AN ACCELERATOR FOR ISOTOPE CONVERSION - THE RECYCLE ACCELERATOR

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Summary. The nature of charged particle - complex nuclei reaction cross sections suggests an isotope conversion accelerator in which a circulating beam kept at constant energy repeatedly traverses a thin target. Statistical deviations in target energy loss and emergent direction induce betatron and synchrotron oscillations. These limit the number of traversals before particle loss, but to useable values. However, the requirements of storage ring injection may reduce the advantage of this accelerator over conventional types.

The cross section for a charged particle - complex nucleus reaction taking place is appreciable only if the incoming particle (p) has an energy of the order of the repulsive potential between the incoming particle and the target nucleus (t), the so-called Coulomb barrier, given by $E_c = (1/4\pi\varepsilon_o)Z_pZ_te^2/R$, $R = (R_p+R_t) = R_o(A_p^{1/3}+A_t^{1/3})$, $R_o \approx (1.4\pm0.1)\times10^{-15}$m, where $R_{p,t}, Z_{p,t}$ are the respective radii and charge numbers.

The cross sections for reactions are predicted fairly well by two complementary theories. The continuum theory predicts a cross section $\sigma_{cont.} \approx \pi(R+\lambda\!\!\!^-)^2(1-U/E)$, $E_b > E_c$; $\sigma_{cont.} \to 0$, $E_b < E_c$, where E_b is the bombarding energy and $U = E_cR/(R+\lambda\!\!\!^-)$, $E = E_b(c.m.) = (\hbar^2/2m_p\lambda\!\!\!^-{}^2) \approx E_b(lab.)$. For $E_b \sim E_c$, $\sigma_{cont.}$ has no simple analytic representation but has a smooth transition between the other two limiting forms. The compound nucleus theory predicts a cross section made up of a sum of Lorentzian resonances with resonant energies E_i somewhat above the energy at which the particle reaction becomes energetically possible.

$$\sigma_{comp.} = \sum_i A_i/[(E-E_i)^2+(\Gamma/2)^2],$$

where E_i is the i th resonant energy. For charged particle bombardment of complex nuclei at $(10 < E_b < 100)$ MeV, $\Gamma/2 \sim (0.1\text{-}0.15)E_b$. With the nuclear radius as an adjustable parameter, $\sigma_{cont.}$ will represent essentially an average of $\sigma_{comp.}$. Figures 1 and 5 are classic and typical examples of the nature of p and α reactions on complex nuclei. The cross sections are seen to have peak values of the order of 1 b and are broad enough to be fairly constant at the peak value over an energy range $E_i \pm 0.5$ MeV. Fig. 5 illustrates the possibility of obtaining a given nuclide in more than one way.

Thinking of the accelerator as a machine for converting a given isotope into another the question arises, as with all machines, as to what the maximum attainable yield is per energy input. If the charged particle reaction is the final desired result, the maximum yield will occur when the bombarding energy corresponds to the peak value of the cross section for the particular reaction. The absolute minimum energy which can be expended is the energy of Coulomb interaction that is required to pass the beam through the target at an energy corresponding to the resonance energy. Letting R be the reaction rate (1/cm^2-sec); n_a, no. of target nuclei (1/cm^2); φ, flux of incident particles (1/cm^2-sec); σ, peak value of the cross section for the reaction (cm^2); P, power input of the target (W/cm^2-sec); W_a, weight of target atoms (g/cm^2); (dE/dx), specific energy loss at $\sim E_r$ (J/(g/cm^2)); A, at. wt. (g); and L, Avogadro's no., $R = n_a\varphi\sigma$, $P = W_a(dE/dx)\varphi = n_a(A/L)(dE/dx)\varphi$; $(R/P) = \sigma(L/A)(dE/dx)^{-1}$

While this seems like a rather unrealistic criterion to measure the performance of a machine by inasmuch as it neglects the acceleration energy, ion source energy, etc., considering it led to the following machine concept, termed the recycle accelerator, illustrated in Figure 3. Neglecting for the moment the details of injection, pulsed beam at energy $E_i + \Delta E/2$ is put into a storage ring intercepted by a target of thickness $\Delta E < \Gamma/2$. After passing through the target, the beam enters an acceleration station which restores ΔE to the surviving particles (the vast majority, by a factor

$$\sim (\sigma_e/\sigma_n) \sim (10^{-15}\ cm^2/10^{-24}\ cm^2) = 10^9:1,$$

where σ_e and σ_n are typical atomic and nuclear interaction cross sections). These then procede in orbit until they intercept the target again. A particle recycling N times will dissipate an amount of energy $N\Delta E$ (supplied by the radio frequency accelerating station), which, for $N \sim (10\text{-}100)$ could be comparable or even greater than E_b.

Coulomb scattering of the charged particle in the target, rather than a nuclear reaction, might be expected by the above argument to play the dominant role in determining the average number of traversals. In fact, it will be demonstrated that while the accelerating radio frequency gap can replace the mean energy loss in the target ΔE, the random angular and energy deviations which cannot be corrected for will eventually limit the total number of traversals by harmfully perturbing the normal betatron and synchrotron oscillations. In the following summary of useful classical Coulomb scattering results, (N_t,N_e) are the number of target nuclei and electrons per unit volume, (e,m_e) are the electron charge and mass, (p,v,m_e) are the bombarding particle's momentum, velocity, mass; Δx is the (geometric) target thickness, (eZ_t,eZ_p) are the target nucleus and particle charge, (b(max), b(min) are the classical maximum and minimum impact parameters, I is the hydrogen atom ionization potential, ε_0 is the dielectric constant of free space, c is the speed of light, (A_p, A_t) is the particle or target nucleon number, and n.r. stands for "non-relativistically". Then it is found that the mean square of the angle by which a particle deviates from its original direction after passing through Δx is

$$\langle\theta^2\rangle \approx \frac{8\pi}{(4\pi\varepsilon_o)^2}\frac{N_t Z_t^2}{p^2}\frac{Z_p^2 e^4}{v^2} B\Delta x,$$

$B \equiv \ln(b(\max)/b(\min))$. The projection of θ on any plane, θ_y, has a mean square $\langle\theta_y^2\rangle = \frac{1}{2}\langle\theta^2\rangle$ and, to a good approximation, is Gaussian distributed:

$$P(\theta_y)d\theta_y = (\pi\langle\theta^2\rangle)^{-\frac{1}{2}} \exp\left(-\theta_y^2/\langle\theta^2\rangle\right)d\theta_y.$$

The mean energy loss is

$$\Delta E \approx \frac{4\pi}{(4\pi\varepsilon_o)^2}\left(\frac{N_e}{m_e}\right)\left(\frac{Z_p^2 e^4}{v^2}\right)B\Delta x,$$

with a mean square deviation from this value of

$$\langle(\delta E)^2\rangle \approx \frac{4\pi}{(4\pi\varepsilon_o)^2} Z_p^2 e^4 N_e \Delta x.$$

From these may be derived

$$\frac{\langle\theta^2\rangle}{\Delta E} \approx \frac{2N_t Z_t^2}{p^2}\frac{m_e}{N_e} \overset{n.r.}{=} \frac{N_t}{N_e}\left(\frac{m_e}{m_p}\right)\frac{Z_t^2}{E_b} = \frac{m_e}{m_p}\frac{Z_t}{E_b},$$

$$\langle(\delta E)^2\rangle = \left(\frac{m_e}{m_p}\right)(m_p v^2)(\Delta E/B) \overset{n.r.}{=} 2\left(\frac{m_e}{m_p}\right)\frac{\Delta E\, E_b}{B}.$$

Some useful approximations are:

$$b(\max)/b(\min)) \approx \frac{2v^2 m_e}{I[1-(v/c)^2]} - (v/c)^2 \overset{n.r.}{=} \frac{4E_b m_e}{I\, m_p},$$

where $I \approx I_o Z_t$, I_o the ionization potential of hydrogen, $Z_t \approx \frac{1}{2}A_t$, and, using $m_p \sim A_p(1837\ m_e)$,

$$\langle\theta^2\rangle/|\Delta E| \sim 5.5\times10^{-4}(Z_t/A_p)E_b^{-1}.$$

The value of E_b used will depend somewhat on the target nucleus, being at least greater than the Coulomb barrier E_c. Since it would seem that the bombarding energy would probably not be greater than $4E_c$, a reasonable value for estimates would be the geometric mean between these or $E_b = 2E_c$; this representative value will be used in subsequent calculations. Figure 2 assumes this in illustrating how $(\langle\theta^2\rangle/-\Delta E)$ changes with A_t for protons and alpha particles.

To consider the betatron oscillations first the axial and radial deviations from a point s on the equilibrium orbit obey equations of motion given to first order respectively by

$$(d^2z/ds^2) = -[n(s)/\rho^2(s)]z,$$

$$(d^2x/ds^2) = -[(1-n(s))/\rho^2(s)]x,$$

which, for $|n(s)| \gg 1$ can both be written as

$$(d^2y/ds^2) = \pm[n(s)/\rho^2(s)]y = \varepsilon(|n|/\rho^2)y,$$

where $n(s) = -(\rho(s)/B)(\partial B/\partial\rho)$, $\rho = (p/qB)$, $q = eZ_p$; $\varepsilon = \pm 1$, depending upon whether the particular region is focusing or not (focusing for axial deviations is defocusing for radial deviations, vv, B is the magnetic field intensity, y stands for either axial or radial deviation. The most general solution is $y(s) = C\beta^{\frac{1}{2}}\cos\nu[\varphi(s)+\varphi_o]$, with, $\nu, \beta(s)$ and $\varphi(s)$ functions of the parameters of the magnet ring. For simplicity, assume a circular ring (no straight sections) with 2M magnets whose radial gradients alternate in sign but are of the same magnitude, with B constant over the (circular) equilibrium orbit. Then $\rho(s) = r$, $|n| = n_o$, $2M = 2\pi(r/\ell)$, (where ℓ is the length of equilibrium orbit through a magnet), and

$$\nu = (r/2\ell)\cos^{-1}[\cos(n_o^{\frac{1}{2}}\ell/r)\cosh(n_o^{\frac{1}{2}}\ell/r)]$$

$$\simeq \frac{n_o}{\sqrt{12}}\left(\frac{\ell}{r}\right) = \frac{\pi}{\sqrt{12}}\left(\frac{n_o}{M}\right),$$

$$\beta^{\frac{1}{2}}(s) \cong (r/\nu)^{\frac{1}{2}}[1 + (n_o/4)(\ell/r)^2\sin(\pi s/\ell)]^{\frac{1}{2}}$$

$$(r/\ell)^{\frac{1}{2}}[1 + (n_o/8)(\ell/r)^2\sin(\pi s/\ell)] \sim (r/\nu)^{\frac{1}{2}},$$

$$\varphi(s) \cong (s/r) + (n_o/\pi)(\ell/r)^3\cos(\pi s/\ell) \simeq (s/r),$$

which expressions are valid assuming the periodicity of $\beta^{\frac{1}{2}}$ is of fairly higher frequency than the argument of the cosine. These frequencies are in the ratio $(\sqrt{12}/n_o)(\ell/r)^2 = (\sqrt{12}/\pi^2)(M^2/n_o)$ and so even with $n_o \sim (100\text{-}300)$, this can be the case; similarly the amplitude of the $\sin(\pi s/\ell)$ term in β, $(n_o/r)(\ell/r)^2 = (\pi^2/4) \times (n_o/M^2)$ can be fairly smaller than 1, say 0.1 to 0.3. Then $y = y_m[1+(n_o/8)(\ell/r)^2\sin(\pi s/\ell)] \times \cos(ks+\varphi_o')$ where $y_m = C(r/\nu)^{\frac{1}{2}}$ is the average amplitude, being modulated so that the amplitude never gets larger than $y_{am} \simeq y_m[1+(n_o/8)(\ell/r)^2] \equiv y_m f$, with f being a sort of form factor, $\varphi_o' = \nu\varphi_o + 0(\ell/r)^3$, and $k \approx (n_o\ell/\sqrt{12}\, r^2) = (\nu/r)$. Choosing the zero of s so that φ_o' is zero, $y \approx y_m f\cos ks$ and neglecting the higher frequency modulation, $y' = dy/ds = -k\, y_m f\sin ks$. Starting with $y(s) = y_o(s)$, suppose that at $s = s_t$ (the target position), $y' \to y' + \theta_y$. Then it is readily shown that $y(s)$ becomes $y(s) = y_o(s) + (\theta_y/k) \times \sin k(s-s_t)$, $(\theta_y/k) = r\theta_y/\nu$. Since these perturbation always occurs at the same point in the orbit, they will add coherently to give after N traversals $y_N = y_m\cos ks + \sin k(s-s_t)\sum_{i=1}^{N}(\delta y)_i$, $(\delta y)_i = (r/\nu)(\theta_y)_i$, where $y_m\cos ks$ represents the originally present betatron oscillation. The number of target traversals a particle makes before being lost through collission with the beam pipe will be determined by its betatron oscillation amplitude becoming so large as to become comparable with a dimension of the beam pipe.

The above equation leads to the following physical picture (see Figure 4). Consider an xz plane in which the overall betatron oscillation amplitude is a maximum. In that plane the successive perturbations of amplitude may be represented as vectors with random azimuthal angles, whose magnitudes are determined by the immediately preceding value of θ given the particle by the target. The components of the individual perturbations may be taken along any particular direction (chosen in Figure 5 to be that for which the aperture of the beam pipe is smallest); these would correspond approximately to the $(\delta y)_i$, and of course also add up to the magnitude of y_N after N traversals. (They would exactly equal $(\delta y)_i$ if y_o were zero, as shown in Figure 5, and the xz plane in question would then be any such that $\sin k(s-s_t) = \pm 1$.)

The problem of how many traversals will be made on the average before loss can be treated as a problem in random walk with absorbing barriers[2], and readily solved if some reasonable approximations are made. The $(\theta_y)_i$ are Gaussian distributed with respect to the incident particle direction, but it will be assumed that the distribution is Gaussian with respect to the direction of the equilibrium orbit at the target location. Then it will be assumed that the Gaussian distribution of the amplitudes $(\delta y)_i$ can be replaced by an appropriate constant value $\overline{\delta y} = \epsilon\langle(\delta y)^2\rangle^{\frac{1}{2}}$, $\epsilon = \pm 1$ randomly. This replacement is exact if the absorbing barriers are at infinity,[3] so it will be assumed a reasonable approximation here.

The equilibrium orbit is assumed to pass through the zero of a y coordinate axis perpendicular to it extending from $-a/2$ to $+a/2$ where a is the smallest dimension of the beam pipe. The interval $-a/2$ to $+a/2$ is divided into $2[(a/2)/\overline{\delta y}]$ intervals ([w] meaning the greatest integer in w). Let $\hat{y} = [y/\overline{\delta y}]$ be the integer corresponding to the interval y is in and $\langle N(\hat{y})\rangle$ the expected or mean number of traversals for a particle "located" at $\hat{y}$ before it reaches for the first time the boundary (thus being "absorbed") at $\hat{y}(\max,\min) = \pm[(a/2)/\overline{\delta y}] = \pm\hat{a}/2$. The relation between $\langle N\rangle$ for a particle at $\hat{y}$ and one to either side of it is $\langle N(\hat{y})\rangle = p\langle N(\hat{y}+1)\rangle + q\langle N(\hat{y}-1)\rangle + 1$, with $\langle N(y(\min))\rangle = \langle N(y(\max))\rangle = 0$, where p is the probability of the particle at $\hat{y}$ taking its next step towards $\hat{y}(\max)$, and q the corresponding probability for the step to go towards $\hat{y}(\min)$, $p+q = 1$. This finite difference equation has as solutions obeying the above boundary conditions

$$\langle N(\hat{y})\rangle = \frac{(\frac{\hat{a}}{2}+\hat{y})}{q-p} - \frac{\hat{a}}{(q-p)}\frac{1-(q/p)^{(\frac{\hat{a}}{2}+\hat{y})}}{1-(q/p)^{\hat{a}}}, \quad p\neq q$$
$$= (\tfrac{\hat{a}}{2})^2 - \hat{y}^2, \quad p=q=\tfrac{1}{2}.$$

It is consistant with the approximation that θ is distributed symmetrically about the equilibrium orbit direction rather than the incident direction to choose the latter solution, i.e., no bias irrespective of previous deviation. Thus letting $(y/\overline{\delta y}) \sim \hat{y}$ and $\langle(\delta y)^2\rangle \sim (\overline{\delta y})^2$,

$$\langle N\rangle_{beta} \approx \frac{(a/2)^2-y^2}{(\overline{\delta y})^2} \approx \frac{(a/2)^2-y_o^2}{\langle(\delta y)^2\rangle}.$$

Here y_o represents the initial displacement in the plane where the betatron oscillation has its greatest amplitude. (An average of $\langle N\rangle_{beta}$ can be made over a known distribution of y_o). Its maximum value is for y=0, i.e., a particle starting out on the equilibrium orbit.

This solution of the random walk in one dimension is the solution of the real problem only if the aperture at right angles to the minimum dimension which y was chosen to represent is infinite. This is because even if deviations in one direction are completely independent of those in the other direction perpendicular to it, the two random walks are coupled in the sense that terminating one automatically ends the other. The $\langle N\rangle$ found is thus conditional on the other walk not having ended. An exact treatment of the problem would require consideration of the shape of the absorbing barrier. One expects, however, the above solution to be a reasonable approximately if it is for the minimum aperture and the other dimension is several times larger. Thus for a particle starting out on the equilibrum orbit

$$\left(\frac{\langle N\rangle}{\nu}_{beta}\right)_{max} \approx \frac{(a/r)^2}{2\langle\theta^2\rangle}.$$

This function is plotted in Figure 6 assuming $(a/r) = 0.125$, i.e., 12.5 cm for an equilibrium orbit radius of 1 m, and $E_b = 2E_c$, for protons and alpha particles, and for assumed target thicknesses of 0.1 and 0.5 MeV. These latter values are smaller than allowed by $(\Gamma/2)$, being limited by the likely maximum acceleration voltage per turn which can be achieved in practice, bearing in mind that the target loss must be made up completely before the particle enters into the magnetic field again, if the particle is not to receive too large a perturbation.

To consider the effect of synchrotron oscillations: assume the particle passes through the target losing an average energy ΔE with variance $\langle(\delta E)^2\rangle^{\frac{1}{2}}$. With the accelerator gap in close proximity to the target, the particle will arrive at the gap with the same phase as if there were no energy loss variance. The second time around, however, there will be a phase error $\xi=\varphi-\varphi_s$, and the non-synchronous φ will oscillate about the synchronous φ_s according to

$$(E_s/h\omega_s^2\Gamma)(d^2\varphi/dt^2) - (qV_m/2\pi)(\sin\varphi-\sin\varphi_s) = 0,$$
$$\Gamma \equiv (\alpha^{-1}-\gamma_s^{-2})[\gamma_s^2/(\gamma_s^2-1)] \approx -(\gamma_s^2-1)^{-1}$$

with E_s, ω_s the synchronous orbit energy and angular frequency, α the momentum compaction, defined below, $\gamma_s = (E_s+m_pc^2)/m_pc^2$. If ξ is small, then the above becomes $(d^2\xi/dt^2)+\Omega^2\xi = 0$, $\Omega^2 \equiv -qV_mh\omega_s^2\Gamma\cos\varphi_s/2\pi E_s$; $\xi = \xi_o\cos\Omega t$, choosing t=0 appropriately. Causing the particle to deviate impulsively from E_s by δE, will change $(d\varphi/dt) = (d\xi/dt)$ by an amount $\delta(d\xi/dt) = h\omega_s\Gamma\delta E/E_s$ resulting in a change of amplitude for ξ given by $\delta(d\xi/dt)/\Omega$, in a manner similar to the previous calculation. After N traversals

$$\xi_N-\xi_o = \varphi_N-\varphi_o = \sin\Omega(t-t_t)(h\omega_s\Gamma/\Omega E_s)\sum_{i=1,N}^{N}(\delta E)_i$$
$$= \sin\Omega(t-t_t)\sum_{i=1,N}^{N}(\delta\varphi)_i.$$

This can be considered a random walk of the variable φ with absorbing boundaries at the limits of phase stability. Letting the acceptable limits be

$\varphi_s \pm \Delta\varphi$, then as in the previous calculation, the average number of traversals before the particle is lost is

$$\langle N\rangle_{synch.,\varphi} \approx [(\Delta\varphi)^2 - (\varphi_o-\varphi_s)^2]/\langle(\delta\varphi)^2\rangle,$$

where $\langle(\delta\varphi)^2\rangle = (h\omega_s\Gamma/\mathcal{E}_s)^2\langle(\delta E)^2\rangle = 2(h\omega_s\Gamma/\mathcal{E}_s)^2 x$ $(m_e/m_p)E_b\Delta E/B$, $E_s = E_b$. Using $\Delta E = qV_m\sin\varphi_s$, $[h\langle N\rangle_{synch.,\varphi}/(\Delta\varphi)^2]_{max} = (4\pi)^{-1}(m_p/m_e)B\cot\varphi_s\Gamma^{-1}$. The basic requirement that $\Delta E = qV_m\sin\varphi_s$ means that φ_s cannot be made arbitrarily small (as the above relation implies would be desirable) since a burden is placed on the accelerating gap to support a correspondingly increased V_m. With present technique, $V_m\sin\varphi_s = (0.5\text{-}1.0)$ MeV/turn is probably the maximum reasonable. In Figure 7 it has been assumed that $\varphi_s = \pi/4$ (this accelerator is always operating below the transition energy); the result is independent of ΔE because of the implicit approximation made above that the energy gained per turn remains $qV_m\sin\varphi_s$ even as $|\xi|$ increases. The effect of synchrotron oscillations on the radial excursion can be also calculated similarly, starting with the definition of α:

$$(\delta r/r) \equiv \alpha^{-1}(\delta p/p) \overset{n\pm r.}{=} \alpha^{-1}(\delta E/2E_s),$$

$$\langle(\delta r)^2\rangle \approx (r/2\alpha E_s)^2\langle(\delta E)^2\rangle.$$

Then $\langle N\rangle_{synch.,r} = [b^2-(\delta r)_o{}^2]/\langle(\delta r)^2\rangle$, where b is the radial aperture, and so $(\langle N\rangle_{synch.,r})_{max} = 2(b/r)^2\alpha^2(E_b/\Delta E)(m_p/m_e)B$. It is easily seen that $\langle N\rangle_{synch.,r} \gg \langle N\rangle_{synch.,\varphi}$ or $\langle N\rangle_{beta}$ so that either of the latter dominate the particle loss process. It can be seen, however, that these are large enough so that isotope conversion would become greatly enhanced compared to a conventional "single-pass" accelerator by a factor N, (or N/2 considering the injector represents a capital investment comparable to the recycle accelerator).

The above calculations for $\langle N\rangle$ were carried out as though the processes were independent of one another. Even if they were, just as in the case of the two dimensional random walk, discussed before, the $\langle N\rangle$ for the whole process must be less than that calculated separately for any of the contributing processes. It is reasonable though to take the smallest value, particularly if appreciably smaller, as a reasonable approximation. As seen in Figures 6 and 7, taking reasonable values for ν, h, and $(\Delta\varphi)^2$ (say 8.75, 1, 0.5), usually gives $\langle N\rangle \gg 1$.

In order for the full potential of the recycle accelerator to be realized the injector would have to deliver a beam with a pulse repetition frequency $\omega_s/2\pi$. Thus single turn injection is ruled out.

While an initial, injection-caused betatron oscillation might cause the inflector septum to be missed the first few cycles, there is no net acceleration per cycle, thus no damping which would allow traversals after the initial few to miss the septum altogether as in some conventional accelerators. Moreover, the high n value counted upon to increase $\langle N\rangle_{beta}$ works against this injection possibility. The conventional storage ring injection scheme has a decelerating gap with a frequency varying periodically (Figure 3) so that the equilibrium orbit goes from r to r-δr adiabatically (assumes injection from outside the ring). One might think the same effect would result if the accelerating gap had V_m varied similarly from $(\Delta E/q\sin\varphi_s)$ to $(\Delta E-\boldsymbol{\delta})/q\sin\varphi_s$, causing a given particle to tend to spiral inward. In either case, in view of the sensitivity exhibited to phase error induced by a δE, these last three methods could only work as compromises which would result in the injection efficiency being appreciably less than unity. No calculations of injection efficiencies have been made yet however.

In conclusion it can be seen that providing injection losses do not nullify the gains obtained by recycling of the bombarding particles through the target there may be an appreciably greater isotope yield per capital investment and operating cost if the recycle accelerator concept is employed, as $\langle N\rangle$ can be a fairly large compared to unity.

We would like to thank Drs. J. W. Bittner and E. D. Courant for enlightening discussions.

References

1. Notation attempts to adhere to usage in the standard references used in preparation of this paper:

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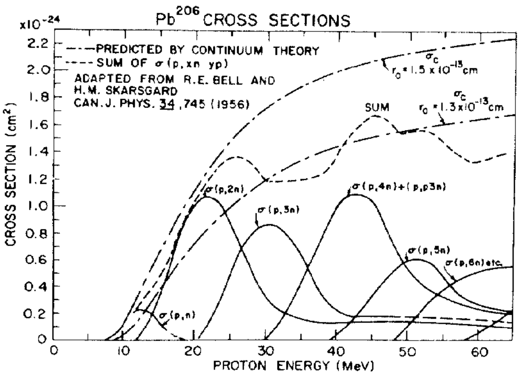


Figure 1.

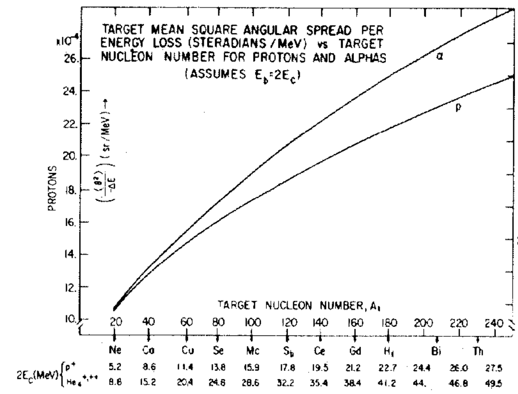
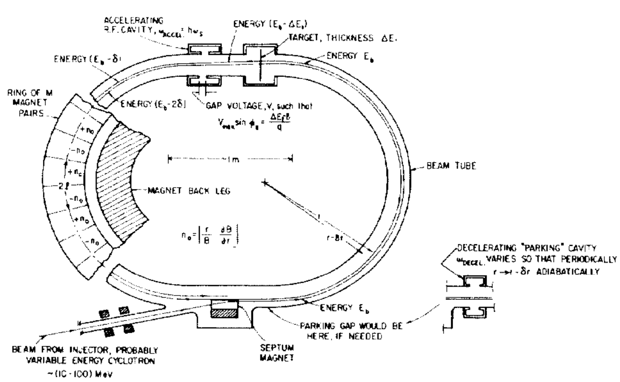
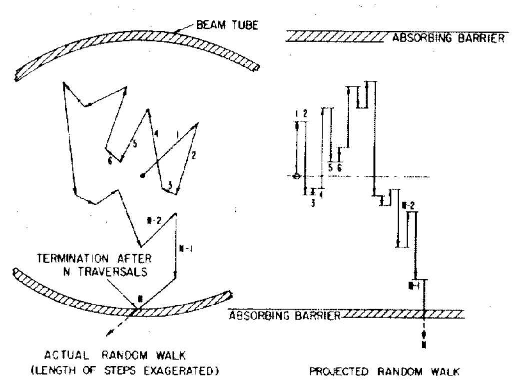


Figure 2.



SCHEMATIC OF THE "RECYCLE ACCELERATOR"

Figure 3.



RANDOM WALK OF BETATRON OSCILLATION AMPLITUDES

Figure 4.

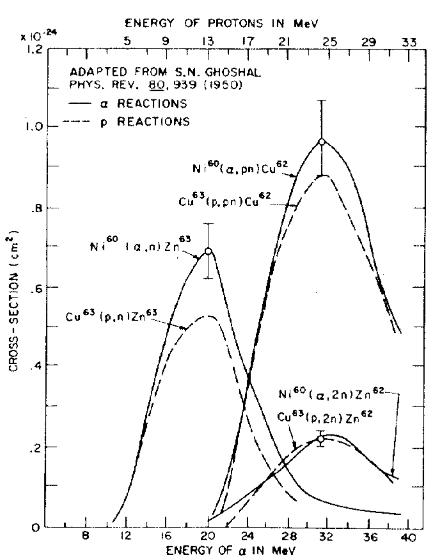


Figure 5.

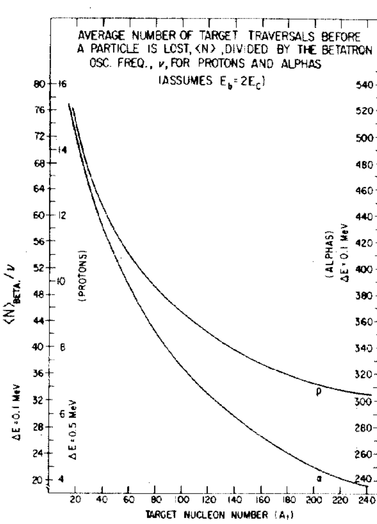


Figure 6.

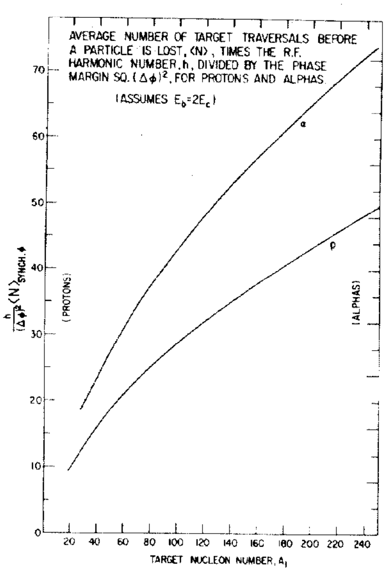


Figure 7.