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AN ABSTRACT MODEL FOR COMPRESSIONS

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Summary. The authors investigate the geometry of a Hilbert space $\mathscr{K}$ which contains two subspaces $\mathscr{A}$ and $\mathscr{B}$ such that the algebraic sum $\mathscr{A} + \mathscr{B}$ is dense in $\mathscr{K}$. The aim is to establish an abstract analogon of the functional model for contractions in the theory of unitary dilations. As a consequence, different known types of functional models can be easily derived in a unified manner.

Keywords: Hilbert space, contraction, unitary dilation.

In the present paper we investigate the geometry of a Hilbert space $\mathscr{K}$ which contains two subspaces $\mathscr{A}$ and $\mathscr{B}$ such that their sum $\mathscr{A} + \mathscr{B}$ is dense in $\mathscr{K}$. The space $\mathscr{K}$ may be isometrically mapped into the formal direct sum $\mathscr{A} \oplus \mathscr{B}$ using an isometry constructed in a fairly simple manner from the orthogonal projections $P(\mathscr{A})$ and $P(\mathscr{B})$ onto $\mathscr{A}$ and $\mathscr{B}$ respectively. (In fact, we shall describe two formally different isometries which turn out to be equivalent.) If U is a bounded linear operator on $\mathscr{K}$ for which $\mathscr{A}$ and $\mathscr{B}$ are reducing subspaces the isometry intertwines the restrictions $U \mid \mathscr{A}$ and $U \mid \mathscr{B} \ominus (\mathscr{B} \cap \mathscr{A})$.

The aim is, of course, to investigate, in full generality, the abstract analogon of the situation which presents itself in the theory of unitary dilations. If T is a completely nonunitary contraction on a Hilbert space $\mathscr{H}$ and U is its minimal unitary dilation on $\mathscr{K}$, the sum of the subspaces $M(\mathscr{L})$ and $M(\mathscr{L}^*)$ is dense in $\mathscr{K}$. Moreover, the following relations obtain.

$$\begin{aligned} \mathscr{K} &= M_-(\mathscr{L}^*) \oplus \mathscr{H} \oplus M_+(\mathscr{L}) \\ &= \mathscr{R} \oplus M(\mathscr{L}^*) \\ &= \mathscr{R}_* \oplus M(\mathscr{L}) . \end{aligned}$$

In the present investigation we imitate the geometric configuration of the subspaces $M(\mathscr{L})$ and $M(\mathscr{L}^*)$ as follows: given $\mathscr{K}$, $\mathscr{A}$, $\mathscr{B}$ and U we define, as the orthocomplement of certain subspaces $\mathscr{A}_+ \subset \mathscr{A}$ and $\mathscr{B}_- \subset \mathscr{B}$, a subspace $\mathscr{H} \subset \mathscr{K}$ and consider the compression $T = P(\mathscr{H})\, U \mid \mathscr{H}$. It turns out that the operator U on $\mathscr{K}$ may be considered as a model for T.

It is surprising how large a portion of the geometry of unitary dilations may be obtained in the generality considered in this paper.

The geometrical considerations of the first section of the present paper indicate a way of constructing a model for contractions using similar ideas but without leaving the framework of the space on which the contraction acts. The first investigation of this type was undertaken by E. Durszt; in fact, the original impetus for our work comes from an interesting lecture describing an elementary method of constructing a functional model for contractions presented by E. Durszt at the 16th Seminar on Functional Analysis in May 1985.

It is possible to derive the results of E. Durszt from the considerations of the first section obtaining, at the same time, some simplifications. This forms the contents of section two; instead of presenting the results as particular cases of the general ideas explained in section one we prefer to give an independent exposition even at the expense of a repetition at a certain point. We feel that this slight overlap is justified by the brevity and simplicity of the argument.

There is yet another approach to functional models using a construction formally quite different from the one discussed thus far; this approach is decribed in the paper [4] of V. I. Vasjunin. In section four we review this approach in the light of the theory presented in section one.

1. NOTATION AND PRELIMINARY REMARKS

Given a Hilbert space $\mathscr{H}$, $\mathscr{A}$ a closed subspace of $\mathscr{H}$ we shall denote, as usual, by $\mathscr{A}^{\perp}$ the orthocomplement of $\mathscr{A}$ in $\mathscr{H}$ and by $P(\mathscr{A})$ the orthogonal projection of $\mathscr{H}$ onto $\mathscr{A}$. The algebra of all bounded operators on $\mathscr{H}$ is denoted by $B(\mathscr{H})$.

In the sequel, we shall use frequently well known facts about the square root of a nonnegative operator. To avoid repetitions in the argument we prefer to state them now in the following form:

(1,1) **Proposition.** *Let B and D be nonnegative operators in $B(\mathscr{H})$, let A_1, A_2, C_1, C_2 be operators from $B(\mathscr{H})$.*

If $A_1B^nA_2 = C_1D^nC_2$ for $n = 0, 1, 2, \dots$ then

$$A_1B^{1/2}A_2 = C_1D^{1/2}C_2 .$$

In particular, $AB = DA$ implies $AB^{1/2} = D^{1/2}A$.

There are different proofs of the existence of a nonnegative square root of a nonnegative operator. The standard argument which avoids the spectral representation can be found, for example, in [2], Problem 95: assume $0 \leqq A \leqq 1$ and let us define a sequence (B_n) of nonnegative operators by the recurrence:

$$B_0 = 0 , \quad B_{n+1} = \tfrac{1}{2}(1 - A + B_n^2) \quad \text{for} \quad n = 0, 1, 2, \dots .$$

Then the sequence (B_n) is nondecreasing, converges strongly and its limit B satisfies $A = (1 - B)^2$. It is easy to see by induction that all operators B_n are polynomials in A. In fact, there exists a sequence (p_n) of polynomials independent of A such that the sequence $p_n(A)$ tends strongly to $A^{1/2}$ for all $0 \leqq A \leqq 1$.

If $A_1 B^k A_2 = C_1 D^k C_2$ for $k \geqq 0$ then $A_1 p_n(B) A_2 = C_1 p_n(D) C_2$ for all $n \geqq 0$ as well. Moreover, multiplying by a suitable constant we may assume $0 \leqq B, D \leqq 1$. Thus

$$\begin{aligned} A_1 B^{1/2} A_2 h &= A_1 \lim p_n(B) A_2 h = \lim A_1 \, p_n(B) A_2 h = \\ &= \lim C_1 \, p_n(D) C_2 h = C_1 \lim p_n(D) C_2 h = \\ &= C_1 D^{1/2} C_2 h \end{aligned}$$

for all $h \in \mathscr{H}$.

2. A GEOMETRIC MODEL FOR COMPRESSIONS

(2,1) *Suppose a Hilbert space $\mathscr{H}$ and two closed subspaces $\mathscr{A}$, $\mathscr{B}$ of $\mathscr{H}$ are given such that*

$$\mathscr{A}^\perp \cap \mathscr{B}^\perp = (0) .$$

Then

1° *the projection $P(\mathscr{B})$ is injective on $\mathscr{A}^\perp$ and*

$$(P(\mathscr{B}) \mathscr{A}^\perp)^- = \mathscr{B} \ominus (\mathscr{B} \cap \mathscr{A});$$

2° *the range of the operator*

$D = (P(\mathscr{A}^\perp) P(\mathscr{B}) P(\mathscr{A}^\perp))^{1/2}$ *is dense in* $\mathscr{A}^\perp$ *and*

$$D = D\,P(\mathscr{A}^\perp)\,P(\mathscr{A}^\perp)\,D = P(\mathscr{A}^\perp)\,DP(\mathscr{A}^\perp);$$

3° *there exists an isometry $V\colon \mathscr{A}^\perp \to \mathscr{B}$ such that*

$$VD = P(\mathscr{B})\,P(\mathscr{A}^\perp)$$

and

$$V\mathscr{A}^\perp = \mathscr{B} \ominus (\mathscr{B} \cap \mathscr{A}).$$

Proof. If $a^\perp \in \mathscr{A}^\perp$ then $P(\mathscr{B})\,a^\perp = 0$ if and only if $a^\perp \in \mathscr{B}^\perp$. Thus $P(\mathscr{B})$ is injective on $\mathscr{A}^\perp$. Its range $P(\mathscr{B})\,\mathscr{A}^\perp$ is contained in $\mathscr{B}$ and $(P(\mathscr{B})\,\mathscr{A}^\perp, b) = 0$ for a given $b \in \mathscr{B}$ if and only if $b \in \mathscr{A}$. This proves the first assertion.

To prove the second assertion, we shall show that Ker $D = \mathscr{A}$; this will also prove that the range of D is dense in $\mathscr{A}^\perp$.

Since $|Dx|^2 = (D^2x, x) = |P(\mathscr{B})\,P(\mathscr{A}^\perp)\,x|^2$ we have $Dx = 0$ if and only if $P(\mathscr{A}^\perp)\,x \in \mathscr{B}^\perp$ so that $P(\mathscr{A}^\perp)\,x \in \mathscr{A}^\perp \cap \mathscr{B}^\perp = (0)$ and this is equivalent to $x \in \mathscr{A}$.

In particular, for $x \in \mathscr{A}^\perp$, we have

$$|Dx| = |P(\mathscr{B})\,x|$$

so that the mapping V_0 defined for elements of the form $Da^\perp$ by the formula

$$V_0 Da^\perp = P(\mathscr{B})\, a^\perp$$

is well defined and isometric. Since $D\mathscr{A}^\perp = D\mathscr{K}$ the domain of definition of V_0 is dense in $\mathscr{A}^\perp$; we shall denote by V the extension of V_0 by density.

Since V_0 is an isometry and $D\mathscr{A}^\perp$ is dense in $\mathscr{A}^\perp$ we have

$$V\mathscr{A}^\perp = (V_0 D\mathscr{A}^\perp)^- = (P(\mathscr{B})\,\mathscr{A}^\perp)^- = \mathscr{B} \ominus (\mathscr{B} \cap \mathscr{A}).$$

Since D annihilates $\mathscr{A}$ we have $D = D\,P(\mathscr{A}^\perp)$ so that also $P(\mathscr{A}^\perp)\,D = D$. According to the definition of V_0 we have $V_0 D = V_0 D\,P(\mathscr{A}^\perp) = P(\mathscr{B})\,P(\mathscr{A}^\perp)$. The proof is complete.

The assumption $\mathscr{A}^\perp \cap \mathscr{B}^\perp = (0)$ implies that the space $\mathscr{A} + \mathscr{B}$ is dense in $\mathscr{K}$. We intend to show now how $\mathscr{K}$ may be imbedded in the direct sum $\mathscr{A} \oplus \mathscr{B}$.

(2,2) *The mapping* $\Phi\colon \mathscr{K} \to \mathscr{A} \oplus \mathscr{B}$ *defined by*

$$\Phi = P(\mathscr{A}) \oplus V P(\mathscr{A}^\perp)$$

is isometric and possesses the following properties:

1° $\Phi\mathscr{A} = \mathscr{A} \oplus (0)$

2° $\Phi b = Qb \oplus (1_{\mathscr{B}} - Q^*Q)^{1/2}\, b$, *for* $b \in \mathscr{B}$, *where* $Q = P(\mathscr{A}) \mid \mathscr{B}$

3° $\Phi\mathscr{K} = \mathscr{A} \oplus (\mathscr{B} \ominus (\mathscr{B} \cap \mathscr{A}))$.

Proof. The equality $\Phi\mathscr{A} = \mathscr{A} \oplus (0)$ as well as the inclusion Range $\Phi \subset \mathscr{A} \oplus$ $\oplus$ Range V are obvious. On the other hand, an arbitrary pair $a \oplus Va^\perp$ lies in the range of Φ: indeed,

$$\Phi(a + a^\perp) = \Phi a + \Phi a^\perp = (a \oplus 0) + (0 \oplus Va^\perp) = a \oplus Va^\perp.$$

It remains to show that $V P(\mathscr{A}^\perp)\, P(\mathscr{B}) = (1_{\mathscr{B}} - Q^*Q)^{1/2}\, P(\mathscr{B})$. Since both $V P(\mathscr{A}^\perp)$ and $(1_{\mathscr{B}} - Q^*Q) = (1 - P(\mathscr{B})\,P(\mathscr{A})) \mid \mathscr{B}$ are zero on $\mathscr{A} \cap \mathscr{B}$ it suffices to consider elements from $\mathscr{B} \ominus (\mathscr{A} \cap \mathscr{B}) = (\text{Range}\, P(\mathscr{B})\,P(\mathscr{A}^\perp))^-$ only. According to the definition of V we have $VP(\mathscr{A}^\perp)\,P(\mathscr{B})\,P(\mathscr{A}^\perp) = VD^2 = P(\mathscr{B})\,P(\mathscr{A}^\perp)\,D$.

The proof will be complete if we show that

$$P(\mathscr{B})\,P(\mathscr{A}^\perp)\,D = (1 - Q^*Q)^{1/2}\,P(\mathscr{B})\,P(\mathscr{A}^\perp).$$

In view of (1,1) it suffices to prove that $P(\mathscr{B})\,P(\mathscr{A}^\perp)$ intertwines D^2 and $1 - Q^*Q$. Indeed, we have

$$\begin{aligned} P(\mathscr{B})\,P(\mathscr{A}^\perp)\,D^2 = P(\mathscr{B})\,D^2 = P(\mathscr{B})\,P(\mathscr{A}^\perp)\,P(\mathscr{B})\,P(\mathscr{A}^\perp) = \\ = P(\mathscr{B})\,(1 - P(\mathscr{A}))\,P(\mathscr{B})\,P(\mathscr{A}^\perp) = \\ = (1 - P(\mathscr{B})\,P(\mathscr{A}) \mid \mathscr{B})\,P(\mathscr{B})\,P(\mathscr{A}^\perp) = (1 - Q^*Q)\,P(\mathscr{B})\,P(\mathscr{A}^\perp). \end{aligned}$$

The proof is complete.

(2,3) *Suppose further we are given a mapping* $U \in B(\mathscr{K})$ *such that both* $\mathscr{A}$ *and* $\mathscr{B}$ *are reducing subspaces for U. Then U is unitarily equivalent to*

$$\tilde{U} = (U \mid \mathscr{A}) \oplus (U \mid \mathscr{B} \ominus (\mathscr{B} \cap \mathscr{A})),$$

more precisely, $\Phi U = \tilde{U}\Phi$.

Proof. It suffices to prove that $VUz = UVz$ for all $z \in \mathscr{A}^\perp$. We shall prove this for elements of the form $Da^\perp$, $a^\perp \in \mathscr{A}^\perp$. If $a^\perp \in \mathscr{A}^\perp$ we have, using the fact that $P(\mathscr{A})$ and $P(\mathscr{B})$ commute with U and, consequently, U commutes with D

$$VUDa^\perp = VDUa^\perp = P(\mathscr{B})\, Ua^\perp = U\, P(\mathscr{B})\, a^\perp = UVDa^\perp .$$

At this point we make the following further assumptions:

Suppose we are given two subspaces $\mathscr{A}_+ \subset \mathscr{A}$ and $\mathscr{B}_- \subset \mathscr{K}$ such that $\mathscr{A}_+ \perp \mathscr{B}_-$. Denote by $\mathscr{H}$ the complement

$$\mathscr{H} = \mathscr{K} \ominus (\mathscr{B}_- \oplus \mathscr{A}_+) .$$

If we set $\mathscr{A}_- = \mathscr{A} \ominus \mathscr{A}_+$ then $\mathscr{H} \subset \mathscr{A}_- \oplus \mathscr{A}^\perp$ so that $P(\mathscr{A})\, h = P(\mathscr{A}_-)\, h$ for all $h \in \mathscr{H}$.

Since $\mathscr{H}^\perp = \mathscr{B}_- \oplus \mathscr{A}_+$ we have

$$P(\mathscr{A}_- \oplus \mathscr{A}^\perp)\, P(\mathscr{H}^\perp)\, \mathscr{B}_-^\perp = P(\mathscr{A}_- \oplus \mathscr{A}^\perp)\, P(\mathscr{B}_- \oplus \mathscr{A}_+)\, \mathscr{B}_-^\perp =$$
$$= P(\mathscr{A}_- + \mathscr{A}^\perp)\, P(\mathscr{A}_+)\, \mathscr{B}_-^\perp = (0) .$$

If we add the assumption $U^*\mathscr{B}_- \subset \mathscr{B}_-$ then $U\mathscr{B}_-^\perp \subset \mathscr{B}_-^\perp$ and $U\mathscr{H} \subset U\mathscr{B}_-^\perp \subset \mathscr{B}_-^\perp$. Using this and the relation $P(\mathscr{A}_- \oplus \mathscr{A}^\perp)\, P(\mathscr{H}^\perp)\, \mathscr{B}_-^\perp = (0)$ we obtain

$$P(\mathscr{A}_- \oplus \mathscr{A}^\perp)\, P(\mathscr{H}^\perp)\, U\mathscr{H} = (0),$$

or equivalently,

$$P(\mathscr{A}_-)\, P(\mathscr{H}^\perp)\, U\mathscr{H} = (0) \quad \text{and} \quad P(\mathscr{A}^\perp)\, P(\mathscr{H}^\perp)\, U\mathscr{H} = (0) .$$

(2,4) *Let* $\mathscr{A}_+ \subset \mathscr{A}$, $\mathscr{B}_- \subset \mathscr{K}$ *be two subspaces such that* $U\mathscr{A}_+ \subset \mathscr{A}_+$, $U^*\mathscr{B}_- \subset \mathscr{B}_-$ *and* $\mathscr{B}_- \perp \mathscr{A}_+$. *Then the compression T of U to* $\mathscr{H} = \mathscr{K} \ominus (\mathscr{B}_- \oplus \mathscr{A}_+)$ *is unitarily equivalent to*

$$\tilde{T} = [(U^* \mid (\mathscr{A} \ominus \mathscr{A}_+))^* \oplus (U \mid \mathscr{B} \ominus (\mathscr{B} \cap \mathscr{A}))] \mid \Phi\mathscr{H} .$$

Proof. It will be sufficient to prove that $\Phi Th = \tilde{T}\Phi h$ for all $h \in \mathscr{H}$.

Since $\mathscr{A}_- = \mathscr{A} \ominus \mathscr{A}_+$ then $U^*\mathscr{A}_- \subset \mathscr{A}_-$ and $U^*\, P(\mathscr{A}_-) = P(\mathscr{A}_-)\, U^*\, P(\mathscr{A}_-)$ so that $P(\mathscr{A}_-)\, U = P(\mathscr{A}_-)\, U\, P(\mathscr{A}_-)$. Since $(U^* \mid \mathscr{A}_-)^* = P(\mathscr{A}_-)\, U \mid \mathscr{A}_-$ we have $P(\mathscr{A}_-)\, U\, P(\mathscr{A}_-) = (U^* \mid \mathscr{A}_-)^*\, P(\mathscr{A}_-)$.

Using this fact, the relations $P(\mathscr{A}_-)\, P(\mathscr{H}^\perp)\, U\mathscr{H} = (0)$, $P(\mathscr{A}^\perp)\, P(\mathscr{H}^\perp)\, U\mathscr{H} = (0)$ and the fact that $UV = VU$ on $\mathscr{A}^\perp$ we obtain, for $h \in \mathscr{H}$,

$$\Phi Th = P(\mathscr{A}_-)\, Th \oplus V P(\mathscr{A}^\perp)\, Th =$$
$$= P(\mathscr{A}_-)\, P(\mathscr{H})\, Uh \oplus V P(\mathscr{A}^\perp)\, P(\mathscr{H})\, Uh =$$

$$= P(\mathscr{A}_-)\, Uh \oplus V P(\mathscr{A}^\perp)\, Uh =$$
$$= P(\mathscr{A}_-)\, U\, P(\mathscr{A}_-)\, h \oplus VU\, P(\mathscr{A}^\perp)\, h =$$
$$= (U^* \mid \mathscr{A}_-)^*\, P(\mathscr{A}_-)\, h \oplus UV P(\mathscr{A}^\perp)\, h = \tilde{T}\Phi h\,.$$

(2,5) *Suppose* $\mathscr{B}_- \subset \mathscr{B}$. *Then*

$$\Phi\mathscr{H} = (\mathscr{A}_- \oplus (\mathscr{B} \ominus (\mathscr{A} \cap \mathscr{B}))) \ominus$$
$$\ominus (Qb_- \oplus (1_{\mathscr{B}} - Q^*Q)^{1/2}\, b_-\,, \quad b_- \in \mathscr{B}_-)\,.$$

Proof. Since $\mathscr{H} = (\mathscr{A}_+ \oplus \mathscr{B}_-)^\perp = \mathscr{A}_+^\perp \cap \mathscr{B}_-^\perp = (\mathscr{A}_- \oplus \mathscr{A}^\perp) \ominus \mathscr{B}_-$ we have

$$\Phi\mathscr{H} = \Phi(\mathscr{A}_- \oplus \mathscr{A}^\perp) \ominus \Phi\mathscr{B}_- = (\mathscr{A}_- \oplus (\mathscr{B} \ominus (\mathscr{A} \cap \mathscr{B}))) \ominus \Phi\mathscr{B}_-\,.$$

Since $\mathscr{B}_- \subset \mathscr{A}_- \oplus \mathscr{A}^\perp$ we have $P(\mathscr{A})\, \mathscr{B}_- \subset \mathscr{A}_-$ so that Q maps $\mathscr{B}_-$ into $\mathscr{A}_-$. Using the additional hypothesis $\mathscr{B}_- \subset \mathscr{B}$ and (2,2) we obtain the desired conclusion.

It is natural to ask whether it would be possible to find an expression of the isometry Φ in terms of operators acting on the space $\mathscr{H}$ only. This is indeed possible: to this end we introduce two operators related to the compression of U to $\mathscr{H}$ in a natural manner as follows.

(2,6) *Define the operators*

$$A \;\,= (P(\mathscr{H})\, P(\mathscr{A}^\perp) \mid \mathscr{H})^{1/2}\,,$$
$$A_* = (P(\mathscr{H})\, P(\mathscr{B}^\perp) \mid \mathscr{H})^{1/2}$$

and

$$B = (1 - AA_*^2A)^{1/2}\,.$$

Then

(1) $$P(\mathscr{H})\, D \mid \mathscr{H} = ABA\,,$$

(2) $$|Ah - BAk| = |P(\mathscr{A}^\perp)\, h - Dk| \quad \textit{for all} \quad h, k \in \mathscr{H}\,,$$

(3) *in particular,* $|BAk| = |Dk|$ *for* $k \in \mathscr{H}$.

Proof. Let us observe first that the inclusion $\mathscr{B}_- \subset \mathscr{B}$ implies

$$P(\mathscr{B}^\perp)\, P(\mathscr{A}^\perp) = P(\mathscr{B}^\perp)\, P(\mathscr{H})\, P(\mathscr{A}^\perp)\,.$$

Indeed,

$$P(\mathscr{B}^\perp)\, P\,(\mathscr{H})\, P(\mathscr{A}^\perp) = P(\mathscr{B}^\perp)\,(1 - P(\mathscr{B}_- \oplus \mathscr{A}_+))\, P(\mathscr{A}^\perp) =$$
$$= P(\mathscr{B}^\perp)\, P(\mathscr{A}^\perp) - P(\mathscr{B}^\perp)\,(P(\mathscr{B}_-) + P(\mathscr{A}_+))\, P(\mathscr{A}^\perp) = P(\mathscr{B}^\perp)\, P(\mathscr{A}^\perp)\,.$$

Using this relation, we prove now, by induction, that

$$P(\mathscr{H})\, P(\mathscr{A}^\perp)\, D^{2n}\, P(\mathscr{A}^\perp)\, P(\mathscr{H}) = AB^{2n}A\, P(\mathscr{H}) \quad \text{for} \quad n = 0, 1, 2, \ldots\,.$$

For $n = 0$ the above relation follows immediately from the definition of A. The induction is based on the following representation

$$D^2 P(\mathscr{A}^\perp) P(\mathscr{H}) = P(\mathscr{A}^\perp)(1 - P(\mathscr{B}^\perp)) P(\mathscr{A}^\perp) P(\mathscr{H}) =$$
$$= P(\mathscr{A}^\perp) P(\mathscr{H}) - P(\mathscr{A}^\perp) P(\mathscr{H}) P(\mathscr{B}^\perp) P(\mathscr{H}) P(\mathscr{A}^\perp) P(\mathscr{H}) =$$
$$= P(\mathscr{A}^\perp) P(\mathscr{H})(1 - A_*^2 A^2) P(\mathscr{H}).$$

Now assume that our relation holds for n. Then

$$P(\mathscr{H}) P(\mathscr{A}^\perp) D^{2n} D^2 P(\mathscr{A}^\perp) P(\mathscr{H}) =$$
$$= P(\mathscr{H}) P(\mathscr{A}^\perp) D^{2n} P(\mathscr{A}^\perp) P(\mathscr{H})(1 - A_*^2 A^2) P(\mathscr{H}) =$$
$$= AB^{2n} A P(\mathscr{H})(1 - A_*^2 A^2) P(\mathscr{H}) =$$
$$= AB^{2n}(1 - AA_*^2 A) A P(\mathscr{H}) = AB^{2n+2} A P(\mathscr{H}).$$

In order to complete the proof of (1), it suffices to apply (1,1) and to observe that $D = P(\mathscr{A}^\perp) D P(\mathscr{A}^\perp)$; this, however, is a consequence of the identity $D = D P(\mathscr{A}^\perp)$ proved in (2,1).

To prove (2), consider an arbitrary pair $h, k \in \mathscr{H}$. Then

$$|Ah - BAk|^2 = |Ah|^2 + |BAk|^2 - 2 \operatorname{Re}(Ah, BAk) =$$
$$= (A^2 h, h) + (AB^2 Ak, k) - 2 \operatorname{Re}(h, ABAk) =$$
$$= (P(\mathscr{A}^\perp) h, h) + (D^2 k, k) - 2 \operatorname{Re}(h, Dk) =$$
$$= |P(\mathscr{A}^\perp) h - Dk|^2.$$

(2,7) *The operator B is injective and $BA\mathscr{H}$ is a dense subset of $(A\mathscr{H})^-$.*

Proof. Since $\operatorname{Ker} B = \operatorname{Ker} B^2$ we have $Bh = 0$ if and only if $h = AA_*^2 Ah$. Since both A and A_* are contractions we have

$$|h| = |AA_*^2 Ah| \leqq |Ah| = |P(\mathscr{A}^\perp) h|$$

so that $h = P(\mathscr{A}^\perp) h = P(\mathscr{H}) P(\mathscr{A}^\perp) h = A^2 h$. Hence $(1 - A) h = (1 + A)^{-1} . (1 - A^2) h = 0$, in other words $h = Ah$ and, consequently, $h = AA_*^2 h$. Again, this implies $|h| \leqq |A_* h| = |P(\mathscr{B}^\perp) h|$ so that $h \in \mathscr{B}^\perp$. Consequently, $h \in \mathscr{B}^\perp \cap \mathscr{A}^\perp = (0)$.

To complete the proof it is sufficient to show that B maps $A\mathscr{H}$ into $(A\mathscr{H})^-$. Since $B = B^*$ the last assertion is equivalent to the inclusion $B \operatorname{Ker} A \subseteq \operatorname{Ker} A$. Consider an $h \in \mathscr{H}$ such that $Ah = 0$. Then $B^2 h = (1 - AA_*^2 A) h = h$, or equivalently $(1 - B^2) h = 0$. Then $(1 - B) h = (1 + B)^{-1}(1 - B^2) h = 0$ so that $Bh = h \in \operatorname{Ker} A$.

(2,8) *The restriction of Φ onto $\mathscr{H}$ can be expressed as follows*

$$\Phi h = P(\mathscr{A}) h \oplus \tilde{V} Ah$$

where $\tilde{V}$ is the isometry on $(A\mathscr{H})^-$ defined on the dense subset $BA\mathscr{H}$ by the formula

$$\tilde{V} BAh = P(\mathscr{B}) P(\mathscr{A}^\perp) h.$$

Proof. Since $|BAh| = |Dh|$ for $h \in \mathscr{H}$ by (3) of Proposition (2,6) we have, for every $h \in \mathscr{H}$,

$$|BAh| = |Dh| = |VDh| = |P(\mathscr{B})\, P(\mathscr{A}^\perp)\, h|$$

so that there exists an isometric mapping $\tilde{V}$ defined for elements of the form BAh by the relation

$$\tilde{V} BAh = P(\mathscr{B})\, P(\mathscr{A}^\perp)\, h = VDh\,.$$

Since $BA\mathscr{H}$ is dense in $(A\mathscr{H})^-$ the mapping $\tilde{V}$ may be extended by density to the whole of $(A\mathscr{H})^-$; we use the same symbol $\tilde{V}$ for this extension.

Let $h \in \mathscr{H}$ and $Ah = \lim BA\, k_n$ for a suitable sequence (k_n). According to (2) of (2,6) we have $P(\mathscr{A}^\perp)\, h = \lim Dk_n$ so that

$$\tilde{V}Ah = \lim \tilde{V}BAk_n = \lim VDk_n = V P(\mathscr{A}^\perp)\, h$$

and

$$\Phi h = P(\mathscr{A})\, h \oplus V P(\mathscr{A}^\perp)\, h = P(\mathscr{A})\, h \oplus \tilde{V}Ah\,.$$

3. THE PARTICULAR CASE OF UNITARY DILATIONS

The results of the preceding section may be considerably sharpened if additional information about the structure of U on the spaces $\mathscr{A}$ and $\mathscr{B}$ is available. This is the case when U is the unitary dilation of a contraction T.

Let $T \in B(\mathscr{H})$ be a contraction and let $U \in B(\mathscr{K})$ be its minimal unitary dilation so that

$$T^n = P(\mathscr{H})\, U^n \mid \mathscr{H} \quad \text{for} \quad n \geqq 0$$

$$\mathscr{K} = \operatorname*{span}_{n \in Z} U^n \mathscr{H}\,.$$

If $\mathscr{L}$ and $\mathscr{L}^*$ stand for $((U - T)\, \mathscr{H})^-$ and $((U^* - T^*)\, \mathscr{H})^-$ respectively then

$$\mathscr{K} = \ldots \oplus U^{*2}\mathscr{L}^* \oplus U^*\mathscr{L}^* \oplus \mathscr{L}^* \oplus \mathscr{H} \oplus \mathscr{L} \oplus U\mathscr{L} \oplus U^2\mathscr{L} \oplus \ldots\,.$$

If we denote by $\mathscr{A}$ and $\mathscr{B}$ the spaces $\mathscr{A} = \bigoplus_{-\infty}^{\infty} U^n\mathscr{L}$, $\mathscr{B} = \bigoplus_{-\infty}^{\infty} U^n\mathscr{L}^*$ then $\mathscr{A}^\perp \cap \cap\, \mathscr{B}^\perp \subset \mathscr{H}$.

Consider a vector $h \in \mathscr{A}^\perp \cap \mathscr{B}^\perp$; since $\mathscr{A}^\perp \cap \mathscr{B}^\perp$ is a reducing subspace for U all vectors of the form $U^n h$, $n \in Z$ belong to $\mathscr{A}^\perp \cap \mathscr{B}^\perp \subset \mathscr{H}$. This means that $T^n h = U^n h$ and $T^{*n} h = U^{*n} h$ for all $n \geqq 0$. In other words,

$$\mathscr{A}^\perp \cap \mathscr{B}^\perp \subset \{h \in \mathscr{H};\ |T^n h| = |T^{*n} h| = |h| \text{ for all } n \in Z\}\,.$$

If T is completely nonunitary, i.e. there is no nonzero h for which $|T^n h| = |T^{*n} h| = |h|$ for all $n \geqq 0$, then $\mathscr{A}^\perp \cap \mathscr{B}^\perp = (0)$ and we may apply the results of Section 2.

If $h \in \mathscr{H}$ then, for $n \geqq 0$,

$$h = U^*(U - T)\, h + U^{*2}(U - T)\, Th + \ldots + U^{*n}(U - T)\, T^{n-1}h + U^{*n}T^n h\,.$$

Since the sums $\sum_0^m |U^{*n}(U - T) T^n h|^2 = |h|^2 - |T^{m+1}h|^2 \leqq |h|^2$ are bounded it follows that $\sum_0^\infty U^{*n+1}(U - T) T^n h$ converges, the limit $\lim U^{*n}T^n h$ exists,

$$P(\mathscr{A}) h = \sum_0^\infty U^{*n+1}(U - T) T^n h$$

and

$$P(\mathscr{A}^\perp) h = \lim U^{*n}T^n h .$$

Analogously

$$P(\mathscr{B}) h = \sum_0^\infty U^{n+1}(U^* - T^*) T^{*n} h$$

and

$$P(\mathscr{B}^\perp) h = \lim U^n T^{*n} h .$$

Moreover,

$$A^2 h = P(\mathscr{H}) P(\mathscr{A}^\perp) h = \lim T^{*n}T^n h$$

and

$$A_*^2 h = \lim T^n T^{*n} h$$

for all $h \in \mathscr{H}$.

Let us compute also $P(\mathscr{B}) P(\mathscr{A}^\perp) h$. We have

$$\begin{aligned} P(\mathscr{B}) P(\mathscr{A}^\perp) h &= P(\mathscr{B}) P(\mathscr{H}) P(\mathscr{A}^\perp) h + P(\mathscr{B}) P(\mathscr{H}^\perp) P(\mathscr{A}^\perp) h = \\ &= P(\mathscr{B}) A^2 h + P(\mathscr{B}) P(\mathscr{H}^\perp) P(\mathscr{A}^\perp) h . \end{aligned}$$

The first summand equals

$$\sum_0^\infty U^{n+1}(U^* - T^*) T^{*n} A^2 h .$$

Now, let us compute the second summand. Since $P(\mathscr{A}^\perp) h = \lim U^{*n}T^n h$ we shall first decompose $U^{*n}T^n h$:

$$\begin{aligned} U^{*n}T^n h = T^{*n}T^n h &+ (U^* - T^*) T^{*n-1}T^n h + \\ + U^*(U^* - T^*) T^{*n-2}T^n h &+ \ldots + U^{*n-1}(U^* - T^*) T^n h . \end{aligned}$$

It follows that, for $n > k$,

$$\begin{aligned} P(U^{*k}\mathscr{L}^*) U^{*n}T^n h &= U^{*k}(U^* - T^*) T^{*n-1-k}T^n h = \\ &= U^{*k}(U^* - T^*) (T^{*n-1-k}T^{n-1-k}) T^{k+1} h \end{aligned}$$

whence

$$P(U^{*k}\mathscr{L}^*) P(\mathscr{A}^\perp) h = \lim_n P(U^{*k}\mathscr{L}^*) U^{*n}T^n h = U^{*k}(U^* - T^*) A^2 T^{k+1} h .$$

Thus

$$P(\mathscr{A}^\perp) h = A^2 h + \sum_0^\infty U^{*k}(U^* - T^*) A^2 T^{k+1} h$$

so that

$$P(\mathscr{B})\, P(\mathscr{H}^{\perp})\, P(\mathscr{A}^{\perp})\, h = \sum_{0}^{\infty} U^{*k}(U^* - T^*)\, A^2 T^{k+1} h .$$

Adding the two summands we obtain

$$P(\mathscr{B})\, P(\mathscr{A}^{\perp})\, h = \sum_{0}^{\infty} U^{n+1}(U^* - T^*)\, T^{*n} A^2 h +$$

$$+ \sum_{0}^{\infty} U^{*k}(U^* - T^*)\, A^2 T^{k+1} h = \sum_{-\infty}^{+\infty} U^{*n-1}(U^* - T^*)\, A_n h$$

where

$$A_n = A^2 T^n \quad \text{for} \quad n \geqq 1$$

$$A_n = T^{*n} A^2 \quad \text{for} \quad n \leqq 0 .$$

Finally, let us make the following identification:

$$J_1 \colon \mathscr{A} \to \bigoplus_{-\infty}^{\infty} \mathscr{D} \quad \text{and} \quad J_2 \colon \mathscr{B} \to \bigoplus_{-\infty}^{\infty} \mathscr{D}_*$$

which are defined by formulae

$$J_1\big(\sum_{-\infty}^{\infty} U^{*n}(U - T)\, h_n\big) = \bigoplus_{-\infty}^{\infty} D h_n , \quad J_2\big(\sum_{-\infty}^{\infty} U^{*n-1}(U^* - T^*)\, h_n\big) = \bigoplus_{-\infty}^{\infty} D_* h_n .$$

Here, as usual, $D = (1 - T^*T)^{1/2}$, $\mathscr{D} = (\text{Range } \mathscr{D})^-$ $D_* = (1 - TT^*)^1{}_/{}^2$ and $\mathscr{D} =$ $= (\text{Range } D_*)^-$.
In particular,

$$J_1\, P(\mathscr{A})\, h = \bigoplus_{0}^{\infty} D T^n h \quad \text{for all} \quad h \in \mathscr{H} ,$$

$$J_2 \tilde{V} B A h = J_2\, P(\mathscr{B})\, P(\mathscr{A}^{\perp})\, h = J_2\big(\sum_{-\infty}^{\infty} U^{*n-1}(U^* - T^*)\, A_n h\big) = \bigoplus_{-\infty}^{\infty} D_* A_n h .$$

If we set $\tilde{V}_2 = J_2 \tilde{V}$ and define Ψ by the formula

$$\Psi h = J_1\, P(\mathscr{A})\, h \oplus \tilde{V}_2 A h$$

then the preceding considerations yield the following

(3,1) **Theorem.**

(1) *The mapping $\tilde{V}_2$ from $(A\mathscr{H})^-$ into $\bigoplus_{-\infty}^{\infty} \mathscr{D}_*$ which is defined on $BA\mathscr{H}$ by the formula*

$$\tilde{V}_2 B A h = \bigoplus_{-\infty}^{\infty} D_* A_n h$$

where

$$A_n = A^2 T^n \quad \text{for} \quad n \geqq 1$$
$$A_n = T^{*n} A^2 \quad \text{for} \quad n \leqq 0$$

is an isometry.

(2) *The mapping* Ψ *from* $\mathscr{H}$ *into* $(\bigoplus_0^\infty \mathscr{D}) \oplus (\bigoplus_{-\infty}^\infty \mathscr{D}_*)$ *defined by the formula*

$$\Psi h = (\bigoplus_0^\infty D T^n h) \oplus \tilde{V}_2 A h$$

is an isometry and

$$\Psi T = (Z \oplus U) \Psi$$

where Z *is the backward shift operator on* $(\bigoplus_0^\infty \mathscr{D})$, U *the bilateral shift operator on* $(\bigoplus_{-\infty}^\infty \mathscr{D}_*)$ *defined by* $(U(h_k)_{-\infty}^\infty)_n = h_{n+1}$.

4. THE CONSTRUCTION OF V. I. VASJUNIN

In this section we intend to use the general results to describe an abstract analogon of the construction used by V. I. Vasjunin to set up a functional model for completely nonunitary contractions.

According to Proposition (2,2) the isometry Φ may be expressed, for elements $b \in \mathscr{B}$, in the form

$$\Phi b = Qb \oplus (1_{\mathscr{B}} - Q^* Q)^{1/2} b$$

where $Q = P(\mathscr{A}) \mid \mathscr{B}$ so that $1_{\mathscr{B}} - Q^* Q = (1 - P(\mathscr{B}) P(\mathscr{A})) \mid \mathscr{B}$. If we set

$$W = (1 - P(\mathscr{B}) P(\mathscr{A}) P(\mathscr{B}))^{1/2}$$

then $1 - W^2 = 0$ on $\mathscr{B}^\perp$ so that $1 - W = (1 + W)^{-1} (1 - W^2)$ is zero on $\mathscr{B}^\perp$ as well. In other words, W maps $\mathscr{B}^\perp$ into itself and, consequently, $W\mathscr{B} \subset \mathscr{B}$. It follows from the uniqueness of nonnegative square roots of nonnegative operators that

$$(W^2 \mid \mathscr{B})^{1/2} = W \mid \mathscr{B} .$$

Thus

$$W \mid \mathscr{B} = ((1 - P(\mathscr{B}) P(\mathscr{A}) P(\mathscr{B})) \mid \mathscr{B})^{1/2} =$$
$$= ((1 - P(\mathscr{B}) P(\mathscr{A})) \mid \mathscr{B})^{1/2}$$

so that

$$\Phi b = P(\mathscr{A})\, b \oplus Wb \quad \text{for} \quad b \in \mathscr{B} .$$

Now $\mathscr{A} + \mathscr{B}$ is dense in $\mathscr{K}$; for elements of the form $a + b$ with $a \in \mathscr{A}$, $b \in \mathscr{B}$ it follows that

$$\Phi(a + b) = \Phi a + \Phi b = (P(\mathscr{A})\, a \oplus 0) + (P(\mathscr{A})\, b \oplus Wb) =$$
$$= P(\mathscr{A})(a + b) \oplus Wb .$$

(5,1) **Proposition.** *Let $T \in B(\mathscr{H})$ be a contraction. Then the limit $T^{*n}T^n h$ exists for all $h \in \mathscr{H}$. Denote by A the selfadjoint square root of this limit, i.e. $A^2 h = \lim T^{*n}T^n h$. The operator A is a contraction which satisfies*

1° $|Ah| = \lim |T^n h|$ *for all* $h \in \mathscr{H}$

2° $A^2 = T^* A^2 T$, *or equivalently,* $|Ah| = |ATh|$ *for all* $h \in \mathscr{H}$

3° $\operatorname{Ker} A = \{h \in \mathscr{H}: T^n h \to 0\}$,
$\operatorname{Ker}(1 - A) = \{h: |Ah| = |h|\} = \{h \in \mathscr{H}; |T^n h| = |h| \text{ for } n \geqq 0\}$

4° $(1 - A^2)\, T^n h \to 0$ *for all* $h \in \mathscr{H}$

5° $|A^2 T^n h| \to |Ah|$ *for all* $h \in \mathscr{H}$.

Proof. Since $|T| \leqq 1$ the sequence $|T^n h|$ is nonincreasing for each $h \in \mathscr{H}$ and, accordingly, has a limit. If $n \geqq m$ then $1 \geqq T^{*m}T^m - T^{*n}T^n \geqq 0$ so that $(T^{*m}T^m - T^{*n}T^n)^{1/2} \leqq 1$ as well. Given an $h \in \mathscr{H}$, $n \geqq m$, we have

$$|(T^{*m}T^m - T^{*n}T^n)\, h|^2 \leqq |(T^{*m}T^m - T^{*n}T^n)^{1/2}|^2\, |(T^{*m}T^m - T^{*n}T^n)^{1/2}\, h|^2 \leqq$$
$$\leqq ((T^{*m}T^m - T^{*n}T^n)\, h, h) = |T^m h| - |T^n h|\,.$$

It follows that the sequence $(T^{*n}T^n)\, h$ is Cauchy and the limit exists.

Properties 1° and 2° follow directly from the definition of A.

If $|Ah|^2 = |h|^2$ then $((1 - A^2)\, h, h) = 0$ and this is equivalent to $(1 - A^2)\, h = 0$. Then

$$(1 - A)\, h = (1 + A)^{-1}\, (1 - A^2)\, h = 0$$

as well. This proves the inclusion $\{h \in \mathscr{H}: |Ah| = |h|\} \subset \operatorname{Ker}(1 - A)$; the other inclusion is obvious.

To prove 4° and 5° let us consider an $h \in \mathscr{H}$. The operator $1 - A^2$ is nonnegative and

$$|(1 - A^2)^{1/2}\, T^n h|^2 = ((1 - A^2)\, T^n h, T^n h) =$$
$$= (T^{*n}T^n h, h) - (T^{*n}A^2 T^n h, h) =$$
$$= (T^{*n}T^n h, h) - (A^2 h, h) \to 0\,.$$

Consequently, $(1 - A^2)\, T^n h = (1 - A^2)^{1/2}\, (1 - A^2)^{1/2}\, T^n h \to 0$ as well. In particular $|T^n h| - |A^2 T^n h| \to 0$; together with $|T^n h| \to |Ah|$ this yields 5°.

(5,2) **Proposition.** *Denote by A_*^2 the strong limit of the sequence $T^n T^{*n}$. The operator $1 - AA_*^2 A$ is nonnegative; write B for $(1 - AA_*^2 A)^{1/2}$. Then*

6° $\operatorname{Ker} B = \operatorname{Ker}(1 - A) \cap \operatorname{Ker}(1 - A_*) = \{h \in \mathscr{H}: |T^n h| = |T^{*n} h| = |h|$
for $n \geqq 0\}$

7° $BA\mathscr{H} \subset (A\mathscr{H})^-$.

Moreover, if T is completely nonunitary, then $\operatorname{Ker} B = (0)$ *so that $BA\mathscr{H}$ is a dense subset in $(A\mathscr{H})^-$.*

8° $T^*AB^2AT = AB^2A$, $T^*ABA\,T = ABA$

9° $|Ah - BAk| = |ATh - BATk|$ *for all* $h, k \in \mathscr{H}$.

Proof. If $Ah = A_*h = h$ for some $h \in \mathscr{H}$ then $B^2h = 0$ and, consequently, $Bh = 0$. On the other hand, if $B^2h = 0$ then $h = AA_*^2Ah$ so that $|h| \leqq |Ah| \leqq |h|$ which, according to 3°, gives $h = Ah$. Now $h = AA_*^2h$ so that $|h| \leqq |A_*h| \leqq |h|$ whence $h = A_*h$ as well.

The inclusion $BA\mathscr{H} \subset (A\mathscr{H})^-$ is equivalent to the inclusion $B \operatorname{Ker} A \subseteq \operatorname{Ker} A$. If $h \in \mathscr{H}$ is such that $Ah = 0$ then $B^2h = (1 - AA_*^2A)\,h = h$, whence $(1 - B)\,h = (1 + B)^{-1}(1 - B^2)\,h = 0$. In other words, $Bh = h \in \operatorname{Ker} A$ which proves the inclusion $B \operatorname{Ker} A \subset \operatorname{Ker} A$. Moreover, if T is completely nonunitary, i.e. $\operatorname{Ker} B = (0)$, then B maps $A\mathscr{H}$ which is dense in $(A\mathscr{H})^-$ onto a dense subset of $(A\mathscr{H})^-$.

For the proof of 8°, it suffices, according to (1,1), to prove the relations $T^*AB^{2n}AT = AB^{2n}A$ for $n \geqq 0$. For $n = 0$, this relation reduces to $T^*A^2T = A^2$. Suppose now that $T^*AB^{2n}AT = AB^{2n}A$. Then

$$\begin{aligned}T^*AB^{2n+2}AT &= T^*AB^{2n}B^2AT = T^*AB^{2n}A(1 - A_*^2A^2)\,T = \\ &= T^*AB^{2n}A(T - TA_*^2T^*A^2T) = T^*AB^{2n}A(T - TA_*^2A^2) = \\ &= T^*AB^{2n}AT(1 - A_*^2A^2) = AB^{2n}A(1 - A_*^2A^2) = AB^{2n+2}A\,.\end{aligned}$$

Finally, let us compute $|Ah - BAk|$ for $h, k \in \mathscr{H}$:

$$\begin{aligned}|Ah - BAk|^2 &= |Ah|^2 + |BAk|^2 - 2\operatorname{Re}(Ah, BAk) = \\ &= (A^2h, h) + (AB^2Ak, k) - 2\operatorname{Re}(h, ABAk) = \\ &= (T^*A^2Th, h) + (T^*AB^2ATk, k) - 2\operatorname{Re}(h, T^*ABATk) = \\ &= |ATh|^2 + |BATk|^2 - 2\operatorname{Re}(ATh, BATk) = \\ &= |ATh - BATk|^2\,.\end{aligned}$$

Notation. As usual, denote by $D = (1 - T^*T)^{1/2}$, $D_* = (1 - TT^*)^{1/2}$, $\mathscr{D} = (\operatorname{Range} D)^-$, $\mathscr{D}_* = (\operatorname{Range} D_*)^-$. Denote by $\mathscr{H}_1 = \bigoplus_0^\infty \mathscr{D}$, $\mathscr{H}_2 = \bigoplus_{-\infty}^\infty \mathscr{D}_*$ nad by Z the backward shift operator on $\mathscr{H}_1$,

$$Z(h_0, h_1, \ldots) = (h_1, h_2, \ldots)\,.$$

Further, denote by U the bilateral shift operator on $\mathscr{H}_2$ such that, for $h = (h_n)_{-\infty}^\infty$,

$$(Uh)_n = h_{n+1} \quad \text{for} \quad n \in Z\,.$$

(5,3) **Proposition.** *Let T be a completely nonunitary contraction.*

1. *The operator V_1 from $\mathscr{H}$ into $\mathscr{H}_1$ defined by*

$$V_1h = \bigoplus_0^\infty DT^nh$$

satisfies

1° $$|V_1 h|^2 = |h|^2 - |Ah|^2,$$

2° $$V_1 T = Z V_1 .$$

2. *The operator* V_2 *from* $(A\mathscr{H})^-$ *into* $\mathscr{H}_2$ *defined on the dense subset* $BA\mathscr{H}$ *of* $(A\mathscr{H})^-$ *by*

$$V_2 BAh = \bigoplus_{-\infty}^{\infty} D_* A_n h$$

where

$$A_n = \begin{cases} A^2 T^n & \textit{for} \quad n \geqq 1 \\ T^{*n} A^2 & \textit{for} \quad n \leqq 0 \end{cases}$$

is an isometry and

$$V_2 AT = U V_2 A .$$

Proof. Take an $h \in \mathscr{H}$. Then

$$|V_1 h|^2 = \sum_0^{\infty} |DT^n h|^2 = \sum_0^{\infty} |T^n h|^2 - |T^{n+1} h|^2 = \\ = |h|^2 - \lim |T^n h|^2 = |h|^2 - |Ah|^2$$

and

$$V_1 Th = \bigoplus_0^{\infty} DT^{n+1} h = Z(\bigoplus_0^{\infty} DT^n h) = Z V_1 .$$

To show that the operator V_2 is well defined let us compute first the norm $|\bigoplus_{-\infty}^{\infty} D_* A_n h|^2$. Using property 5° of (5,1) we get

$$|\bigoplus_{-\infty}^{\infty} D_* A_n h|^2 = \sum_1^{\infty} |D_* A^2 T^n h|^2 + \sum_0^{\infty} |D_* T^{*n} A^2 h|^2 = \\ = \sum_1^{\infty} (|A^2 T^n h|^2 - |T^* A^2 T^n h|^2) + \sum_0^{\infty} (|T^{*n} A^2 h|^2 - |T^{*n+1} A^2 h|^2) = \\ = (\lim |A^2 T^n h|^2 - |A^2 h|^2) + (|A^2 h|^2 - \lim |T^{*n+1} A^2 h|) = \\ = |Ah|^2 - |A_* A^2 h|^2 = (Ah, Ah) - (AA_*^2 A^2 h, Ah) = \\ = ((1 - AA_*^2 A)\, Ah, Ah) = |BAh|^2 .$$

This shows that V_2 is well defined and its extension to the whole $(A\mathscr{H})^-$ is an isometry.

Further,

$$(V_2 BATh)_n = \begin{Bmatrix} D_* A^2 T^{n+1} h & \text{for} \quad n \geqq 1 \\ D_* T^{*-n} A^2 Th = D_* T^{*-n+1} A^2 h & \text{for} \quad n \leqq 0 \end{Bmatrix} = (U V_2 BAh)_n .$$

In other words, $V_2BAT = UV_2BA$. Now, let $Ah = \lim BAh_n$. Using property 9° we also have $ATh = \lim BATh_n$ and

$$V_2ATh = \lim V_2BATh_n = \lim UV_2BAh_n = UV_2 \lim BAh_n = UV_2Ah .$$

The proof is complete.

(5,4) **Theorem.** *The operator* $\Psi: \mathscr{H} \to \mathscr{H}_1 \oplus \mathscr{H}_2$ *defined by*

$$\Psi h = V_1 h \oplus V_2 Ah$$

is an isometry and

$$\Psi T = (Z \oplus U) \Psi .$$

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Souhrn

ABSTRAKTNÍ MODEL PRO KOMPRESE

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Vyšetřuje se geometrie Hilbertova prostoru, který je generován dvěma uzavřenými podporostory. Cílem práce je vybudování abstraktní analogie funkcionálního modelu pro kontrakce v teorii unitárních dilatací. Jako důsledek se dostává jednotný přístup k odvození několika známých typů funkcionálních modelů.

Резюме

АБСТРАКТНАЯ МОДЕЛЬ ДЛЯ СЖИМАЮЩИХ ОПЕРАТОРОВ

VLASTIMIL PTÁK, PAVLA VRBOVÁ

Исследуется геометрия гильбертова пространства, порождённого двумья замкнутыми подпространствами. Целью статьи является построение абстрактного аналога функциональной модели для сжатия в теории унитарных дилатаций. Следствием является единый подход к выводу нескольких известных типов функциональных моделей.

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