a similar manner, the exactness of the octagon at $W_\varepsilon(A,\sigma)$, $W_{-\varepsilon}(A,\sigma)$, $W_\varepsilon(B,\tau_2)$ implies (E1), (E3), (E4), respectively. It remains to show the converse.

(E1) implies exactness at $W_\varepsilon(A,\sigma)$ *(top row).* Suppose that $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma)$ satisfies $[\pi_1(P,f)]=0$ in $W_\varepsilon(B,\tau_1)$. By (E1) and Lemma 3.8, we may replace (P,f) with a Witt equivalent hermitian space to assume that $\pi_1 f$ admits a Lagrangian M with $P=M\oplus M\mu$. Let $g=(\lambda\mu)^{-1}f|_{M\times M}$. Since $\pi_1 f(M,M)=0$, we have $g(M,M)\subseteq(\lambda\mu)^{-1}\ker\pi_1=-\lambda^{-1}\mu^{-1}\mu B=B$. We claim that $g:M\times M\to B$ is a $(-\varepsilon)$-hermitian form over (B,τ_2). Indeed, the sesquilinearily is straightforward, and for all $x,y\in M$, we have

$$\begin{aligned}-\varepsilon g(y,x)^{\tau_2}&=-\varepsilon\mu^{-1}f(y,x)^{\sigma}((\lambda\mu)^{-1})^{\sigma}\mu\\&=-\mu^{-1}f(x,y)\lambda^{-1}\mu^{-1}\mu=\mu^{-1}\lambda^{-1}f(x,y)=g(x,y)\end{aligned}$$

(note that $f(x,y)\in\ker\pi_1=\mu B$ and λ anti-commutes with elements from μB).

Next, we claim that (M,g) is unimodular. Suppose that $g(x,M)=0$. Then $f(x,M)=0$, hence $f(x,P)=f(x,M+M\mu)=0$, and $x=0$ by the unimodularity of f. Now, let $\phi\in\operatorname{Hom}_B(M,B)$. Using $P=M\oplus M\mu$, define $\psi:P\to A$ by

$\psi(x+y\mu) = \lambda\mu\cdot\phi x + \lambda\mu\cdot\phi y\cdot\mu$ for all $x,y\in M$. It is routine to check that $\psi\in\mathrm{Hom}_A(P,A)$. Thus, there exists $x\in P$ such that $\psi y = f(x,y)$ for all $y\in P$. Furthermore, we have $\pi_1 f(x,M) = \pi_1(\lambda\mu\cdot\phi(M)) = 0$, hence $x\in M^{\perp(\pi_1 f)} = M$. Since $g(x,y) = (\lambda\mu)^{-1}f(x,y) = (\lambda\mu)^{-1}\psi y = \phi y$ for all $y\in M$, we have shown that $x\mapsto g(x,-) : M\to\mathrm{Hom}_B(M,B)$ is bijective.

Finally, we show that $\rho_2^{(-\varepsilon)}(M,g) = (M\otimes_B A, \rho_2 g)$ is isomorphic to (P,f). Let φ denote the A-module homomorphism $x\otimes a\mapsto xa : M\otimes_B A\to P$; it is clearly surjective, and it is straightforward to check that $f(\varphi x,\varphi y) = \rho_2 g(x,y)$ for all $x,y\in M\otimes_B A$. The latter means that if $x\in\ker\varphi$, then $\rho_2 g(x, M\otimes_B A) = 0$, and thus $x=0$ because $\rho_2 g$ is unimodular. As a result, φ is injective, and therefore an isometry from $\rho_2^{(-\varepsilon)}(M,g)$ to (P,f).

(E3) implies exactness at $W_{-\varepsilon}(A,\sigma)$ (top row). This similar to the exactness at $W_\varepsilon(A,\sigma)$ with the difference that one defines $g = \lambda^{-1}f|_{M\times M}$.

(E2) implies exactness at $W_\varepsilon(B,\tau_1)$. Let $(Q,g)\in\mathcal{H}^\varepsilon(B,\tau_1)$ be a hermitian space such that $[\rho_1(Q,g)] = 0$ in $W_{-\varepsilon}(A,\sigma)$. By (E2), we may replace (Q,g) with a Witt-equivalent space and assume that $\rho_1 g$ admits a Lagrangian L such that such that $L\oplus(Q\otimes 1) = Q\otimes_B A$. Multiplying both sides with μ yields $L\oplus(Q\otimes\mu) = Q\otimes_B A$. This means that every $x\in Q$ admits a unique element $Jx\in Q$ such that $x\otimes 1 + Jx\otimes\mu\in L$.

The map $J : Q\to Q$ is easily seen to satisfy:

$$J^2x = x\mu^{-2} \qquad (3.2)$$
$$J(xb) = (Jx)(\mu b\mu^{-1})$$

for all $x\in Q$, $b\in B$. We make Q into an A-module by setting

$$x(b_1+\mu b_2) = xb_1 + (Jx)\mu^2 b_2 \qquad (x\in Q,\ b_1,b_2\in B).$$

Using (3.2), it is easy to see that this indeed defines an A-module structure (one has to verify the identities $(x\mu)\mu = x(\mu^2)$ and $(xb)\mu = (x\mu)(\mu^{-1}b\mu)$ for $b\in B$).

Since $\rho_1 g(L,L) = 0$, we have $\rho_1 g(x\otimes 1 + Jx\otimes\mu, y\otimes 1 + Jy\otimes\mu) = 0$ for all $x,y\in Q$. Since $A = B\oplus\mu B$, this means that

$$g(x,y) + \mu g(Jx,Jy)\mu = 0, \qquad (3.3)$$
$$g(x,Jy)\mu + \mu g(Jx,y) = 0$$

for all $x,y\in Q$. Define $f : Q\times Q\to A$ by

$$f(x,y) = g(x,y) - \mu g(Jx,y)\ .$$

We claim that f is an ε-hermitian form over (A,σ). After unfolding the definitions, this comes down to checking that $f(xb,y) = b^\sigma f(x,y)$, $f(x,yb) = f(x,y)b$, $f(x\mu,y) = \mu^\sigma f(x,y)$, $f(x,y\mu) = f(x,y)\mu$ and $f(x,y) = \varepsilon f(y,x)^\sigma$ for all $x,y\in Q$, $b\in B$. The first three identities follow easily from (3.2). For the fourth and fifth identities we also use (3.3):

$$\begin{aligned}
f(x,y\mu) &= g(x,y\mu) - \mu g(Jx,y\mu)\\
&= g(x,(Jy)\mu^2) - \mu g(Jx,(Jy)\mu^2)\\
&= g(x,Jy)\mu^2 - \mu g(Jx,Jy)\mu^2\\
&= -\mu g(Jx,y)\mu + g(x,y)\mu\\
&= (g(x,y) - \mu g(Jx,y))\mu = f(x,y)\mu\ ,
\end{aligned}$$

$$\begin{aligned}\varepsilon f(y,x)^\sigma &= \varepsilon g(y,x)^\sigma - \varepsilon g(Jy,x)^\sigma\mu^\sigma\\ &= g(x,y)+g(x,Jy)\mu\\ &= g(x,y)-\mu g(Jx,y)=f(x,y)\ .\end{aligned}$$

We claim that (Q_A, f) is unimodular. Indeed, suppose $f(x,Q)=0$. Then, the definition of f implies that $g(x,Q)=0$, so $x=0$ by the unimodularity of g. Now let $\phi\in \mathrm{Hom}_A(Q,A)$. Then $\pi_1\circ\phi\in\mathrm{Hom}_B(Q,B)$, hence there exists $x\in Q$ such that $\pi_1\phi y = g(x,y)$ for all $y\in Q$. Define $\psi y := \phi y - f(x,y)$. Then $\psi\in\mathrm{Hom}_A(Q,A)$ satisfies $\mathrm{im}\,\psi\subseteq\ker\pi_1=\mu B$. Since $\mathrm{im}\,\psi$ is a right ideal in A and $b\mu\notin\mu B$ for all $0\neq b\in\mu B$, it follows that $\psi=0$ and $\phi y = f(x,y)$ for all $y\in Q$.

Finally, it is clear that $\pi_1(Q,f)=(Q,g)$, so we have verified the exactness at $W^\varepsilon(B,\tau_1)$.

(E4) implies exactness at $W_\varepsilon(B,\tau_2)$. Let $(Q,g)\in W^\varepsilon(B,\tau_2)$ be a hermitian space such that $[\rho_2(Q,g)]=0$ in $W_\varepsilon(A,\sigma)$. By (E4), we can replace (Q,g) with a Witt equivalent space and assume that $\rho_2 g$ admits a Lagrangian L such that such that $L\oplus(Q\otimes 1)=Q\otimes_B A$.

Define $\sigma_2=\mathrm{Int}(\mu^{-1})\circ\sigma$. Then μ^{-1}-conjugation (see 2G) induces a group isomorphism $(P,f)\mapsto(P,\mu^{-1}f) : W_{-\varepsilon}(A,\sigma)\to W_\varepsilon(A,\sigma_2)$. Furthermore, one readily checks that $\pi_2(P,f)=\pi_1(P,\mu^{-1}f)$. Thus, it is enough to show that there exists $(P,f)\in W_\varepsilon(A,\sigma_2)$ with $\pi_1(P,f)=(Q,g)$. This can be shown exactly as in the proof that (E2) implies exactness at $W_\varepsilon(B,\tau_1)$. □

Remark 3.9. In the course of proving Theorem 3.6, we also showed:

(i) Given $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma)$, there exists $(Q,g)\in\mathcal{H}^{-\varepsilon}(B,\tau_2)$ with $\rho_2 g\cong f$ if and only if $\pi_1 f$ admits a Lagrangian M for which $M\cdot A=P$.

(ii) Given $(Q,g)\in\mathcal{H}^\varepsilon(B,\tau_1)$, there exists $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma)$ with $\pi_1 f\cong g$ if and only if $\rho_1 f$ admits a Lagrangian L for which $L\oplus(Q\otimes 1)=Q\otimes_B A$.

(iii) Given $(P,f)\in\mathcal{H}^{-\varepsilon}(A,\sigma)$, there exists $(Q,g)\in\mathcal{H}^\varepsilon(B,\tau_1)$ with $\rho_1 g\cong f$ if and only if $\pi_2 f$ admits a Lagrangian M for which $M\cdot A=P$.

(iv) Given $(Q,g)\in\mathcal{H}^\varepsilon(B,\tau_2)$, there exists $(P,f)\in\mathcal{H}^{-\varepsilon}(A,\sigma)$ with $\pi_2 f\cong g$ if and only if $\rho_2 f$ admits a Lagrangian L for which $L\oplus(Q\otimes 1)=Q\otimes_B A$.

3C. **Overview of The Proof of Theorem 3.4.** Keep the notation of 3A and suppose that R is semilocal. Thanks to Theorem 3.6 and the antipodal symmetry of (3.1), in order to prove Theorem 3.4, it is enough to establish the conditions (E1)–(E4). The proof is somewhat involved, so we outline the argument first.

Let us consider condition (E2): We are given $(Q,g)\in\mathcal{H}^\varepsilon(B,\tau_1)$ such that $[\rho_1 g]=0$ in $\mathcal{H}^{-\varepsilon}(A,\sigma)$ and need to find a Lagrangian L of $\rho_1 g$ such that $(Q\otimes 1)\oplus L = Q\otimes_B A$, possibly after replacing (Q,g) with a Witt equivalent hermitian space. We abbreviate $Q\otimes_B A$ to QA and identify Q with its copy $Q\otimes 1$ in QA.

Fix a Lagrangian L' of $\rho_1 g$; it exists by Theorem 2.8(ii). We assume that $\mathrm{rrk}_A L'=\frac{1}{2}\mathrm{rrk}_A P$ for simplicity, so that $L'\in\mathrm{Lag}(\rho_1 g)$ (see 2F). Let $\mathfrak{m}_1,\dots,\mathfrak{m}_t$ denote the maximal ideals of R. Suppose that we can find, for every $1\le i\le t$, an isometry $\varphi_i\in U^0(\rho_1 g(\mathfrak{m}_i))$ such that $Q(\mathfrak{m}_i)\oplus\varphi_i(L'(\mathfrak{m}_i))=QA(\mathfrak{m}_i)$. Then, by Theorem 2.18, there exists $\varphi\in U^0(\rho_1 g)$ with $\varphi(\mathfrak{m}_i)=\varphi_i$, and by Lemma 1.7, $Q\oplus\varphi(L')=QA$. We may therefore take $L=\varphi(L')$ and the proof of (E2) reduces into proving the existence of $\varphi_1,\dots,\varphi_t$.

Write $k_1=k(\mathfrak{m}_1)$, $g_1=g(\mathfrak{m}_1)$, $Q_1=Q(\mathfrak{m}_1)$, $L'_1=L'(\mathfrak{m}_1)$ and so on. In ideal circumstances, e.g., when σ_1 is unitary, we have $U^0(\rho_1 g_1)=U(\rho_1 g_1)$ (Proposition 2.16), and the existence of φ_1 can be shown by proving the existence of some $L_1\in\mathrm{Lag}(\rho_1 g_1)$ with $Q_1\oplus L_1=Q_1A_1$ and then using Lemma 2.22 to assert the existence of $\varphi_1\in U(\rho_1 g_1)$ with $\varphi_1(L'_1)=L_1$.

To prove the existence of L_1, we write g_1 as a sum of an anisotropic form and a hyperbolic form (Proposition 2.5) and treat each case separately. In fact, the anisotropic case has already been addressed in Remark 3.7, so only the hyperbolic case should be treated. In addition, when k_1 is infinite, one can use the rationality of the the k_1-variety $\mathbf{U}^0(f_1)$ (see Theorem 2.21) to reduce to the case where k_1 is algebraically closed (Proposition 4.14). Assuming k_1 is algebraically closed or finite, we have $[A_1] = 0$ in $\operatorname{Br} S_1$ and $[B_1] = 0$ in $\operatorname{Br} T_1$, in which case we can further use μ-conjugation and e-transfer (see 2G) to reduce to the case where $\deg B_1 = 1$ and $\deg A_1 = 2$ (Reduction 4.10). After these reductions, establishing the existence of $L_1 \in \operatorname{Lag}(\rho_1 g_1)$ with $Q_1 \oplus L_1 = Q_1 A_1$ becomes a technical check.

Unfortunately, it can happen that φ_1 does not exist. Specifically, in the context of (E2), this can happen when (σ, ε) is orthogonal, $[B] = 0$ and $[A] \neq 0$. In order to understand what goes wrong, it is instructive to view $\mathbf{Lag}(\rho_1 g)$ as an R-scheme on which $\mathbf{U}^0(\rho_1 g)$ acts. Suppose that R is connected for simplicity. Then Propositions 2.23 and 2.25 imply that $\mathbf{Lag}(\rho_1 g)$ is the disjoint union of two components, both being homogeneous $\mathbf{U}^0(\rho_1 g)$-spaces. When $\operatorname{rrk}_B Q$ is even, it turns out that φ_1 exists when L' lives in one of these two components, but not when it lives in the other (Corollary 5.14). Moreover, the former component may have no R-points. To overcome this, we put considerable work into effectively identifying the "good" component of $\mathbf{Lag}(\rho_1 g)$ and understanding when does it have R-points — if it is does not, then (Q, g) must be replaced with a Witt equivalent hermitian space. When $\operatorname{rrk}_B Q$ is odd, φ_1 never exists, but then one can prove that g must be hyperbolic (Proposition 5.10) and thus Witt equivalent to the zero form.

The proofs of conditions (E1), (E3) and (E4) follow a similar strategy and share similar complications, notably when (σ, ε) or (τ_2, ε) are orthogonal. The cases where $\varphi_1, \dots, \varphi_t$ exist are precisely the ones featuring in parts (i)–(iv) of Theorem 7.1 below.

The argument we outlined is carried in Sections 4–7: Section 4 collects some preliminary results, Section 5 establishes conditions (E2) and (E4), Section 6 establishes conditions (E1) and (E3), and the proof of Theorem 3.4 is concluded in Section 7, which also brings some of its by-products.

In order to address (E2) and (E4), resp. (E1) and (E3), simultaneously, *we replace the setting of 3A with the more robust Notation 4.1 below, and use the latter throughout Sections 4–6. We return to the setting of 3A in Section 7.*

4. Preparation for The Proof of Theorem 3.4

This section collects preliminary results that will be used in proving conditions (E1), (E2), (E3), (E4) of Theorem 3.6 when R is semilocal.

We begin with replacing the setting of 3A with a new one — Notation 4.1 — that will be in use until the end of Section 6. The reason for the change of notation is two-fold: first, it will ultimately allow us to treat (E2) and (E4), resp. (E1) and (E3), simultaneously, and second, the new notation is amenable to μ-conjugation and e-transfer in the sense of 2G (see 4D for a precise statement). We will explain how Notation 4.1 specializes to that of 3A (in a few possible ways) in Section 7, where we prove Theorem 3.4.

Notation 4.1. Let (A, σ) be an Azumaya R-algebra with involution and let $\varepsilon \in \mathrm{Z}(A)$ be an element satisfying $\varepsilon^\sigma = \varepsilon$. Write $S = \mathrm{Z}(A)$ and let T be a quadratic étale S-subalgebra of A such that $T^\sigma = T$ and $\operatorname{rk}_T A_A$ is constant along the fibers of $\operatorname{Spec} T \to \operatorname{Spec} S$ (we have $A \in \mathcal{P}(T)$ by Lemma 1.2). Write $B = \mathrm{Z}_A(T)$ and $\tau = \sigma|_B$. The inclusion $S \to T$ is denoted ι, and we let $T_0 = \{t \in T : t^\sigma = t\}$.

We let ρ denote the inclusion map $B \to A$, viewed as a homomorphism of R-algebras with involution $(B,\tau) \to (A,\sigma)$. Given $Q \in \mathcal{P}(B)$, we abbreviate $Q \otimes_B A$ to QA and identify Q as a B-submodule of QA via $x \mapsto x \otimes 1$ (this map is injective because Q_B is flat). If $(Q,g) \in \mathcal{H}^\varepsilon(B,\tau)$, we define $\rho(Q,g) = (QA, \rho g)$ as in 2C.

We let $\pi : A \to B$ denote a homomorphism of (B,B)-bimodules such that $\pi|_B = \mathrm{id}_B$. Given $(P,f) \in \mathcal{H}^\varepsilon(A,\sigma)$, we write $\pi(P,f) = (P, \pi f)$, where $\pi f = \pi \circ f$. We shall see below (Lemma 4.2) that π exists, is unique, and satisfies $\pi \circ \sigma = \tau \circ \pi$. Moreover, $\pi(P,f) \in \mathcal{H}^\varepsilon(B,\tau)$.

By Proposition 1.14, B is Azumaya over T, $[B] = [A \otimes_S T]$ in $\operatorname{Br} T$ and $\deg B = \frac{1}{2}\iota \deg A$. This means that B is separable projective over R, so by Example 1.20, (B,τ) is Azumaya over T_0. Proposition 1.14 also tells us that

$$\iota \operatorname{rrk}_A P = \operatorname{rrk}_B P \qquad \text{and} \qquad \iota \operatorname{rrk}_A QA = 2 \operatorname{rrk}_B Q$$

for all $P \in \mathcal{P}(A)$ and $Q \in \mathcal{P}(B)$ such that $\operatorname{rrk}_B Q$ is constant on the fibers of $\operatorname{Spec} T \to \operatorname{Spec} S$. In addition, $A_B \in \mathcal{P}(B)$ by Lemma 1.2. These facts will be used freely and without comment.

We further note that the assumptions of Notation 4.1 continue to hold if we base change along a ring homomorphism $R \to S$, thanks to Lemma 1.4.

4A. **Existence and Uniqueness of π.**

Lemma 4.2. *With Notation 4.1, the following hold:*

(i) There exists a unique (B,B)-bimodule homomorphism $\pi = \pi_{A,B} : A \to B$ such that $\pi|_B = \mathrm{id}_B$.

(ii) If there exists $\lambda \in T$ such that $\lambda^2 \in S^\times$ and $T = S \oplus \lambda S$, then $\pi a = \frac{1}{2}(a + \lambda^{-1} a \lambda)$ for all $a \in A$.

(iii) $\pi \circ \sigma = \tau \circ \pi$.

(iv) $E := \ker \pi$ satisfies $A = B \oplus E$, $E \cdot E = B$, $EA = A$ and $\operatorname{rrk}_B E_B = \deg B$.

(v) When R is semilocal, there exists λ as in (ii) and $\mu \in A^\times$ such that $E = \mu B = B\mu$, $\lambda\mu = -\mu\lambda$ and $\pi(b_1 + \mu b_2) = b_1$ for all $b_1, b_2 \in B$.[9] *If $\deg A = 2$, then we also have $\mu^2 \in S^\times$.*

(vi) For all $(P,f) \in \mathcal{H}^\varepsilon(A,\sigma)$, we have $(P, \pi f) \in \mathcal{H}^\varepsilon(B,\tau)$, where $\pi f := \pi \circ f$.

Proof. (i) Write $T^e = T^{\mathrm{op}} \otimes_S T$ and $B^e = B^{\mathrm{op}} \otimes_S B$. We view A and B as right B^e-modules using their evident (B,B)-bimodule structure.

Since T is a separable S-algebra, the map $\mu : T^e \to T$ sending $x^{\mathrm{op}} \otimes y$ to xy is split as a morphism of T^e-modules. Let $\xi : T \to T^e$ denote such a splitting, and let $e := \xi(1_T)$. It is well-known that $e^2 = e$ and $e(1 \otimes t) = e(t^{\mathrm{op}} \otimes 1)$ for all $t \in T$, see [39, Lemma III.5.1.2] and its proof.

Note that e is a central idempotent in B^e. Thus, $A = Ae \oplus A(1-e)$, and both summands are B^e-modules. Writing $e = \sum_i u_i^{\mathrm{op}} \otimes v_i$ with $\{u_i, v_i\}_i \subseteq T$, we see that for all $b \in B$, we have

$$be = \sum_i u_i b v_i = b \sum_i u_i v_i = b \cdot 1_T = b,$$

because $\sum_i u_i v_i = \mu(e) = 1_T$. On the other hand, if $a \in Ae$, then for all $t \in T$, we have

$$ta = a(t^{\mathrm{op}} \otimes 1) = ae(t^{\mathrm{op}} \otimes 1) = ae(1 \otimes t) = a(1 \otimes t) = at,$$

hence $a \in B$. We conclude that $B = eA$. This in turn means that $\pi : a \mapsto ae$ is a B^e-module homomorphism, or equivalently, a (B,B)-bimodule homomorphism, which splits the inclusion $B \to A$.

[9]Note that in contrast with 3A, we do no require $\lambda^\sigma = -\lambda$ and $\mu^\sigma = -\mu$. Indeed, this cannot be guaranteed in general. The situations considered in cases (i) and (ii) of Lemma 4.3 below are such examples, the reason being that $\sigma|_B = \mathrm{id}_B$ or $\sigma|_E = \mathrm{id}_E$.

If $\pi' : A \to B$ is another (B,B)-module homomorphism splitting $B \to A$, then $B = Ae \subseteq \ker(\pi - \pi')$. On the other hand, since multiplying on the right by e annihilates $A(1-e)$ while fixing B, we have $A(1-e) \subseteq \ker \pi'$. Since $\ker \pi = A(1-e)$, this means that $\ker(\pi - \pi') \supseteq Ae + A(1-e) = A$, so $\pi = \pi'$.

(ii) Using $\lambda^2 \in S$ and $B = \mathrm{Z}_A(T) = \mathrm{Z}_A(\lambda)$, it is routine to check that $a \mapsto \frac{1}{2}(a + \lambda^{-1}a\lambda)$ is a (B,B)-bimodule homomorphism from A to B which restricts to the identity on B. This map must be π by (i).

(iii) The uniqueness of π implies that $\tau \circ \pi \circ \sigma = \pi$, or rather, $\pi \circ \sigma = \tau \circ \pi$.

(iv) That $A = B \oplus E$ follows from the fact that $\pi : A \to B$ splits the inclusion $A \to B$. Since $\mathrm{rrk}_B A_B = \iota \deg A = 2 \deg B$, this means that $\mathrm{rrk}_B E = \deg B$.

We proceed with checking that $E \cdot E \subseteq B$. It is enough to prove that this holds after localizing at $\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$, so we may assume that R is local. In this case, by Lemma 1.19, there exists $\lambda \in T$ such that $\lambda^2 \in S^\times$ and $T = S \oplus \lambda S$. By (ii), E consists of the elements which anti-commute with λ while $B = \mathrm{Z}_A(\lambda)$. Thus, $E \cdot E \subseteq B$.

Now, since $A = B \oplus E$ and $BEB \subseteq E$, the set $E^2 + E$ is a two sided ideal of A. Since A is Azumaya over S, there exists $I \trianglelefteq S$ such that $E^2 + E = IA = IB + IE$ [20, Lemma II.3.5]. It follows that $E = IE$ and $E^2 = IB$. If $I \neq S$, then $\mathrm{ann}_S E \neq 0$ by Nakayama's Lemma, which is impossible because $\mathrm{rrk}_B E = \deg B > 0$. Thus, $I = S$ and $E^2 = IB = B$. This also means that $EA \supseteq E^2 A = BA = A$.

(v) The existence of λ follows from Lemma 1.19. By (iv), $\mathrm{rrk}_B E_B = \deg B$, so $E_B \cong B_B$ by Lemma 1.24. Let μ be a generator of E_B. By (ii), $0 = \pi(\mu) = \frac{1}{2}(\mu + \lambda^{-1}\mu\lambda)$, so $\lambda\mu = -\mu\lambda$. By (iv), $B = E^2 = \mu B \mu B \subseteq \mu A$, so μ is invertible on the right. This means that $\mathrm{Nrd}_{A/S}(\mu) \in S^\times$, hence $\mu \in A^\times$. Now, since $B = \mathrm{Z}_A(\lambda)$ and E is the set of elements in A which anti-commute with λ (by (ii)), we have $E = \mu B = B\mu$. This means that the map $A = B \oplus \mu B \to B$ sending $b_1 + \mu b_2$ to b_1 ($b_1, b_2 \in B$) is a (B,B)-bimodule homomorphism which restricts to the identity on B. Therefore, it must coincide with π. Finally, if $\deg A = 2$, then $\deg B = 1$, so $B = T$ and A is generated as an S-algebra by λ and μ. Since μ^2 commutes with both λ and μ, we have $\mu^2 \in \mathrm{Z}(A)^\times = S^\times$.

(vi) It is straightforward to check that πf is an ε-hermitian form. We need to show that πf is unimodular. Using (iv), choose $\{u_i, v_i\}_{i=1}^t \subseteq E$ such that $\sum_i u_i v_i = 1$. Given $\phi \in \mathrm{Hom}_B(P,B)$, define $\hat{\phi} : P \to A$ by $\hat{\phi}x = \phi x + \sum_i \phi(x u_i) v_i$. We claim that $\hat{\phi} \in \mathrm{Hom}_A(P,A)$. Indeed, $\hat{\phi}$ is additive, and for all $b \in B$, $b' \in E$ and $x \in P$, we have $\hat{\phi}(xb) = \phi(xb) + \sum_i \phi(xbu_i)v_i = \phi x \cdot b + \sum_{i,j} \phi(xv_j u_j b u_i) v_i = \phi x \cdot b + \sum_{i,j} \phi(x v_j) u_j b u_i v_i = \phi x \cdot b + \sum_j \phi(x v_j) u_j b = \hat{\phi} x \cdot b$ and $\hat{\phi}(xb') = \phi(xb') + \sum_i \phi(xb'u_i)v_i = \sum_i \phi(x u_i v_i b') + \sum_i \phi x \cdot b' u_i v_i = \sum_i \phi(x u_i) v_i b' + \phi x \cdot b' = \hat{\phi} x \cdot b'$. A similar computation shows that $\phi \mapsto \hat{\phi} : \mathrm{Hom}_B(P,B) \to \mathrm{Hom}_A(P,A)$ defines an inverse to $\xi \mapsto \pi \circ \xi : \mathrm{Hom}_A(P,A) \to \mathrm{Hom}_B(P,B)$. The composition of the latter map with $x \mapsto f(x,-) : P \to \mathrm{Hom}_A(P,A)$ is precisely $x \mapsto \pi f(x,-) : P \to \mathrm{Hom}_B(P,B)$, so this map is bijective and πf is unimodular. □

4B. **Some Structural Results.**

Lemma 4.3. *With Notation 4.1, suppose that R is semilocal, $S = R$ and $\deg B = 1$. Then there exist $\lambda, \mu \in A$ such that $\lambda^2, \mu^2 \in R^\times$, $\lambda\mu = -\mu\lambda$, $\{1, \lambda\}$ is an R-basis of B, $\{1, \lambda, \mu, \mu\lambda\}$ is an R-basis of A, and:*

(i) $\lambda^\sigma = \lambda$ and $\mu^\sigma = -\mu$ if $\tau = \mathrm{id}_T$;
(ii) $\lambda^\sigma = -\lambda$ and $\mu^\sigma = \mu$ if τ is unitary and σ is orthogonal;
(iii) $\lambda^\sigma = -\lambda$ and $\mu^\sigma = -\mu$ if σ is symplectic.

Furthermore, $\pi(b_1 + \mu b_2) = b_1$ for all $b_1, b_2 \in B$.

Proof. Let λ and μ be as in Lemma 4.2(v). Then all the requirements are fulfilled except, maybe, (i)–(iii). Note also that we may replace μ with any element of $\mu B^\times$.

Let $E = \ker \pi$. Since $E^\sigma = E$, $B^\sigma = B$ and $A = B \oplus E$, we have $\mathcal{S}_1(A,\sigma) = \mathcal{S}_1(B,\sigma) \oplus \mathcal{S}_1(E,\sigma)$ and $\mathcal{S}_{-1}(A,\sigma) = \mathcal{S}_{-1}(B,\sigma) \oplus \mathcal{S}_{-1}(E,\sigma)$.

Suppose that $\tau = \mathrm{id}_T$. Then $\lambda^\sigma = \lambda$ and $\mathcal{S}_{-1}(A,\sigma) = \mathcal{S}_{-1}(B,\sigma) \oplus \mathcal{S}_{-1}(E,\sigma) = \mathcal{S}_{-1}(E,\sigma)$. By Lemma 1.26, there exists $\mu' \in \mathcal{S}_{-1}(A,\sigma) \cap A^\times$. Since $\mu' \in E = \mu B$, we have $\mu' = \mu t$ for some $t \in T$, so we may replace μ with μ' and finish.

Suppose that τ is unitary and σ is orthogonal. Then τ is the standard R-involution of T, so $\lambda^\sigma = -\lambda$. By Proposition 1.21, we have $1 = \mathrm{rk}_R\, \mathcal{S}_{-1}(A,\sigma) = \mathrm{rk}_R\, \mathcal{S}_{-1}(B,\sigma) + \mathrm{rk}_R\, \mathcal{S}_{-1}(E,\sigma) = 1 + \mathrm{rk}_R\, \mathcal{S}_{-1}(E,\sigma)$, hence $\mathcal{S}_{-1}(E,\sigma) = 0$. Since $E = \mathcal{S}_1(E,\sigma) \oplus \mathcal{S}_{-1}(E,\sigma)$, this means that $E = \mathcal{S}_1(E,\sigma)$ and $\mu^\sigma = \mu$.

Finally, when σ symplectic, using Proposition 1.21 and the fact that R is a summand of B, one finds that $1 = \mathrm{rk}_R\, \mathcal{S}_1(A,\sigma) \geq \mathrm{rk}_R\, \mathcal{S}_1(R,\sigma) + \mathrm{rk}_R\, \mathcal{S}_1(E,\sigma) = 1 + \mathrm{rk}_R\, \mathcal{S}_1(E,\sigma)$, hence $\mathcal{S}_1(E,\sigma) = 0$. This means that $E = \mathcal{S}_{-1}(E,\sigma)$, so $\mu^\sigma = -\mu$ and $(\lambda\mu)^\sigma = -\lambda\mu$. Now, $\lambda = -\mu\lambda\mu^{-1} = (\mu\lambda)^\sigma\mu^{-1} = \lambda^\sigma\mu^\sigma\mu^{-1} = -\lambda^\sigma$. □

Lemma 4.4. *With Notation 4.1, suppose that $T \cong S \times S$ as S-algebras, and let $e, e' \in T$ correspond to $(1_S, 0_S), (0_S, 1_S)$ under this isomorphism. Then:*

(i) $B = eAe + e'Ae'$.
(ii) $\mathrm{rrk}_A\, eA > 0$ *and* $AeA = A$.
(iii) eB *is an Azumaya S-algebra of degree* $\frac{1}{2}\deg A$ *and* $[eB] = [A]$ *in* $\mathrm{Br}\, S$.
(iv) *For every* $Q \in \mathcal{P}(B)$*, we have* $\mathrm{rrk}_A\, QA = \mathrm{rrk}_{eB}\, Qe + \mathrm{rrk}_{e'B}\, Qe'$.
(v) $\pi : A \to B$ *is given by* $\pi a = eae + e'ae'$.

Proof. (i) Since $T = S[e]$, we have $B = \mathrm{Z}_A(e)$. Using the Peirce decomposition of A relative to e (i.e. $A = eAe \oplus eAe' \oplus e'Ae \oplus e'Ae'$ as abelian groups), it routine to check that $\mathrm{Z}_A(e) = eAe + e'Ae'$.

(ii) Since ${}_BB$ is a summand of ${}_BA$ (Lemma 4.2(iv)) and T is a T-summand of B [27, Proposition 2.4.6(1)], $eT \cong S$ is an S-summand of eA, hence $\mathrm{rrk}_A\, eA > 0$. That $AeA = A$ follows from Corollary 1.12.

(iii) By (i), $eB = e(eAe + e'Ae') = eAe$, and by Corollary 1.12 and (ii), eAe is Azumaya over S and $[eAe] = [A]$. That $\deg eB = \frac{1}{2}\deg A$ follows from $\deg B = \frac{1}{2}\iota \deg A$.

(iv) Since $Q = Qe \oplus Qe'$ as B-modules, $QA = QeA \oplus Qe'A$, so it is enough to check that $\mathrm{rrk}_{eB}\, Qe = \mathrm{rrk}_A\, QeA$. By Corollary 1.12 and (ii), $\mathrm{rrk}_A\, QeA = \mathrm{rrk}_{eAe}\, QeAe = \mathrm{rrk}_{eB}\, Qe$.

(v) Using (i), it is easy to check that $a \mapsto eae + e'ae'$ is a homomorphism of (B,B)-bimodules which splits $B \to A$. Thus, it must coincide with π. □

4C. **The Types of** (σ, ε) **and** (τ, ε)**.**

Lemma 4.5. *With Notation 4.1, if σ is unitary, then so is τ.*

Proof. It is enough to prove this when R is a field. Recall that $T_0 = \mathcal{S}_1(T,\sigma)$ and let $T_1 = \mathcal{S}_{-1}(T,\sigma)$. Since $2 \in R^\times$, we have $T = T_0 \oplus T_1$. Since σ is unitary, S is quadratic étale over R and $\sigma|_S$ is the standard R-involution of S. By Lemma 1.19, there exists $\lambda \in S^\times$ such that $\lambda^\sigma = -\lambda$. One readily checks that $t \mapsto \lambda t : T_0 \to T_1$ is a T_0-module isomorphism, so $\mathrm{rk}_{T_0} T = 2$. We observed in the comment after Notation 4.1 that (B,τ) is Azumaya over T_0. Since $\mathrm{rk}_{T_0} T = 2$, Proposition 1.21(iii) implies that τ is unitary. □

Lemma 4.6. *With Notation 4.1, if R is connected, then the type of (τ, ε) is constant on* $\mathrm{Spec}\, T_0$ *(see 1D).*

Proof. By Proposition 1.21(v), (σ, ε) is either orthogonal, symplectic or unitary. If (σ, ε) is unitary, then the lemma follows from Lemma 4.5. Suppose that (σ, ε) is orthogonal or symplectic. Then $S = R$. If T is connected, then the type of (τ, ε) is constant (Proposition 1.21(v)), so assume that T is not connected. Now, by Lemmas 1.16 and 1.17, $T = R \times R$ and $\tau|_T$ is either the standard R-involution of T, or id_T. In the former case, (τ, ε) is unitary, so assume $\tau|_T = \mathrm{id}_T$. Let $e = (1_R, 0_R)$ and $e' = (0_R, 1_R)$. By Lemma 4.4(i), $(B, \tau) = (eAe, \sigma|_{eAe}) \times (e'Ae', \sigma|_{e'Ae'})$, and by Corollary 1.22(ii), $(\sigma|_{eAe}, e\varepsilon)$ and $(\sigma|_{e'Ae'}, e'\varepsilon)$ have the same type as (σ, ε), so the type of (τ, ε) is constant. □

4D. **Simultaneous Conjugation and Transfer.** We check that the setting of Notation 4.1 is compatible with μ-conjugation and e-transfer (see 2G).

Proposition 4.7. *With Notation 4.1, let $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ and $(Q, g) \in \mathcal{H}^\varepsilon(B, \tau)$. Let $\delta \in S$ satisfy $\delta^\sigma \delta = 1$ and let $\mu \in \mathcal{S}_\delta(B, \tau) \cap B^\times$. Then:*

- *(i) The assumptions of Notation 4.1 continue to hold upon replacing $\sigma, \tau, \varepsilon$ with $\mathrm{Int}(\mu) \circ \sigma, \mathrm{Int}(\mu) \circ \tau, \delta\varepsilon$.*
- *(ii) $\rho(\mu g) = \mu(\rho g)$ and $\pi(\mu f) = \mu(\pi f)$ (notation as in 2G).*

Proof. (i) We need to check that $\mathrm{Int}(\mu) \circ \sigma$ is an involution of A which restricts to an involution of T, and $(\delta\varepsilon)^{\mathrm{Int}(\mu)\circ\sigma}(\delta\varepsilon) = 1$. Noting that $\mu \in B = \mathrm{Z}_A(T)$, $\mu^\sigma = \delta^{-1}\mu$ and $T^\sigma = T$, this follows by straightforward computation.

(ii) Let $y, y' \in Q$ and $a, a' \in A$. Then $\rho(\mu g)(y \otimes a, y' \otimes a') = a^{\mathrm{Int}(\mu)\circ\sigma} \mu g(y, y')a' = \mu a^\sigma \mu^{-1} \mu g(y, y')a' = \mu(\rho g)(y \otimes a, y' \otimes a')$, so $\rho(\mu g) = \mu(\rho g)$. Now let $x, x' \in P$. Then $\pi(\mu f)(x, x') = \pi(\mu \cdot f(x, x')) = \mu \cdot \pi(f(x, x')) = \mu(\pi f)(x, x')$, where the second equality holds because π is a left B-module homomorphism and $\mu \in B$. □

Lemma 4.8. *With Notation 4.1, let $e \in B$ be an idempotent with $\mathrm{rrk}_B\, eB > 0$ and let $Q \in \mathcal{P}(B)$. Then the map $\xi = \xi_Q : Qe \otimes_{eBe} eAe \to Q \otimes_B Ae$ determined by $\xi(x \otimes a) = x \otimes a$ ($x \in Qe$, $a \in eAe$) is an isomorphism of eAe-modules.*

Proof. By Proposition 1.12, $BeB = B$. Choose elements $\{u_i, v_i\}_{i=1}^t \subseteq B$ such that $\sum_i u_i e v_i = 1$, and consider the map $\psi : Q \otimes_B Ae \to Qe \otimes_{eBe} eAe$ determined by $x \otimes a \mapsto \sum_i x u_i e \otimes e v_i a$ ($x \in Q$, $a \in Ae$). It is well-defined because for all $b \in B$, we have $\psi(xb \otimes a) = \sum_i xbu_i e \otimes ev_i a = \sum_{i,j} xu_j ev_j bu_i e \otimes ev_i a = \sum_{i,j} xu_j e \otimes ev_j bu_i ev_i a = \sum_j xu_j e \otimes ev_j ba = \psi(x \otimes ba)$. A similar computation shows that ψ is an inverse of ξ. □

Proposition 4.9. *With Notation 4.1, let $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ and $(Q, g) \in \mathcal{H}^\varepsilon(B, \tau)$. Let $e \in B$ be an idempotent such that $e^\sigma = e$ and $\mathrm{rrk}_B\, eB$ is positive and constant along the fibers of $\mathrm{Spec}\, T \to \mathrm{Spec}\, S$. Write $\tau_e = \tau|_{eBe}$, $\sigma_e = \sigma|_{eAe}$, $\rho_e = \rho|_{eBe}$, $\pi_e = \pi|_{eAe}$, $f_e = f|_{Pe \times Pe}$, $g_e = g|_{Qe \times Qe}$ (see 2G). Then:*

- *(i) The assumptions of Notation 4.1 apply upon replacing $A, \sigma, \varepsilon, T, B, \tau, \rho, \pi$ with $eAe, \sigma_e, e\varepsilon, eT, eBe, \tau_e, \rho_e, \pi_e$.*
- *(ii) Upon identifying $Q \otimes_{eBe} eAe$ with $Q \otimes_B Ae$ as in Lemma 4.8, we have $\rho_e g_e = (\rho g)_e$. Furthermore, the map $L \mapsto Le$ defines a bijection from the Lagrangians of ρg to the Lagrangians of $\rho_e g_e$ and, for a Lagrangian L of ρg, we have $L \oplus Q = QA$ (as B-modules) if and only if $Le \oplus Qe = QAe$.*
- *(iii) $\pi_e f_e = (\pi f)_e$, the map $M \mapsto Me$ is a bijection between the Lagrangians of πf and the Lagrangians of $\pi_e f_e$ and, for a Lagrangian M of πf, we have $MA = P$ if and only if $Me \cdot eAe = Pe$.*

Proof. By Proposition 1.12, $BeB = B$, hence $AeA = ABeBA = ABA = A$, and so eA_A is a progenerator. Thus, we can use (t1)–(t6) in 2G for both (B, τ) and (A, σ).

(i) Everything is straightforward except the fact that $\mathrm{rk}_T(eAe_{eAe})$ is constant along the fibers of $\mathrm{Spec}\, T \to \mathrm{Spec}\, S$. To see this, we use Corollary 1.12 to get $\mathrm{rk}_T(eAe_{eAe}) = \deg eBe \cdot \mathrm{rrk}_{eBe}(eAe_{eBe}) = \deg eBe \cdot \mathrm{rrk}_B\, eA_B = \mathrm{rrk}_B\, eB \cdot \iota\, \mathrm{rrk}_A\, eA =$

$\operatorname{rrk}_B eB \cdot 2 \operatorname{rrk}_B eB = 2(\operatorname{rrk}_B eB)^2$. Since $\operatorname{rrk}_B eB$ is constant along the fibers of $\operatorname{Spec} T \to \operatorname{Spec} S$, so is $\operatorname{rk}_T(eAe_{eAe})$.

(ii) This is straightforward; use facts (t1)–(t6) in 2G and Morita theory.

(iii) That $(\pi f)_e = \pi_e f_e$ is straightforward. The second assertion is (t5) in 2G. For the third assertion, note that $MA = P$ implies $Me \cdot eAe = M(AeA)e = MAe = Pe$, and conversely, $Me \cdot eAe = Pe$ implies $MA = M(AeAeA) = Me \cdot eAe \cdot eA = Pe \cdot eA = P(AeA) = PA = P$. □

4E. **Two Important Reductions.**

Reduction 4.10. With Notation 4.1, suppose that R is connected semilocal and $[B] = 0$ in $\operatorname{Br} T$ (note that $[A] = 0$ in $\operatorname{Br} S$ implies $[B] = [A \otimes_S T] = 0$). We claim that verifying statements about hermitian forms over (A, σ) and (B, τ) which are amenable to conjugation and transfer (in the sense of 2G), e.g., the statements $L \oplus Q = QA$ and $MA = P$ from parts (ii) and (iii) of Proposition 4.9, can be reduced into verifying them in the following setting, and without affecting the the types of (σ, ε) and (τ, ε):

- $\deg B = 1$, i.e. $B = T$,
- τ is orthogonal or unitary,
- σ is orthogonal or unitary.

This is done as follows: Applying Proposition 1.21(v) and Lemma 4.6 with $\varepsilon = 1$, we see that the types of σ and τ are constant. If τ is not orthogonal or unitary, then it is symplectic. In this case, by Lemma 1.26, there exists $\mu \in \mathcal{S}_{-1}(B, \tau) \cap B^\times$. By Proposition 4.7, we may apply μ-conjugation and replace $\sigma, \tau, \varepsilon$ with $\operatorname{Int}(\mu) \circ \sigma, \operatorname{Int}(\mu), -\varepsilon$, thus changing τ into an orthogonal involution.

Next, by Theorem 1.30, there exists an idempotent $e \in B$ such that $e^\tau = e$ and $\operatorname{rrk}_B eB = \deg eBe = \operatorname{ind} B = 1$. By Proposition 4.9, we may apply e-transfer and replace $A, \sigma, \varepsilon, T, B, \tau, \rho, \pi$ with $eAe, \sigma_e, e\varepsilon, eT, eBe, \tau_e, \rho_e, \pi_e$ and get $\deg B = \operatorname{ind} B = 1$.

Finally, if σ is not orthogonal or unitary, then it is symplectic. In this case, by Lemma 4.3, there exists $\lambda \in T^\times$ with $\lambda^\sigma = -\lambda$. By Proposition 4.7, we can apply λ-conjugation and replace σ, ε with $\operatorname{Int}(\lambda) \circ \sigma, \operatorname{Int}(\lambda), -\varepsilon$, turning σ into an orthogonal involution and leaving τ unchanged.

Reduction 4.11. Assume that R is connected semilocal and $[A] = 0$ in $\operatorname{Br} S$. After performing Reduction 4.10, Proposition 2.12 implies that σ is adjoint to a unimodular binary δ-hermitian form over $(S, \sigma|_S)$, with $\delta = 1$ if σ is orthogonal. This form can be diagonalized by Proposition 2.13, so, by Example 2.10, we may assume that $A = \mathrm{M}_2(S)$ and σ is given by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} a^\sigma & \alpha c^\sigma \\ \alpha^{-1} b^\sigma & d^\sigma \end{smallmatrix}\right]$ for some $\alpha \in \mathcal{S}_1(S, \sigma) \cap S^\times = R^\times$.

4F. **Miscellaneous Results.**

Lemma 4.12. *With Notation 4.1, if (τ, ε) is orthogonal, then (σ, ε) is orthogonal.*

Proof. It is enough to prove the lemma after specializing R to the algebraic closure of each of its residue fields, so assume R is an algebraically closed field. Then $[B] = 0$. We apply Reduction 4.10 to assume that $\deg A = 2$ and τ is orthogonal. Since (τ, ε) is orthogonal, $\varepsilon = 1$. By Lemma 4.3(i), $\dim_R \mathcal{S}_1(A, \sigma) = 3$, so (σ, ε) is orthogonal by Proposition 1.21. □

Lemma 4.13. *With Notation 4.1, let $P \in \mathcal{P}(A)$.*

(i) *The map $\operatorname{End}_A(P) \otimes_S T \to \operatorname{End}_B(P)$ given by sending $\psi \otimes t$ to $[x \mapsto \psi x \cdot t]$ is an isomorphism of T-algebras.*

(ii) *For all $\psi \in \operatorname{End}_A(P)$, we have $\operatorname{Nrd}_{\operatorname{End}_A(P)/S}(\psi) = \operatorname{Nrd}_{\operatorname{End}_B(P)/T}(\psi)$ in T.*

Proof. We may assume that $\operatorname{rrk}_A P > 0$, otherwise write $R = R_0 \times \operatorname{ann}_R P$ (use [27, Proposition 1.1.15]) and work over R_0.

(i) By Proposition 1.11(i), $\operatorname{End}_A(P)$ is Azumaya over S, $\operatorname{End}_B(P)$ is Azumaya over T and $\deg \operatorname{End}_B(P) = \operatorname{rrk}_B P = \iota \operatorname{rrk}_A P = \iota \deg \operatorname{End}_A(P) = \deg \operatorname{End}_A(P) \otimes_S T$. Thus, the map $\operatorname{End}_A(P) \otimes_S T \to \operatorname{End}_B(P)$ is a homomorphism of Azumaya T-algebras of equal degrees. By [39, Corollary III.5.1.18], such a homomorphism is always an isomorphism.

(ii) This follows from (i) and the fact that reduced norm is preserved under base-change. □

Proposition 4.14. *With Notation 4.1, suppose that R is an infinite field, and let $\overline{R}$ be an algebraic closure of R. Let $(Q, g) \in \mathcal{H}^\varepsilon(B, \tau)$ and let L be a Lagrangian of ρg. Let $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ and let M be a Lagrangian of πf. Then:*

- *(i) If there exists $\varphi \in U^0(\rho g_{\overline{R}})$ such that $Q_{\overline{R}} \oplus \varphi(L_{\overline{R}}) = QA_{\overline{R}}$, then there exists $\psi \in U^0(\rho g)$ such that $Q \oplus \psi L = QA$.*
- *(ii) If there exists $\varphi \in U^0(\pi f_{\overline{R}})$ such that $\varphi M \cdot A_{\overline{R}} = P_{\overline{R}}$, then there exists $\psi \in U^0(\pi f)$ such that $\psi M \cdot A = P$.*

(See 2E for the definition of $U^0(-)$.)

Proof. (i) Consider $\mathbf{U}^0(\rho g)$ as a functor from R-rings to groups and define a subfunctor $R_1 \mapsto \mathbf{X}(R_1)$ of $\mathbf{U}^0(\rho g)$ by

$$\mathbf{X}(R_1) = \{\psi \in U^0(\rho g_{R_1}) \,:\, Q_{R_1} \oplus \psi(L_{R_1}) = QA\}.$$

We claim that $\mathbf{X}$ is represented by open affine subscheme of $\mathbf{U}^0(\rho g)$, also denoted $\mathbf{X}$.

To see this, fix R-bases $\{x_i\}_{i=1}^r$, $\{v_i\}_{i=1}^s$, $\{y_i\}_{i=1}^{r+s}$ to Q, L, QA, respectively. These will also be viewed as R_1-bases of Q_{R_1}, L_{R_1}, QA_{R_1}. The group $U^0(\rho g_{R_1})$ is the zero locus of certain polynomial functions on $\operatorname{End}_{R_1}(QA_{R_1}) \cong R_1^{(r+s)^2}$ with coefficients in R. Thus, it is enough to show that there exists a polynomial $\xi \in R[x_{11}, x_{12}, \dots, x_{(r+s)(r+s)}]$ such that $\mathbf{X}(R_1) = \{\psi \in U^0(\rho g_{R_1}) \,:\, \xi(\psi) \in R_1^\times\}$. To that end, given $y \in QA_{R_1}$, let $[y]$ denote the vector $(\alpha_1, \dots, \alpha_{r+s}) \in R_1^{r+s}$ for which $y = \sum_i y_i \alpha_i$. Then the function sending $a \in R_1^{r^2} \cong \operatorname{End}_{R_1}(Q_{R_1})$ to the determinant of the $(r+s) \times (r+s)$ matrix with columns $[v_1], \dots, [v_s], [ax_1], \dots, [ax_r]$ is a polynomial $\xi \in R[x_{11}, x_{12}, \dots, x_{(r+s)(r+s)}]$ having the desired property.

By Theorem 2.21, the irreducible R-variety $\mathbf{U}^0(\rho g)$ is rational. By the previous paragraph, $\mathbf{X}$ is an open subvariety of $\mathbf{U}^0(\rho g)$ and it is nonempty because $\varphi \in \mathbf{X}(\overline{R})$. Thus, $\mathbf{X}$ is also rational. Since rational varieties have points over any infinite field, $\mathbf{X}(R) \neq \emptyset$ and the existence of ψ follows.

(ii) This is similar to (i), but one uses an open subscheme of $\mathbf{U}^0(\pi f)$ defined as follows: Write $r = \dim_R P$. Since $\varphi M_{\overline{R}} \cdot A_{\overline{R}} = P_{\overline{R}}$, there exist pairs $\{(m_i, a_i)\}_{i=1}^r \subseteq M \times A$ such that $\{\varphi m_i \cdot a_i\}_{i=1}^r$ forms an $\overline{R}$-basis to $P_{\overline{R}}$. Given an R-ring R_1, define $\mathbf{X}(R_1)$ to be the set of $\psi \in U^0(\pi f_{R_1})$ such that $\{\psi m_i \cdot a_i\}_{i=1}^r$ is an R_1-basis to P_{R_1}. □

5. Verification of (E2) and (E4)

Keep the assumptions of Notation 4.1. The purpose of this section is to prove:

Theorem 5.1. *With Notation 4.1, suppose that R is semilocal and let $(Q, g) \in \mathcal{H}^\varepsilon(B, \tau)$. Assume that $[\rho g] = 0$ in $W^\varepsilon(A, \sigma)$. Then:*

- *(i) When T is connected, there exists a Lagrangian L of ρg such that $L \oplus Q = QA$ if and only if:*
  - *(1) (σ, ε) is not orthogonal, or*
  - *(2) (τ, ε) is not unitary, or*

(3) $[A] = 0$ *in* $\mathrm{Br}\, S$, *or*

(4) $[B] \neq 0$ *in* $\mathrm{Br}\, T$, *or*

(5) (τ, ε) *is unitary,* $[B] = 0$, $\mathrm{rrk}_B\, Q$ *is even and* $[D(g)] = \frac{\mathrm{rrk}_B Q}{2}[A]$*; here,* $D(g)$ *is the discriminant algebra of* g*, see 2H.*

When none of (1)–(5) hold, $\mathrm{rrk}_B\, Q$ *is even,* $[D(g)] = (\frac{\mathrm{rrk}_B Q}{2} + 1) \cdot [A]$ *and* g *is isotropic.*

(ii) There exists $(Q', g') \in \mathcal{H}^\varepsilon(B, \tau)$ *with* $[g] = [g']$ *and a Lagrangian* L *of* $\rho g'$ *such that* $L \oplus Q' = Q'A$.

In Section 7, we will use Theorem 5.1 to establish conditions (E2) and (E4) of Theorem 3.6 when R is semilocal. The reader can skip to the next section without loss of continuity.

It is enough to prove Theorem 5.1 when R is connected. Indeed, we can write R as a finite product of connected semilocal rings and work over each factor separately. In this case, by Proposition 1.21(v) and Lemma 4.6, exactly one of the following hold:

(1) (σ, ε) is unitary or symplectic,
(2) (σ, ε) is orthogonal and (τ, ε) is orthogonal or symplectic,
(3) (σ, ε) is orthogonal and (τ, ε) is unitary.

The first two cases will be handled in Theorem 5.9 and the third case will be treated in Theorem 5.21.

5A. **Cases (1) and (2).** We begin by establishing some special cases of Theorem 5.1 in the context of case (1).

Proposition 5.2. *With Notation 4.1, suppose that* R *is a field,* $S = R \times R$ *and* $[A] = 0$*. Let* $(Q, g) \in \mathcal{H}^\varepsilon(B, \tau)$ *be a hermitian space such that* $\mathrm{rrk}_B\, Q$ *is constant. Then* ρg *admits a Lagrangian* L *satisfying* $Q \oplus L = QA$.

Proof. We may apply Reduction 4.10 to assume that $B = T$ and $\deg A = 2$.

Let η denote a nontrivial idempotent of S. Then $\eta^\sigma = 1 - \eta$. By Example 2.4, we may assume that $T = T_1 \times T_1$, $B = B_1 \times B_1^{\mathrm{op}}$ and $A = A_1 \times A_1^{\mathrm{op}}$, with $T_1 \subseteq B_1 \subseteq A_1$, and under these identifications, σ is the exchange involution $(x, y^{\mathrm{op}}) \mapsto (y, x^{\mathrm{op}})$. Furthermore, all hermitian forms over (B, τ) are hyperbolic, and every hermitian space is determined up to isomorphism by its underlying module.

Write ε as $(\alpha, \alpha^{-1}) \in R \times R$ and consider the ε-hermitian form $g_1 : B \times B \to B$ given by $g_1((x_1, x_2^{\mathrm{op}}), (y_1, y_2^{\mathrm{op}})) = (\alpha x_2 y_1, (y_2 x_1)^{\mathrm{op}})$. It is easy to see that $(B, g_1) \in \mathcal{H}^\varepsilon(B, \tau)$. Since $\mathrm{rrk}_B\, B = 1$ and $\mathrm{rrk}_B\, Q$ is constant, we have $Q \cong B^n$ for $n = \mathrm{rrk}_B\, Q$ (Lemma 1.24). As we noted above, this means that $(Q, g) \cong n \cdot (B, g_1)$. It is therefore enough to prove the proposition for $(Q, g) = (B, g_1)$. In this case, the isomorphism $b \otimes a \mapsto ba : B \otimes_B A \to A$ is an isometry from $(QA, \rho g)$ to (A, f_1), where f_1 is given by the same formula as g_1.

Fix an identification $A_1 \cong \mathrm{M}_2(R)$. Since $B_1 = T_1$ is a quadratic étale R-algebra, there exists $t \in T_1$ such that $r := t^2 \in R^\times$ and $T_1 = R \oplus tR$ (Lemma 1.19). Thus, t is conjugate to $\left[\begin{smallmatrix} 0 & r \\ 1 & 0 \end{smallmatrix}\right]$ in A_1. Using this, we choose the identification $A_1 \cong \mathrm{M}_2(R)$ to satisfy $t = \left[\begin{smallmatrix} 0 & r \\ 1 & 0 \end{smallmatrix}\right]$. Now, $B_1 = T_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] R + \left[\begin{smallmatrix} 0 & r \\ 1 & 0 \end{smallmatrix}\right] R$ and one readily checks that $L = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] A_1 \times (A_1 \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right])^{\mathrm{op}}$ is a Lagrangian of $f_1 = \rho g_1$ satisfying $B \oplus L = A$. □

Proposition 5.3. *With Notation 4.1, suppose that* S *is a field,* $[A] = 0$ *and* (σ, ε) *is symplectic or unitary. Let* $(Q, g) \in \mathcal{H}^\varepsilon(B, \tau)$ *be a hyperbolic hermitian space such that* $\mathrm{rrk}_B\, Q$ *is constant. Then* ρg *admits a Lagrangian* L *satisfying* $Q \oplus L = QA$.

Proof. By Reduction 4.10, we may assume that both σ and τ are orthogonal or unitary and $\deg B = 1$. Thus, $B = T$ and $\deg A = 2$. We now split into cases.

Case I. $\operatorname{rrk}_B Q$ *is even.* We apply Reduction 4.11 to assume that $A = \mathrm{M}_2(S)$ and σ is given by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} a^\sigma & \alpha c^\sigma \\ \alpha^{-1} b^\sigma & d^\sigma \end{smallmatrix}\right]$ for some $\alpha \in R^\times$.

Consider the ε-hermitian form $g_1 : B^2 \times B^2 \to B$ given by $g_1((x_1, x_2), (y_1, y_2)) = x_1^\sigma y_2 + \varepsilon x_2^\sigma y_1$. Then (B^2, g_1) is a hyperbolic. Since $\deg B = 1$ and $n := \operatorname{rrk}_B Q$ is constant and even, we have, by Lemma 2.7, $(Q, g) \cong \frac{n}{2} \cdot (B^2, g_1)$. It is therefore enough to prove the proposition for $(Q, g) = (B^2, g_1)$. In this case, $(QA, \rho g)$ can be identified with (A^2, f_1), where $f_1 : A^2 \times A^2 \to A$ is given by the same formula as g_1.

Given an R-subspace E of A, let $\mathcal{S}_\varepsilon(E) = \{a \in E : \varepsilon a^\sigma = a\}$. Suppose that there exists $s \in \mathcal{S}_{-\varepsilon}(A) \setminus S_{-\varepsilon}(B)$ such that $s \in A^\times$. It is routine to check that $L = \{(a, sa) \mid a \in A\}$ is a Lagrangian of A satisfying $L \cap B^2 = 0$, which, by R-dimension considerations, implies $L \oplus B^2 = A^2$. It is therefore enough to establish the existence of s. To that end, we split into subcases.

Subcase I.1. (σ, ε) *is symplectic.* This means that $S = R$, σ is orthogonal and $\varepsilon = -1$. Let $E_1 = \{\left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right] \mid a \in S\}$ and $E_2 = \{\left[\begin{smallmatrix} 0 & \alpha c \\ c & 0 \end{smallmatrix}\right] \mid c \in S\}$. Then E_1 and E_2 are 1-dimensional S-subspaces of $\mathcal{S}_1(A)$. If $E_i \cap \mathcal{S}_1(B) = 0$ for some $i \in \{1, 2\}$, then we can take any $0 \neq s \in E_i$. Otherwise, since $\dim_S B = 2$, we have $B = E_1 + E_2$, so take $s = \left[\begin{smallmatrix} 1 & \alpha \\ 1 & 0 \end{smallmatrix}\right]$.

Subcase I.2. (σ, ε) *is unitary.* Then S is quadratic étale over R and τ is unitary (Lemma 4.5). By Hilbert's Theorem 90, there exists $\delta \in S^\times$ such that $\delta^{-1}\delta^\sigma = \varepsilon$. Since $\mathcal{S}_{-\varepsilon}(A) = \delta^{-1}\mathcal{S}_{-1}(A)$, we reduce into verifying the existence of $s' \in \mathcal{S}_{-1}(A) \setminus \mathcal{S}_{-1}(B)$ with $s' \in A^\times$.

Since σ and τ are unitary involutions, we have $\dim_R \mathcal{S}_{-1}(A) = (\deg A)^2 = 4$ and $\dim_R \mathcal{S}_{-1}(B) = (\deg B)^2 \cdot \operatorname{rk}_R S = 2$ (Proposition 1.21). Let $E_1 = \{\left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right] \mid a \in \mathcal{S}_{-1}(S)\}$ and $E_2 = \{\left[\begin{smallmatrix} 0 & -\alpha c^\sigma \\ c & 0 \end{smallmatrix}\right] \mid c \in S\}$. Then E_1 and E_2 are R-subspaces of $\mathcal{S}_{-1}(A)$ of dimensions 1 and 2, respectively, and $E_2 \setminus \{0\}$ consists of invertible elements. If $\mathcal{S}_{-1}(B) \neq E_2$, then take any $s' \in E_2 \setminus \mathcal{S}_{-1}(B)$. If $\mathcal{S}_{-1}(B) = E_2$, then $\mathcal{S}_{-1}(B) \cap E_1 = E_2 \cap E_1 = 0$ and we can choose any nonzero $s' \in E_1$.

Case II. $\operatorname{rrk}_B Q$ *is odd.* Writing $(Q, g) \cong (U \oplus U^*, \mathbb{h}_U^\varepsilon)$ with $U \in \mathcal{P}(B)$, Lemma 2.6 implies that $\operatorname{rrk}_B Q = \operatorname{rrk}_B U + \sigma(\operatorname{rrk}_B U)$. Since $\operatorname{rrk}_B Q$ is odd, $\operatorname{rrk}_B U$ cannot be σ-invariant. In particular, $\operatorname{rrk}_B U$ is non-constant, forcing $T = S \times S$.

Let e denote a nontrivial idempotent of T. We identify A with $\mathrm{M}_2(S)$ in such a way that the idempotent e corresponds to $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Under this identification, B is the subalgebra of diagonal matrices.

We have $e^\sigma \in \{e, 1 - e\}$. Since $e^\sigma = e$ implies that $\operatorname{rrk}_B U$ is fixed by σ, we must have $e^\sigma = 1 - e$. We conclude that $B = T = S \times S$ and τ is the exchange involution. Now, by Example 2.4, every unimodular ε-hermitian form over (B, τ) is hyperbolic and determined up to isomorphism by its underlying module.

At this point, we claim that we may assume that $\varepsilon = 1$. Indeed, if σ is unitary, then S is a quadratic étale R-algebra and $\sigma|_S$ is its standard involution. Thus, by Hibert's Theorem 90, there exists $\mu \in S^\times$ with $\mu(\mu^{-1})^\sigma = \varepsilon^{-1}$, or rather, $\mu \in \mathcal{S}_{\varepsilon^{-1}}(S, \sigma|_S) \cap S^\times$. Applying μ-conjugation, see 2G and Proposition 4.7, we may assume that $\varepsilon = 1$. If σ is not unitary, then $R = S$, σ is orthogonal and $\varepsilon = -1$, so we can repeat the previous argument with $\mu := (1_S, -1_S) \in S \times S = T$; this will turn σ into a symplectic involution.

Define $g_1 : B \times B \to B$ by $g_1(x, y) = x^\tau y$. Then g_1 is a hyperbolic 1-hermitian form. Since $\operatorname{rrk}_B B = \deg B = 1$, we have $(Q, g) \cong n \cdot (B, g_1)$ for $n = \operatorname{rrk}_B Q$, and so it enough to prove the proposition when $(Q, g) = (Q, g_1)$. In this case, $b \otimes a \mapsto ba : BA \to A_A$ is an isomorphism under which $f_1 := \rho g_1$ is given by $f_1(x, y) := x^\sigma y$. Again, we split into subcases.

Subcase II.1. (σ,ε) is symplectic. Since $\varepsilon = 1$, the involution σ is the unique symplectic involution of $\mathrm{M}_2(S)$, given by $[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]^\sigma = [\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}]$ [40, Proposition 2.21]. Now, it is routine to check that $L = \{[\begin{smallmatrix} \alpha & \beta \\ \alpha & \beta \end{smallmatrix}] \,|\, \alpha,\beta \in S\}$ is a Lagrangian of ρf satisfying $B \oplus L = A$.

Subcase II.2. (σ,ε) is unitary. Since $e^\sigma = 1-e$, there are $\sigma|_S$-linear automorphisms $\sigma_2, \sigma_3 : S \to S$ such that $\sigma : A \to A$ is given by $[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]^\sigma = [\begin{smallmatrix} d^\sigma & \sigma_2 b \\ \sigma_3 c & a^\sigma \end{smallmatrix}]$. Furthermore, $\sigma_2 \circ \sigma_2 = \sigma_3 \circ \sigma_3 = \mathrm{id}_S$.

Since S is a quadratic field extension of R and $2 \in S^\times$, there exists $\delta \in S^\times$ with $\delta^\sigma = -\delta$. Choose some $c \in S^\times$. Then $c = \frac{1}{2}(c - \sigma_3 c) + \frac{1}{2}\delta^{-1}\delta(c + \sigma_3 c)$, hence at least one of $(c-\sigma_3 c)$, $\delta(c+\sigma_3 c)$ is nonzero. Replacing c with $(c-\sigma_3 c)$ or $\delta(c+\sigma_3 c)$, we may assume that $\sigma_3 c = -c$ and $c \neq 0$. Now, it is straightforward to check that $L = [\begin{smallmatrix} 1 & 0 \\ c & 0 \end{smallmatrix}]A = \{[\begin{smallmatrix} \alpha & \beta \\ c\alpha & c\beta \end{smallmatrix}] \,|\, \alpha,\beta \in S\}$ is a Lagrangian of ρf satisfying $B \oplus L = A$. This completes the proof. □

Lemma 5.4. *With Notation 4.1, suppose that S is field. Let $(Q, g) \in \mathcal{H}^\varepsilon(B,\tau)$ be an anisotropic hermitian space such that ρg is hyperbolic. Then $\mathrm{rrk}_B Q$ is constant.*

Proof. This is clear if T is a field, so assume $T = S \times S$ and let e denote a nontrivial idempotent in T. Then $e^\tau \in \{e, 1-e\}$. By Lemma 2.6, $\mathrm{rrk}_B Q$ is τ-invariant, so it constant when $e^\tau = 1-e$. It remains to consider the case $e^\tau = e$. Writing $e' := 1-e$, we need to show that $\mathrm{rrk}_{eB} Qe = \mathrm{rrk}_{e'B} Qe'$.

By Lemma 4.4, $eB = eAe$, $e'B = e'Ae'$ and $AeA = A$. Moreover, $(B,\tau) = (eB, \tau|_{eB}) \times (e'B, \tau|_{e'B})$, because $e^\tau = e$. Thus, we may consider hermitian forms over $(eB,\tau|_{eB})$, resp. $(e'B,\tau|_{e'B})$, as hermitian forms over (B,τ). Given $(V,h) \in \mathcal{H}^\varepsilon(eB,\tau|_{eB})$ (resp. $(V,h) \in \mathcal{H}^\varepsilon(e'B,\tau|_{e'B})$), we write $(VA, \rho h) \in \mathcal{H}^\varepsilon(A,\sigma)$ for the hermitian space obtained by regarding (V,h) as a hermitian space over (B,τ) and then base-changing along the inclusion morphism $\rho : (B,\tau) \to (A,\sigma)$.

For the sake of contradiction, suppose that $\mathrm{rrk}_{eB} Qe \neq \mathrm{rrk}_{e'B} Qe'$. By applying Lemma 4.4(iv) to Qe_B and Qe'_B, we see that $\mathrm{rrk}_A QeA \neq \mathrm{rrk}_A Qe'A$. Viewing g_e and $g_{e'}$ (notation as in 2G) as hermitian forms over (B,τ), we have $(Q,g) = (Qe, g_e) \oplus (Qe', g_{e'})$. Thus, g_e and $g_{e'}$ are anisotropic. Furthermore, $[\rho(g_e)] + [\rho(g_{e'})] = [\rho g] = 0$ in $W_\varepsilon(A,\sigma)$, so $\rho(g_e)$ and $-\rho(g_{e'})$ are Witt equivalent. Since the underlying modules of $\rho(g_e)$ and $-\rho(g_{e'})$, namely, QeA and $Qe'A$, are not isomorphic, either $\rho(g_e)$ or $\rho(g_{e'})$ is isotropic [56, §3.4(2)] (for instance). Without loss of generality, suppose that V is a nonzero A-summand of QeA such that $\rho(g_e)(V,V) = 0$. Then Ve is a nonzero B-module (because $VeA = VAeA = VA = V$) and summand of $QeAe = Qe$ such that $g(Ve,Ve) = 0$, contradicting our assumption that g is anisotropic. □

Lemma 5.5. *With Notation 4.1, let $(Q,g) \in \mathcal{H}^\varepsilon(B,\tau)$, and let L be a Lagragian of ρg satisfying $Q \oplus L = QA$. Suppose that $\mathrm{rrk}_B Q$ is constant. Then $\mathrm{rrk}_A QA$ is even and $\mathrm{rrk}_A L = \frac{1}{2}\mathrm{rrk}_A QA$.*

Proof. We have $\mathrm{rrk}_B QA = \iota\,\mathrm{rrk}_A QA = 2\,\mathrm{rrk}_B Q$, hence $\mathrm{rrk}_B L = \mathrm{rrk}_B QA - \mathrm{rrk}_B Q = \frac{1}{2}\mathrm{rrk}_B QA$. The lemma follows because $\mathrm{rrk}_B L = \iota\,\mathrm{rrk}_A L$ and $\mathrm{rrk}_B QA = \iota\,\mathrm{rrk}_A QA$. □

Proposition 5.6. *With Notation 4.1, suppose that R is a field and (σ,ε) is symplectic or unitary. Let $(Q,g) \in \mathcal{H}^\varepsilon(B,\tau)$ and assume that $\mathrm{rrk}_B Q$ is constant and ρg admits a Lagrangian L with $\mathrm{rrk}_A L = \frac{1}{2}\mathrm{rrk}_A QA$. Then there exists $\varphi \in U^0(\rho g)$ such that $Q \oplus \varphi L = QA$.*

Proof. By Proposition 4.14(i), when R is infinite, it is enough to prove the proposition after base-changing to an algebraic closure of R, in which case $[A] = 0$. On the other hand, if R is finite, then $[A] = 0$ by Wedderburn's theorem. We may therefore assume that $[A] = 0$.

Suppose first that S is a field. Using Proposition 2.5, write $(Q, g) = (Q_1, g_1) \oplus (Q_2, g_2)$ with g_1 anisotropic and g_2 hyperbolic. Then $[\rho g_1] = [\rho g] = 0$ in $W_\varepsilon(A, \sigma)$, so ρg_1 is hyperbolic by Theorem 2.8(ii). Let L_1 be a Lagrangian of ρg_1. By arguing as in Remark 3.7, we see that $Q_1 \oplus L_1 = Q_1 A$. Now, by Lemma 5.4, $\mathrm{rrk}_B\, Q_1$ is constant, and thus, so is $\mathrm{rrk}_B\, Q_2$. With this at hand, Proposition 5.3 says that ρg_2 admits a Lagrangian L_2 such that $Q_2 \oplus L_2 = Q_2 A$.

Let $L' = L_1 \oplus L_2$. Then $Q \oplus L' = QA$. By Lemmas 5.5 and 2.22, there exists $\varphi \in U(\rho g)$ such that $\varphi L = L'$. Since (σ, ε) is symplectic or unitary, we have $U(\rho g) = U^0(\rho g)$ (Proposition 2.16), so we are done.

If S is not a field, Proposition 5.2 implies that there exists $L' \in \mathrm{Lag}(\rho g)$ with $Q \oplus L' = QA$ and we can finish the proof as in the previous paragraph. □

We proceed with showing that Proposition 5.6 also holds in the context of Case (2), namely, when (σ, ε) is orthogonal and (τ, ε) is orthogonal or symplectic. This is similar to the proof of Proposition 5.6, but a few modifications are required in order to account for the possibility that $U^0(\rho g)$ is smaller than $U(\rho g)$.

Proposition 5.7. *With Notation 4.1, suppose S is a field, $[A] = 0$, (σ, ε) is orthogonal, and $\tau|_T = \mathrm{id}_T$. Let $(Q, g) \in \mathcal{H}^\varepsilon(B, \tau)$ be a hyperbolic hermitian space such that $\mathrm{rrk}_B\, Q$ is constant and positive, and let L be a Lagrangian of ρg. Then there exist $\varphi_1, \varphi_{-1} \in U(\rho g)$ such that $\mathrm{Nrd}(\varphi_1) = 1$, $\mathrm{Nrd}(\varphi_{-1}) = -1$ and $Q \oplus \varphi_1 L = Q \oplus \varphi_{-1} L = QA$.*

Proof. As noted in 2F, since (σ, ε) is orthogonal, all Lagrangians of ρg have reduced rank $\frac{1}{2}\,\mathrm{rrk}_A\, P$ and are thus isomorphic as A-modules (Lemma 1.24). Thus, by Lemma 2.22, $U(\rho g)$ acts transitively on $\mathrm{Lag}(\rho g)$. It is therefore enough to prove the proposition for a single Lagrangian L_0 of our choice.

By Reductions 4.10 and 4.11, we may assume that $B = T$, $A = \mathrm{M}_2(S)$, $\varepsilon = 1$ and σ is orthogonal and given by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]^\sigma = \left[\begin{smallmatrix} a & \alpha c \\ \alpha^{-1} b & d \end{smallmatrix}\right]$ for some $\alpha \in S^\times$.

As noted in Case II of the proof of Proposition 5.3, since $\sigma|_T = \mathrm{id}_T$, the reduced rank of Q is even. Thus, arguing as in Case I of that proof, we may assume that $(Q, g) = (B^2, g_1)$ with g_1 given by $g_1((x_1, x_2), (y_1, y_2)) = x_1^\sigma y_2 + x_2^\sigma y_1$. We identify QA with A^2 and $\mathrm{End}_A(QA)$ with $\mathrm{M}_2(A)$ in the obvious way. The form $f_1 := \rho g_1$ is defined by same formula as g_1 and we take $L_0 := A \times 0$ as our fixed Lagrangian.

Existence of φ_1. Let $s = \left[\begin{smallmatrix} 0 & -\alpha \\ 1 & 0 \end{smallmatrix}\right] \in \mathrm{M}_2(S) = A$. Then $s^\sigma = -s$, $s \in A^\times$ and $s \notin B = T$ because $\sigma|_T = \mathrm{id}_T$. It is routine to check that $\varphi_1 := \left[\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right] \in \mathrm{M}_2(A)$ is an isometry of ρg and $\varphi_1 L_0 = \{(a, sa) \,|\, a \in A\}$. Now, as in Case I of the proof of Proposition 5.3, we have $Q \oplus \varphi_1 L_0 = QA$.

Existence of φ_{-1}. Since $B = T$ is a quadratic étale S-algebra, we can write $B = S \oplus \lambda S$ with $\lambda^2 \in S^\times$. The assumption $\sigma|_T = \mathrm{id}_T$ allows us to write $\lambda = \left[\begin{smallmatrix} x_1 & \alpha x_2 \\ x_2 & x_3 \end{smallmatrix}\right]$ with $x_1, x_2, x_3 \in S$. Let

$$\psi := \begin{bmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{bmatrix} \in \mathrm{M}_4(S) = \mathrm{M}_2(A)$$

It is routine to check that $\psi \in U(\rho g)$, $\mathrm{Nrd}(\psi) = -1$, and

$$\psi L_0 = \{([\begin{smallmatrix} 0 & 0 \\ c & d \end{smallmatrix}], [\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}]) \,|\, a, b, c, d \in S\}.$$

Now, if $x_2 \neq 0$, then $B^2 \oplus \psi L_0 = A^2$ and we can take $\varphi_{-1} = \psi$. On the other hand, if $x_2 = 0$, then $B = \left[\begin{smallmatrix} S & 0 \\ 0 & S \end{smallmatrix}\right]$, and hence $\psi(B^2) = B^2$. This means that $B^2 + \psi\varphi_1 L_0 = \psi(B^2 + \varphi_1 L_0) = \psi(A^2) = A^2$, so we can take $\varphi_{-1} = \psi\varphi_1$. □

Proposition 5.8. *With Notation 4.1, suppose that S is a field, (σ,ε) is orthogonal, and $\tau|_T=\mathrm{id}_T$. Let $(Q,g)\in\mathcal{H}^\varepsilon(B,\tau)$ and assume that $\mathrm{rrk}_B\,Q$ is constant and ρg admits a Lagrangian L. Then there exists $\varphi\in U^0(\rho g)$ such that $Q\oplus\varphi L=QA$.*

Proof. As in the proof of Proposition 5.6, we can reduce to the case where $[A]=0$ and write $(Q,g)=(Q_1,g_1)\oplus(Q_2,g_2)$ with g_1 anisotropic and g_2 hyperbolic. Furthermore, $\mathrm{rrk}_B\,Q_2$ is constant, ρg_1 is hyperbolic and any Lagrangian U of ρg_1 satisfies $Q_1\oplus U=Q_1A$.

If $Q_2=0$, then L is a Lagrangian of $(P_1,f_1)=(P,f)$ and we can take $\varphi=\mathrm{id}_P$.

Assume $Q_2\neq 0$, let U be a Lagrangian of ρg_1 and let V be a Lagrangian of ρg_2. By Proposition 5.7, there exist $\varphi_1,\varphi_{-1}\in U(\rho g_2)$ such that $Q_2\oplus\varphi_iV=Q_2A$ and $\mathrm{Nrd}(\varphi_i)=i$ for $i\in\{\pm1\}$. Then $L_i:=U\oplus\varphi_iV$ $(i=\pm1)$ is a Lagrangian of ρg having the same reduced rank as L (see 2F) and satisfying $Q\oplus L_i=QA$. By Lemma 2.22, there exists $\psi\in U(\rho g)$ such that $\psi L=L_1$. If $\mathrm{Nrd}(\psi)=1$, take $\varphi=\psi$. On the other hand, if $\mathrm{Nrd}(\psi)=-1$, then we can take $\varphi:=(\mathrm{id}_{P_1}\oplus\varphi_{-1}\varphi_1^{-1})\psi$, because $\varphi L=U\oplus\varphi_{-1}\varphi_1^{-1}(\varphi_1V)=L_{-1}$ and $\mathrm{Nrd}(\varphi)=\mathrm{Nrd}(\varphi_{-1}\varphi_1)^{-1}\,\mathrm{Nrd}(\psi)=1$. □

We can now establish Theorem 5.1 in cases (1) and (2)

Theorem 5.9. *Assuming R is connected, Theorem 5.1 holds when (σ,ε) is symplectic or unitary, or (τ,ε) is orthogonal or symplectic.*

Proof. We only prove part (ii). It will be clear from the proof that we can take $(Q',g')=(Q,g)$ when T is connected, which is exactly what we need to show in order to prove (i).

Recall that we are given $(Q,g)\in\mathcal{H}^\varepsilon(B,\tau)$ such that $[\rho g]=0$ in $W_\varepsilon(A,\sigma)$. We need to show that ρg admits a Lagrangian L such that $Q\oplus L=QA$, possibly after replacing g with a Witt equivalent form. By Theorem 2.8(ii), ρg is hyperbolic.

We may assume that S is connected. If not, then $S=R\times R$ (Lemma 1.16), and by Example 2.4, g is hyperbolic. We may therefore replace g with the zero form and take $L=0$.

We may also assume that $\mathrm{rrk}_B\,Q$ is constant. Indeed, if $\mathrm{rrk}_B\,Q$ is not constant, then T is not connected. Now, by Lemma 1.16 and our assumption that S is connected, $T\cong S\times S$, so T has exactly two primitive idempotents, denoted e and e'. If σ swaps e and e', then $\mathrm{rrk}_B\,Q$ is constant because it is σ-invariant (Lemma 2.6), so it must be the case that σ fixes e and e'. By Lemma 4.4, $B=eAe\oplus e'Ae'$ and $[eB]=[A]$. Thus, $\mathrm{ind}\,eB=\mathrm{ind}\,A$, and similarly, $\mathrm{ind}\,e'B=\mathrm{ind}\,A$. Let U be a finite projective eB-module of reduced rank $\mathrm{ind}\,eB$ and let V be a finite projective $e'B$-module of reduced rank $\mathrm{ind}\,e'B$; they exist by Theorem 1.25. By Lemma 1.24 and Corollary 1.13, there are $r,s\in\mathbb{Z}$ such that $Q\cong U^r\oplus V^s$, and by Lemma 4.4(iv), $\mathrm{rrk}_A\,QA=\mathrm{rrk}_{eB}\,U^r+\mathrm{rrk}_{e'B}\,V^r=(r+s)\,\mathrm{ind}\,A$. Applying Corollary 2.9(ii) to $(QA,\rho g)$, we see that there exists $W\in\mathcal{P}(A)$ with $\mathrm{rrk}_A\,QA=2\,\mathrm{rrk}_A\,W$. Since $\mathrm{ind}\,A\mid\mathrm{rrk}_A\,W$ (Corollary 1.13), $\mathrm{rrk}_A\,QA$ is an even multiple of $\mathrm{ind}\,A$, and so $r\equiv s \bmod 2$. Now, if $r>s$, we can replace g with $g\oplus(\frac{r-s}{2})\cdot\mathbb{h}_V^\varepsilon$ and if $r<s$, we can replace g with $g\oplus(\frac{s-r}{2})\cdot\mathbb{h}_U^\varepsilon$. After this modification, we get $r=s$, which means that $\mathrm{rrk}_B\,Q$ is constant.

Fix a Lagrangian L of ρg. Since S is connected, $\mathrm{rrk}_A\,L=\frac{1}{2}\,\mathrm{rrk}_A\,QA$ (see 2F). Let $\mathfrak{m}_1,\dots,\mathfrak{m}_t$ denote the maximal ideals of R. By Propositions 5.6 and 5.8, for every $1\le i\le t$, there exists $\varphi_i\in U^0(\rho g(\mathfrak{m}_i))$ such that $Q(\mathfrak{m}_i)\oplus\varphi_i(L(\mathfrak{m}_i))=QA(\mathfrak{m}_i)$. By Theorem 2.18, there exists $\varphi\in U^0(\rho g)$ such that $\varphi(\mathfrak{m}_i)=\varphi_i$. This means that $Q(\mathfrak{m}_i)\oplus(\varphi L)(\mathfrak{m}_i)=QA(\mathfrak{m}_i)$ for all i, so by Lemma 1.7, we have $Q\oplus(\varphi L)=QA$. Since φL is a Lagrangian of QA, we are done. □

5B. **Case (3).** We now turn to prove Theorem 5.1 in Case (3), namely, when R is connected, (σ,ε) is orthogonal and (τ,ε) is unitary. Note that $S=R$.

This case is more subtle than Cases (1) and (2) because the key Propositions 5.6 and 5.8 no longer hold. The proof will therefore consist of characterizing when these propositions fail, and bypassing the failure when they do.

We begin with treating the case where $\operatorname{rrk}_B Q$ is odd; this case is degenerate.

Proposition 5.10. *With Notation 4.1, suppose that R is connected semilocal, (σ,ε) is orthogonal and τ is unitary. Let $(Q,g)\in\mathcal{H}^\varepsilon(B,\tau)$ and assume that ρg is hyperbolic and $\operatorname{rrk}_B Q$ is not constant or not even. Then T is not connected and g is hyperbolic.*

Proof. If T is not connected, then $T\cong R\times R$ by Lemma 1.16, and g is hyperbolic by Example 2.4 (applied to (B,τ)). It is therefore enough to show that T is not connected.

For the sake of contradiction, suppose that T is connected. Then $\operatorname{rrk}_B Q$ is constant and odd. Furthermore, by Corollary 2.9(ii), there exists $V\in\mathcal{P}(A)$ such that $2\iota\operatorname{rrk}_A V=\iota\operatorname{rrk}_A QA=2\operatorname{rrk}_B Q$. Thus, $n:=\operatorname{rrk}_A V$ is odd. By Corollary 1.13, $\operatorname{ind} A\mid n$, so by Theorem 1.8, $n[A]=0$ in $\operatorname{Br} R$. On the other hand, since A has an R-involution, $2[A]=0$, so $[A]=0$.

We now apply Reductions 4.10 and 4.11 to assume that $B=T$, $A=\mathrm{M}_2(S)$, $\varepsilon=1$ and $\sigma:A\to A$ is orthogonal and given by $\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]^\sigma=\left[\begin{smallmatrix} a & \eta c\\ \eta^{-1}b & d\end{smallmatrix}\right]$ for some $\eta\in S^\times$.

By Lemma 1.19, there exists $\lambda\in T^\times$ such that $\lambda^\sigma=-\lambda$ and $T=R\oplus\lambda R$. Then $\lambda=\left[\begin{smallmatrix} 0 & \eta c\\ -c & 0\end{smallmatrix}\right]$ for some $c\in S^\times$, and consequently $T=R\left[\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right]\oplus R\left[\begin{smallmatrix} 0 & \eta\\ -1 & 0\end{smallmatrix}\right]$. Furthermore, by Proposition 2.13, g is diagonalizable, so there exist $\alpha_1,\dots,\alpha_n\in\mathcal{S}_1(T,\tau)\cap T^\times=R^\times$ such that $g\cong\langle\alpha_1,\dots,\alpha_n\rangle_{(T,\tau)}$ (notation as in Example 2.1). Note that $n=\operatorname{rrk}_B Q$ is odd.

Let $e:=\left[\begin{smallmatrix} 1 & 0\\ 0 & 0\end{smallmatrix}\right]$. Then e is an idempotent satisfying $e^\sigma=e$, $eAe=eR$ and $AeA=A$, hence e-transfer (see 2G) induces an group isomorphism $[f]\mapsto[f_e]:W_1(A,\sigma)\to W_1(R,\mathrm{id}_R)$. It is routine to check that upon identifying Ae with R^2 via $\left[\begin{smallmatrix} a & 0\\ c & 0\end{smallmatrix}\right]\mapsto(a,c)$, the bilinear form $(\rho g)_e:A^ne\times A^ne\to eR\cong R$ is just $\langle\alpha_1,\eta\alpha_1,\dots,\alpha_n,\eta\alpha_n\rangle_{(R,\mathrm{id}_R)}$. By assumption, this form is hyperbolic, so it is isomorphic to $n\langle 1,-1\rangle_{(R,\mathrm{id}_R)}$ (Lemma 2.7). Comparing discriminants (using Proposition 2.27(iv)), we find that $(-\eta)^n$ is a square in $R^\times$. Since n is odd, this means that $-\eta$ is a square in $R^\times$, say $-\eta=r^2$. Then $\frac12\left[\begin{smallmatrix} 1 & r\\ r^{-1} & 1\end{smallmatrix}\right]=\frac12\left[\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right]-\frac{1}{2r}\left[\begin{smallmatrix} 0 & \eta\\ -1 & 0\end{smallmatrix}\right]$ is a nontrivial idemptonent in T, contradicting our assumption that T is connected. □

Recall from 2F that $\operatorname{Lag}(f)$ denotes the set of Lagrangians L of $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma)$ with $\operatorname{rrk}_A L=\frac12\operatorname{rrk}_A P$. When (σ,ε) is orthogonal, $\operatorname{Lag}(f)$ consists of all the Lagrangians of f, and when R is semilocal, any two Lagrangians in $\operatorname{Lag}(f)$ are isomorphic (Lemma 1.24). These facts will be used without comment in the sequel.

Proposition 5.11. *With Notation 4.1, suppose that (σ,ε) is orthogonal and τ is unitary. Let $(Q,g)\in\mathcal{H}^\varepsilon(B,\tau)$ and assume that ρg is hyperbolic and $\operatorname{rrk}_B Q$ is even. Then there exists a unique $\mathbf{U}(\rho g)$-equivariant natural transformation of functors from R-rings to sets,*

$$\Phi_g:\mathbf{Lag}(\rho g)\to\boldsymbol{\mu}_{2,R},$$

such that for any R-ring R_1 and any idempotent $e_1\in T_{R_1}$ with $e_1+e_1^\sigma=1$, one has $\Phi_g(P_{R_1}e_1A_{R_1})=1$. The map Φ_g has the following additional properties:

- *(i) If there exists an idempotent $e\in T$ such that $e^\sigma+e=1$, then $\Phi_g=\Phi_{PeA}$ (notation as in Proposition 2.25).*
- *(ii) If $n:=\operatorname{rrk}_B Q$ is constant and $M\in\operatorname{Lag}(g)$, then $\Phi_g(MA)=(-1)^{\frac n2}$.*
- *(iii) If $(Q',g')\in\mathcal{H}^\varepsilon(B,\tau)$ is another hermitian space such that $\rho g'$ is hyperbolic and $\operatorname{rrk}_B Q'$ is even, then $\Phi_{g\oplus g'}(L\oplus L')=\Phi_g(L)\cdot\Phi_{g'}(L')$ for all $L\in\operatorname{Lag}(\rho g)$, $L'\in\operatorname{Lag}(\rho g')$.*

(iv) *If $e \in B$ is an idempotent such that $e^\tau = e$ and $\operatorname{rrk}_B eB$ is positive and constant on the fibers of $\operatorname{Spec} T \to \operatorname{Spec} R$, then $\Phi_g(L) = \Phi_{g_e}(Le)$ for all $L \in \operatorname{Lag}(\rho g)$ (notation as in 2G).*

Note that $Q_{R_1}e_1$ is a Lagrangian of g_{R_1} (see Example 2.4), and therefore $Q_{R_1}e_1A_{R_1}$ is a Lagrangian of ρg_{R_1}. We alert the reader that Φ_g is not defined when $\operatorname{rrk}_B Q$ is not even.

Proof. Fix a Lagrangian L_0 of ρg and let $\Phi_0 := \Phi_{L_0}$ be as in Proposition 2.25.

Let R_1 be an R-ring and let $e, e' \in T_{R_1}$ be two idempotents satisfying $e + e^\sigma = e' + e'^\sigma = 1$. We claim that $\Phi_0(Q_{R_1}eA_{R_1}) = \Phi_0(Q_{R_1}e'A_{R_1})$, or rather, $\Phi_{Q_{R_1}eA_{R_1}}(Q_{R_1}e'A_{R_1}) = 1$ (Proposition 2.25(i)). Base changing along $R \to R_1$, we may assume that $R_1 = R$. Now, by Lemma 2.17, it is enough to show that $\Phi_{QeA}(Qe'A)(\mathfrak{m}) = 1$ in $k(\mathfrak{m})$ for all $\mathfrak{m} \in \operatorname{Max} R$, so assume that R is a field. Since $e \in T$ is an idempotent satisfying $e^\sigma + e = 1$, it is nontrivial and hence $T = R \times R$. Similarly, e' is nontrivial, so $e = e'$ or $e = 1 - e'$. In the first case, we have $QeA = Qe'A$ and $\Phi_{QeA}(Qe'A) = 1$. In the second case, $QA = QeA \oplus Qe'A$, so $\Phi_{QeA}(Qe'A) = (-1)^{\operatorname{rrk}_A QeA} = (-1)^{\operatorname{rrk}_{eB} Qe} = 1$ by Proposition 2.26, Lemma 4.4(iv), and the fact $\operatorname{rrk}_B Q$ is even. This proves the claim.

Write $R_0 = T$. Then $T_{R_0} \cong R_0 \times R_0$ by Lemma 1.18. Let $e_0 \in T_{R_0}$ correspond to $(1_{R_0}, 0_{R_0})$ under this isomorphism. Since $\tau|_T$ is the standard R-involution of T, we have $e_0 + e_0^\sigma = 1$.

Write $\theta := \Phi_0(P_{R_0}e_0A_{R_0}) \in \mu_2(R_0)$. We claim that θ is in fact in $\mu_2(R)$. Let $i_1, i_2 : R_0 \to R_0 \otimes R_0$ denote the maps $r \mapsto r \otimes 1$ and $r \mapsto 1 \otimes r$ respectively. By what we have shown above, $i_1\theta = \Phi_0(P_{R_0\otimes R_0}(i_1e_0)A_{R_0\otimes R_0}) = \Phi_0(P_{R_0\otimes R_0}(i_2e_0)A_{R_0\otimes R_0}) = i_2\theta$. Since $\boldsymbol{\mu}_{2,R}$ is a sheaf on $(\mathcal{A}ff/R)_{\mathrm{fpqc}}$, and since $R \to R_0$ is faithfully flat, this means that $\theta \in \mu_2(R)$.

Define $\Phi_g := \theta^{-1} \cdot \Phi_0$. It is clear that Φ_g is $\mathbf{U}(\rho g)$-equivariant. Let R_1 and $e_1 \in T_{R_1}$ be as in the proposition. Then, in $\mu_2(R_0 \otimes R_1)$, we have $\Phi_g(Q_{R_0\otimes R_1}e_1A_{R_0\otimes R_1}) = \Phi_g(Q_{R_0\otimes R_1}e_0A_{R_0\otimes R_1}) = \Phi_g(Q_{R_0}e_0A_{R_0}) = \theta^{-1}\theta = 1$. Since $R_1 \to R_0 \otimes R_1$ is faithfully flat, this means that $\Phi_g(Q_{R_1}e_1A_{R_1}) = 1$ in $\mu_2(R_1)$.

Suppose that $\Phi' : \mathbf{Lag}(\rho g) \to \boldsymbol{\mu}_{2,R}$ also satisfies the conditions of the Proposition. Then, by Proposition 2.25, both Φ' and Φ must coincide with $\Phi_{Q_{R_0}e_0A_{R_0}}$ on the subcategory of R_0-rings. Since $\boldsymbol{\mu}_2$ and $\mathbf{Lag}(\rho g)$ are sheaves over $(\mathcal{A}ff/R)_{\mathrm{fpqc}}$ and since $R \to R_0$ is faithfully flat, this forces $\Phi' = \Phi_g$.

We finish with verifying (i)–(iv). Since $R \to R_0$ is faithfully it is enough to prove these statements after base-changing to R_0. We may therefore assume that $T = R \times R$ and there exists an idempotent $e_0 \in T$ with $e_0^\sigma + e_0 = 1$.

(i) This is immediate from the uniqueness part of Proposition 2.25.

(ii) We have $Me_0 = M \cap Qe_0$. Since ${}_BA$ is flat, this means that $Me_0A = MA \cap Qe_0A$. By (i) and Proposition 2.26, $\Phi_g(MA) = (-1)^{\operatorname{rrk}_A Qe_0A - \operatorname{rrk}_A Me_0A}$, and $\operatorname{rrk}_A Qe_0A - \operatorname{rrk}_A Me_0A = \operatorname{rrk}_{e_0B} Qe_0 - \operatorname{rrk}_{e_0B} Me_0 = \frac{n}{2}$ by Lemma 4.4(iv).

(iii) This follows from (i) and Proposition 2.25(ii).

(iv) This follows readily from (i), item (t6) in 2G and Proposition 4.9(i). □

Proposition 5.12. *With Notation 4.1, suppose that R is a field, $[A] = 0$, (σ, ε) is orthogonal and τ is unitary. Let $(Q, g) \in \mathcal{H}^\varepsilon(B, \tau)$ be a hyperbolic hermitian space such that $\operatorname{rrk}_B Q$ is constant and even. Then:*

(i) *There exists $L \in \operatorname{Lag}(\rho g)$ such that $Q \oplus L = QA$ and $\Phi_g(L) = 1$.*

(ii) *There is no $L \in \operatorname{Lag}(\rho g)$ such that $Q \oplus L = QA$ and $\Phi_g(L) = -1$.*

Proof. By Reduction 4.10 and Proposition 5.11(iv), we may assume that $B = T$, $A = \mathrm{M}_2(R)$, $\varepsilon = 1$ and σ is orthogonal.

(i) By Reduction 4.11, we may assume that σ is given by $[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]^\sigma = [\begin{smallmatrix} a & \alpha c \\ \alpha^{-1} b & d \end{smallmatrix}]$ for some $\alpha \in R^\times$. Arguing as in Case I of the proof of Proposition 5.3, we may assume that $(Q, g) = (B^2, g_1)$, where $g_1((x_1, x_2), (y_1, y_2)) = x_1^\sigma y_2 + x_2^\sigma y_1$. By Lemma 1.19, there exists $\lambda \in T$ such that $T = R \oplus \lambda R$ and $\lambda^\tau = -\lambda$. This forces $B = T = [\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}]R + [\begin{smallmatrix} 0 & \alpha \\ -1 & 0 \end{smallmatrix}]R$. Now, it is routine to check that $L = \{([\begin{smallmatrix} 0 & 0 \\ c & d \end{smallmatrix}], [\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}]) \,|\, a, b, c, d \in S\}$ is a Lagrangian of ρg satisfying $B^2 \oplus L = A^2$. That $\Phi_g(L) = 1$ will follow once we prove (ii).

(ii) *Step 1.* It is enough to prove the statement after base-changing to an algebraic closure of R, so assume that R is an algebraically closed field. In this case, $B = T = R \times R$ and τ is the exchange involution. By Example 2.4, this means that $g \cong n\langle 1 \rangle_{(B,\tau)}$, where $n = \operatorname{rrk}_B Q$ and $\langle 1 \rangle_{(B,\tau)}$ is the hermitian form $(x, y) \mapsto x^\tau y$ on B. We may therefore assume that $(Q, g) = (B^n, n\langle 1 \rangle_{(B,\tau)})$.

Arguing as in Subcase II.2 of the proof of Proposition 5.3, we may identify A with $\mathrm{M}_2(R)$ in such a way that B is the algebra of diagonal matrices and σ is given by $[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]^\sigma = [\begin{smallmatrix} d & \sigma_2 b \\ \sigma_3 c & a \end{smallmatrix}]^\sigma$, where σ_2, σ_3 are R-linear automorphisms of R of order 2. Since $\sigma_2, \sigma_3 \in \{\pm \mathrm{id}_R\}$ and $\dim_R \mathcal{S}_{-1}(A, \sigma) = 1$ (Proposition 1.21), we must have $\sigma_2 = \sigma_3 = \mathrm{id}_S$, hence σ is given by $[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]^\sigma = [\begin{smallmatrix} d & b \\ c & a \end{smallmatrix}]$.

Step 2. Let $\tilde{A} = \mathrm{M}_n(A)$ and let $\tilde{\sigma} : \tilde{A} \to \tilde{A}$ be given by $(a_{ij})^{\tilde{\sigma}} = (a_{ji}^\sigma)$. Define $\tilde{B}$ and $\tilde{\tau}$ similarly and let $\tilde{\rho} : \tilde{B} \to \tilde{A}$ denote the inclusion map. Let $e \in \tilde{B} = \mathrm{M}_n(B)$ denote the matrix with 1 in the $(1, 1)$-entry and 0 elsewhere, and let $\tilde{g} : \tilde{B} \times \tilde{B} \to \tilde{B}$ denote the diagonal hermitian form $\langle 1 \rangle_{(\tilde{B},\tilde{\tau})}$ (see Example 2.1). It is easy to check that the assumptions of Notation 4.1 apply to $\tilde{A}$, $\tilde{\sigma}$, T (embedded diagonally in $\tilde{A} = \mathrm{M}_n(A)$) and $\tilde{B}$. Furthermore, under the evident isomorphisms $e\tilde{B}e \cong B$, $\tilde{B}e \cong B^n$, one finds that, the e-transfer $\tilde{g}_e$ (see 2G) is just g. Thus, by Propositions 4.9(ii) and 5.11(iv), it is enough to prove that $\tilde{\rho}\tilde{g}$ admits no Lagrangians $\tilde{L}$ with $\tilde{B} \oplus \tilde{L} = \tilde{A}$ and $\Phi_{\tilde{g}}(\tilde{L}) = -1$.

Note that $(\tilde{A}, \tilde{\sigma}) \cong (\mathrm{M}_2(R), \sigma) \otimes (\mathrm{M}_n(R), \mathrm{t})$, where t denotes the transpose involution. Thus, we may identify $\tilde{A}$ with $\mathrm{M}_{2n}(R)$ in such a way that $\tilde{\sigma}$ is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\tilde{\sigma}} = \begin{bmatrix} d^{\mathrm{t}} & b^{\mathrm{t}} \\ c^{\mathrm{t}} & a^{\mathrm{t}} \end{bmatrix},$$

where $a, b, c, d \in \mathrm{M}_n(R)$. Under this identification, $\tilde{B} = \{[\begin{smallmatrix} a & \\ & d \end{smallmatrix}] \,|\, a, d \in \mathrm{M}_n(R)\}$ and $T = \{[\begin{smallmatrix} \alpha 1_n & \\ & \beta 1_n \end{smallmatrix}] \,|\, \alpha, \beta \in R\}$, where 1_n is the $n \times n$ identity matrix.

Overriding previous notation, let $e = [\begin{smallmatrix} 1_n & 0 \\ 0 & 0 \end{smallmatrix}]$. Then $e\tilde{A} = \tilde{B}e\tilde{A}$ is a Lagrangian of $\tilde{\rho}\tilde{g}$, and $\Phi_{\tilde{g}}(e\tilde{A}) = 1$ by the defining property of $\Phi_{\tilde{g}}$. Since $U(\tilde{\rho}\tilde{g})$ acts transitively on $\mathrm{Lag}(\tilde{\rho}\tilde{g})$ (Lemma 2.22), it is enough to prove that for every $\varphi \in U(\tilde{\rho}\tilde{g})$ with $\mathrm{Nrd}(\varphi) = -1$, we have $\tilde{B} + \varphi e\tilde{A} \neq \tilde{A}$. Identifying $\mathrm{End}_{\tilde{A}}(\tilde{A}_{\tilde{A}})$ with $\tilde{A}$ (acting on the left on itself) and writing $\varphi = [\begin{smallmatrix} x & x' \\ y & y' \end{smallmatrix}]$ with $x, x', y, y' \in \mathrm{M}_n(R)$, we get $\varphi e\tilde{A} = \{[\begin{smallmatrix} xa & xb \\ ya & yb \end{smallmatrix}] \,|\, a, b \in \mathrm{M}_n(R)\}$, from which it follows readily that

$$\tilde{B} + \varphi e\tilde{A} = \tilde{A} \qquad \iff \qquad x, y \in \mathrm{GL}_n(R).$$

Step 3. Recall that R is assumed to be algebraically closed. We shall view all finite dimensional R-vector spaces and the group $U(\tilde{A}, \tilde{\sigma}) = U(\tilde{\rho}\tilde{g})$ as varieties over R in the obvious way. Recall from Proposition 2.16 that $U(\tilde{A}, \tilde{\sigma})$ has two (Zariski) connected components — $U^0(\tilde{A}, \tilde{\sigma}) := U^0(\tilde{\rho}\tilde{g})$ and $U^1(\tilde{A}, \tilde{\sigma}) := U(\tilde{A}, \tilde{\sigma}) \setminus U^0(\tilde{A}, \tilde{\sigma})$.

Consider the morphism $\psi : U(\tilde{A}, \tilde{\sigma}) \to \mathrm{M}_n(R) \times \mathrm{M}_n(R)$ given by $[\begin{smallmatrix} x & x' \\ y & y' \end{smallmatrix}] \mapsto (x, y)$. By Step 2, we need to show that $\psi(U^1(\tilde{A}, \tilde{\sigma}))$ does not meet $\mathrm{GL}_n(R) \times \mathrm{GL}_n(R)$. Since $\mathrm{GL}_n(R) \times \mathrm{GL}_n(R)$ is Zariski open in $\mathrm{M}_n(R) \times \mathrm{M}_n(R)$, it is enough to verify this after replacing $U^1(\tilde{A}, \tilde{\sigma})$ with a Zariski dense subset.

Step 4. In what follows, we shall write matrices $a \in \mathrm{M}_n(R)$ in 2×2 block form $\left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{smallmatrix}\right]$, where a_{11} is a 1×1 matrix. With this notation, let

$$u := \left[\begin{array}{cc} \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1_{n-1}\end{smallmatrix}\right] & \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right] \\ \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1_{n-1}\end{smallmatrix}\right] \end{array}\right]$$

and note that $u \in U(\tilde{A},\tilde{\sigma})$ and $\mathrm{Nrd}(u) = -1$.

For all $a, b \in \mathcal{S}_{-1}(\mathrm{M}_n(R),\mathrm{t})$, $c \in \mathrm{GL}_n(R)$, define

$$\xi(a,b,c) = u \cdot \left[\begin{array}{cc} 1 & 0 \\ a & 1\end{array}\right]\left[\begin{array}{cc} c & 0 \\ 0 & (c^{\mathrm{t}})^{-1}\end{array}\right]\left[\begin{array}{cc} 1 & b \\ 0 & 1\end{array}\right] = u \cdot \left[\begin{array}{cc} c & cb \\ ac & acb + (c^{\mathrm{t}})^{-1}\end{array}\right].$$

It is easy to check that ξ is a morphisms of R-varieties from $\mathcal{S}_{-1}(\mathrm{M}_n(R),\mathrm{t}) \times \mathcal{S}_{-1}(\mathrm{M}_n(R),\mathrm{t}) \times \mathrm{GL}_n(R)$ to $U^1(\tilde{A},\tilde{\sigma})$ that is injective on R-points. Since $U^1(\tilde{A},\tilde{\sigma}) \cong U^0(\tilde{A},\tilde{\sigma})$ as R-varieties, and since $U^0(\tilde{A},\tilde{\sigma})$ is just $\mathbf{SO}_{2n}(R)$, it follows that the source and target of ξ have the same dimension (i.e. $\frac{1}{2}(2n)(2n-1) = \frac{1}{2}n(n-1) + \frac{1}{2}n(n-1) + n^2$). Thus, by Chevalley's Theorem, $\mathrm{im}(\xi)$ is dense in $U^1(\tilde{A},\tilde{\sigma})$.

Writing $a = \left[\begin{smallmatrix} 0 & a_{12} \\ a_{21} & a_{22}\end{smallmatrix}\right] \in \mathcal{S}_{-1}(\mathrm{M}_n(S),\mathrm{t})$ and $c = \left[\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{smallmatrix}\right] \in \mathrm{GL}_n(R)$, one readily checks that

$$\xi(a,b,c) = \left[\begin{array}{cc} \left[\begin{smallmatrix} a_{12}c_{21} & a_{12}c_{22} \\ c_{21} & c_{22}\end{smallmatrix}\right] & * \\ * & * \end{array}\right].$$

Since $\left[\begin{smallmatrix} a_{12}c_{21} & a_{12}c_{22} \\ c_{21} & c_{22}\end{smallmatrix}\right]$ is never invertible (multiply by $\left[\begin{smallmatrix} 1 & -a_{12} \\ 0 & 1\end{smallmatrix}\right]$ on the left), we see that $\psi(\mathrm{im}(\xi))$ does not meet $\mathrm{GL}_n(R)\times\mathrm{GL}_n(R)$. Since $\mathrm{im}(\xi)$ is dense in $U^1(\tilde{A},\tilde{\sigma})$, this completes the proof. □

Remark 5.13. In Proposition 5.12, one can similarly show that if $\mathrm{rrk}_B\, Q$ is constant and odd, then there is no $L \in \mathrm{Lag}(\rho f)$ such that $Q \oplus L = QA$: Replace ξ with the maps $\xi_0(a,b,c) = \left[\begin{smallmatrix} 1 & 0 \\ a & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} c & 0 \\ 0 & (c^{\mathrm{t}})^{-1}\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right]$ and $\xi_1(a,b,c) = \left[\begin{smallmatrix} 0 & 1_n \\ 1_n & 0\end{smallmatrix}\right]\xi_0(a,b,c)$ and note that a cannot be invertible when n is odd.

Corollary 5.14. *With Notation 4.1, suppose that (σ,ε) is orthogonal and τ is unitary. Let $(Q,g) \in \mathcal{H}^{\varepsilon}(B,\tau)$ and assume that ρg hyperbolic and $\mathrm{rrk}_B\, Q$ is constant and even. If $L \in \mathrm{Lag}(\rho g)$ satisfies $Q \oplus L = QA$, then $\Phi_g(L) = 1$.*

Proof. Let K be an algebraically closed R-field. Then $T_K \cong K \times K$. Thus, by Example 2.4, g_K is hyperbolic. Now, by Proposition 5.12, $\Phi_g(L_K) = 1$. Thanks to Lemma 2.17, $\Phi_g(L) = 1$ follows by letting K range over the algebraic closures of the residue fields of R. □

Now we can prove an analogue to Propositions 5.6 and 5.8 in Case (3).

Proposition 5.15. *With Notation 4.1, suppose that R is a field, (σ,ε) is orthogonal and τ is unitary. Let $(Q,g) \in \mathcal{H}^{\varepsilon}(B,\tau)$, let $L \in \mathrm{Lag}(\rho g)$ and assume that $\mathrm{rrk}_B\, Q$ is constant and even. Then there exists $\varphi \in U^0(\rho g)$ such that $Q \oplus \varphi L = QA$ if and only if $\Phi_g(L) = 1$.*

Proof. If $Q \oplus \varphi L = QA$ for $\varphi \in U^0(\rho g)$, then $\Phi_g(L) = \mathrm{Nrd}(\varphi)\Phi_g(L) = \Phi_g(\varphi L) = 1$ by Corollary 5.14. We turn to prove the converse.

As in the proof of Proposition 5.6, we can reduce to the case where $[A] = 0$ and write $(Q,g) = (Q_1,g_1) \oplus (Q_2,g_2)$ with g_1 anisotropic and g_2 hyperbolic. Furthermore, there exists a Lagrangian L_1 of ρg_1 such that $Q_1 \oplus L_1 = Q_1 A$.

By Proposition 5.10, $\mathrm{rrk}_B\, Q_1$ is constant and even, and hence so is $\mathrm{rrk}_B\, Q_2$. Thus, by Proposition 5.12(i), there exists $L_2 \in \mathrm{Lag}(\rho g_2)$ with $Q_2 \oplus L_2 = Q_2 A$.

Let $L' := L_1 \oplus L_2$. Then $L' \in \mathrm{Lag}(\rho g)$ and $Q \oplus L' = QA$. By Lemma 2.22, there exists $\varphi \in U(\rho g)$ such that $L' = \varphi L$. By Corollary 5.14, $1 = \Phi_g(L') = \mathrm{Nrd}(\varphi)\Phi_g(L) = \mathrm{Nrd}(\varphi)$, so $\mathrm{Nrd}(\varphi) = 1$ and the proposition follows. □

From Proposition 5.15, we see that in order to apply the proof of Theorem 5.9 to our situation, we have to find $L \in \mathrm{Lag}(\rho g)$ satisfying $\Phi_g(L) = 1$. The purpose of the following propositions is to characterize precisely when such L exists.

We begin by noting that, in many cases, Φ_g is constant on the set $\mathrm{Lag}(\rho g)$.

Proposition 5.16. *With Notation 4.1, suppose that R is connected semilocal, (σ, ε) is orthogonal and τ is unitary. Let $(Q, g) \in \mathcal{H}^\varepsilon(B, \tau)$ be an ε-hermitian space such that ρg is hyperbolic and $\mathrm{rrk}_B Q$ is even. Then $\Phi_g : \mathrm{Lag}(\rho g) \to \mu_2(R) = \{\pm 1\}$ is onto if and only if $[A] = 0$ and $Q \neq 0$.*

Proof. The proposition is clear when $Q = 0$, so assume $Q \neq 0$.

Suppose that Φ_g is onto. Then there are $L_0, L_1 \in \mathrm{Lag}(\rho g)$ such that $\Phi_g(L_0) = 1$ and $\Phi_g(L_1) = -1$. By Lemma 2.22, there exists $\varphi \in U(\rho g)$ such that $\varphi L_0 = L_1$, hence $\mathrm{Nrd}(\varphi) = \mathrm{Nrd}(\varphi)\Phi_g(L_0) = \Phi_g(L_1) = -1$. By Theorem 2.20, this means that $[A] = 0$.

Conversely, if $[A] = 0$, then Theorem 2.20 implies the existence of $\varphi \in U(\rho g)$ with $\mathrm{Nrd}(\varphi) = -1$. Choose some $L \in \mathrm{Lag}(\rho g)$. Then $\Phi_g(\varphi L) = -\Phi_g(L)$, hence Φ_g is onto. □

Proposition 5.17. *Under the assumptions of Proposition 5.16, if T is connected and $[B] \neq 0$, then $\Phi_g(L) = 1$ for all $L \in \mathrm{Lag}(\rho g)$.*

Proof. For the sake of contradiction, suppose that there exists $L \in \mathrm{Lag}(\rho g)$ with $\Phi_g(L) = -1$. By Lemma 1.18, $T_T \cong T \times T$ as T-algebras. Since τ_T is unitary, there exists $e \in T_T$ such that $e^\tau + e = 1$. By the definition of Φ_g, we have $\Phi_g(Q_T e A_T) = 1$, so Φ_g is not constant on $\mathrm{Lag}(\rho g_T)$. Now, applying Proposition 5.16 to g_T (here we need T to be connected), we get $[B] = [A_T] = 0$, a contradiction. □

The next lemmas and proposition concern with the case $[B] = 0$. They will only be needed in proving part (i) of Theorem 5.1. We shall make use of the discriminant algebra $D(g)$ defined in 2H.

Lemma 5.18. *With Notation 4.1, suppose that R is semilocal, $\deg B = 1$, σ is orthogonal, τ is unitary and $\varepsilon = 1$. Define λ, μ as in Lemma 4.3(ii) (so $\lambda^\sigma = -\lambda$ and $\mu^\sigma = \mu$). Let $(Q, g) \in \mathcal{H}^1(B, \tau)$ and assume that ρg is hyperbolic and $\mathrm{rrk}_B Q$ is constant and even. Let $x_1, x_2 \in Q$ and write $x = x_1 + x_2\mu \in QA$.*

- *(i) If $\rho g(x, x) = 0$ and $g(x_1, x_1) \in B^\times$, then $Q_1 := x_1 B + x_2 B$ is a summand of Q with B-basis $\{x_1, x_2\}$. Writing $g_1 = g|_{Q_1 \times Q_1}$, the form g_1 is unimodular, $xA \in \mathrm{Lag}(\rho g_1)$, $Q_1 \oplus xA = Q_1 A$ and $[D(g_1)] = [A]$ in $\mathrm{Br}\, R$.*
- *(ii) If $\mathrm{rrk}_B Q \geq 4$, then there exist x_1, x_2, x as in (i).*

Proof. (i) Write $\alpha := g(x_1, x_1) \in B^\times$. Since g is 1-hermitian and $\mu b = b^\sigma \mu$ for all $b \in B$, we have $0 = \rho g(x, x) = g(x_1, x_1) + 2\mu g(x_2, x_1) + \mu^2 g(x_2, x_2)$, so $g(x_1, x_2) = 0$ and $g(x_2, x_2) = -\mu^2 g(x_1, x_1)$. By examining the Gram matrix of g relative to $\{x_1, x_2\}$, we see that $\{x_1, x_2\}$ is a g-orthogonal basis to Q_1 and g_1 is unimodular and isomorphic to $\langle \alpha, -\mu^2\alpha \rangle_{(B,\tau)}$. Thus, $D(g) = (B/R, \mu^2\alpha^2) \cong (B/R, \mu^2) \cong A$ (see 2H). Let $x' = x_1 - \mu x_2$. One readily checks that $\rho g(x', x') = 0$ and $xA \oplus x'A = Q_1 A$, hence $xA \in \mathrm{Lag}(\rho g_1)$.

We finish by checking that $Q_1 \oplus xA = Q_1 A$. If $y \in Q_1 \cap xA$, then there is $a \in A$ such that $y = xa = x_1 a + x_2 \mu a$. Since $\{x_1, x_2\}$ is a B-basis of Q_1, $\{x_1, x_2\}$ is an A-basis of $Q_1 A$, so $y \in Q_1$ implies that $a, \mu a \in B$. As a result $a \in B \cap \mu^{-1} B = B \cap \mu(\mu^{-2} B) \subseteq B \cap \mu B = 0$ (because $\mu^{-2} \in \mathrm{Z}_A(\lambda) = B$), and $y = xa = 0$. This means that $Q_1 \cap xA = 0$. On the other hand, $Q_1 + xA \supseteq x_1 B + x_2 B + (x_1 + x_2\mu)B + (x_1\mu + x_2\mu^2)B \supseteq x_1 B + x_2 B + x_1\mu B + x_2\mu B = Q_1 A$, so $Q_1 \oplus xA = Q_1 A$.

(ii) *Step 1.* We first prove the claim when R is a field. Using Proposition 2.5, write $(Q, g) = (Q_1, g_1) \oplus (Q_2, g_2)$ with g_1 anisotropic and g_2 hyperbolic. Since

$[\rho g_1] = [\rho g] = 0$, the form ρg_1 is hyperbolic (Theorem 2.8(ii)). Since $\operatorname{rrk}_B Q \geq 4$, either $Q_1 \neq 0$ or $\operatorname{rrk}_B Q_2 \geq 4$.

Assume $Q_1 \neq 0$. Since ρg_1 is hyperbolic, there exists $0 \neq x \in Q_1 A$ such that $\rho g_1(x,x) = 0$. Write $x = x_1 + x_2\mu$ with $x_1, x_2 \in Q_1$. If $x_1 = 0$, replace x with $x\mu$. Since g_1 is an anisotropic and T is semisimple artinian, $g_1(x_1,x_1) \in \mathcal{S}_1(B,\tau)\setminus\{0\} = R^\times$, so x_1, x_2 satisfy the requirements.

Assume $\operatorname{rrk}_B Q_2 \geq 4$. Since g_2 is hyperbolic, it has an orthogonal summand isomorphic to the hyperbolic form $\langle 1, -1, \mu^2, -\mu^2\rangle_{(B,\tau)}$ (Lemma 2.7). Now take x_1 and x_2 to be the elements corresponding to $(0,0,0,1)$ and $(1,0,0,0)$ in Q.

Step 2. We continue to assume that R is field. Let $x, y \in QA$ be two elements such that $\rho g(x,x) = \rho g(y,y) = 0$ and $\operatorname{rrk}_A xA = \operatorname{rrk}_A yA = 2$. We claim that there exists $\varphi \in U^0(\rho g)$ such that $\varphi x = y$.

By Theorem 2.3, there exists $\psi \in U(\rho g)$ such that $\psi x = y$. If $\operatorname{Nrd}(\psi) = 1$, then we can take $\varphi = \psi$, so assume $\operatorname{Nrd}(\psi) = -1$. In this case, $[A] = 0$ by Theorem 2.20.

Since ρg is unimodular and xA is a free summand of QA, there exists $x' \in QA$ such that $\rho g(x,x') = 1$. Write $V = xA + x'A$. Since $\rho g(x,x) = 0$, the restriction of ρg to V is unimodular (the matrix $\left[\begin{smallmatrix} \rho g(x,x) & \rho g(x,x') \\ \rho g(x',x) & \rho g(x',x') \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 1 \\ 1 & * \end{smallmatrix}\right]$ is invertible), so $QA = V \oplus V^\perp$. Let $h = \rho g|_{V^\perp \times V^\perp}$. Since $\iota \operatorname{rrk}_A QA = 2\operatorname{rrk}_B Q \geq 8$ and $\operatorname{rrk}_A V = 4$, we have $V^\perp \neq 0$. Thus, by Theorem 2.20, there exists $\psi_1 \in U(h)$ with $\operatorname{Nrd}(\psi_1) = -1$. Take $\varphi = \psi \circ (\mathrm{id}_V \oplus \psi_1)$.

Step 3. We finally establish the existence of x_1, x_2 in general. Let $L \in \operatorname{Lag}(\rho g)$. Then $\iota \operatorname{rrk}_A L = \frac{1}{2}\iota \operatorname{rrk}_A QA = \operatorname{rrk}_B Q \geq 4$, so L admits a summand isomorphic to A_A (Lemma 1.24). Let y be a generator of such a summand.

Let $\mathfrak{m}_1, \dots, \mathfrak{m}_t$ denote the maximal ideals of R. By Step 1, for all $1 \leq i \leq t$, there exists $x_i = x_{1i} + x_{2i}\mu$, with $x_{1i}, x_{2i} \in Q(\mathfrak{m}_i)$, such that $\rho g(\mathfrak{m}_i)(x_i,x_i) = 0$ and $g(\mathfrak{m}_i)(x_{1i},x_{1i}) \in B(\mathfrak{m}_i)^\times$. We observed in the proof of (i) that $\operatorname{rrk}_{A(\mathfrak{m}_i)} x_i A(\mathfrak{m}_i) = 2$, so by Step 2, there exists $\varphi_i \in U^0(\rho g(\mathfrak{m}_i))$ such that $\varphi_i(y(\mathfrak{m}_i)) = x_i$. By Theorem 2.18, there exists $\varphi \in U^0(\rho g)$ such that $\varphi(\mathfrak{m}_i) = \varphi_i$ for all i. Let $x = \varphi y$ and write $x = x_1 + x_2\mu$ with $x_1, x_2 \in Q$. Since $QA = Q \oplus Q\mu$ (because $A = B \oplus B\mu$), we have $x_1(\mathfrak{m}_i) = x_{1i}$ for all i, hence $g(x_1,x_1) \in B^\times$ (Lemma 1.6). Since $\rho g(x,x) = \rho g(y,y) = 0$, we are done. □

Lemma 5.19. *With Notation 4.1, suppose that R is semilocal, $\deg A = 2$, σ is orthogonal and $\varepsilon = 1$. Let $\alpha, \beta \in R^\times$. If $f := \langle \alpha, \beta\rangle_{(A,\sigma)}$ is hyperbolic, then there exists $x \in A^\times$ such that $x^\sigma x = -\alpha\beta^{-1}$*

Proof. The claim is equivalent to the existence of $x = (x_1, x_2) \in A^\times \times A^\times$ such that $f(x,x) = \alpha x_1 x_1^\sigma + \beta x_2 x_2^\sigma = 0$. Note that if the equality holds, then x_1 is invertible if and only if x_2 is invertible. Since f is hyperbolic, there exists an A-basis $\{u, v\}$ to A^2 such that $f(u,u) = 0$. Write $u = (u_1, u_2) \in A^2$.

Step 1. Suppose R is a field. We claim that there exists $\varphi \in U^0(f)$ such that $\varphi u \in A^\times \times A^\times$. If $u_1 \in A^\times$ or $u_2 \in A^\times$, then we can take $\varphi = \mathrm{id}_{A^2}$, so assume that both u_1 and u_2 are not invertible. In particular, A cannot be a division algebra, hence $A \cong \mathrm{M}_2(R)$ and $\operatorname{rrk}_A Au_1$ and $\operatorname{rrk}_A Au_2$ cannot exceed 1. Since u can be completed to an A-basis of A^2, we must have $Au_1 + Au_2 = A$. Length consideration now force u_1 and u_2 to be rank-1 matrices with $Au_1 \cap Au_2 = 0$. Since $\alpha u_1^\sigma u_1 = -\beta u_2^\sigma u_2$, this means that $u_1^\sigma u_1 = 0$.

Arguing as in Reduction 4.11, we may identify A with $\mathrm{M}_2(R)$ in such a way that σ is given by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]^\sigma = \left[\begin{smallmatrix} a & \gamma c \\ \gamma^{-1} b & d \end{smallmatrix}\right]$ for some $\gamma \in R^\times$. The condition $u_1^\sigma u_1 = 0$ is easily seen to imply that $-\gamma$ is a square. Write $-\gamma = \delta^2$ with $\delta \in R^\times$ and let $c := -\alpha\beta^{-1}$,

$$x_1 = 1_A, \qquad x_2 = \begin{bmatrix} \frac{c+1}{2} & \frac{\delta(c-1)}{2} \\ \frac{c-1}{2\delta} & \frac{c+1}{2} \end{bmatrix},$$

and $x = (x_1, x_2) \in A^2$. It is routine to check that $\det x_2 = c$ and $f(x, x) = 0$. Since xA is a summand of A^2_A, there exists $\varphi \in U(f)$ such that $\varphi u = x$ (Theorem 2.3). If $\operatorname{Nrd}\varphi = 1$, we are done. If not, replace x_2 with $[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}]x_2$ and φ with $\psi\varphi$ where $\psi \in U(f)$ is given by $\psi(z_1, z_2) = (z_1, [\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}]z_2)$.

Step 2. We now prove the general case. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_t$ denote the maximal ideals of R. By Step 1, for all $1 \le i \le t$, there exists $\varphi_i \in U^0(f(\mathfrak{m}_i))$ such that $\varphi_i(u(\mathfrak{m}_i)) \in A(\mathfrak{m}_i)^\times \times A(\mathfrak{m}_i)^\times$. By Theorem 2.18, there exists $\varphi \in U^0(f)$ with $\varphi(\mathfrak{m}_i) = \varphi_i$ for all i. Now, by Lemma 1.6, $x = (x_1, x_2) := \varphi u \in A^\times \times A^\times$ and $f(x, x) = f(u, u) = 0$. □

Proposition 5.20. *With Notation 4.1, suppose that R is semilocal, T is connected, $[A] \neq 0$, $[B] = 0$, (σ, ε) is orthogonal and τ is unitary. Let $(Q, g) \in \mathcal{H}^\varepsilon(B, \tau)$ be a hermitian space such that ρg is hyperbolic and $n := \operatorname{rrk}_B Q$ is even. Then $\Phi_g(L) = 1$ for some $L \in \operatorname{Lag}(\rho g)$ if and only if $[D(g)] = \frac{n}{2} \cdot [A]$. When this fails, $[D(g)] = (\frac{n}{2} + 1) \cdot [A]$ and g is isotropic.*

Proof. By Reduction 4.10 and Proposition 5.11(iv), we may assume that $\deg B = 1$, $\deg A = 2$, σ is orthogonal and $\varepsilon = 1$. Let $\lambda, \mu \in A^\times$ be as in Lemma 4.3(ii) (so $\lambda^\sigma = -\lambda$ and $\mu^\sigma = \mu$). By Proposition 5.16, Φ_g is constant on $\operatorname{Lag}(\rho g)$; we shall denote the value that it attains by $\bar{\Phi}_g \in \{\pm 1\}$. The proposition clear if $n = 0$, so assume $n > 0$.

Suppose $n \ge 4$. Then by Lemma 5.18, we can write $(Q, g) = (Q_1, g_1) \oplus (Q_2, g_2)$, where $\operatorname{rrk}_B Q_1 = 2$, $[D(g_1)] = [A]$ and there exists $L \in \operatorname{Lag}(\rho g_1)$ with $Q_1 \oplus L = Q_1 A$. By Corollary 5.14, $\bar{\Phi}_{g_1} = 1$. Since $\bar{\Phi}_g = \bar{\Phi}_{g_1}\bar{\Phi}_{g_2}$ (Proposition 5.11(iii)), $[D(g)] = [D(g_1)] + [D(g_2)]$ (Proposition 2.28(iii)) and $[\rho_2 g] = [\rho g] = 0$ in $W_\varepsilon(A, \sigma)$ (so $\rho_2 g$ is hyperbolic by Theorem 2.8(ii)), the proposition will hold for (Q, g) if it holds for (Q_2, g_2). Repeating this process, we reduce to the case $n = 2$.

Suppose henceforth that $n = 2$. By Proposition 2.13, we may assume that $g = \langle \alpha, \beta \rangle_{(B,\tau)}$ for some $\alpha, \beta \in B^\times \cap \mathcal{S}_1(B, \tau) = R^\times$, and by Lemma 5.19, there exists $x \in A^\times$ such that $x^\sigma x = -\alpha\beta^{-1}$. Note that $\operatorname{disc}(g) \equiv -\alpha\beta \equiv -\alpha\beta^{-1} \bmod \operatorname{Nr}_{T/R}(T^\times)$, hence $[D(g)] = [(B/R, -\alpha\beta^{-1})]$ (see 2H).

Write $x = b_1 + \mu b_2$ with $b_1, b_2 \in B$. Since $\mu^\sigma = \mu$ and $\mu b = b^\sigma \mu$ for all $b \in B$, we have

$$-\alpha\beta^{-1} = x^\sigma x = (b_1^\sigma b_1 + \mu^2 b_2^\sigma b_2) + 2\mu b_1 b_2.$$

Thus, $b_1 b_2 = 0$ and $b_1^\sigma b_1 + \mu^2 b_2^\sigma b_2 = \alpha\beta^{-1}$. Arguing as in [58, Example 9.4] (for instance), we see that $\operatorname{Nrd}_{A/R}(x) = b_1^\sigma b_1 - \mu^2 b_2^\sigma b_2$. Since $\operatorname{Nrd}_{A/R}(x) \in R^\times$, this means that $b_1 B + b_2 B = B$.

We claim that $b_1 = 0$ or $b_2 = 0$. Indeed, $b_1 B = b_1(b_1 B + b_2 B) = b_1^2 B$, so there exists $c \in B$ with $b_1 = b_1^2 c$. In particular, $b_1 c$ is an idempotent. Since T is connected, $b_1 c = 0$ or $b_1 c = 1$. In the first case, $b_1 = b_1^2 c = 0$, whereas in the second case, $b_1 \in B^\times$, so $b_2 = 0$ because $b_1 b_2 = 0$.

Assume $b_1 = 0$. Then $x = \mu b_2 \in A^\times$ and $-\alpha\beta^{-1} = \mu^2 b_2^\sigma b_2$, hence $[D(g)] = [(B/R, \mu^2)] = [A]$. Let $L = [\begin{smallmatrix} 1 \\ \mu b_2 \end{smallmatrix}]A$ and $L' = [\begin{smallmatrix} -1 \\ \mu b_2 \end{smallmatrix}]A$. One readily checks that $\rho g(L, L) = \rho g(L', L') = 0$ and $L \oplus L' = A^2$, hence $L \in \operatorname{Lag}(\rho g)$. Furthermore, $B^2 \cap L = 0$ and $[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}], [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} \mu \\ \mu^2 b_2^\sigma \end{smallmatrix}], [\begin{smallmatrix} b_2^{-1} \\ \mu \end{smallmatrix}] \in B^2 + L$, so $B^2 \oplus L = A^2$ and $\bar{\Phi}_g = 1$ by Corollary 5.14.

Assume $b_2 = 0$. Then $x = b_1$ and $-\alpha\beta^{-1} = b_1^\sigma b_1$, hence $[D(g)] = [(B/R, 1)] = 0$. Now, Theorem 1.8 and and our assumption that $[A] \neq 0$ imply that $[D(g)] =$

$(\frac{2}{2}+1)[A] \neq [A]$. Furthermore, it is routine to check that $M = [\begin{smallmatrix}1\\ b_1\end{smallmatrix}]B$ is a Lagrangian of g, so g is hyperbolic (and in particular isotropic) and $\Phi_g = -1$ by Proposition 5.11(ii).

Since we cannot have both $b_1 = 0$ and $b_2 = 0$ (because $x \in A^\times$), the proposition follows. □

We finally complete the proof Theorem 5.1 by establishing case (3).

Theorem 5.21. *Theorem 5.1 holds when R is connected, (σ,ε) is orthogonal and τ is unitary.*

Proof. Recall that we are given $(Q,g) \in \mathcal{H}^\varepsilon(B,\tau)$ such that $[\rho g] = 0$ in $W_\varepsilon(A,\sigma)$. By Theorem 2.8(ii), ρg is hyperbolic.

(i) Since T is connected, $n := \operatorname{rrk}_B Q$ is even by Proposition 5.10.

Suppose that there exists $L \in \operatorname{Lag}(\rho g)$ with $Q \oplus L = QA$. By Corollary 5.14, $\Phi_g(L) = 1$. Now, if $[A] \neq 0$ and $[B] = 0$, then we must have $[D(g)] = \frac{n}{2}\cdot[A]$ by Proposition 5.20, as required.

Conversely, suppose that $[A] = 0$, or $[B] \neq 0$, or $[D(g)] = \frac{n}{2}\cdot[A]$. If there exists $L \in \operatorname{Lag}(\rho g)$ with $\Phi_g(L) = 1$, then we can argue as in the last paragraph of the proof of Theorem 5.9, using Proposition 5.15 instead of Propositions 5.6 and 5.8, to prove the existence of $L' \in \operatorname{Lag}(\rho g)$ with $Q \oplus L' = QA$. The existence of L follows from Proposition 5.16 if $[A] = 0$, from Proposition 5.17 if $[B] \neq 0$, and from Proposition 5.20 if $[A] \neq 0$ and $[B] = 0$. Proposition 5.20, also tells us that $[D(g)] = (\frac{n}{2}+1)[A]$ and g is isotropic when $[A] \neq 0$, $[B] = 0$ and $[D(g)] \neq \frac{n}{2}\cdot[A]$.

(ii) We need to prove the existence of $L \in \operatorname{Lag}(\rho g)$ with $Q \oplus L = QA$, possibly after replacing (Q,g) with a Witt-equivalent hermitian space.

If T is not connected, then, as explained in the proof of Proposition 5.10, g is hyperoblic and can thus be replaced with zero form.

Suppose that T is connected. As in the proof of (i), $\operatorname{rrk}_B Q$ is even and it is enough to find $L \in \operatorname{Lag}(\rho g)$ with $\Phi_g(L) = 1$. Moreover, we showed that L exists if $[B] \neq 0$ in $\operatorname{Br} T$, so we only need to consider the case where $[B] = 0$. Let $L \in \operatorname{Lag}(\rho g)$. If $\Phi_g(L) = 1$, we are done, so assume $\Phi_g(L) = -1$. Since $[B] = 0 = [T]$, there exists $N \in \mathcal{P}(B)$ with $\operatorname{rrk}_B N = \deg T = 1$ (Proposition 1.11(iii)). Consider $(Q',g') := (Q,g) \oplus (N \oplus N^*, \mathbb{h}^\varepsilon_N)$, which is Witt-equivalent to (Q,g). By parts (ii) and (iii) of Proposition 5.11, we have $\Phi_{g'}(L \oplus NA) = \Phi_g(L)\cdot\Phi_{\mathbb{h}^\varepsilon_N}(NA) = (-1)\cdot(-1) = 1$. We may therefore replace (Q,g), L with (Q',g'), $L \oplus N$ and finish. □

6. Verification of (E1) and (E3)

Keep the assumptions of Notation 4.1. The purpose of this section is to prove:

Theorem 6.1. *With Notation 4.1, suppose that R is semilocal and let $(P,f) \in \mathcal{H}^\varepsilon(A,\tau)$ be a hermitian spaces such that $[\pi f] = 0$ in $W_\varepsilon(B,\tau)$.*

(i) Assume that T is connected and (τ,ε) is not orthogonal. Then there exists a Lagrangian M of πf such that $MA = P$ if and only if (τ,ε) not unitary, or (σ,ε) is not symplectic, or $4 \mid \operatorname{rrk}_A P$. When these conditions fail, $[A] = 0$ in $\operatorname{Br} S$ and f is hyperbolic.

(ii) Assume that T is connected and (τ,ε) is orthogonal. Then (σ,ε) is orthogonal. Moreover, there exists a Lagrangian M of πf such that $MA = P$ if and only if $[B] \neq 0$ in $\operatorname{Br} T$, or $\operatorname{rrk}_A P$ is even and $\operatorname{disc}(f) = \operatorname{disc}(T/R)^{\frac{1}{2}\operatorname{rrk}_A P}$ (see 2H). When these conditions fail, $[A] = 0$ in $\operatorname{Br} S$, $\operatorname{rrk}_A P$ is even, $\operatorname{disc}(f) = \operatorname{disc}(T/R)^{\frac{1}{2}\operatorname{rrk}_A P+1}$ and f is isotropic.

(iii) There exists $(P',f') \in \mathcal{H}^\varepsilon(A,\sigma)$ with $[f] = [f']$ and a Lagrangian M of $\pi f'$ such that $MA = P'$.

In Section 7, we will use this theorem to establish conditions (E1) and (E3) of Theorem 3.6 when R is semilocal. The reader can skip to the next section without loss of continuity.

As with Theorem 5.1, it is enough to prove the theorem when R is connected. In this case, by Lemma 4.6, exactly one of the following hold:

(1) (τ, ε) is symplectic or unitary,
(2) (τ, ε) is orthogonal.

These cases are treated in Theorems 6.10 and 6.20, respectively.

6A. **Non-Connected Cases.** We begin by addressing the simpler case where T is not connected. Some of the observations made here will be used later.

First, we consider the case where S is not connected.

Proposition 6.2. *Theorem 6.1(iii) holds when R is connected and S is not connected.*

Proof. In this case, $S = R \times R$ (Lemma 1.16) and $\tau|_S$ is the exchange involution. As observed in Example 2.4, this means that every $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ is hyperbolic. Replacing (P, f) with zero form, Theorem 6.1(iii) holds trivially. □

If S is connected and T is not connected, then $T \cong S \times S$ (Lemma 1.16), hence T admits two nontrivial idempotents, call them e and $e' := 1 - e$. We have $e^\sigma = e$ or $e^\sigma = e'$. We devote some attention to the case $e^\sigma = e$, working in slightly greater generality for later reference.

With Notation 4.1, suppose that $T = S \times S$ (but not that S is connected), let $e := (1_S, 0_S)$ and assume $e^\sigma = e$. By Lemma 4.4, we have $A = AeA$, $B = eB \oplus e'B = eAe \oplus e'Ae'$ and $\pi : A \to B$ is given by $a \mapsto eae + e'ae'$. Since $e^\sigma = e$, we may view (B, τ) as $(eB, \tau_e) \times (e'B, \tau_{e'})$, where $\tau_e = \tau|_{eB}$ and $\tau_{e'} = \tau|_{e'B}$. Thus, every hermitian space $(Q, g) \in \mathcal{H}^\varepsilon(B, \tau)$ factors as $(Qe, g_e) \times (Qe', g_{e'})$, where $g_e = g|_{Qe \times Qe}$ and $g_{e'} = g|_{Qe' \times Qe'}$ are ε-hermitian forms over (eB, τ_e) and $(e'B, \tau_{e'})$, respectively. The following simple observation will be important in the sequel.

Proposition 6.3. *Under the previous assumptions and identifications, for every $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$, the hermitian space $(P, \pi f)$ is $(Pe, f_e) \times (Pe', f_{e'})$, where f_e denotes the e-transfer of f (see 2G), and likewise for $f_{e'}$.*

Proof. This is straightforward. □

Proposition 6.4. *Theorem 6.1(iii) holds when S is connected and T is not connected.*

Proof. Write $T = S \times S$ and let $e = (1_S, 0_S)$. By Lemma 4.4(ii), $\operatorname{rrk}_A eA > 0$ and $AeA = A$. Since S is connected, $e^\sigma = e$ or $e^\sigma = 1 - e$.

If $e^\sigma = e$, then f_e is hyperbolic by Proposition 6.3 and the fact that πf is hyperbolic (Theorem 2.8(ii)). By item (t3) in 2G, this means that f is hyperbolic, so we may replace (P, f) with zero form and finish.

Suppose that $e^\sigma = 1 - e$. Then $M := Pe$ is a Lagrangian of πf (Example 2.4). Since $MA = PeA = PAeA = PA = P$, we are done. □

6B. **Case (1).** We now prove Theorem 6.1 in case (1), namely, when R is connected and (B, τ) is unitary or symplectic. As with Theorem 5.1, we first establish some special cases.

Proposition 6.5. *With Notation 4.1, suppose that R is a field, $S \cong R \times R$ and $[A] = 0$. Let $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ be a hermitian space such that $\operatorname{rrk}_A P$ is even. Then there exists $M \in \operatorname{Lag}(\pi f)$ such that $MA = P$ and $\operatorname{rrk}_B M = \frac{1}{2} \operatorname{rrk}_B P$.*

Proof. By Reduction 4.10, we may assume that $B = T$ and $\deg A = 2$. Note that $\mathrm{rrk}_A P$ is constant by Corollary 2.9(i).

By assumption, there exists a nontrivial idempotent $e \in S$ such that $e + e^\sigma = 1$. By Example 2.4, we may assume that $A = A_1 \times A_1^{\mathrm{op}}$ for a central simple R-algebra A_1 and that σ is the exchange involution $(x, y^{\mathrm{op}}) \mapsto (y, x^{\mathrm{op}})$. It is easy to see that there are R-subalgebras $T_1 \subseteq B_1 \subseteq A_1$ such that $T = T_1 \times T_1^{\mathrm{op}}$, $B = B_1 \times B_1^{\mathrm{op}}$ and $B_1 = \mathrm{Z}_{T_1}(A_1)$. Furthermore, $\varepsilon = (\varepsilon_1, \varepsilon_1^{-1})$ for some $\varepsilon_1 \in R^\times$.

Consider the ε-hermitian form $f_1 : A \times A \to A$ given by $f_1((x_1, x_2^{\mathrm{op}}), (y_1, y_2^{\mathrm{op}})) = (\varepsilon_1 x_2 y_1, (y_2 x_1)^{\mathrm{op}})$. Since $\mathrm{rrk}_A P$ is even and constant, and since $\deg A = 2$, there exists $n \in \mathbb{N}$ such that $P \cong A_A^n$. By Example 2.4, this means that $(P, f) \cong n \cdot (A, f_1)$. It is therefore enough to prove the proposition for $(P, f) = (A, f_1)$.

Let $\pi_1 := \pi_{A_1, B_1} : A_1 \to B_1$ be as in Lemma 4.2. The uniqueness of π forces $\pi(x, y^{\mathrm{op}}) = (\pi_1 x, (\pi_1 y)^{\mathrm{op}})$ for all $x, y \in A_1$. Let $E_1 = \ker \pi_1$. Then $M := E_1 \times B_1^{\mathrm{op}}$ and $M' := B_1 \times E_1^{\mathrm{op}}$ are submodules of A_B satisfying $f_1(M, M) = f_1(M', M') = 0$ and $M \oplus M' = A$. Thus, M is a Lagrangian of πf_1. In addition, Lemma 4.2(iv) tells us that $E_1 A_1 = A_1$, so $MA = E_1 A_1 \times A_1^{\mathrm{op}} B_1^{\mathrm{op}} = A$, as required. □

The following proposition holds without restrictions on the type of (τ, ε).

Proposition 6.6. *With Notation 4.1, suppose that S is a field and $[A] = 0$. Let $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ be a hyperbolic hermitian space. If $4 \mid \mathrm{rrk}_A P$, then πf admits a Lagrangian M such that $MA = P$ and $\mathrm{rrk}_B M = \frac{1}{2} \mathrm{rrk}_B P$.*

Proof. By Reduction 4.10, we may assume that $B = T$ and $\deg A = 2$. Consider the hyperbolic ε-hermitian form $f_1 : A^2 \times A^2 \to A$ given by $f_1((x_1, y_1), (x_2, y_2)) = x_1^\sigma y_2 + \varepsilon x_2^\sigma y_1$. Writing $n = \frac{1}{4} \mathrm{rrk}_A P$, we have $n \cdot (A^2, f_1) \cong (P, f)$ by Lemma 2.7. It is therefore enough to prove the proposition for (A^2, f_1). Write $E = \ker \pi$ and let $M := B \times E$ and $M' := E \times B$; both M and M' are right B-submodules of A^2. One readily checks that $\pi f_1(M, M) = \pi f_1(M', M') = 0$ and $A_A^2 = M \oplus M'$. Thus, M is a Lagrangian of πf. By Lemma 4.2(iv), we have $MA = A^2$ and $\mathrm{rrk}_B M = 2 \,\mathrm{rrk}_B B = 2 = \frac{1}{2} \mathrm{rrk}_B A_B^2$. □

Proposition 6.7. *With Notation 4.1, suppose that S is a field, $[A] = 0$ and (τ, ε) is symplectic or unitary. If τ is unitary, we also assume that (σ, ε) is not symplectic. Let $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ be a hyperbolic hermitian space. Then there exists a Lagrangian M of πf such that $MA = P$ and $\mathrm{rrk}_B M = \frac{1}{2} \mathrm{rrk}_B P$.*

Proof. By Reductions 4.10 and 4.11, we may assume that $B = T$, $A = \mathrm{M}_2(S)$, τ is orthogonal or unitary, and σ is orthogonal or unitary and given by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} a^\sigma & \alpha c^\sigma \\ \alpha^{-1} b^\sigma & d^\sigma \end{smallmatrix}\right]$ for some $\alpha \in R^\times$. By Corollary 2.9(ii), $\mathrm{rrk}_A P$ is even.

Let (P_1, f_1) be a hyperbolic hermitian space such that $\mathrm{rrk}_A P_1 = 2$. Then, by Lemma 2.7, $(P, f) \cong n \cdot (P_1, f_1)$ for some $n \in \mathbb{N}$. It is therefore enough to prove the proposition for (P_1, f_1). We now split into cases, making different choices of (P_1, f_1) in each case.

Case I. $\tau|_T$ is not the standard S-involution of T. We may assume that $\varepsilon = -1$. This already holds when $\sigma|_S = \mathrm{id}_S$, because then τ is orthogonal while (τ, ε) is symplectic. When $\sigma|_S \neq \mathrm{id}_S$, by Hilbert's Theorem 90, there exists $\eta \in S$ with $\eta^\sigma \eta^{-1} = -\varepsilon$ and we can apply η-conjugation (see 2G, Proposition 4.7) to replace f, ε with $\eta f, -1$.

Consider the hyperbolic (-1)-hermitian form $f_1 : A \times A \to A$ given by $f_1(x, y) = x^\sigma \left[\begin{smallmatrix} 0 & -\alpha \\ 1 & 0 \end{smallmatrix}\right] y$; note that $\mathrm{rrk}_A A = 2$ and $\left[\begin{smallmatrix} S & S \\ 0 & 0 \end{smallmatrix}\right]$ is a Lagrangian of f_1.

We claim that there exists $\lambda \in T$ such that $\lambda^2 \in S^\times$, $T = S \oplus \lambda S$ and $\lambda^\sigma = \lambda$. If $\sigma|_S = \mathrm{id}_S$, then $\sigma|_T = \mathrm{id}_T$ and the existence of λ follows from Lemma 1.19. If $\sigma|_S \neq \mathrm{id}_S$, then τ is unitary (Lemma 4.5), so T is quadratic étale over $T_0 :=$

$\mathcal{S}_1(T,\tau)$, which is in turn quadratic étale over R and satisfies $T_0 \cap S = \mathcal{S}_1(S,\sigma) = R$. Applying Lemma 1.19 to T_0, we see that there exists $\lambda \in T_0^\times \setminus R$ and $\lambda^2 \in R^\times$. Thus, $\lambda^\sigma = \lambda$, and $T = S \oplus \lambda S$ because $\lambda \notin S$ and $\dim_S T = 2$.

The conditions $\lambda = \lambda^\sigma$ and $\lambda^2 \in S^\times$ force $\lambda = [\begin{smallmatrix} a & \alpha b \\ b & -a \end{smallmatrix}]$ for some $a, b \in S$ such that $a^\sigma = a$ and $a^2 + \alpha b^2 \in S^\times$. Using Lemma 4.2(ii), it is routine to check that

$$\pi([\begin{smallmatrix} 0 & -\alpha \\ 1 & 0 \end{smallmatrix}]) = \tfrac{1}{2}[\begin{smallmatrix} 0 & -\alpha \\ 1 & 0 \end{smallmatrix}] + \tfrac{1}{2}[\begin{smallmatrix} a & \alpha b \\ b & -a \end{smallmatrix}]^{-1}[\begin{smallmatrix} 0 & -\alpha \\ 1 & 0 \end{smallmatrix}][\begin{smallmatrix} a & \alpha b \\ b & -a \end{smallmatrix}] = 0.$$

Thus, for all $x, y \in B$, we have

$$\pi f_1(x,y) = \pi(x^\tau [\begin{smallmatrix} 0 & -\alpha \\ 1 & 0 \end{smallmatrix}] y) = x^\tau \pi([\begin{smallmatrix} 0 & -\alpha \\ 1 & 0 \end{smallmatrix}]) y = 0.$$

It follows that B is a Lagrangian of πf_1. Since $\operatorname{rrk}_B B_B = 1 = \frac{1}{2} \operatorname{rrk}_B A$ and $BA = A = P$, we are done.

Case II. $\tau|_T$ is the standard S-involution of T. Since τ is unitary, (σ,ε) is necessarily orthogonal. Thus, σ is orthogonal and $\varepsilon = 1$.

By Proposition 1.21, $\dim_S \mathcal{S}_{-1}(T,\sigma) = 1 = \dim_S \mathcal{S}_{-1}(A,\sigma)$, so we must have $T = S + \mathcal{S}_{-1}(A,\sigma) = S \oplus \lambda S$ with $\lambda := [\begin{smallmatrix} 0 & \alpha \\ -1 & 0 \end{smallmatrix}]$. Using Lemma 4.2(ii), it is routine to check that $\pi : A \to B$ is given by $\pi[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}] = \frac{1}{2}(x+w)[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}] + \frac{1}{2}(\alpha^{-1}y - z)[\begin{smallmatrix} 0 & \alpha \\ -1 & 0 \end{smallmatrix}]$.

Consider the 1-hermitian hyperbolic form $f_1 : A \times A \to A$ given by $f(x,y) = x^\sigma [\begin{smallmatrix} 0 & \alpha \\ 1 & 0 \end{smallmatrix}] y$ (the space $[\begin{smallmatrix} S & S \\ 0 & 0 \end{smallmatrix}]$ is a Lagrangian). As in Case I, one readily checks that $\pi[\begin{smallmatrix} 0 & \alpha \\ 1 & 0 \end{smallmatrix}] = 0$ and hence B is the required Lagrangian of πf_1. □

The case where τ is unitary and (σ,ε) is symplectic is addressed in the following proposition. Note that we allow R to be semilocal.

Proposition 6.8. *With Notation 4.1, suppose that R is connected semilocal and (σ,ε) is symplectic. Let $(P,f) \in \mathcal{H}^\varepsilon(A,\sigma)$. Then:*

(i) If τ is unitary, T is connected, $[A] \neq 0$ and πf is hyperbolic, then $4 \mid \operatorname{rrk}_A P$.

(ii) If $[A] = 0$, then f is hyperbolic.

(iii) If τ is unitary, T is connected, $[A] = 0$ and πf admits a Lagrangian M with $MA = P$, then $4 \mid \operatorname{rrk}_A P$.

Proof. Note that $S = R$, and thus T quadratic étale over R. By Lemma 1.17 and the connectivity of R, the involution $\tau|_T : T \to T$ is either id_T or the standard R-involution of T. Thus, when $\tau : B \to B$ is unitary, $\tau|_T$ is the standard R-involution of T.

(i) Since τ is unitary, $\tau|_T$ is the standard involution of T. For the sake of contradiction, suppose that $\operatorname{rrk}_A P$ is not divisible by 4. By Corollary 2.9(ii), there exists $V \in \mathcal{P}(B)$ such that $\operatorname{rrk}_B P = 2 \operatorname{rrk}_B V$. Since $\operatorname{rrk}_B P_B = \iota \operatorname{rrk}_A P$, we have $\operatorname{rrk}_A P = 2n$, where $n := \operatorname{rrk}_B V$. Thus, $\operatorname{rrk}_B V$ is odd. By Corollary 1.13 and Theorem 1.8, $n[B] = 0$. On the other hand, $2[B] = 2[A_T] = 0$, because A has an involution of the first kind, so $[B] = 0$. We now apply Reduction 4.10 to assume that $B = T$, $\deg A = 2$, σ is orthogonal and $\varepsilon = -1$.

By Lemma 1.26, there exists $c \in \mathcal{S}_{-1}(T,\tau) \cap T^\times$. We apply c-conjugation, see 2G and Proposition 4.7, to replace σ, f, ε by $\operatorname{Int}(c) \circ \sigma$, cf, $-\varepsilon$. Now, $\varepsilon = 1$ and σ is symplectic. Let $\lambda, \mu \in A^\times$ be as in Lemma 4.3(iii) (so $\lambda^\sigma = -\lambda$ and $\mu^\sigma = -\mu$) and note that $\mathcal{S}_1(A,\sigma) = R$.

Since $\operatorname{rrk}_A P = 2n$, the A-module P is free (Lemma 1.24). Thus, by Proposition 2.13, we may assume that $f = \langle \alpha_1, \dots, \alpha_n \rangle_{(A,\sigma)}$ with $\alpha_1, \dots, \alpha_n \in \mathcal{S}_1(A,\sigma) \cap A^\times = R^\times$. Now, it is routine to check that, upon identifying B_B^2 with A_B via $(b_1, b_2) \mapsto b_1 + \mu b_2$, the form πf is just $\langle \alpha_1, -\eta\alpha_1, \alpha_2, -\eta\alpha_2, \dots, \alpha_n, -\eta\alpha_n \rangle_{(T,\tau)}$, where $\mu^2 = \eta$. Since πf is hyperbolic, $\eta^n \equiv \operatorname{disc}(\pi f) \equiv \operatorname{disc}(n \cdot \langle 1, -1 \rangle_{(T,\tau)}) \equiv 1 \bmod \operatorname{Nr}_{T/R}(T^\times)$ (see 2H). Since n is odd, this means that there exists $t \in T^\times$ with $t^\sigma t = \eta = \mu^2$. As $\mu b = b^\sigma \mu$ for all $b \in B$, the element $e := \frac{1}{2}(1 + \mu^{-1} t)$ is an idempotent. Furthermore, $e \notin \{0,1\}$, otherwise $\mu \in B$. Thus, eA_A is a proper nonzero summand of A_A, so $\mathrm{rrk}_A\, eA = 1$. But this means that $[A] = [eAe] = 0$ (Corollary 1.12), a contradiction.

(ii) By Reduction 4.10, we may assume that σ is orthogonal, hence $\varepsilon = -1$. By Theorem 1.30, there exists $e \in A$ such that $\mathrm{rrk}_A\, eA = 1$ and $e^\sigma = e$. Applying e-transfer (see 2G), we may replace A, σ, f with eAe, $\sigma|_{eAe}$, f_e and assume that $A = R$ and $f : P \times P \to R$ is an anti-symmetric unimodular bilinear form. Every such f is hyperbolic, e.g., apply the argument in [65, Lemma 7.7.2] to a basis element of P.

(iii) Arguing as in (i), we may assume that $B = T$, $\deg A = 2$, σ is symplectic and $\varepsilon = 1$. Let $\lambda, \mu \in A^\times$ be as in Lemma 4.3(iii). Then, writing $\tau_1 := \tau$, $\tau_2 = \mathrm{id}_B$ and $\pi_1 := \pi$, we are in the situation of 3A. Thus, by Remark 3.9(i), there exists $(Q, g) \in \mathcal{H}^{-1}(T, \mathrm{id}_T)$ such that $\rho_2(Q, g) \cong (P, f)$. Since $g : Q \times Q \to T$ is an anti-symmetric unimodular bilinear form, $\mathrm{rrk}_B\, Q$ must be even, and since $\mathrm{rrk}_B\, Q$ is constant (T is connected), we have $\iota\, \mathrm{rrk}_A\, P = \iota\, \mathrm{rrk}_A\, QA = 2\, \mathrm{rrk}_B\, Q$. It follows that $4 \mid \mathrm{rrk}_A\, P$. □

Proposition 6.9. *With Notation 4.1, suppose that R is a field and (τ, ε) is unitary or symplectic. Let $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ be a hermitian space such that $\mathrm{rrk}_A\, P$ is even and πf is hyperbolic. If τ is unitary and (σ, ε) is symplectic, we also assume that $4 \mid \mathrm{rrk}_A\, P$. Let M be a Lagrangian of πf such that $\mathrm{rrk}_B\, M = \frac{1}{2}\mathrm{rrk}_B\, P$. Then there exists $\varphi \in U^0(\pi f)$ such that $\varphi M \cdot A = P$.*

Proof. Thanks to Proposition 4.14, when R is infinite, it is enough to prove the proposition after base-changing to an algebraic closure of R, in which case $[A] = 0$. On the other hand, if R is finite, then $[A] = 0$ by Wedderburn's theorem. We may therefore assume that $[A] = 0$.

We claim that πf admits a Lagrangian M' such that $M'A = P$ and $\mathrm{rrk}_B\, M' = \frac{1}{2}\mathrm{rrk}_B\, P$. To that end, we split into three cases.

Case I. S is not a field. Then M' exists by Proposition 6.5.

Case II. S is a field, τ is unitary and (σ, ε) is symplectic. Then f is hyperbolic by Proposition 6.8(ii) and $4 \mid \mathrm{rrk}_B\, P$ by assumption, so M' exists by Proposition 6.6.

Case III. S is a field, and τ is not unitary or (σ, ε) is not symplectic. Using Proposition 2.5, write $(P, f) = (P_1, f_1) \oplus (P_2, f_2)$ with f_1 anisotropic and f_2 hyperbolic. Then $[\pi f_1] = [\pi f] = 0$ in $W_\varepsilon(B, \tau)$, so $[\pi f_1]$ is hyperbolic by Theorem 2.8(ii).

By virtue of Remark 3.7, any Lagrangian M_1 of πf_1 satisfies $M_1 A = P_1$. We claim that one can choose M_1 such that $\mathrm{rrk}_B\, M_1 = \frac{1}{2}\mathrm{rrk}_B\, P$. Indeed, by Corollary 2.9(ii), $\mathrm{rrk}_A\, P_2$ is even, so $\mathrm{rrk}_A\, P_1$ is also even. Thus, $\mathrm{rrk}_B\, P_1$ is even. Since $[B] = [A \otimes_S T] = [T]$, there exists $V \in \mathcal{P}(B)$ such that $\mathrm{rrk}_B\, V = \deg T = 1$ (Proposition 1.11(iii)). By Lemmas 2.7 and 2.6, there is an isometry $\mathbb{h}^\varepsilon_{V^n} \to \pi f_1$, where $n = \frac{1}{2}\mathrm{rrk}_B\, T$. Take M_1 to be the image of V^n in P_1.

Next, by Proposition 6.7, there exists a Lagrangian M_2 of πf_2 with $M_2 A = P_2$ and $\mathrm{rrk}_B\, M_2 = \frac{1}{2}\mathrm{rrk}_B\, P_2$. Take $M' = M_1 \oplus M_2$.

Now, since $\mathrm{rrk}_B\, M' = \frac{1}{2}\mathrm{rrk}_B\, P = \mathrm{rrk}_B\, M$, Lemma 2.22 and Proposition 2.16 imply that there exists $\varphi \in U^0(\pi f)$ such that $\varphi M = M'$, so $\varphi M \cdot A = P$. □

Theorem 6.10. *Theorem 6.1 holds when R is connected and (τ, ε) is symplectic or unitary.*

Proof. Recall that we are given $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ such that $[\pi f] = 0$. By Theorem 2.8(ii), πf is hyperbolic.

(i) Suppose that that πf admits a Lagrangian M such that $MA = P$. If τ is unitary and (σ, ε) is symplectic, then parts (i) and (iii) of Proposition 6.8 imply that $4 \mid \mathrm{rrk}_A\, P$. Moreover, part (i) of this proposition implies that $[A] = 0$ when τ is unitary, (σ,ε) is symplectic and $4 \nmid \operatorname{rrk}_A P$. In this case, part (ii) of that proposition says that f is hyperbolic.

Conversely, suppose that (τ,ε) is symplectic, or (σ,ε) is not symplectic, or $4 \mid \operatorname{rrk}_A P$. Let M be a Lagrangian of πf and let $\mathfrak{m}_1,\dots,\mathfrak{m}_t$ denote the maximal ideals of R. By Proposition 6.9, for all $1 \le i \le t$, there exists $\varphi_i \in U^0(\pi f(\mathfrak{m}_i))$ such that $\varphi_i(M(\mathfrak{m}_i)) \cdot A(\mathfrak{m}_i) = P(\mathfrak{m}_i)$. By Theorem 2.18, there exists $\varphi \in U^0(\pi f)$ such that $\varphi(\mathfrak{m}_i) = \varphi_i$ for all i. Thus, $M' := \varphi M$ is a Lagrangian of πf such that $M'A + P\mathfrak{m}_i = P$ for all i. By Nakayama's Lemma $\operatorname{ann}_R(P/M'A)$ is not contained in any maximal ideal of R, so it must be R and $M'A = P$.

(ii) This statement is vacuous under our assumptions.

(iii) By Proposition 6.2, Proposition 6.4 and (i), we only need to consider the case where T is connected, τ is unitary, (σ,ε) is symplectic and $[A] = 0$. In this case, f is hyperbolic by Proposition 6.8(ii), so we may take f' to be the zero form and let $M = 0$. □

6C. **Case (2).** We now prove Theorem 6.1 in Case (2), namely, when R is connected and (τ,ε) is orthogonal. The main difference with Case (1) is the failure of Proposition 6.9. Thus, the majority of the argument will be dedicated to effectively characterizing the Lagrangians M of πf for which Proposition 6.9 fails.

Throughout this subsection, we assume, on top of Notation 4.1, that (τ,ε) is orthogonal, hence $\tau|_T = \mathrm{id}_T$ and $S = R$. This also means that (σ,ε) is orthogonal (Lemma 4.12).

Following Remark 2.14, given $(Q,g) \in \mathcal{H}^\varepsilon(B,\tau)$, we write $\mathbf{U}_T(g)$ for the group T-scheme of isometries of g, and $\mathbf{U}(g) = \mathbf{U}_R(g)$ for the R-scheme of isometries of g. The corresponding neutral components are denoted $\mathbf{U}^0_T(g)$ and $\mathbf{U}^0(g) = \mathbf{U}^0_R(g)$. It was observed in Remark 2.14 that $\mathcal{R}_{T/R}\mathbf{U}_T(g) = \mathbf{U}(g)$ and $\mathcal{R}_{T/R}\mathbf{U}^0_T(g) = \mathbf{U}^0(g)$, where $\mathcal{R}_{T/R}$ is the Weil restriction. Combining this with Proposition 2.16, we see that $\mathbf{U}^0(g)$ is the scheme-theoretic kernel of

$$\mathcal{R}_{T/R}(\mathrm{Nrd}) : \mathbf{U}(g) = \mathcal{R}_{T/R}\mathbf{U}_T(g) \to \mathcal{R}_{T/R}\boldsymbol{\mu}_{2,T}.$$

We abbreviate $\mathcal{R}_{T/R}(\mathrm{Nrd})$ to Nrd. The norm map $\mathrm{Nr}_{T/R} : T \to R$ induces a morphism of affine group R-schemes,

$$\mathrm{Nr}_{T/R} : \mathcal{R}_{T/R}\boldsymbol{\mu}_{2,T} \to \boldsymbol{\mu}_{2,R},$$

and its kernel is $\boldsymbol{\mu}_{2,R}$, viewed as a subgroup R-scheme of $\mathcal{R}_{T/R}\boldsymbol{\mu}_{2,T}$ via the inclusion $R \to T$. We write

$$N := \mathrm{Nr}_{T/R} \circ \mathrm{Nrd} : \mathbf{U}(g) \to \boldsymbol{\mu}_{2,R}.$$

Given $(P,f) \in \mathcal{H}^\varepsilon(A,\sigma)$, Lemma 4.13(ii) implies that the diagram

$$\begin{array}{ccc} U(f) & \hookrightarrow & U(\pi f) \\ \downarrow{\scriptstyle \mathrm{Nrd}} & & \downarrow{\scriptstyle \mathrm{Nrd}} \\ \mu_2(R) & \hookrightarrow & \mu_2(T) \end{array}$$

commutes. Thus, given $\varphi \in U(f)$, we may speak about the reduced norm of φ without specifying if we view φ as an isometry of f or πf.

Finally, recall from 2F that $\mathrm{Lag}(\pi f)$ denotes the set of Lagrangians M of πf with $\operatorname{rrk}_B M = \frac{1}{2}\operatorname{rrk}_B P$, and these are all the Lagrangians of πf because $\tau|_T = \mathrm{id}_T$. In particular, if πf is hyperbolic, then $\iota \operatorname{rrk}_A P = \operatorname{rrk}_B P$ must be even. Recall also the sheaf $\mathbf{Lag}(\pi f)$ over $(\mathcal{A}ff/T)_{\mathrm{fppf}}$; we write $\mathcal{R}_{T/R}\mathbf{Lag}(\pi f)$ for its Weil restriction, which is the sheaf on $(\mathcal{A}ff/R)_{\mathrm{fppf}}$ mapping an R-ring S to $\mathrm{Lag}(\pi f_S)$.

Lemma 6.11. *With Notation 4.1, suppose that (τ,ε) is orthogonal. Let $(P,f) \in \mathcal{H}^\varepsilon(A,\sigma)$ and assume that πf is hyperbolic. Then (P_T, f_T) is hyperbolic.*

Proof. By Lemma 1.18, we have $T_T \cong T \times T$. Let $e := (1_T, 0_T) \in T_T$. By assumption, πf_T is hyperbolic, so by Proposition 6.3, the e-transfer of f_T (see 2G) is also hyperbolic. Thus, f_T is hyperbolic. □

Proposition 6.12. *With Notation 4.1, suppose that (τ, ε) is orthogonal. Let $(P, f) \in \mathcal{H}^\varepsilon(A, \tau)$ and assume that πf is hyperbolic. Let $\mathbf{U}(\pi f)$ act on $\boldsymbol{\mu}_{2,R}$ via N. Then there exists a unique $\mathbf{U}(\pi f)$-equivariant natural transformation,*

$$\Psi_f : \mathcal{R}_{T/R}\mathbf{Lag}(\pi f) \to \boldsymbol{\mu}_{2,R},$$

such that for any R-ring R_1 and any $L_1 \in \mathrm{Lag}(f_{R_1})$, one has $\Psi_f(L_1) = 1$ in $\mu_2(R_1)$. The map Ψ_f has the following additional properties:

- *(i) If f is hyperbolic and $L \in \mathrm{Lag}(f)$, then $\Psi_f = \mathrm{Nr}_{T/R} \circ \mathcal{R}_{T/R}\Phi_L^{(\pi f)}$ (notation as in Proposition 2.25).*
- *(ii) If $(P', f') \in \mathcal{H}^\varepsilon(A, \sigma)$ and $\pi f'$ is hyperbolic, then $\Psi_{f \oplus f'}(M \oplus M') = \Psi_f(M) \cdot \Psi_{f'}(M')$ for all $M \in \mathrm{Lag}(\pi f)$, $M' \in \mathrm{Lag}(\pi f')$.*
- *(iii) Let $e \in B$ be a σ-invariant idempotent such that $\mathrm{rrk}_B\, eB$ is positive and constant on the fibers of $\mathrm{Spec}\, T \to \mathrm{Spec}\, R$. Then $\Psi_f(M) = \Psi_{f_e}(Me)$ for all $M \in \mathrm{Lag}(\pi f)$ (notation as in 2G).*

Note that L_1 is a Lagrangian of πf_{R_1} because we can find $L_1' \in \mathrm{Lag}(f_{R_1})$ such that $L_1 \oplus L_1' = P$ as A-modules (see 2B) and $\pi f(L_1, L_1) = \pi f(L_1', L_1') = 0$.

Proof. Fix some $M_0 \in \mathrm{Lag}(\pi f)$, write $\Phi_0 = \mathcal{R}_{T/R}\Phi_{M_0}^{(\pi f)} : \mathcal{R}_{T/R}\mathbf{Lag}(\pi f) \to \mathcal{R}_{T/R}\boldsymbol{\mu}_2$ (see Proposition 2.25 for the definition of Φ_{M_0}), and let $\Psi_0 := \mathrm{Nr}_{T/R} \circ \Phi_0$. It is clear that $\Psi_0 : \mathcal{R}_{T/R}\mathbf{Lag}(\pi f) \to \boldsymbol{\mu}_{2,R}$ is $\mathbf{U}(\pi f)$-equivariant.

We claim that for any R-ring R_1 and $V, W \in \mathrm{Lag}(f_{R_1})$, we have $\Psi_0(V) = \Psi_0(W)$ in $\mu_2(R_1)$. Since $\boldsymbol{\mu}_{2,R}$ is a sheaf on $(\mathcal{A}ff/R)_{\mathrm{fpqc}}$, it is enough to check that $\Psi_0(V) = \Psi_0(W)$ after base-changing along a faithfully flat ring homomorphism $R_1 \to R_2$. By Proposition 2.23, we can choose R_2 such that there exists $\varphi \in U(f_{R_2})$ with $V \otimes_{R_1} R_2 = \varphi(W \otimes_{R_1} R_2)$. Since $\mathrm{Nrd}(\varphi) \in \mu_2(R_2)$ and Ψ_0 is $\mathbf{U}(\pi f)$-equivariant, we have $\Psi_0(V \otimes_{R_1} R_2) = \mathrm{Nr}_{T/R}(\mathrm{Nrd}(\varphi)) \cdot \Psi_0(W \otimes_{R_1} R_2) = \Psi_0(W \otimes_{R_1} R_2)$ in $\mu_2(R_2)$, as required.

Let $R_0 := T$. Then f_{R_0} is hyperbolic by Lemma 6.11. Fix some $L_0 \in \mathrm{Lag}(f_{R_0})$ and write $\theta := \Psi_0(L_0) \in \mu_2(R_0)$. We claim that θ is in fact in $\mu_2(R)$. To that end, let $i_1, i_2 : R_0 \to R_0 \otimes R_0$ denote the homomorphisms $r \mapsto r \otimes 1$, $r \mapsto 1 \otimes r$. By the previous paragraph, we have $i_1\Psi_0(L_0) = \Psi_0(L_0 \otimes_{i_1} (R_0 \otimes R_0)) = \Psi_0(L_0 \otimes_{i_2} (R_0 \otimes R_0)) = i_2\Psi_0(L_0)$ in $\mu_2(R_0 \otimes R_0)$. Since $\boldsymbol{\mu}_{2,R}$ is a sheaf on $(\mathcal{A}ff/R)_{\mathrm{fpqc}}$, this means that $\theta \in \mu_2(R)$.

Define $\Psi_f = \theta^{-1} \cdot \Psi_0$. Then $\Psi_f : \mathcal{R}_{T/R}\mathbf{Lag}(\pi f) \to \boldsymbol{\mu}_2$ is $\mathbf{U}(\pi f)$-equivariant and $\Psi_f(L_0) = 1$ in $\mu_2(R_0)$. Let R_1 be an R-ring and let $L_1 \in \mathrm{Lag}(f_{R_1})$. By what we have shown above, $\Psi_0(L_1 \otimes_{R_1} (R_0 \otimes R_1)) = \Psi_0(L_0 \otimes_{R_0} (R_0 \otimes R_1)) = \theta$ in $\mu_2(R_0 \otimes R_1)$. Since $R_1 \to R_0 \otimes R_1$ is faithfully flat, this means that $\Psi_0(L_1) = \theta$ in $\mu_2(R_1)$, so $\Psi_f(L_1) = 1$. Thus, Ψ_f satisfies the condition in the proposition.

If $\Psi' : \mathcal{R}_{T/R}\mathbf{Lag}(\pi f) \to \boldsymbol{\mu}_{2,R}$ also satisfies the condition in the proposition, then $\Psi'(L_0) = 1 = \Psi(L_0)$. If R_1 is an R-ring and $M \in \mathrm{Lag}(\pi f_{R_1})$, then, by Proposition 2.23, there exists a faithfully flat $R_0 \otimes R_1$-ring R_2 and $\varphi \in U(\pi f_{R_2})$ such that $\varphi(L_0 \otimes_{R_0} R_2) = M \otimes_{R_1} R_2$. Thus, $\Psi'(M) = N(\varphi)\Psi'(L_0) = N(\varphi)\Psi_f(L_0) = \Psi_f(M)$ in $\mu_2(R_2)$. Since $R_1 \to R_0 \otimes R_1 \to R_2$ is faithfully flat, this means that $\Psi'(M) = \Psi_f(M)$ in $\mu_2(R_1)$, so $\Psi' = \Psi_f$.

We finally verify the additional properties (i)–(iii).

(i) Take $M_0 = L$ and $L_0 = L_{R_0}$ in the construction of Ψ_f; one gets $\theta = 1$.

(ii) It is enough to prove the equality after base-changing to R_0. It is then a consequence of (i) (take $L = L_0$) and Proposition 2.25(ii).

(iii) Again, we may base change to R_0 first. The claim then follows from (i) and item (t6) in 2G. $\square$

It turns out that Ψ_f is often constant on $\mathrm{Lag}(\pi f)$.

Lemma 6.13. *With Notation 4.1, suppose that R is connected semilocal and (τ, ε) is orthogonal. Let $(P, f) \in \mathcal{H}^\varepsilon(A, \tau)$ and assume that πf is hyperbolic. Then $\Psi_f : \mathrm{Lag}(\pi f) \to \mu_2(R) = \{\pm 1\}$ is onto if and only if $T \cong R \times R$, $[A] = 0$ and $P \neq 0$.*

Proof. The lemma is clear if $P = 0$, so assume $P \neq 0$.

Let $M, M' \in \mathrm{Lag}(\pi f)$. By Lemma 2.22, there exists $\varphi \in U(\pi f)$ such that $\varphi M = M'$, hence $\mathrm{Nr}_{T/R}(\mathrm{Nrd}(\varphi))\Psi_f(M) = \Psi_f(M')$. From this we see that the condition that $\Psi_f : \mathrm{Lag}(\pi f) \to \mu_2(R)$ is onto is equivalent to the existence of $\varphi \in U(\pi f)$ with $\mathrm{Nr}_{T/R}\mathrm{Nrd}(\varphi) = -1$ in $\mu_2(R)$.

Suppose that $[A] = 0$ and $T = R \times R$, and let $e = (1_R, 0_R)$ and $e' = (0_R, 1_R)$. By Proposition 6.3, we may identify $U(\pi f)$ with $U(f_e) \times U(f_{e'})$, and under this identification, $\mathrm{Nrd} : U(\pi f) \to \mu_2(T)$ is just $\mathrm{Nrd} \times \mathrm{Nrd} : U(f_e) \times U(f_{e'}) \to \mu_2(R) \times \mu_2(R)$. Since $[Be] = [Be'] = [A] = 0$ (Lemma 4.4(iii)), this map is onto by Theorem 2.20. One readily checks that $\mathrm{Nr}_{T/R} : \mu_2(T) \to \mu_2(R)$ is also onto, so we conclude that there exists $\varphi \in U(\pi f)$ with $\mathrm{Nr}_{T/R}\mathrm{Nrd}(\varphi) = -1$.

Conversely, suppose that $\varphi \in U(\pi f)$ satisfies $\mathrm{Nr}_{T/R}\mathrm{Nrd}(\varphi) = -1$. If T were connected, then we would have $\mathrm{Nr}_{T/R}(\mu_2(T)) = \mathrm{Nr}_{T/R}(\{\pm 1\}) = 1$, so we must have $T \cong R \times R$ (Lemma 1.16) and $\mathrm{Nrd}(\varphi) \in \{(1, -1), (-1, 1)\}$. Let e and e' denote the nontrivial idempotents of T. Appealing to Proposition 6.3 as in the previous paragraph, we see that $\varphi|_{Pe} \in U(f_e)$ and $\varphi|_{Pe'} \in U(f_{e'})$, and either $\mathrm{Nrd}(\varphi|_{Pe}) = -1$ or $\mathrm{Nrd}(\varphi|_{Pe'}) = -1$. Thus, by Theorem 2.20, $[eAe] = 0$ or $[e'Ae'] = 0$. Since $[A] = [eAe] = [e'Ae']$ (Lemma 4.4(iii)), $[A] = 0$. $\square$

Proposition 6.14. *With Notation 4.1, suppose that R is a field, $T \cong R \times R$, $[A] = 0$ and (τ, ε) is orthogonal. Let $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ be a hyperbolic hermitian space. Then there exists $M \in \mathrm{Lag}(\pi f)$ with $MP = A$. Every such M satisfies $\Psi_f(M) = (-1)^{\frac{1}{2}\mathrm{rrk}_A P}$.*

Proof. By Reduction 4.10, Corollary 1.12 and Proposition 6.12(iii), we may assume that $B = T$, $\deg A = 2$ and τ is orthogonal. As a result, $\varepsilon = 1$. Recall that σ is also orthogonal in this case (Lemma 4.12).

Let e denote a nontrivial idempotent of T. We identify A with $\mathrm{M}_2(R)$ in such a way that $e = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Thus, $B = T = R + Re$ consist of the diagonal matrices, and $\pi : A \to B$ is given by $\pi\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]$. Since $e^\sigma = e$, there exist $\alpha \in R^\times$ such that σ is given by $\sigma : \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} a & \alpha c \\ \alpha^{-1} b & d \end{smallmatrix}\right]$.

Let $f_1 : A \times A \to A$ be the hyperbolic 1-hermitian form given by $f_1(x, y) = x^\sigma \left[\begin{smallmatrix} 0 & \alpha \\ 1 & 0 \end{smallmatrix}\right] y$ ($\left[\begin{smallmatrix} R & R \\ 0 & 0 \end{smallmatrix}\right]$ is a Lagrangian). Since $\mathrm{rrk}_A A = 2$ and $\mathrm{rrk}_A P$ is even (because πf is hyperbolic), we have $(P, f) \cong \frac{\mathrm{rrk}_A P}{2} \cdot (A, f_1)$ (Lemma 2.7). Thus, it is enough to prove the existence of M when $(P, f) = (A, f_1)$. To that end, take $M := B = \left[\begin{smallmatrix} R & 0 \\ 0 & R \end{smallmatrix}\right]$; it is a Lagrangian because $A = M \oplus M'$ and $f_1(M', M') = 0$ for $M' = \left[\begin{smallmatrix} 0 & R \\ R & 0 \end{smallmatrix}\right]$.

We proceed with proving the second statement of the proposition. Suppose that $M \in \mathrm{Lag}(\pi f)$ satisfies $MA = P$. Write $e' = 1 - e$. Using Proposition 6.3, we shall view πf as $f_e \times f_{e'}$ and identify $U(\pi f)$ and $\mathrm{Lag}(\pi f)$ with $U(f_e) \times U(f_{e'})$ and $\mathrm{Lag}(f_e) \times \mathrm{Lag}(f_{e'})$, respectively.

Since $M \in \mathrm{Lag}(\pi f)$, we have $Me \in \mathrm{Lag}(f_e)$, and so $MeA \in \mathrm{Lag}(f)$ (see item (t5) in 2G). Similarly, $Me'A \in \mathrm{Lag}(f)$. Since $MA = P$, we have $MeA + Me'A = P$ and A-length considerations force $P = MeA \oplus Me'A$.

By Lemma 2.22, there exists $\varphi \in U(f)$ such that $\varphi(MeA) = Me'A$. Write $\varphi_e = \varphi|_{Pe}$. Then, viewing $(\varphi_e, 1)$ as an element of $U(f_e) \times U(f_{e'}) = U(\pi f)$ and working in $\mathrm{Lag}(\pi f) = \mathrm{Lag}(f_e) \times \mathrm{Lag}(f_{e'})$, we have

$$\begin{aligned}(\varphi_e, 1) \cdot M &= (\varphi_e, 1)(Me, Me') = (\varphi_e(Me), Me') \\ &= (\varphi(MeA) \cdot e, Me') = (Me'Ae, Me'Ae') = Me'A.\end{aligned}$$

By Proposition 6.12, we have $N(\varphi_e, 1) \cdot \Psi_f(M) = \Psi_f(Me'A) = 1$, because $Me'A \in \mathrm{Lag}(f)$. Furthermore, by Proposition 2.26, $MeA \oplus Me'A = P$ implies that $\mathrm{Nrd}(\varphi) = (-1)^{\frac{1}{2}\mathrm{rrk}_A P}$. Together, this gives $\Psi_f(M) = N(\varphi_e, 1)^{-1} = \mathrm{Nrd}(\varphi_e) \cdot \mathrm{Nrd}(1) = \mathrm{Nrd}(\varphi) = (-1)^{\frac{1}{2}\mathrm{rrk}_A P}$, as required. □

Corollary 6.15. *With Notation 4.1, suppose that (τ, ε) is orthogonal. Let $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ and let $M \in \mathrm{Lag}(\pi f)$. If $MA = P$, then $\Psi_f(M) = (-1)^{\frac{1}{2}\mathrm{rrk}_A P}$.*

Proof. By Lemma 2.17, it is enough to prove the corollary after specializing to an algebraic closure of $k(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}\, R$, so assume that R is an algebraically closed field. Then $[A] = 0$ and $T \cong R \times R$. We claim that f is hyperbolic. Indeed, f_T is hyperbolic by Lemma 6.11 and $T \cong R \times R$, so f is also hyperoblic. The corollary therefore follows from Proposition 6.14. □

Proposition 6.16. *With Notation 4.1, suppose that (τ, ε) is orthogonal and R is a field. Let $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ and let $M \in \mathrm{Lag}(\pi f)$. Then there exists $\varphi \in U^0(\pi f)$ such that $\varphi M \cdot A = P$ if and only if $\Psi_f(M) = (-1)^{\frac{1}{2}\mathrm{rrk}_P A}$.*

Proof. If $\varphi M \cdot A = P$ for $\varphi \in U^0(\pi f)$, then $\Psi_f(M) = N(\varphi)\Psi_f(M) = \Psi_f(\varphi M) = (-1)^{\frac{1}{2}\mathrm{rrk}_P A}$ by Corollary 6.15. We turn to prove the converse.

Using Proposition 2.5, write $(P, f) = (P_1, f_1) \oplus (P_2, f_2)$ with f_1 anisotropic and f_2 hyperbolic. Since $[\pi f_1] = [\pi f] = 0$ in $W_\varepsilon(B, \tau)$, the form πf_1 is hyperoblic by Theorem 2.8(ii). Let $M_1 \in \mathrm{Lag}(\pi f_1)$. Arguing as in Remark 3.7, we see that $M_1 A = P_1$, and $\Psi_{f_1}(M_1) = (-1)^{\frac{1}{2}\mathrm{rrk}_A P_1}$ by Corollary 6.15. We now split into cases.

Case I. $[A] = 0$ and $T \cong R \times R$. By Proposition 6.14, there exists $M_2 \in \mathrm{Lag}(\pi f_2)$ such that $M_2 A = P_2$. Write $M' = M_1 \oplus M_2$. Since $M'A = P$, we have $\Psi_f(M') = (-1)^{\frac{1}{2}\mathrm{rrk}_A P}$ by Corollary 6.15. By Lemma 2.22, there exists $\psi \in U(\pi f)$ such that $\psi M = M'$. Since $\Psi_f(M') = (-1)^{\frac{1}{2}\mathrm{rrk}_A P} = \Psi_f(M)$, this means that $\mathrm{Nrd}(\psi) \in \ker(\mathrm{Nr}_{T/R} : \mu_2(T) \to \mu_2(R)) = \mu_2(R)$.

If $\mathrm{Nrd}(\psi) = 1$, take φ to be ψ. If $\mathrm{Nrd}(\psi) = -1$, then $P \neq 0$. Since $[A] = 0$, Theorem 2.20 implies that there exists $\xi \in U(f)$ with $\mathrm{Nrd}(\xi) = -1$. Then ξ is an A-linear isometry of πf, hence $\xi\psi M$ is a Lagrangian of πf satisfying $\xi\psi M \cdot A = \xi(\psi M \cdot A) = \xi P = P$, so take $\varphi = \xi\psi$.

Case II. $[A] = 0$ and T is a field. Let $L_2 \in \mathrm{Lag}(f_2)$. By definition, we have $\Psi_{f_2}(L_2) = 1$, hence $\Psi_f(M_1 \oplus L_2) = \Psi_{f_1}(M_1) \cdot \Psi_{f_2}(L_2) = (-1)^{\frac{1}{2}\mathrm{rrk}_A P_1}$ (Proposition 6.12(ii)). On the other hand, by Lemma 6.13, $\Psi_f(M_1 \oplus L_2) = \Psi_f(M) = (-1)^{\frac{1}{2}\mathrm{rrk}_A P}$, so $\frac{1}{2}\mathrm{rrk}_A P_2 = \frac{1}{2}(\mathrm{rrk}_A P - \mathrm{rrk}_A P_1)$ must be even. Now, by Proposition 6.6, there exists $M_2 \in \mathrm{Lag}(\pi f_2)$ such that $M_2 A = P_2$. Proceed as in Case I.

Case III. $[A] \neq 0$. By Wedderburn's Theorem, R is infinite. Therefore, thanks to Proposition 4.14(ii), we are reduced into proving the proposition when R is algebraically closed. This is covered by Case I. □

From Proposition 6.16, we see that in order to apply the proof of Theorem 6.10 to our situation, we need to find a Lagrangian $M \in \mathrm{Lag}(\pi f)$ with $\Psi_f(M) = (-1)^{\frac{1}{2}\mathrm{rrk}_A P}$. The following two propositions, which address the cases $[B] \neq 0$ and $[B] = 0$ respectively, characterize precisely when such M exists.

Proposition 6.17. *With Notation 4.1, suppose that R is semilocal, T is connected and (τ,ε) is orthogonal. Let $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma)$ and assume that πf is hyperbolic. If $[B]\neq 0$, then $\Psi_f(M)=(-1)^{\frac{1}{2}\operatorname{rrk}_P A}$ for all $M\in\operatorname{Lag}(\pi f)$.*

Proof. For the sake of contradiction, suppose that there exists $M\in\operatorname{Lag}(\pi f)$ with $\Psi_f(M)=(-1)^{\frac{1}{2}\operatorname{rrk}_P A+1}$. By Lemma 6.11, there exists $L\in\operatorname{Lag}(f_T)$, and $\operatorname{rrk}_{A_T} L$ is constant because it equals $\frac{1}{2}\operatorname{rrk}_{A_T} P_T$.

Suppose that $\operatorname{rrk}_{A_T} L$ is odd. Then $(\operatorname{rrk}_{A_T} L)\cdot[A_T]=0$ by Corollary 1.13 and Theorem 1.8, and $2[A_T]=0$ because A_T has a T-involution. Thus, $[B]=[A_T]=0$, a contradiction.

Suppose that $\operatorname{rrk}_{A_T} L$ is even. Then $\frac{1}{2}\operatorname{rrk}_A P$ is also even. Now, $\Psi_f(M_T)=\Psi_f(M)=-1$ while $\Psi_f(L)=1$. By Lemma 6.13, this means that $[B]=[A_T]=0$, so again, we have reached a contradiction. □

Lemma 6.18. *With Notation 4.1, suppose that R is semilocal,* $\deg B=1$, $\tau=\mathrm{id}_B$ *and $\varepsilon=1$. Let $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma)$ and assume that πf is hyperbolic and* $\operatorname{rrk}_A P$ *is constant and greater than* 2. *Then there exists $x\in P$ such that $f(x,x)\in A^\times$ and $\pi f(x,x)=0$.*

Proof. Define λ,μ as in Lemma 4.3(i) (so $\lambda^\sigma=\lambda$ and $\mu^\sigma=-\mu$).

Step 1. We first prove the existence of x when R is a field. Write $(P,f)=(P_1,f_1)\oplus(P_2,f_2)$ with f_1 anisotropic and f_2 hyperbolic (Proposition 2.5). As in the proof of Proposition 6.16, πf_1 and πf_2 are hyperbolic, so both $\operatorname{rrk}_A P_1$ and $\operatorname{rrk}_A P_2$ are even. By assumption, $\operatorname{rrk}_A P_1>0$ or $\operatorname{rrk}_A P_2\geq 4$.

If $\operatorname{rrk}_A P_1>0$, then there exists nonzero $x\in P_1$ such that $\pi f_1(x,x)=0$. Thus, $f_1(x,x)\in\mathcal{S}_1(A,\sigma)\cap\ker\pi=\mu\lambda R$. Since f_1 is anisotropic, $f_1(x,x)\neq 0$, so $f(x,x)=f_1(x,x)\in\mu\lambda R^\times\subseteq A^\times$.

If $\operatorname{rrk}_A P_2\geq 4$, then f_2 has an orthogonal summand isomorphic to $\langle\mu\lambda,-\mu\lambda\rangle_{(A,\sigma)}$ (Lemma 2.7). Now take x to be the vector corresponding to $(1_A,0_A)\in A^2$ in P.

Step 2. We continue to assume that R is a field. Let $x,y\in P$ be two elements such that $\pi f(x,x)=\pi f(y,y)=0$ and $\operatorname{rrk}_B xB=\operatorname{rrk}_B yB=1$. We claim that there exists $\varphi\in U^0(\pi f)$ such that $\varphi x=y$.

Suppose first that T is a field. Our assumptions imply that $xB\cong yB$ as B-modules. Since πf is unimodular, there exists $x'\in P$ such that $\pi f(x,x')=1$. In particular, the restriction of f to $Q=xB\oplus xB'$ is unimodular, so $P=Q\oplus Q^\perp$. Since $\operatorname{rrk}_B P=\iota\operatorname{rrk}_A P\geq 4$, we have $Q^\perp\neq 0$. By Theorem 2.20, there exists $\psi_0\in U(f|_{Q^\perp\times Q^\perp})$ with $\operatorname{Nrd}(\psi_0)=-1$. Let $\psi=\mathrm{id}_Q\oplus\psi_0\in U(\pi f)$. By Theorem 2.3, there exists $\varphi\in U(\pi f)$ with $\varphi x=y$. If $\operatorname{Nrd}(\varphi)=1$, we are done. If not, $\operatorname{Nrd}(\varphi)=-1$ (because T is a field) and we can replace φ with $\varphi\psi$ to get $\operatorname{Nrd}(\varphi)=1$.

When T is not a field, we have $T=R\times R$ and we can apply the argument of the previous paragraph separately over each factor of T.

Step 3. We now prove the proposition for all R. Since $\operatorname{rrk}_B P=\iota\operatorname{rrk}_A P\geq 4$ and πf is hyperbolic, there exists $y\in P$ such that $\pi f(y,y)=0$ and yB is a summand of P_B of reduced rank 1.

Let $\mathfrak{m}_1,\dots,\mathfrak{m}_t$ denote the maximal ideals of R. By Step 1, for all $1\leq i\leq t$, there exists $x_i\in P(\mathfrak{m}_i)$ such that $\pi f(\mathfrak{m}_i)(x_i,x_i)=0$ and $f(\mathfrak{m}_i)(x_i,x_i)\in A(\mathfrak{m}_i)^\times$. The latter condition implies that $\operatorname{ann}_{B(\mathfrak{m}_i)} x_i=0$, so $\operatorname{rrk}_{B(\mathfrak{m}_i)} x_iB(\mathfrak{m}_i)=1$. Thus, by Step 2, there exists $\varphi_i\in U^0(\pi f(\mathfrak{m}_i))$ such that $\varphi_i y=x_i$.

By Theorem 2.18, there exists $\varphi\in U(\pi f)$ such that $\varphi(\mathfrak{m}_i)=\varphi_i$ for all $1\leq i\leq t$. Let $x=\varphi y$. Then $f(x,x)=f(y,y)=0$ and $f(x,x)(\mathfrak{m}_i)=f(\mathfrak{m}_i)(x_i,x_i)\in A(\mathfrak{m}_i)^\times$ for all i. By Lemma 1.6, $f(x,x)\in A^\times$, as required. □

The next proposition makes use of the discriminant of hermitian forms over (A,σ), see 2H.

Proposition 6.19. *With Notation 4.1, suppose that R is semilocal, T is connected, $[B]=0$ and (τ,ε) is orthogonal. Let $(P,f)\in\mathcal{H}^{\varepsilon}(A,\sigma)$ and assume that πf is hyperbolic. Then $\Psi_f(M)\equiv(-1)^{\frac12\operatorname{rrk}_A P}$ for some $M\in\operatorname{Lag}(\pi f)$ if and only if $\operatorname{disc}(f)=\operatorname{disc}(T/R)^{\frac12\operatorname{rrk}_A P}$. When this fails, $[A]=0$, $\operatorname{disc}(f)=\operatorname{disc}(T/R)^{\frac12\operatorname{rrk}_A P+1}$ and f is isotropic.*

Proof. Recall that $\operatorname{rrk}_A P$ is even because πf is hyperbolic. Furthermore, $\operatorname{rrk}_A P$ is constant because T is connected. By Reduction 4.10, we may assume that $\deg B=1$, $\deg A=2$, $\tau=\mathrm{id}_T$ and $\varepsilon=1$. Define λ,μ as in Lemma 4.3(i) (so $\lambda^\sigma=\lambda$ and $\mu^\sigma=-\mu$). By Lemma 6.13, Ψ_f is constant on $\operatorname{Lag}(\pi f)$ and we denote the value that it attains by $\bar\Psi_f$. The proposition is clear if $P=0$, so assume $P\neq 0$.

Suppose first that $\operatorname{rrk}_A P>2$. By Lemma 6.18, there exists $x\in P$ with $f(x,x)\in A^\times$ and $\pi f(x,x)=0$. Write $P_1=xA$, $P_2=P_1^\perp$ and let $f_i=f|_{P_i\times P_i}$ $(i=1,2)$. Since $f(x,x)\in A^\times$, we have $(P,f)=(P_1,f_1)\oplus(P_2,f_2)$. Moreover, $f(x,x)\in\ker\pi\cap\mathcal{S}_1(A,\sigma)\cap A^\times=\mu\lambda R^\times$, so $(P_1,f_1)\cong\langle\alpha\mu\lambda\rangle_{(A,\sigma)}$ for some $\alpha\in R^\times$. Thus, $\operatorname{disc}(f_1)\equiv-\operatorname{Nrd}(\mu)\operatorname{Nrd}(\alpha\mu\lambda)\equiv\alpha^2\lambda^2\equiv\operatorname{disc}(T/R)\bmod (R^\times)^2$ (Proposition 2.27(v)). Since $\pi f(x,x)=\pi f(x\mu,x\mu)=0$ and $xA=xB\oplus x\mu B$, we have $xB\in\operatorname{Lag}(\pi f_1)$. Moreover, $xB\cdot A=xA$ implies that $\bar\Psi_{f_1}=-1$ (Corollary 6.15). Since πf_1 is hyperbolic, $[\pi f_2]=[\pi f]=0$ in $W_\varepsilon(B,\tau)$, so πf_2 is hyperbolic by Theorem 2.8(ii). Now, $\operatorname{disc}(f)=\operatorname{disc}(f_1)\operatorname{disc}(f_2)=\operatorname{disc}(T/R)\operatorname{disc}(f_2)$ and $\bar\Psi_f=\bar\Psi_{f_1}\bar\Psi_{f_2}=-\bar\Psi_{f_2}$ (Proposition 6.12(ii)), so the proposition holds for (P,f) if and only if it holds for (P_2,f_2). Replacing (P,f) with (P_2,f_2) and repeating this process, we eventually reduce to the case where $\operatorname{rrk}_A P=2$.

Suppose henceforth that $\operatorname{rrk}_A P=2$. We may assume that $P=A_A$ and $f=\langle a\rangle_{(A,\sigma)}$ for some $a\in\mathcal{S}_1(A,\sigma)\cap A^\times$. Write $a=b_1+\mu b_2$ with $b_1,b_2\in B$ and let θ denote the standard R-involution of T (so $\lambda^\theta=-\lambda$). Note that $a^\sigma=a$ implies $b_2^\theta=-b_2$ and, by Proposition 2.27(v), $\operatorname{disc}(f)\equiv-\operatorname{Nrd}(\mu)\operatorname{Nrd}(b_1+\mu b_2)\equiv\mu^2(b_1^\theta b_1-\mu^2 b_2^\theta b_2)\equiv\mu^2(b_1^\theta b_1+\mu^2 b_2^2)\bmod (R^\times)^2$.

Straightforward computation shows that the Gram matrix of πf relative to the B-basis $\{1,\mu\}$ is

$$X=\begin{bmatrix} b_1 & -\mu^2 b_2\\ -\mu^2 b_2 & -\mu^2 b_1^\theta\end{bmatrix}\in\mathrm{GL}_2(T).$$

Thus, $\operatorname{disc}(\pi f)\equiv\mu^2 b_1^\theta b_1+\mu^4 b_2^2\equiv\operatorname{disc}(f)\bmod (T^\times)^2$. Since πf is hyperbolic, $\operatorname{disc}(\pi f)=(T^\times)^2$, so there exists $d\in T^\times$ such that

$$d^2=\mu^2(b_1^\theta b_1+\mu^2 b_2^2)\equiv\operatorname{disc}(f)\bmod (R^\times)^2.$$

Identifying A_B with B^2 via the basis $\{1,\mu\}$, let $M=\left[\begin{smallmatrix}-\mu^2 b_2+d\\ -b_1\end{smallmatrix}\right]T+\left[\begin{smallmatrix}\mu^2 b_1^\theta\\ -\mu^2 b_2-d\end{smallmatrix}\right]T$ and $M'=\left[\begin{smallmatrix}-\mu^2 b_2-d\\ -b_1\end{smallmatrix}\right]T+\left[\begin{smallmatrix}\mu^2 b_1^\theta\\ -\mu^2 b_2+d\end{smallmatrix}\right]T$. Viewing M a subset of A, we have

$$M=(-\mu^2 b_2+d-\mu b_1)T+(\mu^2 b_1^\theta-\mu^3 b_2-\mu d)T.$$

We claim that $M+M'=B^2$. To see this, observe that $\left[\begin{smallmatrix}-2\mu^2 b_2\\ -2b_1\end{smallmatrix}\right]=\left[\begin{smallmatrix}-\mu^2 b_2+d\\ -b_1\end{smallmatrix}\right]+\left[\begin{smallmatrix}-\mu^2 b_2-d\\ -b_1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}-2\mu^2 b_1^\theta\\ 2\mu^2 b_2\end{smallmatrix}\right]=-\left[\begin{smallmatrix}\mu^2 b_1^\theta\\ -\mu^2 b_2-d\end{smallmatrix}\right]-\left[\begin{smallmatrix}\mu^2 b_1^\theta\\ -\mu^2 b_2+d\end{smallmatrix}\right]$ live in $M+M'$, and they generate B^2_B because they are the columns of the matrix $\left[\begin{smallmatrix}0 & 2\\ -2 & 0\end{smallmatrix}\right]X\in\mathrm{GL}_2(T)$. Furthermore, we have $M\cap M'=0$ because

$$\begin{bmatrix} b_1 & -\mu^2 b_2+d\\ -\mu^2 b_2-d & -\mu^2 b_1^\theta\end{bmatrix}M=0\qquad\text{and}\qquad\begin{bmatrix} b_1 & -\mu^2 b_2-d\\ -\mu^2 b_2+d & -\mu^2 b_1^\theta\end{bmatrix}M'=0,$$

while $\left[\begin{smallmatrix} b_1 & -\mu^2 b_2+d\\ -\mu^2 b_2-d & -\mu^2 b_1^\theta\end{smallmatrix}\right]+\left[\begin{smallmatrix} b_1 & -\mu^2 b_2-d\\ -\mu^2 b_2+d & -\mu^2 b_1^\theta\end{smallmatrix}\right]=2X\in\mathrm{GL}_2(T)$. It is routine to check that $\pi f(M,M)=\pi f(M',M')=0$, so we conclude that $M\in\operatorname{Lag}(\pi f)$.

We claim that $d \in R^\times$ or $d \in \lambda R^\times$. Indeed, write $d = \alpha + \beta\lambda$ with $\alpha, \beta \in R$. Then $(\alpha^2 + \lambda^2\beta^2) + 2\alpha\beta\lambda = d^2 \in R^\times$, so $\alpha\beta = 0$ and $\alpha R + \beta R = R$ (because $d^2 = \alpha^2 + \lambda^2\beta^2 \in \alpha R + \beta R$). Multiplying the latter equation by α, we get $\alpha^2 R = \alpha R$, so there exists $c \in R$ with $c\alpha^2 = \alpha$. The element αc is an idempotent and R is connected, hence $\alpha c \in \{0, 1\}$. If $\alpha c = 0$, then $\alpha = c\alpha^2 = 0$ and $d \in \lambda R^\times$. On the other hand, if $\alpha c = 1$, then $\beta = 0$, because $\alpha\beta = 0$, and $d \in R^\times$.

Now, if $d = \alpha\lambda$ for some $\alpha \in R^\times$, then $\operatorname{disc}(f) = \lambda^2(R^\times)^2 = \operatorname{disc}(T/R)$, and

$$\begin{aligned} 2\alpha\lambda\mu &= \mu^3 b_2 + \alpha\lambda\mu - \mu^2 b_1^\theta + \mu^2 b_1^\theta - \mu^3 b_2 - \alpha\mu\lambda \\ &= (-\mu^2 b_2 + d - \mu b_1)\mu + (\mu^2 b_1^\theta - \mu^3 b_2 - \mu d) \in MA \end{aligned}$$

(note that $b_2\mu = \mu b_2^\theta = -\mu b_2$). Thus, $MA = A$ and $\bar{\Psi}_f = -1$ by Corollary 6.15.

On the other hand, if $d \in R^\times$, then $\operatorname{disc}(f) = (R^\times)^2 = \operatorname{disc}(T/R)^2$ and

$$\begin{aligned} (-\mu^2 b_2 + d - \mu b_1)\mu &= -(\mu^2 b_1^\theta - \mu^3 b_2 - \mu d) \in M, \\ (\mu^2 b_1^\theta - \mu^3 b_2 - \mu d)\mu &= -(-\mu^2 b_2 + d - \mu b_1)\mu^2 \in M. \end{aligned}$$

Thus, M is an A-module, and it follows that $\pi(f(M, M)\mu) = \pi f(M, M\mu) = \pi f(M, M) = 0$, hence $f(M, M) = 0$. Similarly, M' is also an A-module with $f(M', M') = 0$, so f is hyperbolic and M is a Lagrangian of f. In particular, f is isotropic. Now, by the characterizing property of Ψ_f in Proposition 6.12, $\bar{\Psi}_f = \Psi_f(M) = 1$, and by Corollary 6.15, there is no $M' \in \operatorname{Lag}(\pi f)$ with $M'A = P$. Moreover, $\operatorname{rrk}_A M = \frac{1}{2}\operatorname{rrk}_A A = 1$, so $[A] = [\operatorname{End}_A(M)] = [R] = 0$ by Proposition 1.11(i).

The proposition follows because only one of the previous cases can hold. Indeed, we cannot have $\bar{\Psi}_f = 1$ and $\bar{\Psi}_f = -1$ simultaneously. □

Theorem 6.20. *Theorem 6.1 holds when (τ, ε) is orthogonal.*

Proof. Recall that we are given $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$ such that $[\pi f] = 0$ in $W_\varepsilon(B, \tau)$. By Theorem 2.8(ii), πf is hyperoblic. As explained in the introduction to this subsection, this means that $\operatorname{rrk}_A P$ is even.

(i) This part is vacuous under our assumptions.

(ii) The pair (σ, ε) is orthogonal by Lemma 4.12.

Suppose that there exists $M \in \operatorname{Lag}(\pi f)$ with $MA = P$. Then $\Psi_f(M) = (-1)^{\frac{1}{2}\operatorname{rrk}_A P}$ by Corollary 6.15. Now, by Proposition 6.19, either $[B] \neq 0$, or $\operatorname{disc}(f) = \operatorname{disc}(T/R)^{\frac{1}{2}\operatorname{rrk}_A P}$, as required.

Conversely, suppose that $[B] \neq 0$, or $\operatorname{disc}(f) = \operatorname{disc}(T/R)^{\frac{1}{2}\operatorname{rrk}_A P}$. If there exists $M \in \operatorname{Lag}(\pi f)$ with $\Psi_f(M) = (-1)^{\frac{1}{2}\operatorname{rrk}_A P}$, then we can argue as in the second paragraph of the proof of Theorem 6.10(i), replacing Proposition 6.9 with Proposition 6.16, to show that there exists $M' \in \operatorname{Lag}(\pi f)$ with $M'A = P$. The existence of M follows from Proposition 6.17 if $[B] \neq 0$ and from Proposition 6.19 if $[B] = 0$.

Proposition 6.19 also implies that $[A] = 0$, $\operatorname{disc}(f) = \operatorname{disc}(T/R)^{\frac{1}{2}\operatorname{rrk}_A P + 1}$ and f is isotropic if $[B] = 0$ and $\operatorname{disc}(f) \neq \operatorname{disc}(T/R)^{\frac{1}{2}\operatorname{rrk}_A P}$.

(iii) By Propositions 6.2 and 6.4, we may assume that T is connected. By (ii), we may further assume that $[B] = 0$ and $\operatorname{disc}(f) \neq \operatorname{disc}(T/R)^{\frac{1}{2}\operatorname{rrk}_A P}$, in which case $[A] = 0$ and $\Psi_f(M) = (-1)^{\frac{1}{2}\operatorname{rrk}_A P + 1}$ for all $M \in \operatorname{Lag}(\pi f)$.

Fix some $M \in \operatorname{Lag}(\pi f)$. Since $[A] = [R]$, there exists $V \in \mathcal{P}(A)$ such that $\operatorname{rrk}_A V = \deg R = 1$ (Proposition 1.11(iii)). Then $M \oplus V$ is a Lagrangian of $\pi(f \oplus \mathbb{h}_V^\varepsilon)$ which satisfies $\Psi_{f \oplus \mathbb{h}_V^\varepsilon}(M \oplus V) = (-1)^{\frac{1}{2}\operatorname{rrk}_A P + 1}\Psi_{\mathbb{h}_V^\varepsilon}(V) = (-1)^{\frac{1}{2}\operatorname{rrk}_A P + 1} = (-1)^{\frac{1}{2}\operatorname{rrk}_A(P \oplus V \oplus V^*)}$ (Proposition 6.12(ii), Lemma 2.6). Replacing (P, f) and M with $(P \oplus V \oplus V^*, f \oplus \mathbb{h}_V^\varepsilon)$ and $M \oplus V$, we may assume that $\Psi_f(M) = (-1)^{\frac{1}{2}\operatorname{rrk}_A P}$ and proceed as in the proof of the "if" part of (ii). □

7. Proof of Theorem 3.4

We can now prove Theorem 3.4. We use the notation of 3A.

Proof of Theorem 3.4. Assume R is semilocal, and recall from 3A that (A,σ) is an Azumaya R-algebra with involution, $\lambda,\mu\in A^\times$ satisfy $\lambda^2\in S:=\mathrm{Z}(A)$, $\lambda\mu=-\mu\lambda$, $\lambda^\sigma=-\lambda$, $\mu^\sigma=-\mu$, and we have $B=\mathrm{Z}_A(\lambda)$, $T=S[\lambda]$, $\tau_1=\sigma|_B$, $\tau_2=\mathrm{Int}(\mu^{-1})\circ\sigma|_B$. To avoid later ambiguity, we henceforth write σ_1 in place of σ.

By Lemma 3.1, R,S,T,B,A satisfy the assumptions of Notation 4.1. Furthermore, π_1 coincides with $\pi=\pi_{A,B}$ of Lemma 4.2. We shall consider four possibilities for the involution σ in Notation 4.1, namely, σ_1 (i.e. σ from 3A), $\mathrm{Int}(\lambda^{-1})\circ\sigma_1$, $\mathrm{Int}(\mu^{-1})\circ\sigma_1$ and $\mathrm{Int}((\lambda\mu)^{-1})\circ\sigma_1$. The involution $\tau=\sigma|_B$ from Notation 4.1 is τ_1 in the first two cases and τ_2 in the last two cases, because λ commutes with elements from B. Recall that $\rho:(B,\tau)\to(A,\sigma)$ is the inclusion map.

According to Theorem 3.6, we need prove conditions (E1), (E2), (E3), (E4).

Proof of (E1). Let $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma_1)$ and assume $[\pi_1 f]=0$. Then existence of the required Lagrangian of $\pi_1 f$ follows by applying Theorem 6.1(iii) to (P,f) with σ_1 in place of σ.

Proof of (E2). Let $(Q,g)\in\mathcal{H}^\varepsilon(B,\tau_1)$ and assume $[\rho_1 g]=0$ in $W_{-\varepsilon}(A,\sigma)$. Put $\sigma=\mathrm{Int}(\lambda^{-1})\circ\sigma_1$ and $\tau=\tau_1$. Then $\rho_1 g=\lambda\rho g$, where the right hand side is λ-conjugation (see 2G) of the base-change of g along ρ (see 2C). Thus, $[\rho g]=0$ in $W_\varepsilon(A,\sigma)$, so by Theorem 5.1(ii), there exists $(Q',g')\in\mathcal{H}^\varepsilon(B,\tau_1)$ with $[g]=[g']$ and a Lagrangian L of $\rho g'$ such that $L\oplus Q'=Q'A$. Since L is also a Lagrangian of $\rho_1 g'=\lambda\rho g'$, we have established (E2) for (Q,g).

Proof of (E3). Let $(P,f)\in\mathcal{H}^{-\varepsilon}(A,\sigma_1)$ and assume $[\pi_2 f]=0$. Put $\sigma=\mathrm{Int}(\mu^{-1})\circ\sigma_1$, $\tau=\tau_2$ and let $f_2=\mu^{-1}f\in\mathcal{H}^\varepsilon(A,\sigma)$. Then $\pi_2 f=\pi f_2$. By applying Theorem 6.1(iii) to (P,f_2), we see that there exists $(P',f_2')\in\mathcal{H}^\varepsilon(A,\sigma)$ with $[f_2]=[f_2']$ and a Lagrangian M of $\pi_2 f_2'$ such that $MA=P'$. Put $f'=\mu f_2'$. Then $(P',f')\in\mathcal{H}^{-\varepsilon}(A,\sigma_1)$, $[f']=[f]$ and M is a Lagrangian of $\pi_2 f'=\pi f_2'$ with $MA=P'$.

Proof of (E4). Let $(Q,g)\in\mathcal{H}^\varepsilon(B,\tau_2)$ and assume $[\rho_2 g]=0$. Put $\sigma=\mathrm{Int}((\lambda\mu)^{-1})\circ\sigma_1$ and $\tau=\tau_2$. Then $\rho_2 g=(\lambda\mu)\rho g$, and the proof proceeds as in the case of (E2).

This completes the proof of Theorem 3.4. □

In fact, Theorems 5.1 and 6.1 allow us to describe the image of the functors $\pi_1,\pi_2,\rho_1,\rho_2$ when T is connected. This description is given in the following theorem, which can be regarded as a refinement of Theorem 3.4; it will be needed for some of the applications.

Theorem 7.1. *With notation as in 3A, suppose that R is semilocal and T is connected.*

(i) Let $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma)$. Then there exists $(Q,g)\in\mathcal{H}^{-\varepsilon}(B,\tau_2)$ with $\rho_2 g\cong f$ if and only if $[\pi_1 f]=0$ and one of the following hold:
  (1) (σ,ε) is not symplectic;
  (2) $4\mid \mathrm{rrk}_A P$.
  If $[\pi_1 f]=0$ and conditions (1)–(2) fail, then $[A]=0$ and f is hyperbolic.

(ii) Let $(Q,g)\in\mathcal{H}^\varepsilon(B,\tau_1)$. Then there exists $(P,f)\in\mathcal{H}^\varepsilon(A,\sigma)$ with $\pi_1 f\cong g$ if and only if $[\rho_1 g]=0$ and one of the the following hold:
  (1) $(\sigma,-\varepsilon)$ is not orthogonal;
  (2) $[A]=0$;
  (3) $[B]\neq 0$;
  (4) $(\sigma,-\varepsilon)$ is orthogonal, $[B]=0$, $\mathrm{rrk}_B Q$ is even and $[D(g)]=\frac{\mathrm{rrk}_Q B}{2}[A]$ (see 2H; τ_1 is unitary).

If $[\rho_1 g] = 0$ *and conditions (1)–(4) fail, then* $\operatorname{rrk}_B Q$ *is even,* $[D(g)] = (\frac{\operatorname{rrk}_Q B}{2} + 1)[A]$ *and* g *is isotropic.*

(iii) Let $(P, f) \in \mathcal{H}^{-\varepsilon}(A, \sigma)$*. Then there exists* $(Q, g) \in \mathcal{H}^{\varepsilon}(B, \tau_1)$ *with* $\rho_1 g \cong f$ *if and only if* $[\pi_2 f] = 0$ *and one of the following hold:*

(1) (τ_2, ε) *is not orthogonal;*

(2) $[B] \neq 0$*;*

(3) (τ_2, ε) *is orthogonal,* $\operatorname{rrk}_A P$ *is even, and* $\operatorname{disc}(f) = \operatorname{disc}(T/R)^{\frac{1}{2}\operatorname{rrk}_A P}$ *(see 2H;* $(\sigma, -\varepsilon)$ *is orthogonal in this case).*

If $[\pi_2 f] = 0$ *and conditions (1)–(3) fail, then* $[A] = 0$*,* $\operatorname{rrk}_A P$ *is even,* $\operatorname{disc}(f) = \operatorname{disc}(T/R)^{\frac{1}{2}\operatorname{rrk}_A P + 1}$ *and* f *is isotropic.*

(iv) Let $(Q, g) \in \mathcal{H}^{\varepsilon}(B, \tau_2)$*. Then there exists* $(P, f) \in \mathcal{H}^{\varepsilon}(A, \sigma)$ *with* $\pi_2 f \cong g$ *if and only if* $[\rho_2 g] = 0$*.*

Proof. We will use the notation and observations from the first two paragraphs of the proof of Theorem 3.4. In particular, we write σ_1 for σ of 3A.

(i) By Remark 3.9(i), (Q, g) exists if and only if $\pi_1 f$ admits a Lagrangian M with $MA = P$. Put $\sigma = \sigma_1$ and $\tau = \tau_1$, and observe that $\tau : B \to B$ is unitary because $\tau_1|_T \neq \mathrm{id}_T$ and T is connected (see Proposition 1.21). Applying Theorem 6.1(i) to f with σ_1 in place of σ now gives the required statement.

(ii) By Remark 3.9(ii), (P, f) exists if and only if $\rho_1 g$ admits a Lagrangian L with $Q \oplus L = QA$. Put $\sigma = \operatorname{Int}(\lambda^{-1}) \circ \sigma_1$ and $\tau = \tau_1$. Then $\rho_1 g = \lambda \rho g$ (notation as in 2C, 2G), so $\rho g \in \mathcal{H}^{\varepsilon}(A, \sigma)$ has the same Lagrangians as $\rho_1 g$. The statement therefore follows by applying Theorem 5.1(i) to (Q, g); note that (σ, ε) and $(\sigma_1, -\varepsilon)$ have the same type by Corollary 1.22(i).

(iii) By Remark 3.9(iii), (Q, g) exists if and only if $\pi_2 f$ admits a Lagrangian M with $MA = P$. Put $\sigma = \operatorname{Int}(\mu^{-1}) \circ \sigma_1$, $\tau = \tau_2$ and let $f_2 = \mu^{-1} f \in \mathcal{H}^{\varepsilon}(A, \sigma_2)$. Then $\pi_2 f = \pi f_2$ and both forms have the same Lagrangians. In addition, f is isotropic if and only if f_2 is isotropic, and $\operatorname{disc}(f) = \operatorname{disc}(f_2)$ (Proposition 2.27(ii)). The statement therefore follows by applying parts (i) and (ii) of Theorem 6.1 to (P, f_2). Note that (σ, ε) and $(\sigma_1, -\varepsilon)$ have the same type by Corollary 1.22(i), and (τ, ε) cannot be unitary if (σ, ε) is symplectic, because $\tau_2|_T = \mathrm{id}_T$ when $\sigma_1|_S = \mathrm{id}_S$.

(iv) By Remark 3.9(iv), (P, f) exists if and only if $\rho_2 g$ admits a Lagrangian L with $Q \oplus L = QA$. Put $\sigma = \operatorname{Int}((\lambda\mu)^{-1}) \circ \sigma_1$ and $\tau = \tau_2$. Then, as in (ii), $\rho_2 g = (\lambda\mu)\rho g$, and the statement follows by applying Theorem 5.1(i) to (Q, g). Note that (τ, ε) is not unitary when (σ, ε) is orthogonal, again because $\tau_2|_T = \mathrm{id}_T$ when $\sigma|_S = \mathrm{id}_S$. □

Corollary 7.2. *With the notation of 3A, suppose that* R *is a semilocal and* T *is connected. Let* $(P, f) \in \mathcal{H}^{\varepsilon}(A, \sigma)$ *be anisotropic and assume that* $[\pi_1 f] = 0$ *in* $W_{\varepsilon}(B, \tau)$*. Then there exists* $(Q, g) \in \mathcal{H}^{-\varepsilon}(B, \tau_2)$ *with* $\rho_2 g \cong f$*.*

Similar statements hold for the image of π_1*,* ρ_1*,* π_2*.*

Corollary 7.3. *With the notation of 3A, suppose that* R *is a regular semilocal domain with fraction field* K *and* T *is connected. Let* $(P, f) \in \mathcal{H}^{\varepsilon}(A, \sigma)$ *and assume that* $[\pi_1 f] = 0$ *in* $W_{\varepsilon}(B, \tau)$*. Then there exists* $(Q, g) \in \mathcal{H}^{-\varepsilon}(B, \tau_2)$ *with* $\rho_2 g \cong f$ *if and only if there exists* $(Q', g') \in \mathcal{H}^{-\varepsilon}(B_K, \tau_{2,K})$ *with* $\rho_2 g' \cong f_K$*.*

Similar statements hold for the image of π_1*,* ρ_1*,* π_2*.*

Proof. Since R is a regular domain and T is connected and finite étale over R, the ring T is also a regular domain [68, Tag 03PC]. In particular, T_K is a field. By the Auslander–Goldman theorem [3, Theorem 7.2], the natural maps $\operatorname{Br} R \to \operatorname{Br} K$ and $\operatorname{Br} T \to \operatorname{Br} T_K$ are injective. Thus, $[A] = 0$ if and only if $[A_K] = 0$ and $[B] = 0$ if and only if $[B_K] = 0$. Furthermore, since R is integrally closed, the map $R^{\times}/(R^{\times})^2 \to K^{\times}/(K^{\times})^2$ is injective. Finally, since T is connected, the type of

(τ_2, ε) is the same as the type of $(\tau_{2,K}, \varepsilon)$ and the type of $(\sigma, \pm\varepsilon)$ is the same as the type of $(\sigma_K, \pm\varepsilon)$. The corollary now follows readily from Theorem 7.1. □

8. Applications

8A. **Quadratic Étale and Quaternion Azumaya Algebras.** When R is a field, Grenier-Boley and Mahmoudi [30] noted that the octagon (3.1) recovers two exact sequences of Lewis [43] involving Witt groups associated to separable quadratic field extensions and quaternion division algebras. We generalize these sequences to quadratic étale algebras and quaternion (i.e. degree-2) Azumaya algebras over semilocal rings.

Before we begin, recall from Lemma 1.19 that when R is semilocal (and $2 \in R^\times$), every quadratic étale R-algebra is of the form $R[\lambda \mid \lambda^2 = \alpha]$ for some $\alpha \in R^\times$, and in this case, the standard R-involution sends λ to $-\lambda$. Quaternion R-algebras admit a similar description, which is well-known when R is a field.

Lemma 8.1. *Let A be a quaterion Azumaya algebra over a semilocal ring R. Then there exist $\lambda, \mu \in A$ such that $\lambda^2, \mu^2 \in R^\times$, $\lambda\mu = -\mu\lambda$ and $\{1, \lambda, \mu, \mu\lambda\}$ is an R basis of A. Furthermore, A admits a unique symplectic involution, σ, which satisfies $\lambda^\sigma = -\lambda$ and $\mu^\sigma = -\mu$.*

Proof. It is enough to consider the case where R is connected. Otherwise, write R as a product of connected rings and work over each factor separately.

The map $\sigma : a \mapsto \mathrm{Trd}_{A/R}(a) - a$ is a symplectic involution of A, see [62, Theorem 4.1]. It is unique by [39, Proposition I.1.3.4]. By Lemma 1.26, there exists $\lambda \in \mathcal{S}_{-1}(A, \sigma) \cap A^\times$. Then $-\lambda = \lambda^\sigma = \mathrm{Trd}(\lambda) - \lambda$, hence $\mathrm{Trd}(\lambda) = 0$. Thus, $\lambda^2 = \mathrm{Trd}(\lambda)\lambda - \mathrm{Nrd}(\lambda) = -\mathrm{Nrd}(\lambda) \in R^\times$. Since $R \cap \lambda R \subseteq \mathcal{S}_1(A, \sigma) \cap \mathcal{S}_{-1}(A, \sigma) = 0$, it follows that $T := R[\lambda]$ is a quadratic étale R-algebra with R-basis $\{1, \lambda\}$. By Corollary 1.15, $\mathrm{rrk}_T A_A = 2$, so we are in the setting of Notation 4.1 and the lemma follows from Lemma 4.3. (Of course, there are more direct proofs.) □

Corollary 8.2. *Let R be a semilocal ring, let A be a quaternion Azumaya R-algebra and let λ, μ, σ be as in Lemma 8.1. Write $B = R[\lambda]$ and $\tau = \sigma|_B$. Then the sequence*

$$0 \to W_1(A, \sigma) \xrightarrow{\pi_1} W_1(B, \tau) \xrightarrow{\rho_1} W_{-1}(A, \sigma) \xrightarrow{\pi_2} W_1(B, \mathrm{id}_B) \xrightarrow{\rho_2} W_{-1}(A, \sigma)$$
$$\xrightarrow{\pi_1} W_{-1}(B, \tau) \xrightarrow{\rho_1} W_1(A, \sigma) \to 0$$

in which the maps are defined as in 3A (with $\tau_1 = \tau$ and $\tau_2 = \mathrm{id}_B$) is exact.

Proof. By Proposition 6.8(ii), $W_{-1}(B, \mathrm{id}_B) = 0$. The corollary therefore follows from Theorem 3.4. □

Corollary 8.3. *Let R be a semilocal ring and let T be a quadratic étale R-algebra with standard involution θ. Let $\rho : R \to T$ denote the inclusion map, viewed as a morphism from (R, id_R) to (T, θ) or (T, id_T), and let $\lambda \in T$ be an element such that $\lambda^2 \in R^\times$ and $T = R \oplus \lambda R$ (λ always exists by Lemma 1.19). Then the sequence*

$$0 \to W_1(T, \theta) \xrightarrow{\mathrm{Tr}} W_1(R, \mathrm{id}_R) \xrightarrow{\lambda\rho} W_1(T, \mathrm{id}_T) \xrightarrow{\mathrm{Tr}} W_1(R, \mathrm{id}_R) \xrightarrow{\lambda\rho} W_{-1}(T, \theta) \to 0$$

with maps given by $\mathrm{Tr}(g) = \mathrm{Tr}_{T/R} \circ g$, $\lambda\rho(f) = \lambda \cdot \rho f$ (notation as in 2C, 2G) is exact.

Baeza [5, Korollar 2.9] and Mandelberg [46, Proposition 2.1] established the exactness at the left-to-middle and middle terms, respectively. Baeza [6, Theorem V.5.8] later proved the exactness at these terms without assuming that $2 \in R^\times$ and gave an alternative ending to the exact sequence.

Proof. Let $A = \mathrm{M}_2(R)$ and let $\sigma : A \to A$ be the symplectic involution $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$. Write $\alpha := \lambda^2$. We embed (T, θ) in (A, σ) by identifying λ with $\left[\begin{smallmatrix} 0 & \alpha \\ 1 & 0 \end{smallmatrix}\right]$. Let $\mu := \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. Then $A, \sigma, \lambda, \mu, B := T$ and $\tau := \theta$ satisfy the assumptions of Corollary 8.2. By Proposition 6.8(ii), $W_1(A, \sigma) = 0$, so the exact sequence of Corollary 8.2 reduces to:

(8.1)

$$0 \to W_1(T, \theta) \xrightarrow{\rho_1} W_{-1}(A, \sigma) \xrightarrow{\pi_2} W_1(T, \mathrm{id}_T) \xrightarrow{\rho_2} W_{-1}(A, \sigma) \xrightarrow{\pi_1} W_{-1}(T, \theta) \to 0$$

Let $u = 2\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, $e = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, and let t denote the transpose involution on A. Then u-conjugation induces an isomorphism $W_{-1}(A, \sigma) \to W_1(\mathrm{M}_2(R), \mathrm{t})$ and e-transfer induces an isomorphism $W_1(\mathrm{M}_2(R), \mathrm{t}) \to W_1(R, \mathrm{id}_R)$; see 2G. We claim that under the resulting isomorphism $W_{-1}(A, \sigma) \to W_1(R, \mathrm{id}_R)$, the maps $\rho_1, \pi_2, \rho_2, \pi_1$ in (8.1) become $\mathrm{Tr}, (-\lambda)\rho, \mathrm{Tr}, \lambda\rho$, respectively. This will imply that the sequence in the corollary is exact (the sign change in the second map does not matter).

To see that ρ_1 and ρ_2 correspond to Tr, note that for every $Q \in \mathcal{P}(B)$, the map $x \mapsto xe : Q \to QAe$ is a natural isomorphism of R-modules. Indeed, it is routine to check this for $Q = B_B$ and the general case follows from the naturality and the fact that every $Q \in \mathcal{P}(B)$ is a summand of B_B^n for some n. One readily checks that $e^\tau u(\lambda x)e = e^\tau u(\lambda\mu x)e = e\,\mathrm{Tr}_{T/R}(x)$ for all $x \in T$. Using this, it is routine to check that, upon identifying eAe with R, the isomorphism $x \mapsto xe : Q \to QAe$ is an isometry from $\mathrm{Tr}\,g$ to $(u(\rho_1 g))_e$, resp. $(u(\rho_2 g))_e$, which is what we want.

We now show that π_1 and π_2 correspond to $\lambda\rho$ and $(-\lambda)\rho$, respectively. Given $V \in \mathcal{P}(R)$, we view V^2 as a right A-module by considering pairs in V^2 as 1×2 matrices and letting $A = \mathrm{M}_2(R)$ act by matrix multiplication. If $(V, f) \in \mathcal{H}^1(R, \mathrm{id}_R)$, let $f' : V^2 \times V^2 \to A$ be given by $f'((x, y), (z, w)) = \left[\begin{smallmatrix} f(x,z) & f(x,w) \\ f(y,z) & f(y,w) \end{smallmatrix}\right]$. Then (V^2, f') is a 1-hermitian space over (A, t) and $f'_e \cong f$. It is enough to show that $\pi_1(u^{-1}f') \cong \lambda\rho f$ and $\pi_2(u^{-1}f') \cong (-\lambda)\rho f$. The A-module V^2 inherits a T-module structure, and the map $x \otimes (a + \lambda b) \mapsto (2xa, 2x\alpha b) : V_T \to V^2$ $(x \in V,\ a, b \in R)$ is an isomorphism of T-modules. (As in the previous paragraph, it is enough to check this for $V = R_R$.) Note also that $\pi_1\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \frac{1}{2}(a + d) + \frac{1}{2}(\alpha^{-1}b + c)\lambda$ and $\pi_2\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \frac{1}{2}(a - d) + \frac{1}{2}(\alpha^{-1}b - c)\lambda$ (e.g. use Lemma 4.2(ii)). It is now routine to check that the isomorphism $V_T \to V^2$ is an isometry from $\lambda\rho f$ to $\pi_1(u^{-1}(f'))$, resp. from $(-\lambda)\rho f$ to $\pi_2(u^{-1}(f'))$, which is what we want. □

Corollary 8.4. *Let A be a quaternion Azumaya algebra over a semilocal ring R and let $\sigma : A \to A$ be the unique symplectic involution of A. Then the map*

$$[f] \mapsto [\mathrm{Trd}_{A/R} \circ f] : W_1(A, \sigma) \to W_1(R, \mathrm{id}_R)$$

is injective.

Proof. Let λ, μ, σ be as in Lemma 8.1 and let B, τ be as in Corollary 8.2. Corollaries 8.2 and 8.3 imply that the maps $\pi_1 : W_1(A, \sigma) \to W_1(B, \tau)$ and $\mathrm{Tr}_{B/R} : W_1(B, \tau) \to W_1(R, \mathrm{id}_R)$ are injective. Their composition is the map in the corollary. □

We now generalize a theorem of Jacobson [36] (see [65, Theorems 10.1.1, 10.1.7] for a modern restatement) from fields to semilocal rings. Here we need Corollary 7.2.

Theorem 8.5. *Let A be a quadratic étale (resp. quaternion Azumaya) algebra over a semilocal ring R and let $\sigma : A \to A$ be the standard involution (resp. unique symplectic involution) of A. Write $\mathrm{Tr} = \mathrm{Tr}_{A/R}$ (resp. $\mathrm{Tr} = \mathrm{Trd}_{A/R}$) and let $(P, f), (P', f') \in \mathcal{H}^1(A, \sigma)$. Then:*

- *(i) (P, f) is isotropic if and only if $(P, \mathrm{Tr} \circ f) \in \mathcal{H}^1(R, \mathrm{id}_R)$ is isotropic.*
- *(ii) $(P, f) \cong (P', f')$ if and only if $(P, \mathrm{Tr} \circ f) \cong (P', \mathrm{Tr} \circ f')$ in $\mathcal{H}^1(R, \mathrm{id}_R)$.*

Proof. (i) By writing R as a product of connected rings and working over each factor separately, we may assume that R is connected.

We begin with the case where A is quadratic étale over R and $\mathrm{Tr} = \mathrm{Tr}_{A/R}$. If A is not connected, then $A = R \times R$ and σ is the exchange involution $(x, y) \mapsto (y, x)$ (Lemma 1.16). In this case, (P, f) is always hyperbolic (Example 2.4) and thus so is $(P, \mathrm{Tr}\, f)$. A hyperbolic space is isotropic if and only if its underlying module is nonzero, so the equivalence holds.

Suppose now that A is connected. It is clear that if (P, f) is isotropic, then so is $(P, \mathrm{Tr}\, f)$. Conversely, assume that $(P, \mathrm{Tr}\, f)$ is isotropic. By Proposition 2.5, there is anisotropic $(Q, g) \in \mathcal{H}^1(R, \mathrm{id}_R)$ and a nonzero $U \in \mathcal{P}(R)$ such that $\mathrm{Tr}\, f \cong g \oplus \mathbb{h}^1_U$. By Corollary 7.2 and the isomorphism between (8.1) and the exact sequence in the Corollary 8.3, there is $(P'', f'') \in \mathcal{H}^1(A, \sigma)$ such that $\mathrm{Tr}\, f'' \cong g$. Corollary 8.3 also tells us that $\mathrm{Tr} : W_1(A, \sigma) \to W_1(R, \mathrm{id}_R)$ is injective, so $[f''] = [f]$. Since $\mathrm{Tr}\, f \cong \mathrm{Tr}\, f'' \oplus \mathbb{h}^1_U$, we have $\mathrm{rrk}_A P'' < \mathrm{rrk}_A P$. Thus, Theorem 2.8(i) implies that there is a nonzero $V \in \mathcal{P}(A)$ such that $f \cong f'' \oplus \mathbb{h}^1_V$, so f is isotropic.

We now consider with the case where A is quaternion Azumaya. Define B, τ_1 and π_1 as in Corollary 8.2. Since we proved part (i) for quadratic étale algebras, and since $\mathrm{Trd}_{A/R} = \mathrm{Tr}_{B/R} \circ \pi_1$, it is enough to show that (P, f) is isotropic if and only if $(P, \pi_1 f)$ is isotropic. The "only if" part is clear so we turn to the "if" part. Suppose that $(P, \pi_1 f)$ is isotropic. Using Proposition 2.5, choose anisotropic $(Q, g) \in \mathcal{H}^\varepsilon(B, \tau_1)$ and nonzero $U \in \mathcal{P}(B)$ such that $\pi_1 f \cong g \oplus \mathbb{h}^1_U$. If B is connected, then arguing as in the previous paragraph shows that f is isotropic. If B is not connected, then $B = R \times R$ by Lemma 1.16. Let $e = (1, 0) \in B$. By Lemma 4.4(iii), we have $[A] = [eB] = [R] = 0$ in $\mathrm{Br}\, R$, so by Lemma 6.8(ii), f is hyperbolic and therefore isotropic ($P \neq 0$ because $\pi_1 f$ is isotropic).

(ii) The map $\mathrm{Tr} : W_1(A, \sigma) \to W_1(R, \mathrm{id}_R)$ is injective by Corollary 8.3 in the quadratic étale case and by Corollary 8.4 in the quaternion Azumaya case. The statement now follows from Theorem 2.8(iii). □

Remark 8.6. When R is a field, Theorem 8.5(i) is straightforward. Indeed, if there is some nonzero $x \in P$ with $\mathrm{Tr}\, f(x, x) = 0$, then $f(x, x) = 0$ because $f(x, x) \in \mathcal{S}_1(A, \sigma) = R$. Since R is a field, xA is an A-summand of P, so the form f is isotropic. However, this argument does not work when R is a semilocal ring, because xA may not be an A-summand of P even when xR is an R-summand of P. For example, take $R = \mathbb{Z}_{(5)}$, $A = \mathbb{Z}_{(5)}[i]$ where $i = \sqrt{-1}$, let $\sigma : A \to A$ act by complex conjugation and consider $f = \langle 1, -1 \rangle_{(A,\sigma)}$. Put $x = (1 + 2i, 1 + 2i) \in A^2$. Then $\mathrm{Tr}\, f(x, x) = 0$ and xR is an R-summand of A^2_A, but A/xA has nonzero 5-torsion elements, which means that xA cannot be an A-summand of A^2_A.

8B. **The Grothendieck–Serre Conjecture.** Let R be a regular local ring with fraction field K. Recall from the introduction that the Grothendieck–Serre conjecture asserts that for every reductive (connected) group R-scheme $\mathbf{G}$, the restriction map $\mathrm{H}^1_{\text{ét}}(R, \mathbf{G}) \to \mathrm{H}^1_{\text{ét}}(K, \mathbf{G})$ has trivial kernel. We now use the corollaries of 8A and results of Balmer–Walter [9] and Balmer–Preeti [8] to establish the conjecture for some outer forms of $\mathbf{GL}_n$ and $\mathbf{Sp}_{2n}$ when $\dim R \leq 4$. (Note that in contrast to many sources discussing the conjecture, R is not assumed to contain a field.)

In order to translate statements about Witt groups to cases of the Grothendieck–Serre conjecture, we use the following proposition. The case $A = R$ is contained in [17, Proposition 1.2].

Proposition 8.7. *Let R be a semilocal regular domain with fraction field K, let (A, σ) be an Azumaya R-algebra with involution and let $\varepsilon \in \mathrm{Z}(A)$ be an element such that $\varepsilon^\sigma \varepsilon = 1$. Then the following conditions are equivalent:*

(a) The restriction map $W_\varepsilon(A,\sigma) \to W_\varepsilon(A_K,\sigma_K)$ *is injective.*
(b) Every two hermitian spaces $(P,f),(P',f') \in \mathcal{H}^\varepsilon(A,\sigma)$ *such that* $f_K \cong f'_K$ *are isomorphic.*
(c) The restriction map $\mathrm{H}^1_{\text{ét}}(R,\mathbf{U}(f)) \to \mathrm{H}^1_{\text{ét}}(K,\mathbf{U}(f))$ *has trivial kernel for all* $(P,f) \in \mathcal{H}^\varepsilon(A,\sigma)$*,*
(d) The restriction map $\mathrm{H}^1_{\text{ét}}(R,\mathbf{U}^0(f)) \to \mathrm{H}^1_{\text{ét}}(K,\mathbf{U}^0(f))$ *has trivial kernel for all* $(P,f) \in \mathcal{H}^\varepsilon(A,\sigma)$*.*

Proof. (a) $\Longrightarrow$ (b): By (a), $[f]=[f']$ in $W_\varepsilon(A,\sigma)$. Since $P_K \cong P'_K$, we have $\operatorname{rrk}_A P \cong \operatorname{rrk}_A P'$, so $f \cong f'$ by Theorem 2.8(iii).

(b) $\Longrightarrow$ (a): Let $(P,f) \in \mathcal{H}^\varepsilon(A,\sigma)$ and suppose that $[f_K]=0$. Then, by Theorem 2.8(ii), f_K is hyperbolic, say $(P_K,f_K) \cong (Q \oplus Q^*, \mathbb{h}^\varepsilon_Q)$ with $Q \in \mathcal{P}(A_K)$. Since R is regular, $\operatorname{ind} A = \operatorname{ind} A_K$ [1, Proposition 6.1]. Thus, by Theorem 1.25 and Corollary 1.13, there is $L \in \mathcal{P}(A)$ with $\operatorname{rrk}_A L = \operatorname{rrk}_{A_K} Q$. By Lemma 1.24, $Q \cong L_K$. This means that $f_K \cong \mathbb{h}^\varepsilon_Q \cong (\mathbb{h}^\varepsilon_L)_K$, so by (b), $f \cong \mathbb{h}^\varepsilon_L$ and $[f]=0$.

(b) $\Longleftrightarrow$ (c): It is well-known that $\mathrm{H}^1_{\text{ét}}(R,\mathbf{U}(f))$ is in correspondence with isomorphism classes of hermitian spaces $(P',f') \in \mathcal{H}^\varepsilon(A,\sigma)$ that become isomorphic to (P,f) after base-changing to some faithfully flat finitely presented (i.e. fppf) R-algebra, see [12, Proposition 5.1], for instance. By [11, Proposition A.1], (P',f') is such a space if and only if $\operatorname{rrk}_A P = \operatorname{rrk}_A P'$, or equivalently, $\operatorname{rrk}_{A_K} P_K = \operatorname{rrk}_{A_K} P'_K$. The equivalence now follows from the fact that the correspondence is compatible with base change.

(c) $\Longleftrightarrow$ (d): If (σ,ε) is symplectic or unitary, then $\mathbf{U}^0(f)=\mathbf{U}(f)$ (Proposition 2.16) and the statement is trivial.

Assume (σ,ε) is orthogonal. Then $1 \to \mathbf{U}^0(f) \to \mathbf{U}(f) \xrightarrow{\mathrm{Nrd}} \boldsymbol{\mu}_2 \to 1$ is a short exact sequence of sheaves on $(\mathcal{A}ff/R)_{\text{ét}}$. (To see that the last map is surjective, pass to the stalks and apply Theorem 2.20.) This induces a commutative diagram of pointed sets,

$$\begin{array}{ccccccccc}
U(f) & \xrightarrow{\mathrm{Nrd}} & \{\pm1\} & \longrightarrow & \mathrm{H}^1_{\text{ét}}(R,\mathbf{U}^0(f)) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(R,\mathbf{U}(f)) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(R,\boldsymbol{\mu}_2) \\
\downarrow & & \| & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\
U(f_K) & \xrightarrow{\mathrm{Nrd}} & \{\pm1\} & \longrightarrow & \mathrm{H}^1_{\text{ét}}(K,\mathbf{U}^0(f)) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(K,\mathbf{U}(f)) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(K,\boldsymbol{\mu}_2)
\end{array}$$

in which the rows are cohomology exact sequences and the vertical arrows are restriction maps. We need to prove that α has trivial kernel if and only if β has trivial kernel.

The map γ has trivial kernel by [18, Proposition 2.2], for instance. Thus, the Four Lemma implies that β has trivial kernel whenever α has trivial kernel.

Next, since R is regular, $[A]=0$ if and only if $[A_K]=0$ [3, Theorem 7.2]. Thus, by Theorem 2.20, $\mathrm{Nrd} : U(f) \to \{\pm1\}$ and $\mathrm{Nrd} : U(f_K) \to \{\pm1\}$ have the same image. Using this, an easy diagram chase shows that if β has trivial kernel, then so does α. □

Conditions (a)–(d) of Proposition 8.7 are conjectured to hold under the assumptions of the proposition. This was affirmed by Gille [29, Theorem 7.7] (see also section 3.3 of that paper) when R is regular local and contains a field. Furthermore, we have:

Theorem 8.8 (Balmer, Preeti, Walter)**.** *Let* R *be a semilocal regular domain with* $\dim R \leq 4$ *and let* K *denote the fraction field of* R*. Then the restriction map* $W_1(R,\mathrm{id}_R) \to W_1(K,\mathrm{id}_K)$ *is injective.*

Proof. Balmer and Walter [9, Corollary 10.4] proved the theorem when R is local, and Balmer and Preeti [8, p. 3] showed that it is enough to require that R is semilocal. □

We establish the following additional cases.

Theorem 8.9. *Let R be a regular semilocal domain with $\dim R \leq 4$, let K denote the fraction field of R and let (A, σ) be an Azumaya R-algebra with involution. Assume that*

(1) $\operatorname{ind} A = 1$ and σ is unitary, or
(2) $\operatorname{ind} A \leq 2$ and σ is symplectic.

Then restriction map $W_1(A, \sigma) \to W_1(A_K, \sigma_K)$ is injective.

Proof. When $\operatorname{ind} A = 1$ and σ is symplectic, we have $W_1(A, \sigma) = 0$ (Proposition 6.8(ii)). We may therefore assume that $\operatorname{ind} A = 2$ when σ is symplectic.

By Theorem 1.30, there exists a σ-invariant idempotent $e \in A$ such that $\deg eAe = \operatorname{ind} A$. Applying e-transfer (see 2G), we may assume that $\deg A = \operatorname{ind} A$, namely, that A is quadratic étale or quaternion Azumaya over R.

Consider the commutative square

$$\begin{array}{ccc} W_1(A, \sigma) & \longrightarrow & W_1(R, \mathrm{id}_R) \\ \big\downarrow & & \big\downarrow \\ W_1(A_K, \sigma_K) & \longrightarrow & W_1(K, \mathrm{id}_K) \end{array}$$

in which the vertical arrows are restriction maps and the horizonal arrows are given by $[f] \mapsto [\mathrm{Tr}_{A/R} \circ f]$ if A is quadratic étale, or $[f] \mapsto [\mathrm{Trd}_{A/R} \circ f]$ if A is quaternion Azumaya. The right vertical arrow is injective by Theorem 8.8 and the top horizontal arrow is injective by Corollary 8.3 when A is quadratic étale, and by Corollary 8.4 when A is quaternion Azumaya. Thus, the left vertical arrow is also injective. □

As a corollary, we verify some cases of the Grothendieck–Serre conjecture.

Corollary 8.10. *With notation and assumptions as in Theorem 8.9, the restriction map $\mathrm{H}^1_{\text{ét}}(R, \mathbf{U}(A, \sigma)) \to \mathrm{H}^1_{\text{ét}}(K, \mathbf{U}(A, \sigma))$ has trivial kernel.*

Proof. We have $\mathbf{U}(A, \sigma) \cong \mathbf{U}(f)$, where $f : A \times A \to A$ is the 1-hermitian form $f(x, y) = x^\sigma y$, so the corollary follows from Proposition 8.7 and Theorem 8.9. □

8C. **Purity.** Let R be a regular domain with fraction field K and let $\mathbf{G}$ be a reductive (connected) group R-scheme. Recall from the introduction that we say that purity holds for $\mathbf{G}$ if

$$\operatorname{im}\left(\mathrm{H}^1_{\text{ét}}(R, \mathbf{G}) \to \mathrm{H}^1_{\text{ét}}(K, \mathbf{G})\right) = \bigcap\nolimits_{\mathfrak{p} \in R^{(1)}} \left(\mathrm{H}^1_{\text{ét}}(R_{\mathfrak{p}}, \mathbf{G}) \to \mathrm{H}^1_{\text{ét}}(K, \mathbf{G})\right),$$

where $R^{(1)}$ denotes the set of height-1 primes in $\operatorname{Spec} R$. The local purity conjecture asserts that purity holds for $\mathbf{G}$ whenever R is regular semilocal.

The following proposition allows us to prove purity for some group schemes by establishing certain results about hermitian forms.

Proposition 8.11. *Let $R, A, \sigma, \varepsilon$ and K be as in Proposition 8.7. Suppose that:*

(1) every anisotropic ε-hermitian space over (A, σ) remains anisotropic after base changing along $R \to K$, and
(2) $\operatorname{im}\left(W_\varepsilon(A, \sigma) \to W_\varepsilon(A_K, \sigma_K)\right) = \bigcap_{\mathfrak{p} \in R^{(1)}} \operatorname{im}\left(W_\varepsilon(A_{\mathfrak{p}}, \sigma_{\mathfrak{p}}) \to W_\varepsilon(A_K, \sigma_K)\right)$.

Then the equivalent conditions of Proposition 8.7 hold, and purity holds for $\mathbf{U}(f)$ for every $(P, f) \in \mathcal{H}^\varepsilon(A, \sigma)$.

Both (1) and (2) are conjectured to hold when R is a regular semilocal domain.

Proof. We first prove that condition (a) of Proposition 8.7 holds. Let w be a Witt class in $\ker(W_\varepsilon(A,\sigma) \to W_\varepsilon(A_K,\sigma_K))$. By Proposition 2.5, w is represented by an anisotropic $(P,f) \in \mathcal{H}^\varepsilon(A,\sigma)$. Since $[f_K]=0$, the form f_K is hyperoblic (Theorem 2.8(ii)). If $P \neq 0$, then f_K is isotropic, and by assumption (1), so is f, contradicting our choice of f. Thus, $P=0$ and $w=0$.

Next, let $(P,f) \in \mathcal{H}^\varepsilon(A,\sigma)$. Recall from the proof of Proposition 8.7 that $\mathrm{H}^1_{\text{ét}}(R,\mathbf{U}(f))$ classifies isomorphism classes of hermitian spaces $(P',f') \in \mathcal{H}^\varepsilon(A,\sigma)$ with $\mathrm{rrk}_A P = \mathrm{rrk}_A P'$. Thus, purity for $\mathbf{U}(f)$ is the equivalent to saying that if $(P_0,f_0) \in \mathcal{H}^\varepsilon(A_K,\sigma_K)$ is such that for every $\mathfrak{p} \in R^{(1)}$, there is $(P',f') \in \mathcal{H}^\varepsilon(A_\mathfrak{p},\sigma_\mathfrak{p})$ with $f'_K \cong f_0$, then there is $(P'',f'') \in \mathcal{H}^\varepsilon(A,\sigma)$ such that $f''_K \cong f_0$.

Given $(P_0,f_0) \in \mathcal{H}^\varepsilon(A_K,\sigma_K)$, assumption (2) implies that there is $(\tilde{P},\tilde{f}) \in \mathcal{H}^\varepsilon(A,\sigma)$ such that $[f_0]=[\tilde{f}_K]$. By Proposition 2.5, we may take $\tilde{f}$ to be anisotropic. By (1), $\tilde{f}_K$ is also anisotropic. By Corollary 2.9(i), $\mathrm{rrk}_{A_K}\tilde{P}_K$ and $\mathrm{rrk}_{A_K} P_0$ are constant. If $\mathrm{rrk}_{A_K}\tilde{P}_K > \mathrm{rrk}_{A_K} P_0$, then by Theorem 2.8(i), there is a nonzero $V \in \mathcal{P}(A_K)$ such that $\tilde{f}_K \cong f_0 \oplus \mathbb{h}^\varepsilon_V$, contradicting the fact that $\tilde{f}_K$ is anisotropic. Thus, $\mathrm{rrk}_{A_K}\tilde{P}_K \le \mathrm{rrk}_{A_K} P_0$. Applying Theorem 2.8(i) again, we get $V \in \mathcal{P}(A_K)$ such that $\tilde{f}_K \oplus \mathbb{h}^\varepsilon_V \cong f_0$. As in the proof of Proposition 8.7, there is $L \in \mathcal{P}(A)$ with $V \cong L_K$, so $f_0 \cong (\tilde{f} \oplus \mathbb{h}^\varepsilon_L)_K$. □

Condition (1) of Proposition 8.11 is known as *purity for* $W_\varepsilon(A,\sigma)$. Provided R is regular local and contains a field, it was established by Ojanguren and Panin [49] when $A=R$ and $\varepsilon=1$, and by Gille [29, Theorem 7.7] for general A,σ,ε.

Condition (2) was proved by Panin and Pimenov [54, Theorem 1.1] for $(A,\sigma)=(R,\mathrm{id}_R)$ when R is a regular semilocal domain containing an infinite field k, and Scully [66, Theorem 5.1] eliminated the assumption that k is infinite.

We use these results together with Theorem 8.5 to prove purity for some outer forms of $\mathbf{GL}_n$ and $\mathbf{Sp}_{2n}$.

Theorem 8.12. *Let R be a regular local ring containing a field and let K denote the fraction field of R. Let (A,σ) be a quadratic étale R-algebra with its standard involution or a quaternion Azumaya R-algebra with its symplectic involution. Let $(P_0,f_0) \in \mathcal{H}^1(A_K,\sigma_K)$ be a hermitian space such that for every $\mathfrak{p} \in R^{(1)}$, there exists $(P^{(\mathfrak{p})},f^{(\mathfrak{p})}) \in \mathcal{H}^1(A_\mathfrak{p},\sigma_\mathfrak{p})$ such that $(P_0,f_0) \cong (P^{(\mathfrak{p})}_K,f^{(\mathfrak{p})}_K)$. Then there exists $(P,f) \in \mathcal{H}^1(A,\sigma)$ such that $(P_0,f_0) \cong (P_K,f_K)$. Equivalently, for every $(P,f) \in \mathcal{H}^1(A,\sigma)$, purity holds for $\mathbf{U}(f)$.*

Proof. We need to prove conditions (1) and (2) of Proposition 8.11. We noted above that (1) holds in our situation, see Gille [29, Theorem 7.7], so it remains to prove (2). Suppose that $(P,f) \in \mathcal{H}^1(A,\sigma)$ is anisotropic and let Tr be as in Theorem 8.5. By part (i) of that theorem, $(P,\mathrm{Tr}\circ f)$ is an anisotropic 1-hermitian space over (R,id_R), and by [66, Theorem 5.1], so is $(P_K,\mathrm{Tr}\circ f_K)$. Applying Theorem 8.5(i) again, shows that (P_K,f_K) is anisotropic, which is what we want. □

8D. **The Kernel of The Restriction Map.** In our final application we characterize the kernel of the restriction map $W_1(R,\mathrm{id}_R) \to W_1(S,\mathrm{id}_S)$ when R is a 2-dimensional regular domain (not necessarily semilocal) and S is a quadratic étale R-algebra. When R is a field, this is a celebrated theorem of Pfister, see [65, Theorem I.5.2], for instance.

The proof makes use of Colloit-Thélène and Sansuc's purity theorem in dimension 2 [16, Corollary 6.14] and a theorem of Pardon [55, Theorem 5] asserting that $W_1(R,\mathrm{id}_R) \to W_1(K,\mathrm{id}_K)$ is injective when R is regular of dimension 2 with fraction field K.

Theorem 8.13. *Let R be a regular domain of dimension ≤ 2 and let S be a quadratic étale R-algebra with standard involution θ. Then the sequence*

$$W_1(S,\theta) \xrightarrow{[g]\mapsto[\mathrm{Tr}_{S/R}\circ g]} W_1(R,\mathrm{id}_R) \xrightarrow{[f]\mapsto[f_S]} W_1(S,\mathrm{id}_S)$$

is exact in the middle.

Proof. When S is not connected, $S = R\times R$ (Lemma 1.16) and the theorem is straightforward. Assume that S is a domain henceforth. We abbreviate $\mathrm{Tr}_{S/R}$ to Tr and let K denote the fraction field of R.

The sequence is a chain complex in the middle by virtue of Proposition 3.5 and the proof of Corollary 8.3; this can also be checked directly.

Let $(P,f)\in\mathcal{H}^1(R,\mathrm{id}_R)$ and assume that $[f_S]=0$ in $W_1(S,\mathrm{id}_S)$. Then $[f_{S\otimes K}] = 0$ in $W_1(S_K,\mathrm{id}_{S_K})$. By virtue of Corollary 8.3, there exists $(Q_0,g_0)\in\mathcal{H}^1(S_K,\theta_K)$ such that $[\mathrm{Tr}\, g_0]=[f_K]$. Adding a hyperbolic space to (Q_0,g_0), we may assume that $\dim_K Q_0 > \dim_K P_K$.

Write $f_0' = \mathrm{Tr}\, g_0$. Then there exists a K-vector space V such that $f_0'\cong f_K\oplus \mathbb{h}^1_V$. Choose $U\in\mathcal{P}(R)$ with $U_K\cong V$ and let $f' = f\oplus \mathbb{h}^1_U$. Then $f'_K\cong f'_0 = \mathrm{Tr}\, g_0$.

Let $\mathfrak{p}\in R^{(1)}$. By Corollary 7.3 and the proof of Corollary 8.3, there exists $(Q^{(\mathfrak{p})},g^{(\mathfrak{p})})\in\mathcal{H}^1(S_\mathfrak{p},\theta_\mathfrak{p})$ such that $\mathrm{Tr}\, g^{(\mathfrak{p})}\cong f'_\mathfrak{p}$. In particular, $\mathrm{Tr}\, g^{(\mathfrak{p})}_K \cong f'_K = \mathrm{Tr}\, g_0$. Since $\mathrm{Tr}: W_1(S_K,\theta_K)\to W_1(K,\mathrm{id}_K)$ is injective (Corollary 8.3), $[g^{(\mathfrak{p})}_K]=[g_0]$, and since $\dim_K Q^{(\mathfrak{p})}_K = \dim_K Q_0$, this means that $g^{(\mathfrak{p})}_K\cong g_0$.

Fix some $(W,h)\in\mathcal{H}^1(S,\theta)$ with $\mathrm{rk}_S W = \dim_{S_K} Q_0$. Recall from the proof of (b) $\Longleftrightarrow$ (c) in Proposition 8.7, that $\mathrm{H}^1_{\text{ét}}(R,\mathbf{U}(h))$ classifies isomorphism classes of unimodular 1-hermitian forms (W',h') over (S,θ) with $\mathrm{rk}_S W = \mathrm{rk}_S W'$. Furthermore, $\mathbf{U}(h)\to\mathrm{Spec}\, R$ is reductive (see 2E). Thus, by Colloit-Thélène and Sansuc's theorem on purity in dimension 2 [16, Corollary 6.14], there exists $(Q,g)\in\mathcal{H}^1(S,\theta)$ such that $g_K\cong g_0$.

Note that $[\mathrm{Tr}\, g_K]=[\mathrm{Tr}\, g_0]=[f_K]$. By [55, Theorem 5] or [9, Corollary 10.2] (here we need $\dim R\leq 3$), the map $W_1(R,\mathrm{id}_R)\to W_1(K,\mathrm{id}_K)$ is injective, so $[\mathrm{Tr}\, g]=[f]$. □

Remark 8.14. Theorem 8.13 also holds if $R\to S$ is replaced with a quadratic étale covering of regular integral schemes $Y\to X$; the proof is exactly the same. For the definition of the Witt group in this more general setting, consult [7, §1.2.1] and [28, §1.5, §1.6].

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