An accurate approach based on the orthonormal shifted discrete Legendre polynomials for variable-order fractional Sobolev equation

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Abstract

This paper applies the Heydari–Hosseininia nonsingular fractional derivative for defining a variable-order fractional version of the Sobolev equation. The orthonormal shifted discrete Legendre polynomials, as an appropriate family of basis functions, are employed to generate an operational matrix method for this equation. A new fractional operational matrix related to these polynomials is extracted and employed to construct the presented method. Using this approach, an algebraic system of equations is obtained instead of the original variable-order equation. The numerical solution of this system can be found easily. Some numerical examples are provided for verifying the accuracy of the generated approach.

Keywords: Variable-order time fractional Sobolev equation; Orthonormal shifted discrete Legendre polynomials; Nonsingular variable-order fractional derivative

1 Introduction

Over the past decades, the subject of fractional calculus (as a generalization of the classical calculus) has been widely studied [1–3]. In fact, fractional derivative and integral operators, due to higher degree of freedom in comparison to the classical operators as well as their memory and nonlocal properties, have received many applications in various problems [4]. For instance, some important works related to recent developments in fractional calculus and its applications can be found in [5–10]. The reader should note that the most important issue about problems involving such operators is finding their exact solutions, which is often very difficult and may even be impossible. This fact has led to the use of numerical methods as a convenient alternative to solve this drawback. Some numerical methods that have recently been applied to solve such problems can be found in [11–18].

Given that the order of fractional operators is permissible to take any value, a more general generalization is that the order of fractional operators be a definite function of the variables in the problem [19]. In fact, fractional operators of variable order (VO) can be utilized for more accurate modeling of real-world phenomena [20, 21]. The remarkable point about such operators is that their memory property is more evident [22]. Some

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problems that have recently been modeled by such operators can be found in [23, 24]. However, similar to constant-order fractional equations, the major challenge in dealing with VO fractional equations is finding their analytical solutions, which is often impossible. For this reason, in recent years, many numerical approaches have been constructed to solve this category of problems. For instances, see [25–29].

The Sobolev equation is a well-studied partial differential equation which has been frequently utilized in the fluid dynamics to express the fluid motion through rock or soil, and other media [30]. This equation is a special form of the Benjamin–Bona–Mahony–Burgers problem, where the coefficients of nonlinear term and both first-order derivatives are zero [31]. Many applications of the Sobolev equation have been reported in moisture migration in soil [32], thermodynamics [33], and fluid motion [33]. There are many approaches that have been applied to solve various types of the Sobolev equation in recent years. For instances, see [30, 31, 34–37].

Recently, the author of [38] introduced a new nonsingular VO fractional derivative, where the Mittag-Leffler function is its kernel. As far as we know, there is no previous VO fractional version of the Sobolev problem. This motivates us to pursue the following goals:

- Defining a VO fractional prescription of the Sobolev equation using the nonsingular fractional derivative expressed in [38].
- Constructing a highly accurate method based upon the orthonormal shifted discrete Legendre polynomials (DLPs) for this equation.

So, we concentrate on the problem

$$
{}_{0}^{HH}\partial_{\tau}^{\zeta(\tau)}\theta(y,\tau) - \mu {}_{0}^{HH}\partial_{\tau}^{\zeta(\tau)}\theta_{yy}(y,\tau) - \nu\theta_{yy}(y,\tau) = \varphi(y,\tau),
$$
$$
\zeta(\tau) \in (0,1), (y,\tau) \in [0,y_b] \times [0,\tau_b], \tag{1.1}
$$

under the initial and boundary conditions

$$
\theta(y,0) = \hat{\theta}(y), \qquad \theta(0,\tau) = \tilde{\theta}_0(\tau), \qquad \theta(y_b,\tau) = \tilde{\theta}_1(\tau), \tag{1.2}
$$

where $\theta(\cdot,\cdot)$ is the undetermined solution, μ and ν are positive constants, $\zeta(\cdot)$ is a continuous function in its domain, and $\varphi(\cdot,\cdot)$, $\hat{\theta}(\cdot)$, $\tilde{\theta}_0(\cdot)$, and $\tilde{\theta}_1(\cdot)$ are given functions. Also, ${}_{0}^{HH}\partial_{\tau}^{\zeta(\tau)}\theta(y,\tau)$ is the VO fractional derivative of order $\zeta(\tau)$ with respect to τ in the Heydari–Hosseininia (HH) sense of the functions $\theta(y,\tau)$ [38]. This equation can have useful applications in many applied problems, such as the transport phenomena of humidity in soil, the heat conduction phenomena in different media, and the porous theories concerned with percolation into rocks with cracks. Note that in the case of $\zeta(\tau) = 1$, this problem reduces to the classical Sobolev problem.

One good idea for solving fractional functional equations is employing polynomials as basis functions to construct numerical methods. This is important for two reasons: First, the computation of the fractional derivative and integral of these functions is easy; and second, if the solution of the problem under study is sufficiently smooth, high-precision solutions can be achieved. Basis orthogonal polynomials are classified into discrete and continuous kinds regarding the method of calculating their expansion coefficients [39]. Unlike continuous polynomials, the expansion coefficients of which are calculated by integrating (in most cases numerically), the expansion coefficients of discrete polynomials

are calculated accurately using a finite summation. In recent years, discrete polynomials have been extensively applied for solving diverse problems. For instances, see [39–47].

This study applies the orthonormal shifted DLPs for solving the Sobolev equation (1.1) subject to conditions (1.2). To this end, a new fractional matrix related to the VO fractional differentiation of these polynomials is obtained and applied for generating a numerical technique for this problem. The intended approach is constructed using these polynomials expansion and the tau technique. This technique converts the VO fractional problem into an algebraic system of equations that readily can be handled. Note that since it is easier to obtain the operation matrix of VO fractional derivative of the orthonormal shifted DLPs than continuous polynomials, we have considered these discrete polynomials as basis functions for solving this VO fractional problem.

Organization of this article is as follows: The VO fractional derivative in the HH sense is reviewed in Sect. 2. The orthonormal shifted DLPs are reviewed in Sect. 3. Some matrix equalities are obtained in Sect. 4. The computational approach is explicated in Sect. 5. Numerical examples are given in Sect. 6. Conclusion of this study is provided in Sect. 7.

2 Preliminaries

Here, we review the definition of the VO fractional differentiation used in this study. First of all, we express the definition of the Mittag-Leffler function that is given in [4] by

$$\mathbf{E}_{a,b}(\tau) = \sum_{j=0}^{\infty} \frac{\tau^j}{\Gamma(ja+b)}, \quad a, b \in \mathbb{R}^+, \tau \in \mathbb{C}. \tag{2.1}$$

Please remember that for $b = 1$ it is considered as $\mathbf{E}_a(\tau) = \mathbf{E}_{a,1}(\tau)$. The VO fractional derivative of order $\zeta(\tau) \in (0,1)$ (where $\zeta(\tau)$ is a continuous function on its domain) in the HH sense of the function $\theta(\tau)$ is given in [38] as follows:

$$_0^{HH}D_\tau^{\zeta(\tau)}\theta(\tau) = \frac{1}{1-\zeta(\tau)} \int_0^\tau \mathbf{E}_1\left(\frac{-\zeta(\tau)(\tau-s)}{1-\zeta(\tau)}\right)\theta'(s)\,ds, \quad \tau > 0. \tag{2.2}$$

The above definition results in

$$_0^{HH}D_\tau^{\zeta(\tau)}\tau^r = \begin{cases} 0, & r = 0, \\ \dfrac{r!}{1-\zeta(\tau)}t^r\mathbf{E}_{1,r+1}\left(-\dfrac{\tau\zeta(\tau)}{1-\zeta(\tau)}\right), & r \geq 1, \end{cases} \tag{2.3}$$

where $r \in \mathbb{Z}^+ \bigcup \{0\}$.

3 Orthonormal shifted discrete Legendre polynomials

The orthonormal shifted DLPs are defined over $[0, \tau_b]$ as follows [44]:

$$L_{\tau_b,i}(\tau;N) = \frac{1}{\sqrt{\sigma(i,N)}} \sum_{k=0}^{i} \sum_{m=0}^{k} (-1)^k \binom{i}{k}\binom{i+k}{k}\left(\frac{N}{\tau_b}\right)^m \frac{S_k^{(m)}}{N^{(k)}}\tau^m, \quad i = 0, 1, \dots, N, \tag{3.1}$$

where

$$\sigma(i,N) = \frac{(N+i+1)^{(i+1)}}{(2i+1)N^{(i)}}, \tag{3.2}$$

$S_k^{(m)}$s are the first type Stirling numbers,

$$
N^{(k)} = \begin{cases} 1, & k = 0, \\ N(N-1)(N-2)\dots(N-k+1), & k \ge 1, \end{cases}
\tag{3.3}
$$

and $\binom{i}{k}$ is the binomial coefficient. These polynomials can be utilized for approximating any continuous function $\theta(\tau)$ over $[0, \tau_b]$ as follows:

$$
\theta(\tau) \simeq \sum_{i=0}^{N} e_i L_{\tau_b, i}(\tau; N) \triangleq \mathbf{E}^T \Psi_{\tau_b, N}(\tau),
\tag{3.4}
$$

where

$$
\mathbf{E} = [e_0 \ e_1 \ \dots \ e_N]^T,
$$

in which

$$
e_i = \sum_{r=0}^{N} \theta\left(\frac{\tau_b}{N} r\right) L_{\tau_b, i}\left(\frac{\tau_b}{N} r; N\right),
\tag{3.5}
$$

and

$$
\Psi_{\tau_b, N}(\tau) = \left[L_{\tau_b, 0}(\tau; N) \ L_{\tau_b, 1}(\tau; N) \ \dots \ L_{\tau_b, N}(\tau; N)\right]^T.
\tag{3.6}
$$

Likely, a continuous function $\theta(y, \tau)$ defined over $[0, y_b] \times [0, \tau_b]$ can be approximated by the orthonormal shifted DLPs as

$$
\theta(y, \tau) \simeq \sum_{i=0}^{M} \sum_{j=0}^{N} \theta_{ij} L_{y_b, i}(y; M) L_{\tau_b, j}(\tau; N) \triangleq \Psi_{y_b, M}(y)^T \mathbf{\Theta} \Psi_{\tau_b, N}(\tau),
\tag{3.7}
$$

in which $\mathbf{\Theta} = [\theta_{(i-1)(j-1)}]$ is a matrix with $(M+1) \times (N+1)$ entries as

$$
\theta_{(i-1)(j-1)} = \sum_{r=0}^{M} \sum_{l=0}^{N} \theta\left(\frac{y_b}{M} r, \frac{\tau_b}{N} l\right) L_{y_b, i}\left(\frac{y_b}{M} r; M\right) L_{\tau_b, j}\left(\frac{\tau_b}{N} l; N\right),
$$

$$
1 \le i \le M+1, 1 \le j \le N+1.
\tag{3.8}
$$

4 Matrix relationships

Here and in what follows, we give some matrix relationships related to the orthonormal shifted DLPs.

Theorem 4.1 ([44]) *Differentiation of the vector $\Psi_{\tau_b, N}(\tau)$ introduced in (3.6) satisfies the relation*

$$
\frac{d\Psi_{\tau_b, N}(\tau)}{d\tau} = \mathbf{D}_N^{(1, \tau_b)} \Psi_{\tau_b, N}(\tau),
\tag{4.1}
$$

where $\mathbf{D}_N^{(1,\tau_b)} = [d_{ij}^{(1,\tau_b)}]$ is a matrix of order $(N + 1)$ with entries

$$
d_{ij}^{(1,\tau_b)} = \begin{cases}
\dfrac{1}{\sqrt{\sigma(i-1,N)}} \displaystyle\sum_{l=0}^{N}\sum_{k=1}^{i-1}\sum_{r=1}^{k}(-1)^k \binom{i-1}{k}\binom{i+k-1}{k} \\
\qquad \times \dfrac{N}{\tau_b}\dfrac{S_k^{(r)}}{N^{(k)}}rl^{r-1}L_{\tau_b,j-1}\left(\dfrac{\tau_b}{N}l;N\right), & \begin{aligned}&2 \le i \le N+1,\\&1 \le j \le i-1,\end{aligned} \\[2mm]
0, & \text{otherwise.}
\end{cases}
$$

Moreover, for any integer n, we have

$$
\frac{d^n \Psi_{\tau_b,N}(\tau)}{d\tau^n} = \underbrace{\mathbf{D}_N^{(1,\tau_b)} \times \mathbf{D}_N^{(1,\tau_b)} \times \cdots \times \mathbf{D}_N^{(1,\tau_b)}}_{n\ times}\Psi_{\tau_b,N}(\tau) \triangleq \mathbf{D}_N^{(n,\tau_b)}\Psi_{\tau_b,N}(\tau). \tag{4.2}
$$

Theorem 4.2 *Suppose that $\zeta : [0,\tau_b] \longrightarrow (0,1)$ is a given continuous function and $\Psi_{\tau_b,N}(\tau)$ is the vector expressed in* (3.7). *Then we have*

$$
{}_0^{HH}D_\tau^{\zeta(\tau)}\Psi_{\tau_b,N}(\tau) \simeq \mathbf{Q}_N^{(\zeta,\tau_b)}\Psi_{\tau_b,N}(\tau), \tag{4.3}
$$

where $\mathbf{Q}_N^{(\zeta,\tau_b)} = [q_{ij}^{(\zeta,\tau_b)}]$ is a matrix of order $(N + 1)$ with entries

$$
q_{ij}^{(\zeta,\tau_b)} = \begin{cases}
a_{ij}^{(\zeta,\tau_b)}, & 2 \le i \le N+1, 1 \le j \le N+1, \\
0, & \text{otherwise,}
\end{cases}
$$

in which

$$
\begin{aligned}
a_{ij}^{(\zeta,\tau_b)} &= \frac{1}{\sqrt{\sigma(i-1,N)}}\sum_{l=0}^{N}\sum_{k=1}^{i-1}\sum_{m=1}^{k}(-1)^k\binom{i-1}{k}\binom{i+k-1}{k} \\
&\quad \times \frac{S_k^{(m)}}{N^{(k)}}\frac{m!l^m}{1-\zeta(\frac{\tau_b}{N}l)}\mathbf{E}_{1,m+1}\left(\frac{-(\frac{\tau_b}{N}l)\zeta(\frac{\tau_b}{N}l)}{1-\zeta(\frac{\tau_b}{N}l)}\right)L_{\tau_b,j-1}\left(\frac{\tau_b}{N}l;N\right).
\end{aligned}
$$

Proof Regarding (2.3), we have ${}_0^{HH}D_\tau^{\zeta(\tau)}L_{\tau_b,0}(\tau;N) = 0$. So, in the matrix $\mathbf{Q}_N^{(\zeta,\tau_b)}$, the first row should be zero. Assume $\hat{i} \ge 1$ and $\zeta : [0,\tau_b] \longrightarrow (0,1)$ be a continuous function. From (2.3) and (3.1), we get

$$
\begin{aligned}
{}_0^{HH}D_\tau^{\zeta(\tau)}L_{\tau_b,\hat{i}}(\tau;N) &= \frac{1}{\sqrt{\sigma(\hat{i},N)}}\sum_{k=0}^{\hat{i}}\sum_{m=0}^{k}(-1)^k\binom{\hat{i}}{k}\binom{\hat{i}+k}{k}\left(\frac{N}{\tau_b}\right)^m\frac{S_k^{(m)}}{N^{(k)}}{}_0^{HH}D_\tau^{\zeta(\tau)}\tau^m \\
&= \frac{1}{\sqrt{\sigma(\hat{i},N)}}\sum_{k=1}^{\hat{i}}\sum_{m=1}^{k}(-1)^k\binom{\hat{i}}{k}\binom{\hat{i}+k}{k}\left(\frac{N}{\tau_b}\right)^m\frac{S_k^{(m)}}{N^{(k)}}\frac{m!}{1-\zeta(\tau)} \\
&\quad \times \tau^m\mathbf{E}_{1,m+1}\left(-\frac{\tau\zeta(\tau)}{1-\zeta(\tau)}\right).
\end{aligned}
$$

The above result can be approximated as

$$
{}_0^{HH}D_\tau^{\zeta(\tau)}L_{\tau_b,\hat{i}}(\tau;N) \simeq \sum_{\hat{j}=0}^{N} \hat{q}_{\hat{i}\hat{j}}^{(\zeta,\tau_b)} L_{\tau_b,\hat{j}}(\tau;N), \tag{4.4}
$$

where, regarding (3.5), we have

$$
\begin{aligned}
\hat{q}_{\hat{i}\hat{j}}^{(\zeta,\tau_b)} &= \sum_{l=0}^{N} \left({}_0^{HH}D_\tau^{\zeta(\tau)}L_{\tau_b,\hat{i}}(\tau;N)\right) L_{\tau_b,\hat{j}}(\tau;N)\big|_{\tau=\frac{\tau_b}{N}l} \\
&= \frac{1}{\sqrt{\sigma(\hat{i},N)}} \sum_{l=0}^{N}\sum_{k=1}^{\hat{i}}\sum_{m=1}^{k}(-1)^k \binom{\hat{i}}{k}\binom{\hat{i}+k}{k}\frac{S_k^{(m)}}{N^{(k)}}\frac{l^m m!}{1-\zeta(\frac{\tau_b}{N}l)} \\
&\quad \times \mathbf{E}_{1,m+1}\left(-\frac{(\frac{\tau_b}{N}l)\zeta(\frac{\tau_b}{N}l)}{1-\zeta(\frac{\tau_b}{N}l)}\right) L_{\tau_b,\hat{j}}\left(\frac{\tau_b}{N}l;N\right).
\end{aligned}
$$

Eventually, via the change of indices $\hat{i}=i-1$ and $\hat{j}=j-1$, and considering $q_{ij}^{(\zeta,\tau_b)}$ instead of $\hat{q}_{i-1j-1}^{(\zeta,\tau_b)}$, we obtain

$$
\begin{aligned}
q_{ij}^{(\zeta,\tau_b)} &= \frac{1}{\sqrt{\sigma(i-1,N)}} \sum_{l=0}^{N}\sum_{k=1}^{i-1}\sum_{m=1}^{k}(-1)^k \binom{i-1}{k}\binom{i+k-1}{k}\frac{S_k^{(m)}}{N^{(k)}}\frac{m!l^m}{1-\zeta(\frac{\tau_b}{N}l)} \\
&\quad \times \mathbf{E}_{1,m+1}\left(\frac{-(\frac{\tau_b}{N}l)\zeta(\frac{\tau_b}{N}l)}{1-\zeta(\frac{\tau_b}{N}l)}\right) L_{\tau_b,j-1}\left(\frac{\tau_b}{N}l;N\right)
\end{aligned}
$$

for $2 \leq i \leq N+1$ and $1 \leq j \leq N+1$. Thus, the expressed claim is proved. $\qquad\square$

For example, whenever $\zeta(\tau)=0.5+0.25\sin(\tau)$, we obtain

$$
\mathbf{Q}_5^{(\zeta,2)} = \begin{bmatrix}
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-1.45297090 & 0.61950634 & 0.34505555 & 0.10604946 & -0.02074431 & -0.01425841 \\
-0.07721547 & -1.86890658 & 1.68369319 & 0.62293619 & 0.00763093 & -0.05902572 \\
-0.63012499 & 0.20031328 & -1.56749609 & 2.41761593 & 0.51704726 & -0.09569810 \\
0.16799373 & -0.53602371 & 0.52366685 & -1.58744491 & 2.84614045 & 0.31438495 \\
-0.10105451 & 0.93727364 & -1.13652072 & 0.69242364 & -1.92755571 & 3.29194139
\end{bmatrix}.
$$

5 Computational method

In order to use the orthonormal shifted DLPs for problem (1.1) with initial and boundary conditions (1.2), we express the unknown solution as

$$
\theta(y,\tau) \simeq \Psi_{y_b,M}(y)^T \mathbf{\Theta} \Psi_{\tau_b,N}(\tau), \tag{5.1}
$$

where $\mathbf{\Theta}$ is an $(M+1)\times(N+1)$ matrix, and its elements are undetermined. Theorem 4.1 results in

$$
\theta_{yy}(y,\tau) \simeq \Psi_{y_b,M}(y)^T \left(\mathbf{D}_M^{(2,y_b)}\right)^T \mathbf{\Theta} \Psi_{\tau_b,N}(\tau). \tag{5.2}
$$

Besides, Theorem 4.2 together with the above relations yields

$$
{}_0^{HH}\partial_\tau^{\zeta(\tau)}\theta(y,\tau) \simeq \Psi_{y_b,M}(y)^T \mathbf{\Theta}\mathbf{Q}_N^{(\zeta,\tau_b)}\Psi_{\tau_b,N}(\tau) \tag{5.3}
$$

and

$$
{}^{HH}_{0}\partial_\tau^{\zeta(\tau)}\theta_{yy}(y,\tau) \simeq \Psi_{y_b,M}(y)^T\left(\mathbf{D}_M^{(2,y_b)}\right)^T\mathbf{\Theta}\mathbf{Q}_N^{(\zeta,\tau_b)}\Psi_{\tau_b,N}(\tau). \tag{5.4}
$$

In addition, we represent $\varphi(y,\tau)$ using the orthonormal shifted DLPs as follows:

$$
\varphi(y,\tau) \simeq \Psi_{y_b,M}(y)^T\mathbf{\Phi}\Psi_{\tau_b,N}(\tau), \tag{5.5}
$$

where $\mathbf{\Phi}$ is an $(M+1)\times(N+1)$ given matrix, and its elements are evaluated like in (3.8). By inserting (5.2)–(5.5) into (1.1), we obtain

$$
\Psi_{y_b,M}(y)^T\underbrace{\left(\mathbf{\Theta}\mathbf{Q}_N^{(\zeta,\tau_b)} - \mu\left(\mathbf{D}_M^{(2,y_b)}\right)^T\mathbf{\Theta}\mathbf{Q}_N^{(\zeta,\tau_b)} - \nu\left(\mathbf{D}_M^{(2,y_b)}\right)^T\mathbf{\Theta} - \mathbf{\Phi}\right)}_{\mathbf{\Lambda}}\Psi_{\tau_b,N}(\tau) \simeq 0. \tag{5.6}
$$

The functions given in (1.2) can also be approximated via the orthonormal shifted DLPs as

$$
\hat{\theta}(y) \simeq \Psi_{y_b,M}(y)^T\hat{\mathbf{\Theta}}, \tag{5.7}
$$

and

$$
\begin{aligned}
\tilde{\theta}_0(\tau) &\simeq \tilde{\mathbf{\Theta}}_0^T\Psi_{\tau_b,N}(\tau), \\
\tilde{\theta}_1(\tau) &\simeq \tilde{\mathbf{\Theta}}_1^T\Psi_{\tau_b,N}(\tau),
\end{aligned} \tag{5.8}
$$

in which $\hat{\mathbf{\Theta}}$ is an $(M+1)$-order column vector, $\tilde{\mathbf{\Theta}}_0$ and $\tilde{\mathbf{\Theta}}_1$ are $(N+1)$-order column vectors, and their elements are evaluated like in (3.5). Now, from (1.2), (5.1), (5.7), and (5.8), we obtain

$$
\Psi_{y_b,M}(y)^T\underbrace{\left(\mathbf{\Theta}\Psi_{\tau_b,N}(0) - \hat{\mathbf{\Theta}}\right)}_{\Pi_1} \simeq 0 \tag{5.9}
$$

and

$$
\underbrace{\left(\Psi_{y_b,M}(0)^T\mathbf{\Theta} - \tilde{\mathbf{\Theta}}_0^T\right)}_{\Pi_2}\Psi_{\tau_b,N}(\tau) \simeq 0, \qquad \underbrace{\left(\Psi_{y_b,M}(y_b)^T\mathbf{\Theta} - \tilde{\mathbf{\Theta}}_1^T\right)}_{\Pi_3}\Psi_{\tau_b,N}(\tau) \simeq 0. \tag{5.10}
$$

Utilizing (5.6), (5.9), and (5.10), we generate the following system:

$$
\begin{cases}
[\mathbf{\Lambda}]_{ij} = 0, & i = 1,2,\ldots,M-1, j = 2,3,\ldots,N+1, \\
(\Pi_1)_i = 0, & i = 1,2,\ldots,M+1, \\
(\Pi_2)_j = 0, \quad (\Pi_3)_j = 0, & j = 2,3,\ldots,N+1.
\end{cases} \tag{5.11}
$$

Finally, by solving (5.11) and finding the elements of the matrix $\mathbf{\Theta}$, we find a numerical solution for the primary VO fractional problem by inserting $\mathbf{\Theta}$ into (5.1).

6 Numerical examples

The approach generated using the orthonormal shifted DLPs is applied in this section for solving some numerical examples. The L_2-error of the numerical results is measured as

$$e_\theta = \left(\int_0^{\tau_b} \int_0^{y_b} \left(\theta(y,\tau) - \tilde{\theta}(y,\tau) \right)^2 dy\, d\tau \right)^{1/2},$$

where θ and $\tilde{\theta}$ are the analytic and numerical solutions, respectively. The convergence order (CO) of this approach is computed as follows:

$$\text{CO} = \log_{\frac{\bar{N}_1}{\bar{N}_2}} \left(\frac{\varepsilon_2}{\varepsilon_1} \right),$$

where ε_i and ε_2 are the first and second L_2-error values, respectively. Furthermore, $\bar{N}_i = (M_i + 1) \times (N_i + 1)$ for $i = 1, 2$ is the number of the orthonormal shifted DLPs utilized in the ith implementation. In addition, we have applied Maple 18 (with 15 digits precision) for obtaining the results. Meanwhile, the series generating the Mittag-Leffler function is applied for 25 terms.

Example 1　Consider problem (1.1) on $[0,3] \times [0,1]$ with $\mu = \nu = 1$ and

$$\varphi(y,\tau) = \sin(\tau)\sinh(y - 3).$$

This example has the analytic solution

$$\theta(y,\tau) = \sin(\tau)\sinh(3 - y).$$

So, we have

$$\theta(y,0) = 0, \qquad \theta(0,\tau) = \sin(\tau)\sinh(3), \qquad \theta(3,\tau) = 0.$$

We have applied the expressed method for this example with three choices of $\zeta(\tau)$. The extracted results are listed in Table 1. This table shows the high-precision of the proposed approach in solving this example. It also confirms that the results have a high degree of convergence. The last column of this table confirms the low computational works of the presented algorithm. Graphical behaviors of the extracted results for $\zeta(\tau) = 0.50 + 0.25\sin(\tau)$ where $(M = 9, N = 8)$ are illustrated in Fig. 1. This figure shows the high accuracy of the presented method for obtaining a smooth solution for this example.
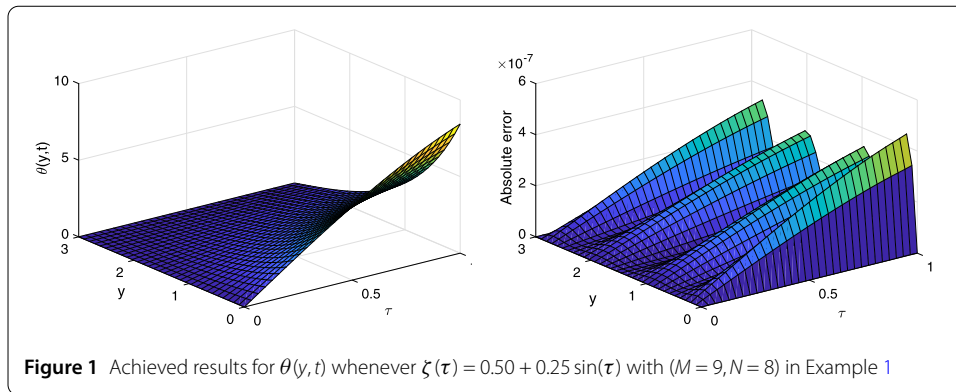
Table 1 Results extracted via the presented approach for Example 1 with three choices of $\zeta(\tau)$

M	N	$\zeta(\tau) = 0.50 + 0.25\sin(\tau)$		$\zeta(\tau) = 0.85 - 0.25e^{-\tau}$		$\zeta(\tau) = 0.65 + 0.25\tau^3\cos(\tau)$		CPU time
		e_θ	CO	e_θ	CO	e_θ	CO	
5	4	4.8716E-03	–	4.8703E-03	–	4.8704E-03	–	02.54
6	5	4.3022E-04	07.2127	4.3018E-04	07.2122	4.3018E-04	07.2122	05.29
7	6	3.2230E-05	09.0078	3.2171E-05	09.0139	3.2179E-05	09.0130	12.71
8	7	2.6413E-06	09.9541	2.6412E-06	09.9470	2.6379E-06	09.9529	31.04
9	8	1.9235E-07	11.7400	2.1424E-07	11.2568	2.1424E-07	11.2512	64.64

Figure 1 Achieved results for $\theta(y, t)$ whenever $\zeta(\tau) = 0.50 + 0.25 \sin(\tau)$ with ($M = 9, N = 8$) in Example 1
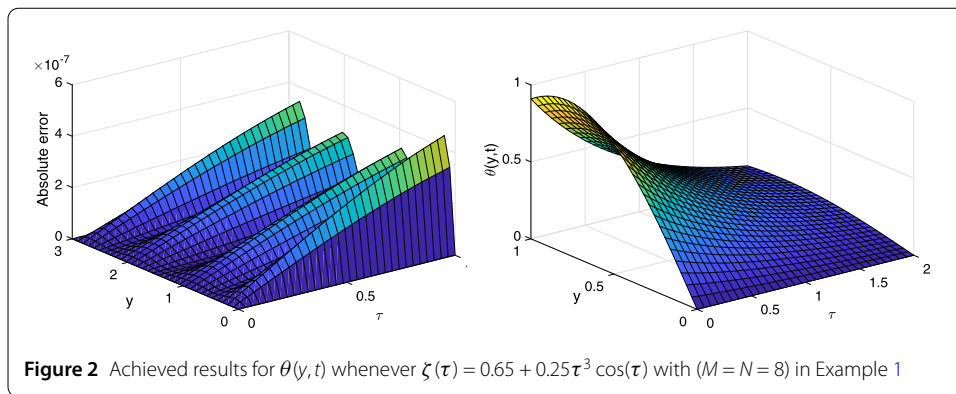
Table 2 Results extracted via the presented approach for Example 2 with three choices of $\zeta(\tau)$

M	N	$\zeta(\tau) = 0.50 + 0.25 \sin(\tau)$		$\zeta(\tau) = 0.85 - 0.25 e^{-\tau}$		$\zeta(\tau) = 0.65 + 0.25\tau^3 \cos(\tau)$		CPU time
		e_θ	CO	e_θ	CO	e_θ	CO	
4	4	1.6710E-03	–	1.6731E-03	–	1.6713E-03	–	05.70
5	5	1.6538E-04	06.3430	1.6555E-04	06.3436	1.6540E-04	06.3431	12.96
6	6	6.7121E-06	10.3935	6.7288E-06	10.3887	6.7221E-06	10.3890	23.78
7	7	6.8627E-07	08.5388	6.8598E-07	08.5496	6.8298E-07	08.5623	51.53
8	8	1.5974E-08	15.9628	1.5855E-08	15.9928	1.5414E-08	16.0939	92.15

Figure 2 Achieved results for $\theta(y, t)$ whenever $\zeta(\tau) = 0.65 + 0.25\tau^3 \cos(\tau)$ with ($M = N = 8$) in Example 1

Example 2 Consider problem (1.1) on $[0, 1] \times [0, 2]$ with $\mu = \frac{1}{2}$, $\nu = 1$ and

$$\varphi(y, \tau) = \left(4e^{-\tau} - \frac{1}{1 - \zeta(\tau)} \sum_{l=0}^{\infty} (-\tau)^{l+1} \mathbf{E}_{1, l+2} \left(-\frac{\tau \zeta(\tau)}{1 - \zeta(\tau)} \right) \right) \sin(2y).$$

This example has the analytic solution

$$\theta(y, \tau) = e^{-\tau} \sin(2y).$$

Thus, we have

$$\theta(y, 0) = \sin(2y), \qquad \theta(0, \tau) = 0, \qquad \theta(1, \tau) = e^{-\tau} \sin(2).$$

The technique established upon the orthonormal shifted DLPs is implemented for this example. The gained results are provided in Table 2, and they confirm the high-precision and

low computations of the approach. It can also be seen that as the number of the orthonormal shifted DLPs increases, the accuracy of the results increases rapidly. The obtained results with ($M = N = 8$) whenever $\zeta(\tau) = 0.65 + 0.25\tau^3 \cos(\tau)$ are shown in Fig. 2. This figure illustrates that the proposed method can provide a highly accurate solution for this example across the domain.

7 Conclusion

In this study, the Heydari–Hosseininia fractional differentiation as a kind of nonsingular variable-order (VO) fractional derivative was utilized for generating a VO fractional version of the Sobolev equation. The orthonormal shifted discrete Legendre polynomials (DLPs) as a convenient family of basis functions were employed to generate a numerical algorithm for this equation. A new fractional operational matrix related to VO fractional differentiation of these polynomials was obtained. The established scheme converts solving the problem under consideration into solving an algebraic system of equations. The validity of this technique was investigated by solving two numerical examples. The obtained results confirmed that the established method is able to generate numerical solutions with high accuracy for such problems even by applying a small number of the orthonormal shifted DLPs. As future research direction, the VO fractional derivative applied in this study can be utilized for generating VO fractional version of other applicable problems, such as Schrödinger equation and advection-diffusion equation.

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