11-1-1982

# An active filter design program (theory and application)

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AN ACTIVE FILTER DESIGN PROGRAM
(theory and application)

by

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A Thesis Submitted
in
Partial Fulfillment
of the
Requirements for the Degree of
MASTER OF SCIENCE
in
Electrical Engineering

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ROCHESTER INSTITUTE OF TECHNOLOGY
ROCHESTER, NEW YORK
NOVEMBER, 1982

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## ACKNOWLEDGEMENTS

I wish to thank all those who supported the development of this thesis. My advisor, Dr. Edward Salem, showed much patience and wisdom in guiding me through the entire thesis development. I also thank him for providing my first exposure to this field. His enthusiasm and knowledge provided the catalyst for my interest in filter design. I thank Elizabeth Russo who typed the first manuscript of the thesis. My deepest appreciation to Dorothy Haggerty who expertly typed the entire final thesis text. Her relentless efforts and valuable comments were an impetus for the completion of the thesis. To all of my family especially my wife Marcy and our children, I dedicate this thesis. Their value of education and loving support were the foundation of this effort. Most importantly, I thank the one person who made this all possible, our maker.

ABSTRACT

AN ACTIVE FILTER DESIGN PROGRAM
(theory and application)

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This thesis deals with the design of filters in the frequency domain. The intention of the thesis is to present an overview of the concepts of filter design along with two significant developments: a comprehensive filter design computer program and the theoretical development of an Nth order elliptic filter design procedure.

The overview is presented in a fashion which accents the filter design process. The topics discussed include defining the attenuation requirements, normalization, determining the poles and zeros, denormalization and implementation. For each of these topics the text addresses the fundamental filter types (low pass through band stop). Within the topic of determining the poles and zeros, three classical approximations are discussed: the Butterworth, Chebyshev and elliptic function. The overview is concluded by illustrating selected methods of implementing the basic filter types using infinite gain multiple feedback (IGMF) active filters.

The second major portion of the thesis discusses the structure, use and results of a computer program called FILTER. The program is very extensive and encompasses all the design processes developed within the thesis. The user of the program experiences an interactive session where the design of the filter is guided from parameter entries through the response evaluation and finally the determination of component values for each stage of the active filter. Complete examples are given.

Included within the program is an algorithm for determining the transfer function of an Nth order elliptic function filter. The development of the theory and the resulting design procedure are presented in the appendices. The elliptic theory and procedure represent an important result of the thesis effort. The significance of this development stems from the fact that methods of elliptic filter design have previously been too disseminated in the literature or inconclusive for an Nth order design approach. Included in this development is a rapidly converging method of determining the precise value of the elliptic sine function. This function is an essential part of the elliptic design process.

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# CHAPTER I

# INTRODUCTION

## 1.1 Historical Introduction and Comments

The science of signal filtering has evolved from its infancy 66 years ago to a present day design technology. Inherent in this growth has been the determination of network theorists to develop new filter concepts and design procedures. A multitude of information sources and computer programs have evolved to such purpose. For some engineers this represents a variety of solutions while to others an informational dilemma. Even with all this, there are still areas in filter design which remain undeveloped. Let us briefly examine the historical background of network and filter design synthesis and then examine the needs which exist in this field today.

The concept of an electric filter was initially proposed in 1915 by K. Wagner of Germany and G. Campbell of the United States. The concepts were a result of their initial work, performed independently, which related to loaded transmission lines and classical theories of vibrating systems. The first practical method of filter design became available when, in 1923, O. Zobel proposed a synthesis method using multiple reactances [1]. This method was used until the 1950's when W. Cauer and S. Darlington published new network synthesis concepts related to the use of rational functions (in particular the Chebyshev rational function) [2]. Later, their concepts were recognized as the foundation for modern day filter design. In the 1960's, network synthesis concepts and procedures expanded and were proliferated by the classic texts written by M. Van Valkenberg [3], L. Weinberg [4], A. Zverev [5] and many others.

These network synthesis concepts, which resulted in filter design techniques, became practical with the advent of the digital computer. Prior to this, complex mathematics related to solving the insertion loss characteristics of some filters were tedious and not exact. Such was the case for solutions to the Chebyshev rational function as described by Darlington and Cauer [2]. With the computer, these solutions were no longer a barrier to filter design. As a result, tables and graphs became available to the engineer for implementing these complex filter designs [6], [7]. Many filter forms and classes have since evolved from these original ideas. Examples of these filters are the Butterworth, Bessel, Chebyshev, image parameter, helical, crystal, etc.

Today, the electric filter manifests itself not only in electrical and electronic fields but in most of the scientific community. For the layman, the devices that have resulted from the remarkable developments in network synthesis are literally packed into many consumer products as special features (consider the controls on stereo players and recorders as an example). Yet the development of filter concepts and their applications do not end here.

New technologies are rapidly developing which combine the computational power of the 16 bit and forthcoming 32 bit microprocessors with filter synthesis concepts [8]. Image processing and voice analysis/synthesis are some examples. These applications become increasingly practical with advances in integrated circuits which implement complex filter functions. (As an example, consider the FLT-U2 universal active filter chip produced by Datel Intersil). Indeed, filter design principles have set today's standard in technology. Their concepts and devices are used as research tools by many fields in the scientific community. The concepts of network synthesis are therefore essential as a basic tool for the engineer. There remains however a realistic problem between filter concepts and filter design.

Not all engineers are conversant with the different types, terminology and techniques of filter design. Certainly, the basic ideas used, such as those derived from the theory of linear systems (Fourier, Laplace transforms, etc.), have been presented to engineers. However, the everyday involvement of engineering responsibilities does not always afford the opportunity to develop and make direct use of network synthesis and filter design concepts. This suggests that something additional, perhaps a computer program, is needed to assist these engineers. Although design aids such as tables and graphs are available in a variety of texts, the design methodology often remains buried in complex analyses. Those who are involved with filter design realize that such types as the elliptic function filters are not practical without a computer due to the complexity and precision of the mathematics involved. In light of these problems, computer programs have been written to aid the filter design process [9],[10]. Many of these programs are useful but very limited. Most consider only a particular form or type of filter such as the low pass Butterworth or Chebyshev approximations. There are other filter types remaining such as the high pass, band pass and band stop (reject). Still, other approximations exist for each of the above filter types. Examples are the Bessel and elliptic function filters. A person then realizes that there remains a need for a computer program which combines the filter types and approximations used often by engineers. This program should assimilate all the facts, parameter variability, precision and design techniques into one source which could be drawn upon as a practical tool. The days of research involved in evaluating various filter types and their performance would then be minimized. Such is the purpose of this thesis.

## 1.2 Thesis Objective and Scope

The objective of this thesis is to develop a computer program which could be used by engineers as a practical design tool for electronic active filter analysis and design in the frequency domain. The result of this effort is a computer program called FILTER which assimilates into one comprehensive algorithm the filter designs used often by engineers. The program FILTER implements the low pass, high pass, band pass and band stop magnitude responses using any of three classical approximations: Butterworth, Chebyshev and elliptic (Cauer). The complex mathematics, sorting of parameters and numerous iterations involved in these designs are transparent to the user of the program. Instead, the user experiences a simple interactive session which guides the design process from the initial step of defining the parameters of the magnitude response to the final step of component selection for an active filter circuit configuration.

To supplement this design program, this thesis text is provided. The scope of the text is limited to the design concepts of filters in the frequency domain. The thesis text begins with the basic definitions and descriptions of filters; then it progresses through the theory of approximations and culminates with a design methodology for the basic filter types. These design concepts are well established with the exception of the elliptic function design procedure.

One of the major efforts in this thesis was the development of an elliptic function design method. After analyzing the various sources of literature, it was evident that the elliptic design procedures were quite complex, incomplete and disseminated. Much remained between the theory and a practical design procedure. Because the mathematics are complex, involving elliptic integrals and elliptic trigonometric functions, a computer program would be a necessity for implementing the elliptic filter on a practical basis. Programs have been written for elliptic function filters but again they are limited in scope, remain undocumented in analysis and method or are just unavailable due to some propriety. A clearly stated elliptic design procedure, even if complex, is presently needed. This is especially true for active filter design. R. W. Daniels has presented active filter concepts in his text[10]. His text is a most informative source on elliptic design since it includes both programs and analyses pertaining to elliptic parameters. Yet a definitive procedure with analytical support still remains somewhat nebulous for his solutions to the elliptic integral and elliptic sine. These uncertainities have been clarified and developed within this text.

An elliptic design method is presented in this thesis and is implemented by the FILTER program. The methods are based upon the elliptic function theory presented by A. J. Grossman in his article "Synthesis of Tchebycheff Parameter Symmetrical Filters"[11]. Interestingly, his article describes the ideas presented by S. Darlington and then follows through with the determination of the poles and zeros. This is one of the most important steps in filter design. His discussion however was limited to odd order functions and did not derive solutions to the elliptic sine or integral. His interpretation, nonetheless, was clearly presented. In this thesis, the concepts presented by Grossman are expanded to include the total design process of any order low pass through band stop filters. The active filter implementation technique used is based upon the concepts presented by R. W. Daniels[10]. His concepts were expanded here to include multiple iterations of reasonable component values as a solution to implementing a larger range and optimization of elliptic filters.

In summary, it is hoped that the program FILTER along with this text, offer a practical design tool and reference for those filter types most often used by engineers in small signal processing. In addition, the development of the elliptic function design procedure will hopefully promote its use.

The following is a discussion on the organization and purpose of the chapters within the thesis text.

## 1.3 Thesis Organization

This text starts with the fundamental concepts of filters and builds into a final design methodology. With this in mind, the contents of the remaining text, 6 chapters in all, will now be discussed.

Chapter II provides an overview of the basic filter concepts which are commonly drawn upon throughout the text. The terminology and graphical representations of the four filter types are presented. These types are the low pass, high pass, band pass and band stop. Initially these types are discussed from an ideal point of view. Subsequently the realistic forms and their terminology are presented. This leads to the transfer function, H(s), which conveniently describes the total filter response (magnitude and phase).

Chapter III introduces the concept of the normalized low pass filter. The importance of the normalized low pass filter as the focal point of the design process is established. In this normalization process, methods are shown for converting each of the four filter types into the normalized low pass form. The concept of the insertion loss function is also presented. These concepts are then expanded to consider the Butterworth, Chebyshev and elliptic function approximations in their normalized low pass form. Their magnitude and phase responses are considered along with the pole/zero locations on the s-plane. Comparisons are then made among these filter characteristics. As a result of these discussions, the reader should develop an intuitive feeling for the differences in the magnitude responses and their advantages or disadvantages. In the design process, where the normalized low pass filter is used, a person then realizes the choice among the three approximations to meet his/her design specifications.

Chapter IV describes how the normalized low pass filter relates to the other basic magnitude types such as high pass, band pass and band stop filters. These relationships, described by two processes, normalization and transformation, are concisely and mathematically presented. Having determined the poles and zeros, the process of denormalization is presented as the step required to return from this normalized low pass domain. The denormalized poles/zeros are the final pieces of information needed to begin the hardware implementation. The theoretical development of the design process is completed in this chapter. The steps of normalization, transformation, deriving the poles/zeros and denormalization have all been developed and presented in chapters II-IV. What is needed at this point is a section which pulls together all these concepts into practical design methods.

Chapter V is a section on active filter design methodology. Here the concepts of normalization through denormalization are applied for each filter type (low pass through band stop). These methods present the total design process which starts by defining the desired filter response parameters and culminates with an active filter circuit.

Chapter VI introduces the program FILTER. This program is a comprehensive algorithm which implements all the concepts previously developed. This chapter then describes how the program is structured to handle the design variabilities and filter responses. A program map and general flowchart aid this discussion.

Chapter VII is the user manual for the program FILTER. The specifications, definitions of parameters and examples are given. The reader should at least glance at this section and see the example program executions.

In the appendix, the development of the elliptic function design method is presented with an example and theory of elliptic parameters. This appendix is informative if a person desires an in-depth interpretation of the elliptic parameters and how they relate to the design procedure.

# CHAPTER II

# BASIC FILTER CONCEPTS

## 2.1 Introduction

This chapter provides an overview of the fundamental concepts which are used to describe the characteristic responses of the basic filter types. The basic filter types are the low pass, high pass, band pass and band stop. This overview begins with a discussion of the magnitude and phase characteristics of the ideal frequency domain filter. By using the ideal form, the terms, definitions and analyses of the response characteristics can be presented without great complexity. Following this, a discussion is presented which explains how the magnitude shape of the ideal frequency domain filter is altered due to practical limitations. This leads to the straight line transition diagram which represents the non-ideal or realistic magnitude response of the filter. These diagrams are presented for each of the four basic filter types along with the symbols and definitions which are to be used commonly throughout the text. The transfer function, H(s), is then introduced as a convenient mathematical representation of the realistic response of the filter. The transfer function describes the magnitude response as represented with the straight line transition diagram, as well as the phase response. The general form of H(s) is presented. Then the particular forms of the transfer functions and their representations on the s-plane are presented for each of the four filter types.

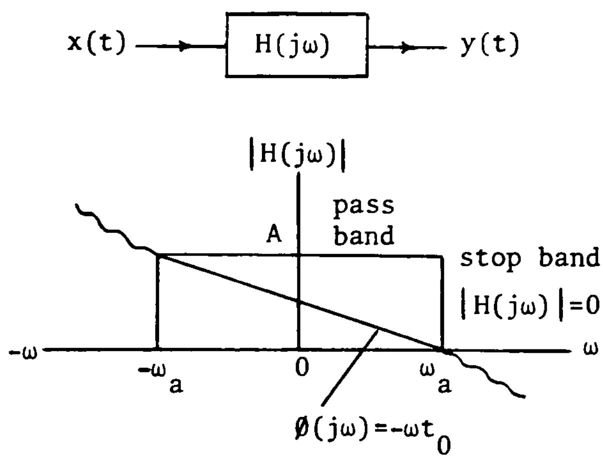
By the end of this chapter, the reader should have an understanding of the parameter definitions which describe the magnitude response of the four filter types. In addition, the reader should be acquainted with the representations of the filter types using the transfer function, H(s).

## 2.2 The Ideal Frequency Domain Filter

In this section the four basic filter types (low pass, high pass, band pass and band stop) are introduced in their ideal form. The purpose here is to review the magnitude shapes in the frequency domain and introduce the terminology associated with these filters. Emphasis is placed on examining the theory related to the low pass filter characteristics. The remaining filter types are simply presented without theoretical involvement. This is because their ideal nature is similar to the low pass and the same analytical thought process applies.

### 2.2.1 The Ideal Low Pass Filter

Figure 2.1 illustrates the magnitude and phase characteristics of the ideal low pass filter. The characteristics of the filter are represented by $H(j\omega)$. The input and output signals are $x(t)$ and $y(t)$ respectively. The ideal low pass filter and any ideal filter has a constant magnitude in the frequency band of interest (pass band). This is represented by $|H(j\omega)| = A$ in the pass band as




Ideal Frequency
Domain Filter
(low pass)

Figure 2.1

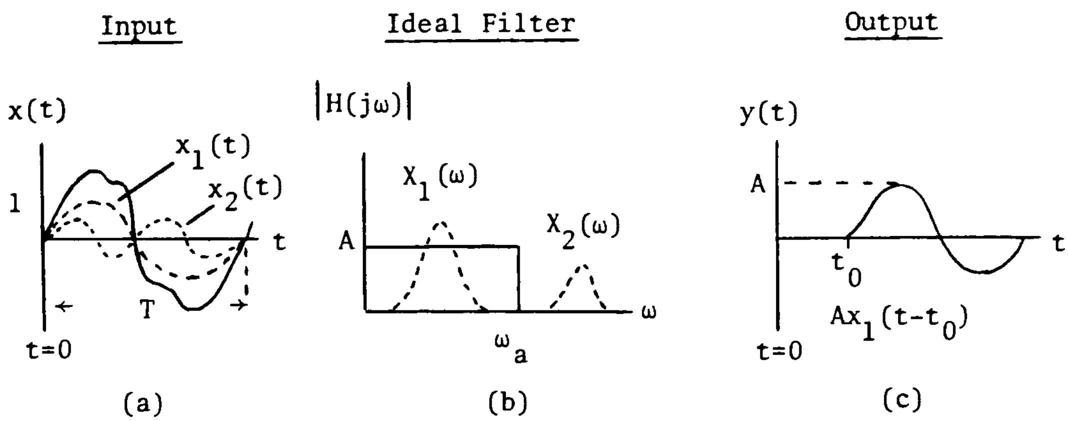
shown in figure 2.1. Outside of the pass band the magnitude is zero. This region is defined as the stop band with $|H(j\omega)|=0$ as shown in figure 2.1. The ideal filter also responds with a linear phase shift, $\emptyset(j\omega)$, throughout the pass band. Beyond the pass band the phase shift is of no concern since, ideally, there is no output signal. Therefore, any phase shift can exist. These ideal characteristics can be summarized for the low pass filter as follows:

$$\text{Magnitude} = |H(j\omega)| = A \qquad \text{for } -\omega_a<\omega<\omega_a$$
$$= 0 \qquad \text{for } |\omega|>\omega_a$$

$$\text{Phase} = \emptyset(j\omega) = -\omega t_0 \qquad \text{for } -\omega_a<\omega<\omega_a$$
$$= \text{any} \qquad \text{for } |\omega|>\omega_a$$

where $\omega_a$ refers to the cutoff frequency. Let us examine the reason why the phase shift, $\emptyset(j\omega)$, is $-\omega t_0$ and then derive the function, $H(j\omega)$, by using the Fourier transform.

If it is assumed that the filter removes energy from the input signal, $x(t)$, by using some combination of reactive elements, then the output signal, $y(t)$, will have a phase shift which results in a time delay relative to the input. Let this time delay be designated by $t_0$ seconds. This delay, $t_0$, is illustrated in figure 2.2c where an input signal $x(t)$ has been put through an ideal low pass filter to produce $y(t)$. The input signal, $x(t)$, is composed of two separate signals $x_1(t)$ and $x_2(t)$ as shown in figure 2.2a. $x_1(t)$ consists of frequencies within the pass band of the ideal low pass filter. This is represented by $X_1(\omega)$ in figure 2.2b. The signal $x_2(t)$ consists of frequencies in the stop band, shown as $X_2(\omega)$. The output signal, $y(t)$, is the filtered




Time Delay ($t_0$) Characteristics
of the
Ideal Low Pass Filter

Figure 2.2

pass band signal component, $x_1(t)$, altered in magnitude to A and time delayed by $t_0$ seconds (figure 2.2c). The magnitude, A, is usually less than the input magnitude unless the filter device is active. The frequency components in the stop band, $X_2(\omega)$ as shown in figure 2.2b, have been totally removed by the filter. The time delay, $t_0$, can be related to the period, T, of any frequency component in $x_1(t)$ as follows:

$$\text{Delay} = \frac{-t_0}{T} \qquad \text{(cycles)}$$

or in terms of phase,

$$\phi = -\frac{2\pi}{T} \cdot t_0 \qquad \text{(radians)}$$

The phase shift can then be described as a function of frequency since

$\omega = 2\pi/T$. Therefore,

$$\phi(j\omega) = -\omega t_0 \tag{2.1}$$

where $\omega$ is any pass band frequency and $t_0$ is the time delay. Equation (2.1) can also be rearranged to determine the time delay, $t_0$, in terms of phase shift, $\phi(j\omega)$, as shown below.

$$t_0 = \frac{\phi(j\omega)T}{2\pi} = \frac{\phi(j\omega)}{2\pi} \cdot \frac{1}{f}$$

Or in terms of degrees, we have

$$\text{Time delay} = t_0 = \frac{\phi(j\omega)}{360^\circ} \cdot \frac{1}{f} \qquad \text{(seconds)} \tag{2.2}$$

where $\phi(j\omega)$ is the phase shift associated with any pass band frequency, f.

The group delay, $T_g$, is the slope of the phase shift, $\phi(j\omega)$, at a particular frequency, $\omega_x$. This is mathematically represented as

$$\text{Group delay} = T_g = -\left.\frac{d\phi(j\omega)}{d\omega}\right|_{\omega=\omega_x} \tag{2.3}$$

Since the ideal filter illustrated in figure 2.1 has the same phase slope throughout the pass band, the group delay is the same for all frequencies and is found as follows:

for

$$\phi(j\omega) = -\omega t_0$$

the group delay is

$$T_g = -\frac{d\phi(j\omega)}{d\omega} = t_0$$

where $t_0$ is the delay time previously specified and shown in figure 2.2c. Having specified the pass band magnitude as A and the time delay as $t_0$ seconds, the output signal y(t) can be related to the input x(t) by

$$y(t) = A \cdot x_1(t-t_0) + 0 \cdot x_2(t-t_0)$$

which simplifies to

$$\text{Output signal} = y(t) = A \cdot x(t-t_0) \tag{2.4}$$

The input and output signals can be represented in the frequency domain by use of the Fourier transform. This will lead to the transfer function, $H(j\omega)$, for the ideal low pass filter. For the input signal $x_1(t)$ existing for time T, the Fourier transform is

$$\mathcal{F}[x(t)] = X(\omega) = \int_0^T x(t)\ e^{-j\omega t} dt$$

Since

$$x(t) = x_1(t) + x_2(t)$$

and by use of the superposition principle, we have

$$\mathcal{F}[x_1(t) + x_2(t)] = \int_0^T x_1(t)\ e^{-j\omega t} dt + \int_0^T x_2(t)\ e^{-j\omega t} dt \tag{2.5}$$

$$= X_1(\omega) + X_2(\omega)$$

For the output signal, $y(t)$, the transform is

$$F[y(t)] = Y(\omega) = \int_{t_0}^{T+t_0} y(t)\ e^{-j\omega t} dt$$

Substituting $y(t) = Ax_1(t-t_0)$ from equation (2.4) into $Y(\omega)$, we have

$$Y(\omega) = \int_{t_0}^{T+t_0} Ax_1(t-t_0)\ e^{-j\omega t} dt \tag{2.6}$$

Letting $T = t-t_0$, then

$$t = T+t_0$$

and

$$dt = dT$$

If the expressions $T = t-t_0$ and $dt = dT$ are substituted into equation (2.6) for $Y(\omega)$ then

$$Y(\omega) = \int_{0}^{T} Ax_1(T)\ e^{-j\omega(T+t_0)} dT$$

or

$$Y(\omega) = \underbrace{A \cdot e^{j\omega t_0}}_{H(j\omega)} \cdot \underbrace{\int_0^T x_1(T)\, e^{-j\omega T}\, dT}_{X_1(\omega)} \qquad (2.7)$$

The right hand portion of equation (2.7) was previously shown to be $X_1(\omega)$ in equation (2.5). Thus the Fourier transform of the output signal is

$$Y(\omega) = Ae^{-j\omega t_0} X_1(\omega) \qquad (2.8)$$

where the time delay is represented in the frequency domain as $e^{-j\omega t_0}$. The transfer function of the ideal low pass filter is then found from

$$H(j\omega) = \frac{Y(\omega)}{X(\omega)} = |H(j\omega)|\ e^{j\emptyset(j\omega)}$$

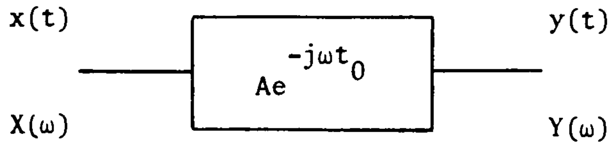
which yields

$$H(j\omega) = Ae^{j(-\omega t_0)} \qquad (2.9)$$

The representation of the ideal low pass filter with its transfer function is depicted in the following section in figure 2.3.

### 2.2.2 Summary of the Ideal Low Pass Filter Characteristics

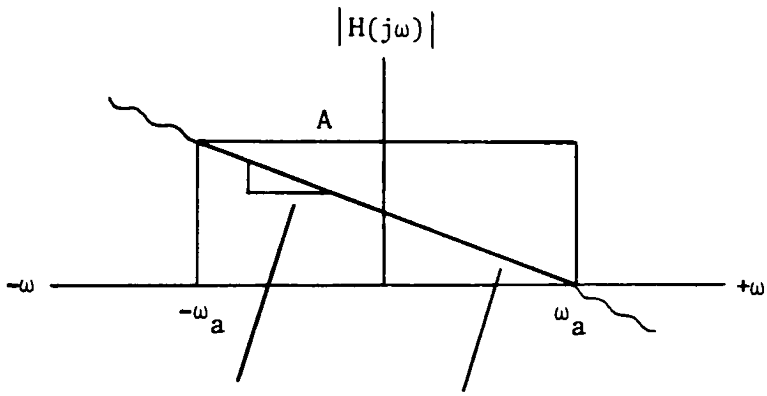
Let us summarize the characteristics which have been determined for the ideal low pass filter. Figure 2.3 illustrates these characteristics. This figure is similar to figure 2.1 and is presented here in order to complement the summary.




$$H(j\omega) = |H(j\omega)|e^{j\phi(j\omega)} = Ae^{-j\omega t_0}$$

The Transfer Function

(a)




$$-\frac{d\phi(j\omega)}{d\omega} = t_0 \qquad \phi(j\omega) = -\omega t_0$$

Magnitude and Phase
Characteristics

(b)

Summary of the Ideal Low
Pass Filter Characteristics

Figure 2.3

The magnitude characteristics, as shown in figure 2.3b, were described as

$$|H(j\omega)| = A \qquad \text{for } -\omega_a<\omega<\omega_a$$
$$= 0 \qquad \text{for } |\omega|>\omega_a$$

where $\omega_a$ is the cutoff frequency. The linear phase shift was described as

$$\emptyset(j\omega) = -\omega t_0 \qquad \text{for } -\omega_a<\omega<\omega_a$$
$$= \text{any} \qquad \text{for } |\omega|>\omega_a$$

As a result of the linear phase shift, the phase delay was described as

$$\emptyset_d = \emptyset(\omega_a) \cdot \frac{\omega_x}{\omega_a}$$

where $\emptyset(\omega_a)$ is the phase shift at the cutoff frequency and $\omega_x$ is any pass band frequency. The group delay was then described as

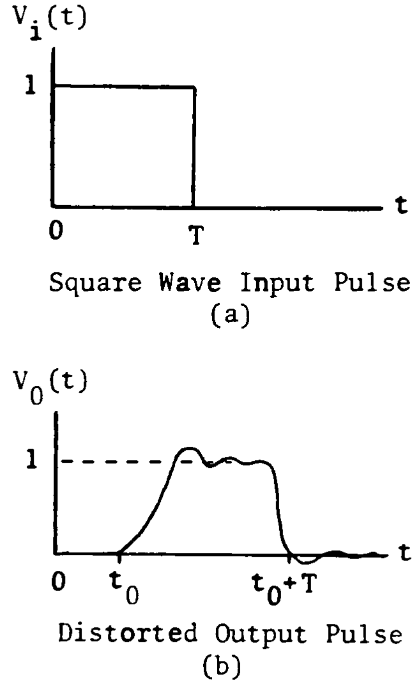
$$T_g = -\frac{d\emptyset(j\omega)}{d\omega}$$
$$= t_0 \qquad \text{for } -\omega_a<\omega<\omega_a$$

The Fourier transform of the output signal was then found as

$$Y(\omega) = Ae^{-j\omega t_0} \cdot X(\omega)$$

Having derived the ideal characteristics, now their interpretation will be examined.

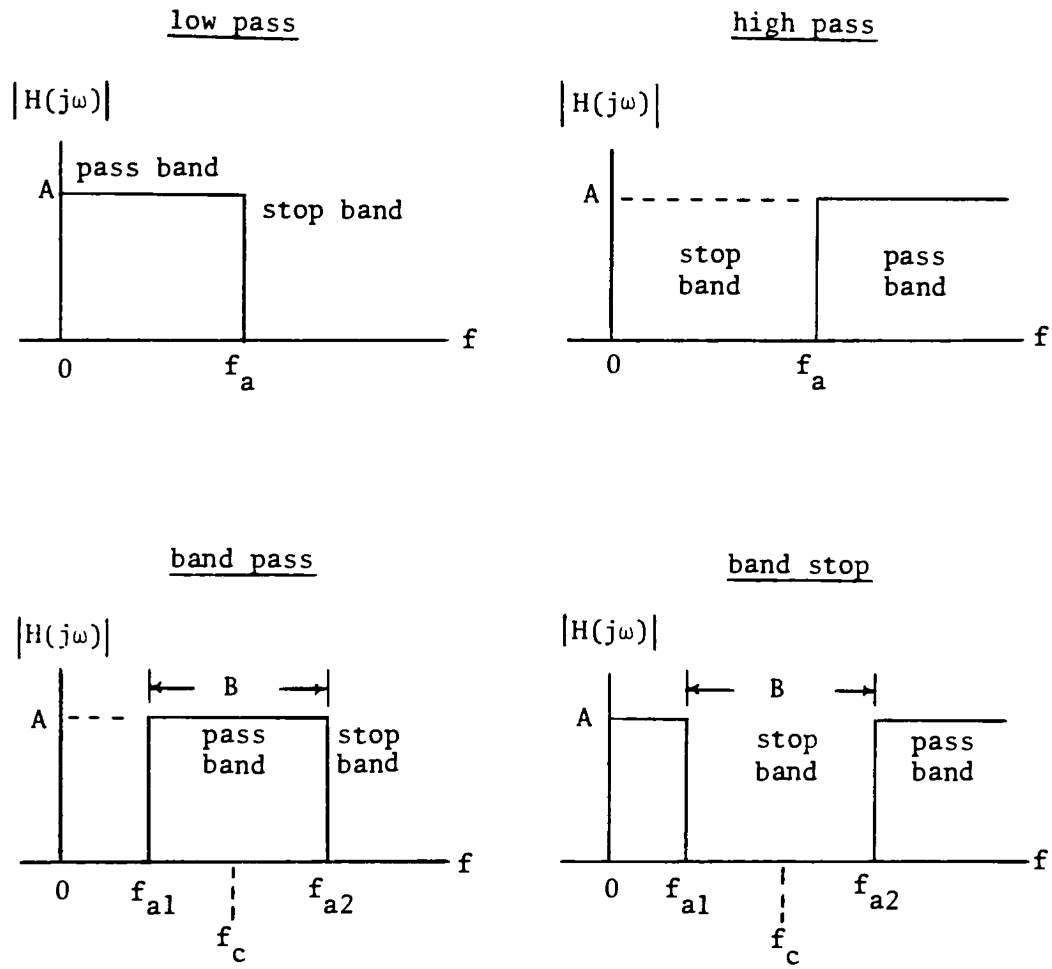
The linear phase shift and the constant group delay indicate that there is no phase distortion at any frequency in the pass band. As an example of how phase distortion affects a signal, consider the harmonics which constitute a square wave pulse as depicted in figure 2.4a. The ideal low pass filter would pass the frequency components within the pass band with equal time delay, $t_0$. If the frequency constituents are passed with unequal time delay, the output signal would contain harmonic distortion as depicted in figure 2.4b. For the ideal filter, harmonic distortion does not occur. Also the gain constant, A, indicates that the ideal filter responds equally for all frequencies in the pass band. These characteristics are ideal and as such do not really exist. These limitations are examined in detail in section 2.3. For the moment however, let us accept these ideal characteristics and use them to briefly describe the magnitude responses of the remaining filter types.




Effect of Phase Distortion on a Square Wave Pulse

Figure 2.4

The magnitude shapes of the four ideal filter types are illustrated in figure 2.5. Notice the terminology used and the parameters described at the bottom. Since they are similar to those parameters previously described for the low pass filter, we will not discuss them in great detail but simply examine some of their salient points. The




$|H(j\omega)|$ ----- magnitude of the filter response
A -------- pass band magnitude
$f_a$ ------ cutoff frequency
$f_{a1}$ ------- lower cutoff frequency
$f_{a2}$ ------- upper cutoff frequency
B -------- bandwidth = $f_{a2} - f_{a1}$
$f_c$ ------- center frequency = $\sqrt{f_{a1} \cdot f_{a2}}$

Magnitude Response and Parameter
Definitions of the Ideal Filter Types

Figure 2.5

filters all have a pass band magnitude which is constant and arbitrarily designated as A. Outside of the pass band the magnitude is zero. The main difference among these ideal filters is where their pass bands and stop bands occur on the frequency axis. The frequency band of interest describes the type of filter to be used. For the low pass and high pass filters, the pass band edge is described by one cutoff frequency, $f_a$, as shown in figure 2.5. For the band pass filter there are two cutoff frequencies, $f_{a1}$ and $f_{a2}$, which describe the region of the pass band. For the band stop filter, the region between the two cutoff frequencies, $f_{a1}$ and $f_{a2}$, describes the region of attenuation or stop band. The center frequency, $f_c$, for both the band pass and band stop filters is usually defined as the geometric mean of the two cutoff frequencies. That is

$$\text{center frequency} = f_c = \sqrt{f_{a1} \cdot f_{a2}} \tag{2.10}$$

We have thus far examined the magnitude and phase characteristics of the ideal low pass filter. We then determined the transfer function of the ideal filter which followed as a result of the time shifted output signal. The remaining filter types were then simply presented in order to illustrate their selective frequency bands of interest. Next we will be concerned with the practical restraints which are imposed upon the ideal response. As a result, the ideal magnitude response diagrams are altered to represent the realistic filter response.

## 2.3 Limitations Affecting the Ideal Low Pass Filter Response

The ideal frequency domain filter was discussed mainly to introduce the magnitude characteristics and terminology of the four basic filter types. Now the physical and temporal limitations which restrain the ideal characteristics will be examined. Specifically we will examine how these limitations affect the ideal magnitude response. Let us initially derive the time domain impulse response, h(t), of the ideal frequency domain filter. This will exemplify the temporal restraints imposed upon an ideal response.

Recall from the previous section that the response of the ideal filter can be described by its transfer function, $H(j\omega)$ (equation 2.9) where

$$H(j\omega) = Ae^{-j\omega t_0}$$

The impulse response can be determined by using the inverse Fourier transform.

That is,

$$h(t) = \mathcal{F}^{-1}\,[H(j\omega)] = \frac{1}{2\pi}\int_{-\omega_a}^{\omega_a} H(j\omega)\,e^{j\omega t}d\omega \qquad (2.11)$$

If equation (2.9) for $H(j\omega)$ is substituted into equation (2.11) we have,

$$h(t) = \frac{1}{2\pi}\int_{-\omega_a}^{\omega_a} Ae^{-j\omega t_0}\,e^{j\omega t}\,d\omega$$

or

$$h(t) = \frac{A}{2\pi} \int_{-\omega_a}^{\omega_a} e^{j\omega(t-t_0)} d\omega$$

where the limits of integration are defined as the pass band cutoff angular frequencies. This yields

$$h(t) = \frac{A}{2\pi} \cdot \frac{1}{j(t-t_0)} e^{j\omega(t-t_0)} \Bigg|_{-\omega_a}^{\omega_a}$$

or

$$h(t) = \frac{A}{2\pi} \left[ \frac{e^{j\omega_a(t-t_0)} - e^{-j\omega_a(t-t_0)}}{j(t-t_0)} \right]$$

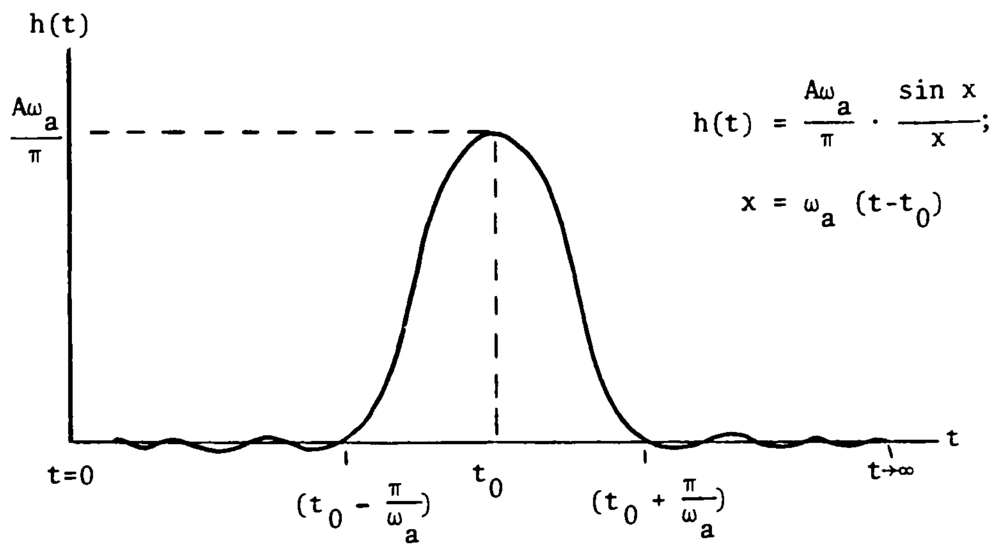
Letting $x = t-t_0$ and multiplying numerator and denominator by $\omega_a$, we then have

$$h(t) = \frac{A\omega_a}{\pi} \cdot \frac{\sin(x)}{x} \tag{2.12}$$

where

$$x = (t-t_0)\,\omega_a$$

The impulse response, $h(t)$, as described by equation (2.12) is a sinc function with a maximum amplitude of $(A\omega_a/\pi)$. This is illustrated in figure 2.6.




Impulse Response of the
Ideal Frequency Domain Filter
(Low Pass)

Figure 2.6

The impulse response curve crosses the abscissa ($h(t)=0$) whenever $x = n\pi$. Since $x = \omega_a(t-t_0)$, the impulse response is 0 when

$$\omega_a(t-t_0) = n\pi$$

or when

$$t = t_0 + \frac{n\pi}{\omega_a} \tag{2.13}$$

Some interesting characteristics are found when $\omega_a$, the cutoff frequency, is allowed to vary. As $\omega_a$ approaches infinity, the amplitude of $h(t)$ approaches infinity as can be seen by equation (2.12). Also the impulse response occurs only at $t=t_0$ as seen in equation (2.13) for $\omega_a$ equal to infinity. Therefore, as $\omega_a$ becomes extremely large, the impulse response approaches a delta function. Now we are in a position to re-examine and summarize the restrictions which prevent us from obtaining an ideal filter response. The effects of these limitations are shown in figure 2.7 and are discussed below.

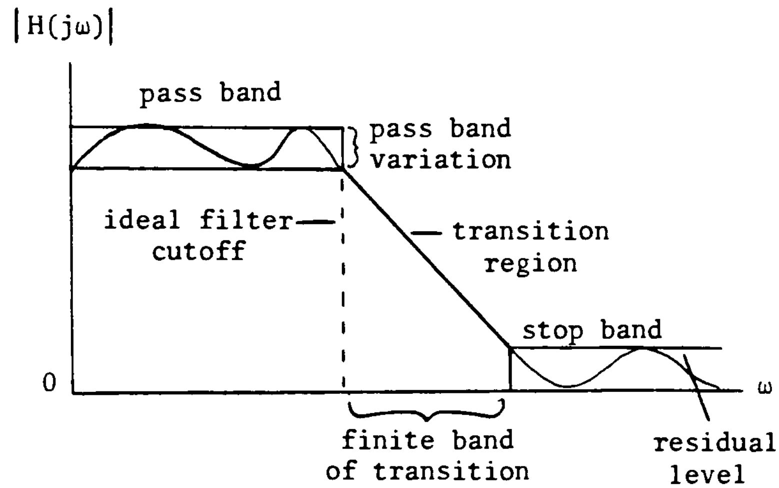
The impulse response extends from $-\infty$ to $+\infty$ in time. This infinite time span cannot actually exist. The existence of h(t) for values of t less than 0 requires the filter to respond with an output signal before the impulse input to the filter takes place.

Aside from the impulse response, imperfect filter components pose restrictions on an ideal filter response. Let us briefly discuss a few of these based upon our intuitive reasoning.

The real filter cannot respond with infinite attenuation in the stop band. This would be required in order to achieve $|H(j\omega)|=0$ in the stop band. That is, all of the energy outside of the pass band would have to be completely attenuated by the filter. This is not possible since reactive elements contain some internal resistance. Some residual noise always exists in the stop band area. Note the non-zero level in the stop band region in figure 2.7.

The pass band magnitude cannot be constant. Some pass band variation exists even if it is minute. A constant magnitude would require the reactive elements in the filter to react with equal impedance for all pass band frequencies.

The transition from the pass band to stop band cannot occur at one frequency. This would require an infinite slope at the cutoff frequency. Interpreted another way, the sensitivity of the filter components would have to be infinite near the cutoff frequency. This is not possible. Some finite transition width is required as shown in figure 2.7.




Restrictions Imposed
on the Ideal Filter
(Straight Line Transition Diagram)

Figure 2.7

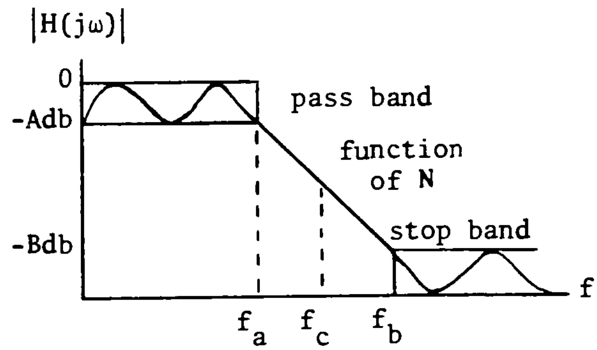
To summarize, the limitations imposed upon the ideal filter result in pass band variation, transition width and finite stop band attenuation. These effects are graphically illustrated by use of a straight line transition diagram such as shown in figure 2.7. These diagrams and their parameter definitions for the four basic filter types are presented in the next section.

## 2.4 Filter Parameters

Having examined the limitations which alter the ideal filter response, the straight line transition diagram was presented as a graphical representation of the magnitude response of the realistic filter. In this section the non-ideal magnitude shapes and parameters are presented for each of the four filter types (low pass, high pass, band pass and band stop). The mathematical representations of these responses will be addressed in section 2.5.

### 2.4.1 The Low Pass Filter Parameters

The straight line transition diagram for the low pass filter is illustrated in figure 2.8. This diagram is similar to figure 2.7 with the addition of the specific filter parameters. These parameter definitions will now be examined. From here on, when we refer to the response of a filter, it will be of the non-ideal form. Refer to figure 2.8. Within the pass band the magnitude varies from 0 db to -Adb. If it is assumed that the magnitude of the filter is at best equal to one (non-attenuating), the maximum value of $|H(j\omega)|$ is then 0 db. There exists




Transition width = $(f_b - f_a)$

The Low Pass Filter

Figure 2.8

some pass band attenuation for which the magnitude is less than one and the boundary of this variation is represented as -Adb. The pass band variation can occur as a ripple, such as shown in figure 2.8 or as a continually decreasing (monotonic) magnitude. The methods of achieving these shapes will be discussed in Chapter III. The attenuation, -Adb, is defined as the maximum pass band variation and is associated with the pass band edge (cutoff) frequency designated as $f_a$. The stop band edge frequency is represented as $f_b$. Associated with $f_b$ is the minimum stop band attenuation -Bdb. The transition width, TW, is defined as the difference between the pass and stop band edge frequencies.

That is,

$$TW = \left| f_b - f_a \right| \tag{2.14}$$

The transition width and slope are directly related to the filter order, N. As N increases, the transition width becomes narrower and results in a faster attenuation rate. The method of determining the filter order, N, is described in Chapter III. At this point it will simply be stated that N is a function of $f_a$, $f_b$, -Adb and -Bdb. The frequency, $f_c$, is defined as the center frequency of the transition region where

$$f_c = \sqrt{f_a \cdot f_b} \tag{2.15}$$

The center frequency did not hold any significance for the ideal filter since the transition rate was infinite. We will find this term, $f_c$, useful in discussing the elliptic function filter in Chapter III.

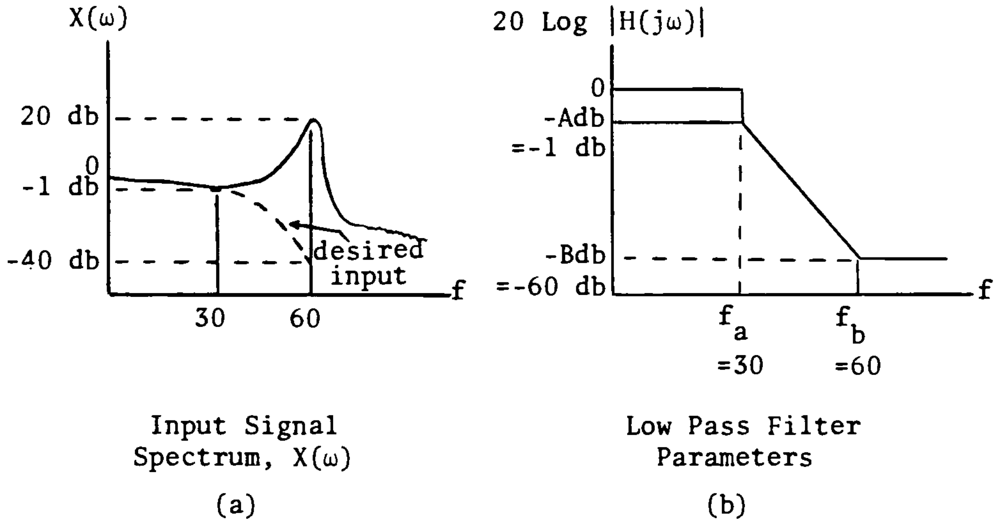
The parameters just described are summarized in Table 2.1. It should be noted that these are the parameters usually given by the designer when specifying the magnitude response of the low pass filter. The same symbols and terms will also apply to the high pass filter discussed in the following section.

Parameter Definitions
of the
Magnitude Response
(Low Pass and High Pass)

| | | |
|---|---|---|
| $f_a$ | : | pass band edge or cutoff frequency |
| $f_b$ | : | stop band edge frequency |
| -Adb | : | maximum pass band variation occurring at $f_a$ |
| -Bdb | : | minimum stop band attenuation at $f_b$ |
| $f_c$ | : | center frequency of the transition region $= \sqrt{f_a \cdot f_b}$ |
| TW | : | transition width $= \lvert f_b - f_a \rvert$ |

Table 2.1

Consider the following example where we wish to specify the magnitude response parameters as shown in Table 2.1. A system has an input signal as shown in figure 2.9a. It is desired to pass all of the low frequency components up to 30 Hz with no greater than a -1 db attenuation. This defines -Adb = -1 db at the cutoff frequency, $f_a$ = 30 Hz for a low pass filter as shown in figure 2.9b. The system also requires that the 60 Hz component within the input signal be suppressed by 40 db relative to the signals of interest. Since the 60 Hz component has an ampli-




Specification of the Low Pass Filter Parameters (example in section 2.4.1)

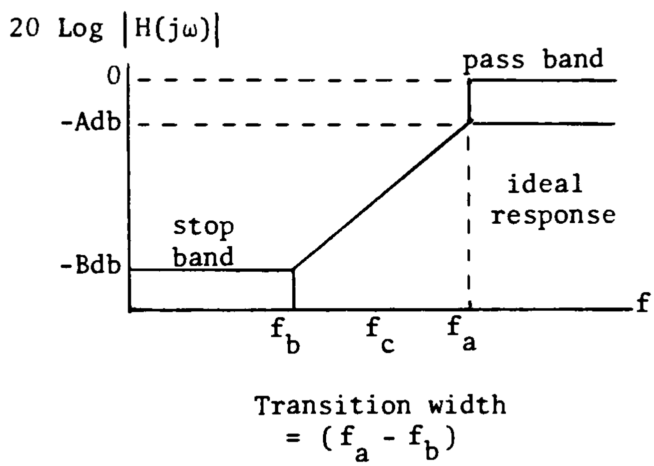
Figure 2.9

tude of 20 db above the pass band frequencies, the required attenuation by the low pass filter is a minimum of (-20 db) plus (-40 db) or -60 db. This then defines the minimum stop band attenuation required, -Bdb = -60 db, at the stop band edge frequency, $f_b$ = 60 Hz. All the parameters ($f_a$, $f_b$, -Adb, -Bdb) have therefore been defined and are illustrated along with the straight line transition diagram in figure 2.9b.

There are two points which should be noted from the example just discussed. The first is that the input or source signal is well defined in terms of its frequency spectrum. A designer must be aware of the frequency content prior to determining the filter requirements. In addition, the source signal will appear to contain various noise levels depending upon the filter input characteristics. Noise versus input impedance is one example. The second point is that the stop band attenuation is determined by the system's need for a minimum signal to noise ratio. (More accurately (S+N)/N).

### 2.4.2 The High Pass Filter Parameters

The straight line transition diagram for the high pass filter is illustrated in figure 2.10. The terms and symbols are identical to that of the low pass filter previously described. The definitions of the filter parameters are therefore the same as those in table 2.1. We will not go into describing these parameters but simply point out one possible misinterpretation. The lower frequency $f_b$ should not be mistakenly identified as the lower frequency $f_a$ such as in the case of the low pass filter. For the high pass filter, $f_b$ is a lower frequency which defines the stop band edge. This may be a trivial point but it emphasizes the convention of the terminology used.

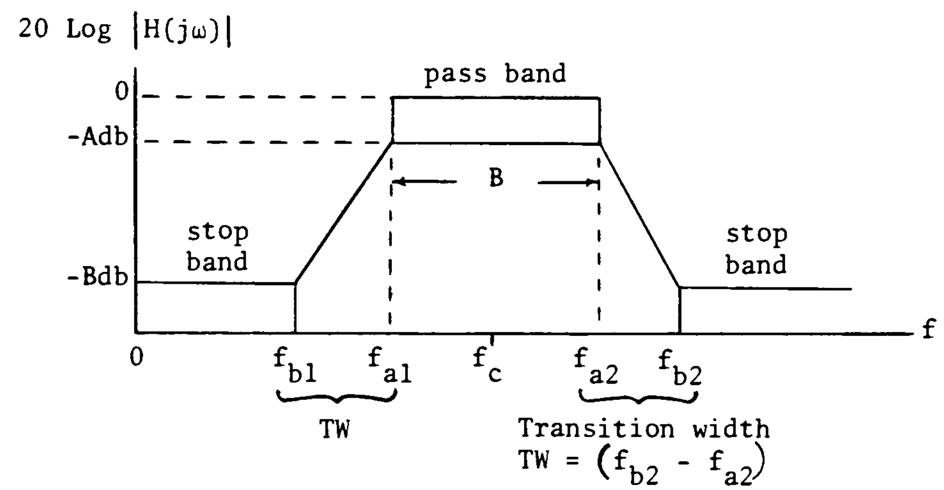



The High Pass Filter

Figure 2.10

### 2.4.3 The Band Pass Filter Parameters

The magnitude response of the band pass filter is illustrated in figure 2.11. Since there are two stop band regions, multiple pass band and stop band edge frequencies are required. If we refer to figure 2.11,




The Band Pass Filter

Figure 2.11

we see that there are two cutoff frequencies, $f_{a1}$ and $f_{a2}$. The frequency, $f_{a1}$, is referred to as the lower pass band edge frequency; $f_{a2}$ is referred to as the upper pass band edge frequency. Associated with the pass band edge (or cutoff) frequencies is the maximum pass band variation, -Adb, as shown in figure 2.11. Similarly, $f_{b1}$ and $f_{b2}$ are the lower and upper stop band edge frequencies respectively. Associated with $f_{b1}$ and $f_{b2}$ is the minimum stop band attenuation, -Bdb. The center frequency, $f_c$, is usually defined as the geometric mean of the band edge frequencies. That is,

$$\text{Center frequency} = f_c = \sqrt{f_{a1} \cdot f_{a2}}$$
$$= \sqrt{f_{b1} \cdot f_{b2}} \tag{2.16}$$

The bandwidth, B, is defined as the range of frequencies in the pass band. That is,

$$B = (f_{a2} - f_{a1}) \tag{2.17}$$

Care should be taken when specifying or interpreting the bandwidth. A person might assume that the bandwidth is associated with the -3 db pass band edge frequencies. This is not necessarily the case. Rather, the specification given should be properly associated with the design specifications $f_{a1}$, $f_{a2}$ and -Adb where -Adb is totally arbitrary. The significance of the -3 db pertains only to the Butterworth filter in the event that -Adb is unspecified. Then it is normally assumed that -Adb = -3 db and that the frequencies at -3 db define the bandwidth. A transition width (TW) can also be described for the band pass filter as

$$TW = \left| f_{b2} - f_{a2} \right| = \left| f_{b1} - f_{a1} \right| \tag{2.18}$$

This however is seldom used when specifying the band pass filter. We will see the reason for this when the normalization process is examined. The definitions and symbols just discussed are summarized in Table 2.2. Note the similarity to that of table 2.1.

When specifying the parameters for the band pass filter, a number of combinations of parameters are possible. For example, given the pass and stop band edge frequencies, the center frequency and bandwidth can be calculated using equations 2.16 and 2.17. Conversely, the pass band edge frequencies can be derived from these equations when given the center

| | | |
|---|---|---|
| $f_{a1}$, $f_{a2}$ | : | pass band edge frequencies (lower and upper respectively) |
| $f_{b1}$, $f_{b2}$ | : | stop band edge frequencies (lower and upper respectively) |
| -Adb | : | maximum pass band variation occurring at $f_{a1}$, $f_{a2}$ |
| -Bdb | : | minimum stop band attenuation at $f_{b1}$, $f_{b2}$ |
| $f_c$ | : | center frequency $= \sqrt{f_{a1} \cdot f_{a2}} = \sqrt{f_{b1} \cdot f_{b2}}$ |
| B | : | bandwidth $= (f_{a2} - f_{a1})$ |
| TW | : | transition width $= \lvert f_{b2} - f_{a2} \rvert = \lvert f_{b1} - f_{a1} \rvert$ |

Parameter Definitions of the Magnitude Response (Band Pass and Band Stop)

Table 2.2

frequency and bandwidth. The following equations describe $f_{a2}$ and $f_{a1}$ and are derived by simple manipulation of equations 2.16 and 2.17.

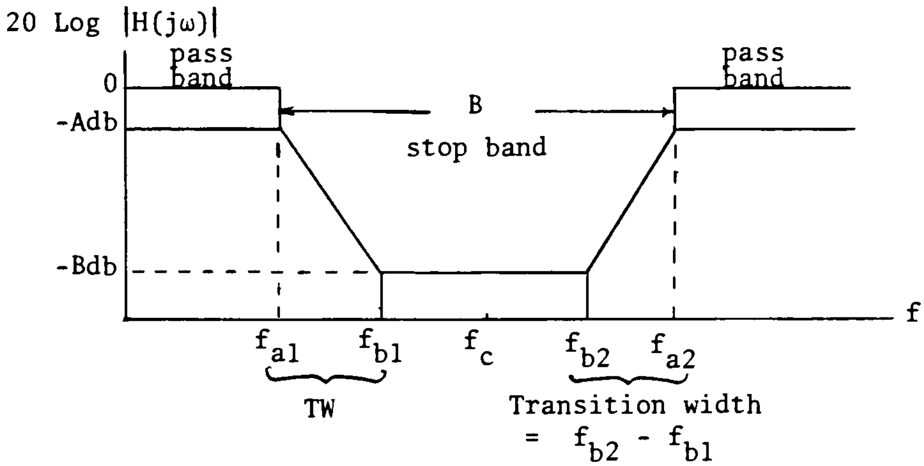
$$f_{a1} = \tfrac{1}{2}\left(-B + \sqrt{B^2 + 4 f_c^{\,2}}\right)^{\frac{1}{2}}$$

$$f_{a2} = \tfrac{1}{2}\left(B + \sqrt{B^2 + 4 f_c^{\,2}}\right)^{\frac{1}{2}} \tag{2.19}$$

The parameters which are commonly given when specifying a band pass filter are $f_c$, B, -Adb, $f_{b2}$, and -Bdb.

### 2.4.4 The Band Stop Filter Parameters

The straight line transition diagram for the band stop filter is illustrated in figure 2.12. The terms and symbols are identical with




The Band Stop Filter

Figure 2.12

those defined for the band pass filter in table 2.2. One point should be noted. The frequency difference ($f_{b2}$ - $f_{b1}$) does not describe the bandwidth. This mistake can easily occur. The bandwidth is defined as in table 2.2, that is, $B = (f_{a2} - f_{a1})$.

### 2.5 The Transfer Function, H(s)

In section 2.4 we examined the representations of the non-ideal filters using straight line transition diagrams. These diagrams illustrate the magnitude characteristics of the transfer function, that is, $|H(j\omega)|$. Now we need to consider the representation of both the magnitude and phase characteristics of the filter. This is accomplished by use of the Laplacian complex frequency, $s = \sigma + j\omega$. The relationship between the input and output of a filter is

$$H(s) = \frac{Y(s)}{X(s)} \tag{2.20}$$

where Y(s) and X(s) are found by use of the Laplace transform.

That is,

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)\, e^{-st} dt$$

where f(t) represents a time dependent function such as the filter input, x(t), or the output y(t). Notice that H(s) is shown as a rational function. We will initially examine the general form of H(s). Subsequently, the particular transfer functions associated with each of the filter types are discussed. This section will also develop the s-plane diagrams which represent the transfer function, H(s), in a graphical manner.

### 2.5.1 The General Form of H(s)

The general form of H(s) is represented as a ratio of two polynomials. That is

$$H(s) = K \frac{a_m s^m + a_{m-1} s^{m-1} + \dots + a_o}{b_n s^n + b_{n-1} s^{n-1} + \dots + b_o}$$

where the term K is a constant. If H(s) is factored, we have the following general form:

$$H(s) = \frac{(s-z_1)(s-z_2)(s-z_3) \;\ldots\; (s-z_m)}{(s-s_1)(s-s_2)(s-s_3) \;\ldots\; (s-s_n)}$$

or

$$H(s) = K \cdot \frac{\prod_{i=1}^{m} (s-z_i)}{\prod_{i=1}^{n} (s-s_i)} \qquad (2.21)$$

The roots of the numerator polynomial are $z_1$, $z_2$, .... $z_m$ and are referred to as the "zeros" of H(s). The roots of the denominator polynomial of equation 2.21 are called the "poles" of H(s). Let us examine equation 2.21 more closely and see how the terms relate to the magnitude and phase of the filter response, H(s).

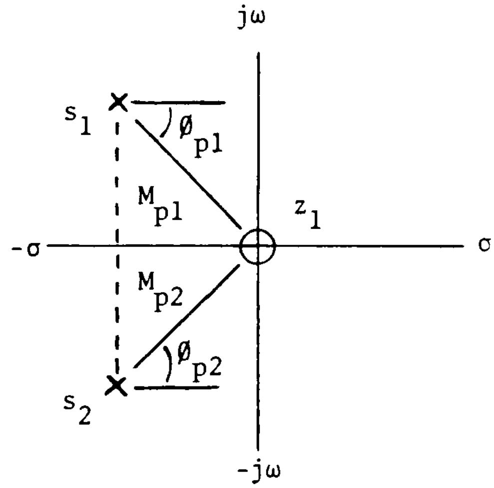
When the complex variable s, as in H(s), equals a zero value, $z_i$, the magnitude of H(s) becomes zero. In a similar manner, when s is equal to a pole value, $s_i$, the magnitude response becomes infinite. We can see that the poles and zeros control the response of the filter. Properly selected, they provide for the magnitude and phase response desired as in the case for example of a high pass filter. We will see an example of this shortly. The general form of the complex zero is represented by

$$z_i = \sigma_{zi} + j\omega_{zi} \qquad \text{for } i = 1 \text{ to } m$$

and for the poles,

$$s_i = \sigma_{pi} + j\omega_{pi} \qquad \text{for } i = 1 \text{ to } n$$

The method of determining the pole/zero values will not be developed here (see Chapter III). It will simply be stated for now that their values are a function of the parameters derived using the straight line transition diagram (-Adb, -Bdb, $f_a$, $f_b$, etc.). Since the poles and zeros are complex containing both real ($\sigma_i$) and imaginary ($j\omega_i$) terms, they can be represented as a coordinate on a two dimensional diagram called the s-plane such as shown in figure 2.13. The poles of H(s) are represented with an "x" on the s-plane; the zeros of H(s) are represented with a "○". The magnitude and phase of the individual pole/zero factors within H(s) can be determined by letting $s = j\omega$ where $\omega = 2\pi f$ and f is any arbitrary frequency at which the function H(s) is to be evaluated. That is, given the pole or zero value, $\sigma_i + j\omega_i$, and the frequency $s = j\omega$, each factor of H(s) is then



s-Plane
Diagram

Figure 2.13

$$[j\omega + \underbrace{(\sigma_i + j\omega_i)}_{s_i \text{ or } z_i}]$$

The magnitude is then found as

$$M_i = \sqrt{\sigma_i^2 + (\omega - \omega_i)^2} \tag{2.22}$$

and the phase is

$$\emptyset_i = \tan^{-1}\left(\frac{\omega - \omega_i}{\sigma_i}\right) \tag{2.23}$$

The pole or zero factor can then be represented at a particular frequency ($s = j\omega$) as

$$(s - (\sigma_i + j\omega_i)) = M_i \angle \emptyset_i$$

For equations 2.22 and 2.23 the subscripts "p" and "z" will be added to differentiate the pole values from the zero values. The total magnitude, designated as $|H(j\omega)|$, is determined by magnitudes of the individual poles ($M_p$) and zeros ($M_z$) as follows:

$$|H(j\omega)| = \frac{M_{z1} \cdot M_{z2} \cdots\cdot M_{zm}}{M_{p1} \cdot M_{p2} \cdots\cdot M_{pn}} \tag{2.24}$$

The total phase, $\emptyset(j\omega)$, of the transfer function, $H(j\omega)$, is simply the sum of the individual pole/zero phase angles. That is

$$\emptyset(j\omega) = (\emptyset_{z1} + \emptyset_{z2} + \cdots\cdot \emptyset_{zm}) - (\emptyset_{p1} + \emptyset_{p2} + \cdots\cdot \emptyset_{pn}) \tag{2.25}$$

The expressions for the total magnitude, $|H(j\omega)|$, and total phase, $\emptyset(j\omega)$, can now be combined to form a general expression for the transfer function $H(j\omega)$. That is,

$$|H(j\omega)| = K \cdot \frac{\prod_{i=1}^{m} M_{zi}}{\prod_{i=1}^{n} M_{pi}} \tag{2.26}$$

and

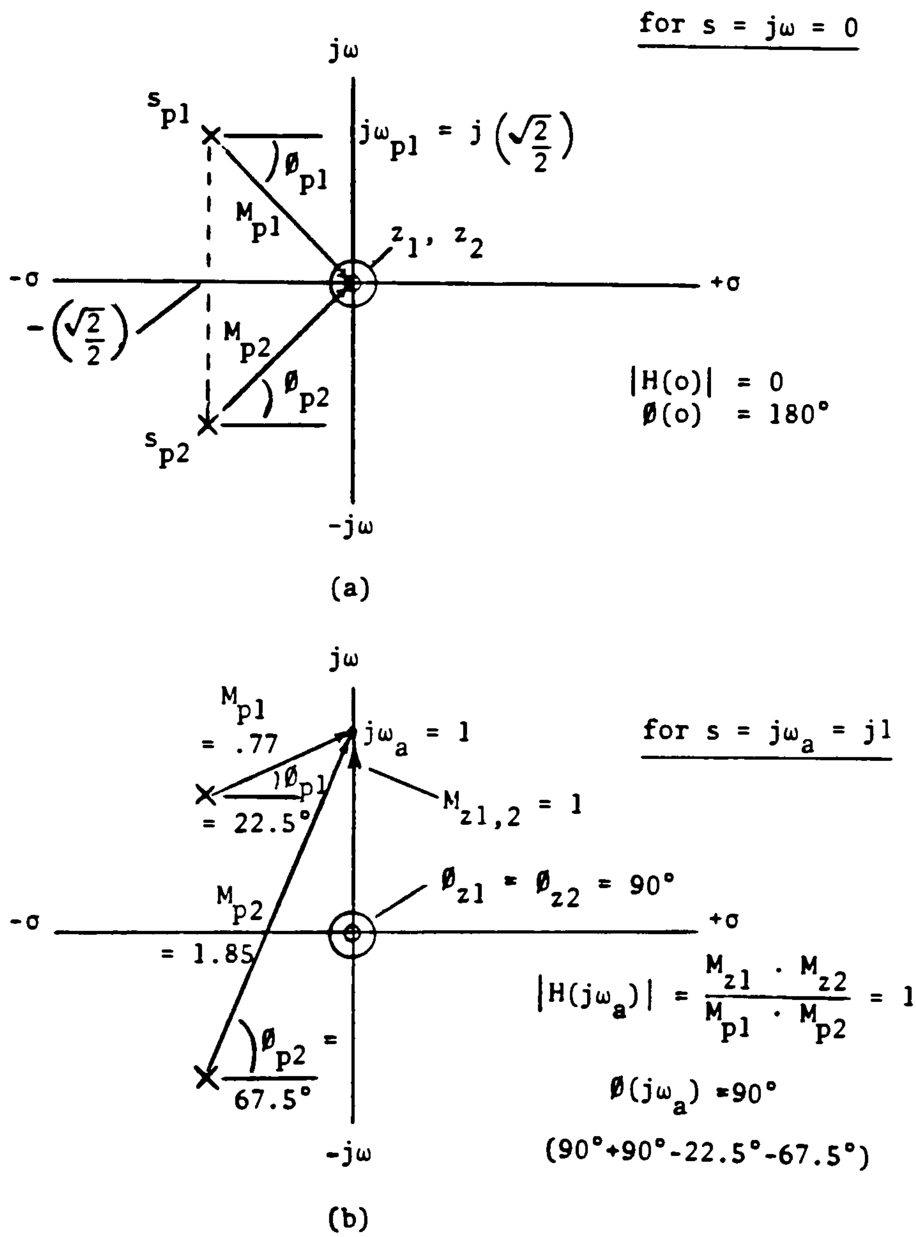
$$\emptyset(j\omega) = \sum_{i=1}^{m} \emptyset_{zi} - \sum_{i=1}^{n} \emptyset_{pi} \tag{2.27}$$

where $|H(j\omega)| \underline{/\emptyset(j\omega)}$ is equal to $H(s)$ for $s = j\omega$. Notice that the general equations above are equivalent to the general equation 2.21 previously developed for $H(s)$. The difference is that the variable s has been substituted with $s = j\omega$ for evaluation of magnitude and phase.

### 2.5.2 Evaluation of the Transfer Function, H(s), on the s-Plane

In the last section the s-plane was introduced as a means of representing the poles and zeros of the transfer function, $H(s)$. In this section we will examine how the s-plane diagram can be used to evaluate the response of the filter. The equations previously derived for the magnitude and phase will be used here.

Figure 2.14 illustrates an s-plane diagram for a transfer function with two poles and two zeros. That is,




Evaluation of the Transfer Function, H(s), on the s-Plane

(Example for Section 2.5.2)

Figure 2.14

$$H(s) = K \frac{s^2}{(s-s_1)(s-s_2)}$$

$$= K \frac{s^2}{\left(s - \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right)\left(s - \frac{\sqrt{2}}{2} \quad j\frac{\sqrt{2}}{2}\right)}$$

The poles are complex conjugates and are located at $\sigma \pm j\omega = -\frac{\sqrt{2}}{2} \pm j\frac{\sqrt{2}}{2}$. The zeros, $s_{z1}$ and $s_{z2}$, are both located at the origin ($s_{z1} = s_{z2} = 0$) which accounts for the $s^2$ in the numerator. The response of this filter can be found by determining the values of magnitude and phase for each pole/zero value as we travel vertically along the $+j\omega$ axis on the s-plane. Consider figure 2.14a where $s = j\omega = 0$ (that is, the frequency chosen to evaluate $H(j\omega)$ is dc). The magnitude vectors for each of the poles, $s_{p1}$ and $s_{p2}$, are directed towards the origin ($s = j\omega = 0$). The magnitude of each of the vectors is determined using equation 2.22. That is for $j\omega = 0$ and

$$M_i = \sqrt{\sigma_i^2 + (\omega - \omega_i)^2}$$

the magnitudes for the poles are

$$M_{p1} = M_{p2} = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(0 - \frac{\sqrt{2}}{2}\right)^2} = 1$$

For the zeros, the magnitudes are

$$M_{z1} = M_{z2} = 0$$

The total magnitude at $j\omega = 0$ is found using equation (2.24). That is,

$$\left|H(j\omega)\right|_{\omega=0} = \frac{M_{z1} \cdot M_{z2}}{M_{p1} \cdot M_{p2}} = \frac{0}{1} = 0$$

or in terms of decibels,

$$20 \log \left|H(j\omega)\right| = \infty$$

The phase of the pole/zero vectors is found using equation 2.23. It is assumed for the purpose of the two zeros that the frequency at which $H(j\omega)$ is being evaluated is 0+. Then for $s = j\omega = j0+$, the phase angles are

$$\phi_{p1} = \tan^{-1}\left(\frac{0 - \omega_{p1}}{\sigma_{p1}}\right)$$

$$= \tan^{-1}\left(\frac{-\sqrt{2}/2}{-\sqrt{2}/2}\right) = -45°$$

$$\phi_{p2} = \phi_{p1} = 45°$$

$$\phi_{z1} = \phi_{z2} = +90°$$

The total phase by equation 2.25 is then

$$\phi(j\omega) = (\phi_{z1} + \phi_{z2}) - (\phi_{p1} + \phi_{p2})$$

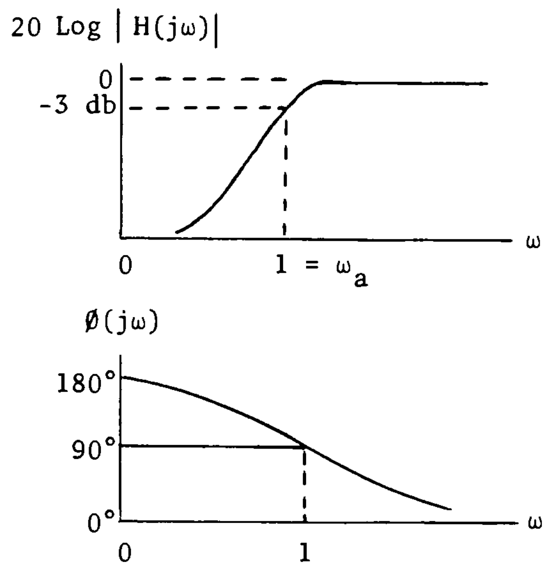
$$= (90° + 90°) - (45° - 45°)$$

$$= 180°$$

Therefore at a frequency of 0, the transfer function, $H(j\omega)$, has a magnitude of 0 and a phase shift of 180°. That is,

$$H(j\omega) = |H(j\omega)| \underline{/\emptyset(j\omega)} = 0 \underline{/180°}$$

This is illustrated in figure 2.15 at $\omega = 0$. As $j\omega$ becomes larger with increasing frequency, we cross the cutoff frequency, $\omega_a$. For the case at hand, $\omega_a = 1$ as shown in figure 2.15.




High Pass Response
(Two Pole Filter)

Figure 2.15

Then, at $s = j\omega = j1$, the magnitude and phase of $H(j\omega)$ are found as follows:

$$M_{p1} = \left[\left(\frac{\sqrt{2}}{2}\right)^2 + \left(1 - \frac{\sqrt{2}}{2}\right)^2\right]^{\frac{1}{2}} = .765$$

$$M_{p2} = \left[\left(\frac{\sqrt{2}}{2}\right)^2 + \left(1 + \frac{\sqrt{2}}{2}\right)^2\right]^{\frac{1}{2}} = 1.848$$

$$M_{z1} = M_{z2} = 1$$

The total magnitude is then

$$|H(j\omega)|_{\omega=\omega_a=1} = \frac{M_{z1} \cdot M_{z2}}{M_{p1} \cdot M_{p2}} = \frac{1}{(.765)(1.848)}$$

$$= .707$$

or

$$20 \text{ Log } |H(j\omega)| = -3 \text{ db}$$

as shown in figure 2.15 at $\omega_a$. The phase angles, as shown in figure 2.14b are

$$\phi_{p1} = \tan^{-1}\left(\frac{1 - \sqrt{2}/2}{\sqrt{2}/2}\right) = 22.5°$$

$$\phi_{p2} = \tan^{-1}\left(\frac{1 - (-\sqrt{2}/2)}{\sqrt{2}/2}\right) = 67.5°$$

$$\phi_{z1} = \phi_{z2} = +90°$$

Therefore at s = j1, the total phase is

$$\phi(j\omega) = (90° + 90°) - (22.5° + 67.5°) = 90°$$

The transfer function is then described as

$$H(j\omega)\Big|_{\omega=1} = -3 \text{ db } \underline{/90°}$$

This is shown in figure 2.15 for $\omega = \omega_a = 1$. This response is obviously that of a high pass filter. We have seen previously for a high pass filter that the pass band is the range of frequencies beyond the cutoff, $\omega_a$. For the example being discussed we would then expect the magnitude of $H(j\omega)$ to be a maximum near $\omega = \infty$. Let us see if this is indeed the case.

At $s = j\omega = j\infty$ all of the pole/zero vectors on the s-plane are pointing vertically (angles = 90°) and have equal magnitudes approaching $\infty$. The value of $H(j\omega)$ is then found as

$$\left|H(j\omega)\right|_{\omega=\infty} = \frac{M_{z1} \cdot M_{z2}}{M_{p1} \cdot M_{p2}} \simeq 1 = 0 \text{ db}$$

and

$$\emptyset(j\omega) = (\emptyset_{z1} + \emptyset_{z2}) - (\emptyset_{p1} - \emptyset_{p2})$$

$$= (90° + 90°) - (90° + 90°) = 0°$$

Combining the magnitude and phase we have

$$H(j\omega)\Big|_{\omega=\infty} = 0 \text{ db } \underline{/0°}$$

This confirms the magnitude response as that of a high pass filter. Notice that the range of $j\omega$ below $j\omega_a$ is the stop band and beyond $j\omega_a$ is the pass band. Thus the s-plane is a translation of $H(s)$ onto a complex plane which represents the magnitude as described by the straight line transition diagram as well as the phase. In regards to the straight line transition diagram, we found in this example that $f_a = 1$ and $-Adb = -3$ db. The terms $-Bdb$ and $f_b$ were not specified.

As we have just seen, the s-plane diagram can provide a simple and complete representation of the filter response. Since there are different types of filters (low pass, band pass, etc.), the s-plane diagrams must vary accordingly. In fact, the s-plane provides a clear pictorial representation of the unique characteristics of one filter versus another given the same type of filter. These unique characteristics are discussed in Chapter III.

The following section illustrates the typical form of the transfer function, H(s), and the s-plane diagram associated with each filter type. We will not go into any analysis of these forms as we just did with the high pass example but simply discuss a few of the important points.

### 2.5.3 The Transfer Functions

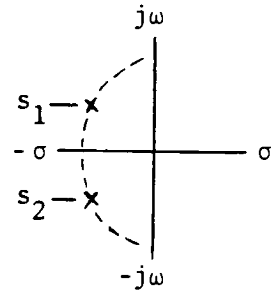
The previous sections examined the general form of the transfer function, H(s), and a means of interpreting the filter response using the s-plane diagram. In this section the typical forms of the transfer functions are given for each filter type. The magnitude response curves will not be presented here since the straight line transition diagrams of section 2.4 serve this purpose.

The common forms of the second order transfer functions are illustrated for each filter type in figure 2.16. At first glance, there are apparent differences in the transfer function equation, H(s), and in the s-plane diagrams. This relates directly to the fact that they all have different magnitude and phase responses. Let us briefly discuss the salient points regarding the transfer function of each filter type.

Figure 2.16a illustrates the low pass filter characteristics. The transfer function shown, $H(s)_{LP}$, is for a general two pole (N=2) filter as seen by the two factors $(s-s_i)$ and the two x's on the s-plane. The number of poles can generally be any number greater than or equal to

## Low Pass

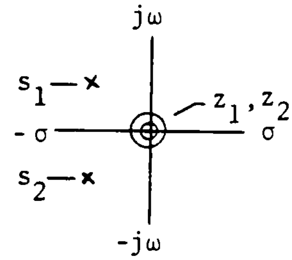
$$H(s)_{LP} = \frac{K}{(s-s_1)(s-s_2)}$$




( a )

## High Pass

$$H(s)_{HP} = \frac{K's^2}{(s-s_1)(s-s_2)}$$

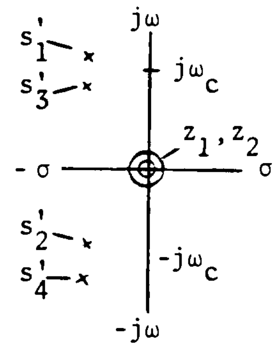



( b )

## Band Pass

$$H(s)_{BP} = \frac{K\ s^2}{(s^2 - Bss_1 + \omega_c^2)(s^2 - Bss_2 + \omega_c^2)}$$

$$= \frac{K\ s^2}{(s-s_1')(s-s_2')(s-s_3')(s-s_4')}$$



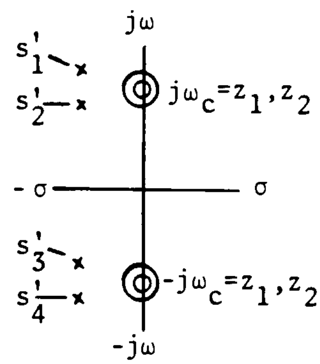

( c )

## Band Stop

$$H(s)_{BS} = \frac{K'\ (s^2 + \omega_c^2)^2}{(s^2 - Bss_1^{-1} + \omega_c^2)(s^2 - Bss_2^{-1} + \omega_c^2)}$$

$$= \frac{K'\ (s-z_1)^2\ (s-z_2)^2}{(s-s_1')(s-s_2')(s-s_3')(s-s_4')}$$

( d )




Typical Transfer Functions

Figure 2.16

1. Their positions on the s-plane usually occur as a pattern in the form of a circle or an ellipse. As mentioned earlier, the greater the number of poles, the faster the transition rate from the pass band to the stop band. The region of $j\omega$ enclosed by the poles is the pass band and beyond this region is the stop band. Variations of the form of H(s) do exist for example in the case of the elliptic function low pass filter (refer to Chapter III).

The high pass filter, shown in figure 2.16b, has already been discussed in section 2.5.2. We will not elaborate any further here.

The band pass response is illustrated in figure 2.16c. Notice that there are two pair of poles vertically displaced from the real ($\sigma$) axis by $j\omega_c$. Recall from the discussion on the low pass filter that the region enclosed by the poles is the pass band. Therefore the band pass magnitude response is similar to that of the low pass only displaced along the frequency axis. The transfer function shown contains two terms of the form

$$(s^2 - Bss_1 + \omega_c^2)$$

from which four poles of the form $(s-s_i)$ are derived. We will briefly discuss these terms. The term, $\omega_c$, is the center frequency of the filter described by equation 2.16; B is the bandwidth of the filter occurring at the pass band edge frequencies associated with -Adb. The terms $s_1$, $s_2$ are the normalized low pass poles. The concept of normalization and the relationship of the low pass filter to other filters will be discussed in Chapter III. The order of the band pass filter is usually described by its normalized low pass equivalent. For the case shown the low pass order is N=2. The translation to the band pass (denormalization) yields two quadratic terms such as those in the denominator of $H(s)_{BP}$. Thus for a band pass filter, the number of poles equals 2N.

The band stop response is illustrated in figure 2.16d. Its characteristics are similar to that of the band pass which accounts for a band translation along the frequency axis. Notice however that in the central region of the poles there exists two zeros, $jz_1$ and $jz_2$ on the $+j\omega$ axis. These zeros and their conjugates, $-jz_1$ and $-jz_2$, are derived from $(s^2 + \omega_c^2)^2$ in the numerator of $H(s)_{BS}$. When the frequency variable, $j\omega$, crosses through this region of the zeros, the magnitude of $H(j\omega)_{BS}$ is decreasing and at the zeros, $H(j\omega) = -\infty$. This then defines the stop band region. The terms in the denominator of $H(s)_{BS}$ are defined the same as that of the band pass.

## 2.6 Summary

We started off Chapter II by discussing the ideal filter characteristics. In doing so the basic purpose of each filter, concerning the magnitude response, was identified. Some terminology associated with the ideal filters was introduced. Essentially the concepts of the ideal filter served as a basis for the non-ideal responses.

In consideration of the realistic filter response, the ideal filter was examined to show the areas which were limited in a real sense. This led to the straight line transition diagram as a representation of the magnitude response of the real filter. Then the transfer function was introduced in its general form as a mathematical representation of not only the magnitude response but phase as well. From this, the particular characteristics of the transfer function were shown for each filter type.

In the discussion of the transfer functions for the band pass and band stop filters, a relationship to the normalized low pass filter became evident. It turns out that all of the filter types do indeed have a relationship to the normalized low pass filter. This relationship provides for a very convenient manner with which the poles and zeros of the various filter types are determined. This process of normalization and the method of determining the poles/zeros is the topic of Chapter III.

# CHAPTER III

# NORMALIZATION AND THE CLASSICAL APPROXIMATIONS

## 3.1 Introduction

In this chapter we examine the methods of determining the poles and zeros of the transfer function, H(s). It was shown in chapter II that the transfer function represented the complete response (magnitude and phase) of the filter by way of the poles and zeros. It was also found that when the transfer function is evaluated along the $j\omega$ axis on the s-plane, the magnitude response (phase also) could be determined. This same magnitude response is described by the parameters associated with the straight line transition diagram. Thus there exists a relationship between these magnitude parameters and the pole/zero values on the s-plane. The objective of this chapter is to analyze and demonstrate the methods of finding the poles and zeros using the parameters from the straight line transition diagram. Of importance also is describing the unique magnitude response characteristics and the pole/zero patterns as they relate to the classical approximations (Butterworth, Chebyshev and elliptic).

We approach the solution of deriving the poles and zeros by discussing three main topics. These are 1) normalization, 2) the characteristic equation and 3) the classical approximations. The first topic, normalization, deals with converting the straight line transition diagrams of the four filter types into one simple low pass form, normalized along the frequency axis. The resulting normalized low pass filter is the basis of all filter design techniques. This form simplifies the task of analyzing the variety of response requirements. Inherent in this process of normalization is establishing a relationship between the four filter types (low pass, high pass, band pass and band stop) and the normalized low pass form. The normalization equations are briefly presented. Then examples are given to demonstrate the normalization process for each filter type.

The second topic deals with the mathematical representation of the normalized low pass filter. This representation is called the characteristic equation. Here we examine how the equation describes the normalized low pass filter and the manner in which it approximates the ideal filter. An examination is also made of the various magnitude shapes of the normalized low pass filter. The concept of insertion loss is related to the characteristic equation. Then a discussion is presented which relates the solution of the insertion loss function to the normalized poles and zeros. As a result of these discussions we see the relationship between the normalized low pass filter, the characteristic equation and the pole/zero values of the transfer function, H(s).

The third topic, classical approximations, deals with three approximations to the ideal normalized low pass filter. These classical functions are the Butterworth, Chebyshev and elliptic. Each has its unique properties and objective in achieving one or more characteristics of the ideal low pass response. For each of these we discuss the magnitude response as it relates to the approximating function. Then the methods of evaluating the filter order, N, along with the poles and zeros are discussed. Lastly a comparison is made among the three classical responses.

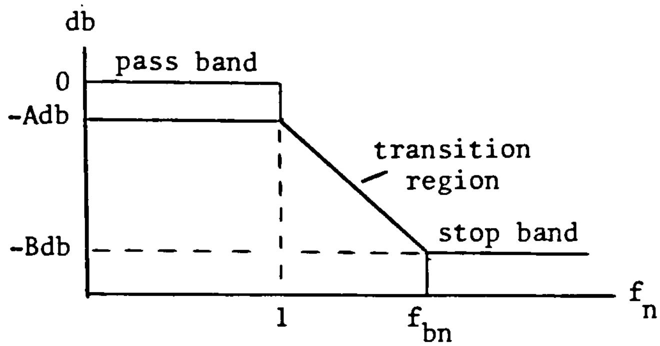
As a result of this chapter, the reader should develop an appreciation of how the filter design requirements, representented by the straight line transition diagram, relate to the poles and zeros of the transfer function, H(s). In addition the reader should be acquainted with the unique properties of each approximation to the ideal low pass filter and how they suit the design needs.

## 3.2 The Normalized Low Pass Filter

In this section we establish the normalized low pass filter as a basis for all filter designs. We begin by defining the normalized low pass filter and its parameters. Then the normalization process is examined for each of the four filter types (low pass through band stop). As a result we are left with one normalized form which can be used to determine the poles and zeros. The use of the normalized low pass filter eliminates the need for an abundance of design data in that only the low pass design tables are needed.

### 3.2.1 Definition and Parameters of the Normalized Low Pass Filter

The straight line transition diagram for the normalized low pass filter is illustrated in figure 3.1. The frequency axis ($f_n$) is normalized such that the cutoff frequency is equal to 1. The normalized pass band shown is the frequency range $0 \leq f_n \leq 1$. (The term, frequency, is used although it is improper since the $f_n$ axis is really unitless.) The parameters of the magnitude response, as shown in figure 3.1, are defined in the same manner as we have previously done in chapter II with the addition of the subscript, n, on frequency values to indicate that they are normalized. These parameter definitions are summarized in figure 3.1. The shape of the magnitude response can vary de-




<u>Normalized Parameters</u>

cutoff frequency: 1
stop band edge frequency: $f_{bn}$
maximum pass band variation: -Adb
minimum stop band attenuation: -Bdb
filter order: N

Parameters of the
Normalized Low Pass Filter

Figure 3.1

pending upon the choice of Butterworth, Chebyshev and elliptic responses as we will discuss later in sections 3.4-3.7.

The parameters of the normalized low pass filter (-Adb, -Bdb, $f_{bn}$) are all that we need to evaluate the filter order, N, and the pole/zero values. The determination of these values is accomplished using equations unique to each classical response (Butterworth, Chebyshev and elliptic). These equations and examples will be given in the sections to follow. For the remainder of this section we concern ourselves with the methods of transforming the designer specified parameters of the straight line transition diagrams into the normalized low pass filter.

### 3.2.2 Normalization of the Low Pass Response

Figure 3.2a illustrates the straight line transition diagram of the non-normalized (denormalized) low pass filter. The objective is to obtain the normalized form shown in figure 3.2b. Since the magnitude response shape in figure 3.2a is already in the low pass form, the normalization process simply requires dividing the frequency values of figure 3.2a by the cutoff frequency, $f_a$, to obtain a normalized frequency axis, $f_n$.

That is,

$$f_{xn} = \frac{f_x}{f_a} \tag{3.1}$$

where

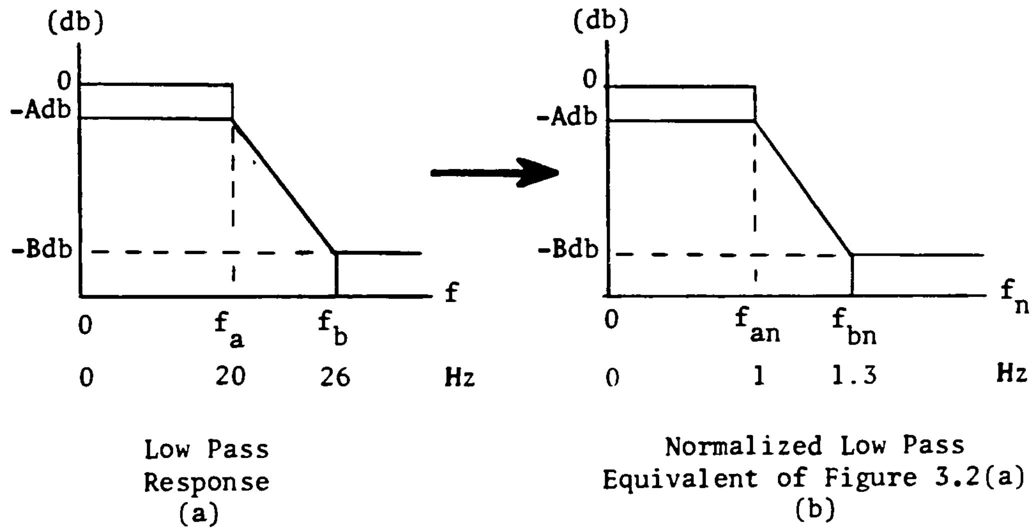
$f_x$ is any arbitrary frequency,

$f_a$ is the cutoff frequency,

$f_{xn}$ is the normalized value of $f_x$.

Thus the normalized cutoff frequency, using equation 3.1, is $f_a/f_a = 1$ and the normalized stop band edge frequency is $f_b/f_a = f_{bn}$.

Consider an example. Shown in figure 3.2a are the following parameters: cutoff frequency ($f_a$) = 20 Hz, stop band edge frequency ($f_b$) = 26 Hz. The normalized frequency values are found using equation 3.1 as follows:




Normalization of the
Low Pass Response

Figure 3.2

$$f_{an} = \frac{f_a}{f_a} = \frac{20}{20} = 1$$

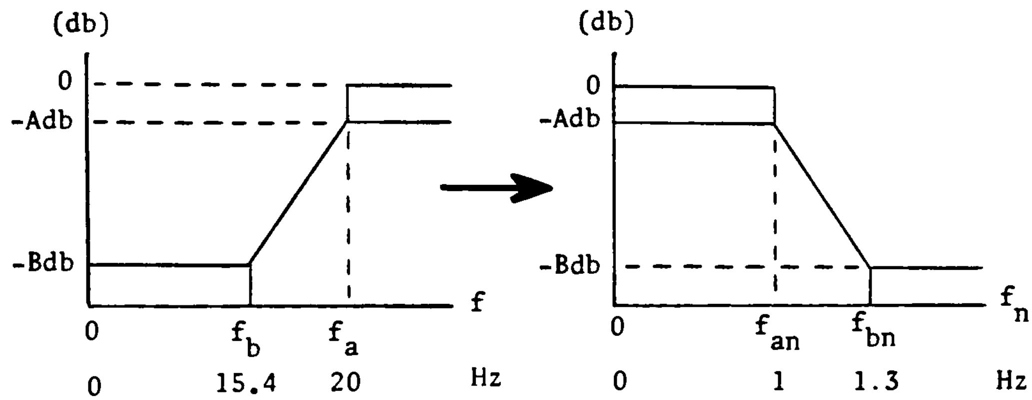
and

$$f_{bn} = \frac{f_b}{f_a} = \frac{26}{20} = 1.3$$

These values are shown in figure 3.2b. Note that the corresponding magnitude parameters (-Adb, -Bdb) maintain their relationship to the cutoff and stopband frequencies. This completes the normalization process for the low pass filter.

### 3.2.3 Normalization of the High Pass Response

The normalized low pass filter parameters can be found for a high pass response by initially considering a simple transformation of the frequency values. Refer to figure 3.3. By comparing the magnitude responses we see that there is a reciprocal relationship in frequency between the high pass response (figure 3.3a) and the low pass response (figure 3.3b). For example, the magnitude is -Adb at a frequency equal to ∞ for the high pass response compared to zero (1/∞) for the low pass response. This suggests that the normalized low pass filter parameters can be found from the high pass parameters by first normalizing the high pass values using the cutoff frequency and then inverting the value.




High Pass Response (a)

Normalized Low Pass Equivalent of Figure 3.3(a) (b)

Normalization of the High Pass Response

Figure 3.3

That is,

$$f_{xn} = \left(\frac{f_x}{f_a}\right)^{-1} \tag{3.2}$$

where

$f_x$ is any high pass frequency,

$f_a$ is the high pass cutoff frequency,

$f_{xn}$ is the normalized low pass frequency value.

As an example, consider the frequency values shown in figure 3.3a, $f_a$ = 20 Hz and $f_b$ = 15.4 Hz. The corresponding normalized low pass values are shown in figure 3.3b and derived using equation 3.2 as shown below.

$$f_{an} = \left(\frac{f_a}{f_a}\right)^{-1} = \left(\frac{20}{20}\right)^{-1} = 1$$

and

$$f_{bn} = \left(\frac{f_b}{f_a}\right)^{-1} = \left(\frac{15.4}{20}\right)^{-1} = 1.3$$

Notice that the magnitude parameters, -Adb and -Bdb, remain the same as shown in figure 3.3b. It is interesting to compare this normalized low pass filter response with the one previously found in section 3.2.2, figure 3.2. The normalized parameters are the same and at this level the high pass and low pass responses are equivalent. Analysis of the pole/zero values using this normalized form will therefore apply to both the high pass and low pass responses.

This completes the process of normalization as it pertains to the high pass filter response. Now we will examine a slightly different interpretation of normalization as it applies to the band pass filter response.

### 3.2.4 Normalization of the Band Pass Response

The magnitude response of the band pass filter is similar in shape to that of the low pass filter except that the passband has been shifted to a higher center frequency. One might consider the normalization process of the band pass response to the normalized low pass filter as a "band normalization". Instead of normalizing frequency values with respect to the cutoff frequency as in the case of the low pass or high pass, we simply normalize any arbitrary bandwidth with respect to the bandwidth, B, defined by the passband edge frequencies.

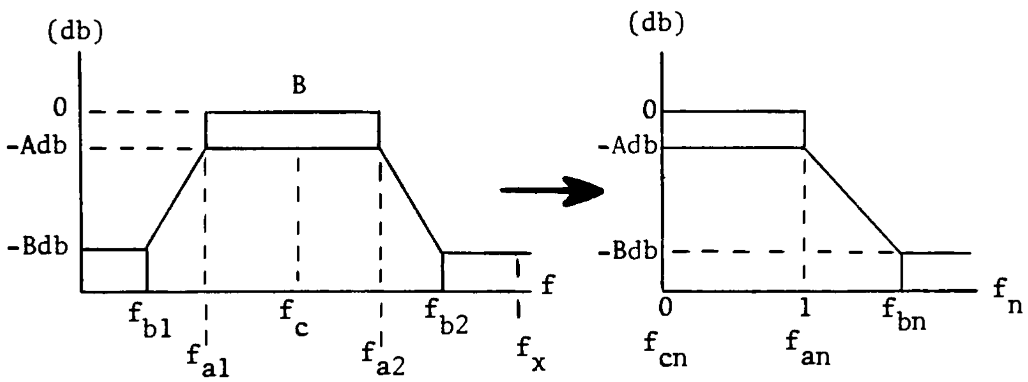
That is,

$$f_{xn} = \frac{B_x}{B} \tag{3.3}$$

where

- $B$ is the bandwidth at the cutoff frequencies equal to $f_{a2} - f_{a1}$,
- $B_x$ is any arbitrary bandwidth,
- $f_{xn}$ is the resulting normalized value

As an example, consider the normalized value of the bandwidth defined by the cutoff frequencies $f_{a2}$ and $f_{a1}$ as shown in figure 3.4a. The width of the band at the cutoff frequencies is $f_{a2} - f_{a1}$. Therefore




Band Pass Response

(a)

Normalized Low Pass Equivalent of Figure 3.4(a)

(b)

Normalization of the Band Pass Response

Figure 3.4

the normalized value derived from these frequencies is found using equation 3.3 as follows:

$$B_a = f_{a2} - f_{a1}$$

$$B = f_{a2} - f_{a1}$$

and

$$f_{an} = \frac{B_a}{B} = 1$$

As a second example consider the normalized value corresponding to the center frequency $f_c$ as shown in figure 3.4a. The bandwidth at $f_c$ is zero since this is the center of the passband. Therefore the value $B_x$ of equation 3.3 is 0 and the normalized value is

$$f_{cn} = \frac{0}{B} = 0$$

From these examples we see that the cutoff-frequency band $(f_{a2} - f_{a1})$ corresponds to the normalized value of 1 and the center frequency, $f_c$, corresponds to 0 on the normalized response. These are shown in figure 3.4b.

Now consider some arbitrary frequency value, $f_x$, occurring in the stop band region as shown in figure 3.4a. The bandwidth associated with $f_x$ is not $2(f_x - f_c)$. This was the error commonly occurring when one defines the center frequency as $\frac{1}{2}\left(f_{a2} - f_{a1}\right)$. We can however modify equation 3.3 to consider the geometric relationship between the center frequency, $f_c$, and the frequency band at any arbitrary frequency. Suppose we define this arbitrary and known frequency as $f_{x2}$, an upper band frequency. Corresponding to $f_{x2}$ there is a lower band frequency $f_{x1}$ such that

$$f_c^2 = f_{x2} \cdot f_{x1} \tag{3.4}$$

This simply defines two frequencies which have a geometric center frequency of $f_c$, such as $f_c^2 = f_{a2} \cdot f_{a1}$.

From equation 3.4 we have,

$$f_{x1} = \frac{f_c^2}{f_{x2}}$$

We can substitute this into

$$f_{xn} = \frac{B_x}{B} = \frac{f_{x2} - f_{x1}}{B}$$

and arrive at

$$f_{xn} = \frac{f_{x2} - \left(f_c^2/f_{x2}\right)}{B} = \frac{f_{x2}^2 - f_c^2}{f_{x2} \cdot B}$$

or finally the general equation,

$$f_{xn} = \frac{f_x^2 - f_c^2}{f_x \cdot B} \tag{3.5}$$

Equation 3.5 says that given the bandwidth, B, defined by the two cutoff frequencies and knowing the center frequency, $f_c$, we can normalize any frequency value, $f_x$, to obtain the corresponding normalized low pass value, $f_{xn}$. This is a more convenient form since it allows the use of one frequency instead of an unknown bandwidth.

Consider the following example. Figure 3.5 illustrates a band pass response with the center frequency, $f_c$, at 100 kHz and the upper stop band edge frequency, $f_{b2}$, at 106 kHz. The bandwidth, B, is shown as $f_{a2} - f_{a1} = 9$ kHz.

The normalized low pass value, $f_{bn}$, corresponding to $f_{b2}$ is found using equation 3.5 as follows:

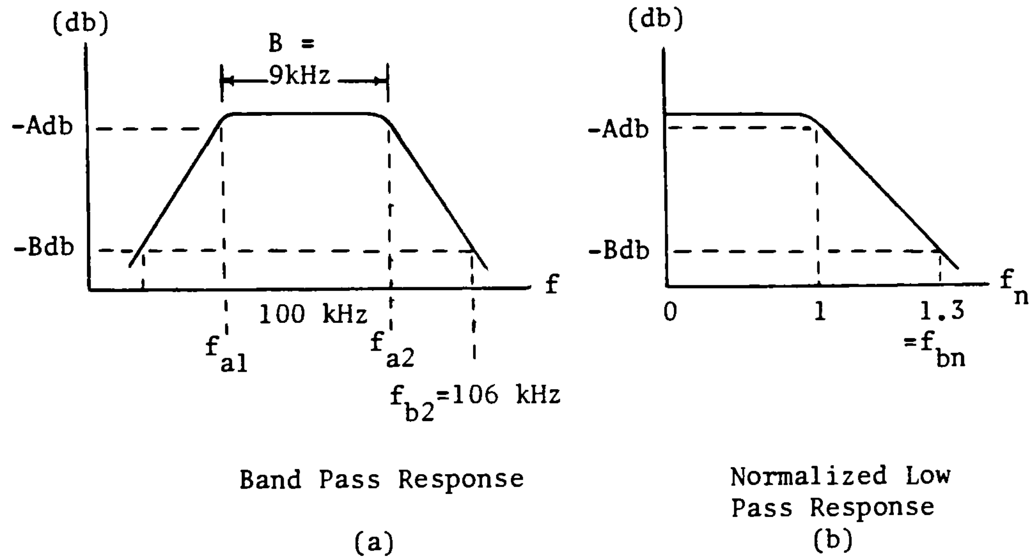
for

$$B = 9 \text{ kHz},\ f_{b2} = 106 \text{ kHz},\ f_c = 100 \text{ kHz}$$

we have

$$f_{bn} = \frac{f_{b2}^2 - f_c^2}{f_{b2} \cdot B} = \frac{(106\ \text{k})^2 - (100\ \text{k})^2}{106\ \text{k} \cdot 9\ \text{k}}$$

$$= 1.3$$




Example of Band Pass Normalization

Figure 3.5

This value is shown in figure 3.5b for the normalized low pass response. Also shown in figure 3.5b is the cutoff frequency equal to 1 which corresponds to the non-normalized bandwidth at $f_{a2}$. The magnitude response parameters, -Adb and -Bdb, remain unchanged.

As a final point, it should be noted that the normalized frequency axis in the case of a band pass response carries a different meaning than that of the normalized frequency axis for the low or high pass response. For the band pass response, the normalized frequency axis actually represents a proportion of frequency bands. For the low pass and high pass, the normalized frequency axis was related to a direct ratio of frequencies.

In the next section we consider a combination of band normalization as well as band transformation. This relates to the normalization process of the band stop filter response.

### 3.2.5 Normalization of the Band Stop Response

When comparing a band stop magnitude response with that of a band pass, we can see that the functions within the bands are inversely related. This is somewhat similar to the relationship between the normalized low pass and the high pass responses. The band stop normalization process is then performed by a transformation to the band pass followed by the same normalization technique used for the band pass response.

That is,

$$f_{xn} = \left(\frac{B_x}{B}\right)^{-1} \tag{3.6}$$

where

- $B$ is the bandwidth defined by the two cutoff frequencies, $f_{a2}$, $f_{a1}$,
- $B_x$ is any arbitrary bandwidth,

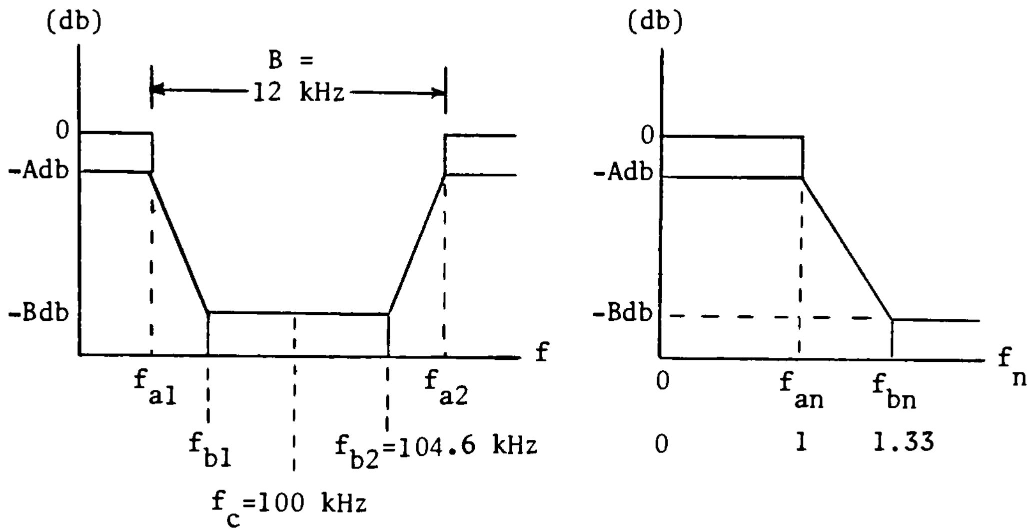
and

- $f_{xn}$ is the corresponding normalized low pass value.

Any arbitrary frequency, $f_x$, can be normalized by using the following equation:

$$f_{xn} = \left(\frac{f_x^2 - f_c^2}{B\ f_x}\right)^{-1} \tag{3.7}$$

Equation 3.7 is the same as that of the band pass normalization equation 3.5, only inverted.




Band Stop Response (a)

Normalized Low Pass Equivalent of Figure 3.6(a) (b)

Normalization of the Band Stop Response

Figure 3.6

Consider the following example. We wish to find the normalized low pass filter values given the band stop frequency parameters shown in figure 3.6a. The normalized stop band value, $f_{bn}$, is found using equation 3.6 as shown below.

For

$$f_{b2} = 104.6 \text{ kHz}, \ f_c = 100 \text{ kHz}, \ B = 12 \text{ kHz}$$

we have,

$$f_{bn} = \left(\frac{f_{b2}^2 - f_c^2}{f_{b2} \cdot B}\right)^{-1} = \left(\frac{104.6 \text{ k}^2 - 100 \text{ k}^2}{104.6 \text{ k} \cdot 12 \text{ k}}\right)^{-1}$$

$$f_{bn} = 1.33$$

The value of $f_{bn}$ is shown in figure 3.6b. Also shown are the normalized values corresponding to $f_c$ and $f_{a2}$ (0 and 1 respectively). The magnitude values, -Adb and -Bdb, remain unchanged.

The normalized parameters just obtained are the same as those previously found for the low, high and band pass examples. Since they are equivalent at the normalized level, we only need to determine one set of normalized poles and zeros. The next section describes a general equation for the normalized low pass response and how it relates to the poles and zeros.

## 3.3 The Low Pass Characteristic Function

In the previous sections, the parameters of the straight line transition diagram were converted to a normalized low pass form. Now we wish to describe the normalized low pass response by way of a mathematical function in order to analyze the poles and zeros. In this section a characteristic equation is introduced as a representation of the normalized response. Then the terms of this equation are examined to show the relationship to the poles and zeros of the transfer function, H(s).

### 3.3.1 The Characteristic Equation

The general equation which describes the normalized low pass filter response is

$$H(j\omega) \cdot H(-j\omega) = \frac{1}{1 + \varepsilon^2 K^2 (\omega)} \tag{3.8}$$

where $K(\omega)$ represents a function used to achieve certain ideal low pass approximations (such as Butterworth, Chebyshev and elliptic), $\varepsilon$ is a constant and $\omega$ is a normalized frequency value.

An intuitive look at equation 3.8 shows why it describes the normalized low pass response. If $K(\omega)$ is zero, the magnitude of $H(j\omega)$ is one or 0 db; if $K(\omega)$ is infinite, the magnitude of $H(j\omega)$ is zero or $-\infty$ db. Although these are extreme cases, one can rationalize that for properly chosen $K(\omega)$, there is some finite attenuation process occurring from $\omega = 0$ to $\omega = \infty$. The manner in which this attenuation occurs is characterized by $K(\omega)$. Such characteristics are ripple (or lack of it) in the pass or stop bands, various attenuation rates in the transition region, etc.

Having introduced the characteristic function we will now briefly examine its relationship to the normalized poles and zeros of $H(s)$.

### 3.3.2 Relating the Characteristic Function to the Normalized Poles and Zeros

The general form of the characteristic function, $K(\omega)$, is a rational function described simply as

$$K(\omega) = \frac{N(\omega)}{D(\omega)}$$

If this rational form of $K(\omega)$ is substituted into the characteristic equation, 3.8, we have

$$H(j\omega) \cdot H(-j\omega) = \frac{1}{1 + \varepsilon^2 \dfrac{N^2(\omega)}{D^2(\omega)}}$$

This equation can be rearranged to show how the denominator of $K(\omega)$ relates to the zeros of $H(j\omega)$.

That is,

$$H(-j\omega) \cdot H(j\omega) = \frac{D^2(\omega)}{D^2(\omega) + \varepsilon^2 N^2(\omega)}$$

We can substitute $s/j$ for $\omega$ to arrive at

$$H(s) \cdot H(-s) = \frac{D^2(s/j)}{D^2(s/j) + \varepsilon^2 N^2(s/j)} \tag{3.9}$$

If the above equation for the transfer function is factored, we have the following:

$$H(s) = \frac{(s-z_1)(s-z_2) \;....\; (s-z_m)}{(s-s_1)(s-s_2) \;....\; (s-s_n)} \tag{3.10}$$

Equation 3.10 is the general form of $H(s)$ where $z_1 \;....\; z_m$, $s_1 \;....\; s_n$ are the normalized zero and pole values respectively. Notice that the zeros ($z$) of $H(s)$ are derived from the denominator ($D(\omega)$) of the characteristic function, $K(\omega)$. The pole values of $H(s)$ are derived from setting the denominator of $H(s)$, as in equation 3.9, to zero.

That is,

$$D^2\ (s/j) + \varepsilon^2\ N^2\ (s/j) = 0$$

This simplifies to

$$1 + \varepsilon^2 \frac{N^2\ (s/j)}{D^2\ (s/j)} = 0$$

or

$$\frac{N^2\ (s/j)}{D^2\ (s/j)} = \frac{-1}{\varepsilon^2}$$

Therefore the values of the poles for the transfer function H(s) can be determined by finding the values of s at which

$$K^2\ (\omega)\Big|_{\omega = s/j} = \frac{-1}{\varepsilon^2} \tag{3.11}$$

In the next three sections the characteristic function, $K(\omega)$, is related to three classical approximations: the Butterworth, Chebyshev and elliptic. In these sections to follow, the objective is to determine the normalized pole and zero values for the transfer function H(s).

## 3.4 <u>The Butterworth Low Pass Approximation</u>

In the previous section it was shown that the response of the normalized low pass filter is dependent upon a characteristic function, $K(\omega)$. In this section $K(\omega)$ is described as it pertains to the classical Butterworth approximation of the ideal normalized low pass filter.

We begin by defining the function $K(\omega)$ for the Butterworth response. Subsequently a method is presented for determining the normalized Butterworth s-plane poles.

### 3.4.1 The Butterworth Polynomial

The characteristic function, $K(\omega)$, for the Butterworth approximation is a polynomial described as

$$K(\omega) = (\omega)^N \tag{3.12}$$

where N is the order of the normalized low pass filter and $\omega$ is a normalized low pass frequency value. If $K(\omega)$ is substituted into equation 3.8, we have the characteristic equation for the normalized low pass Butterworth response.

That is,

$$H(j\omega) \cdot H(-j\omega) = \frac{1}{1 + \left(\omega^N\right)^2} \tag{3.13}$$

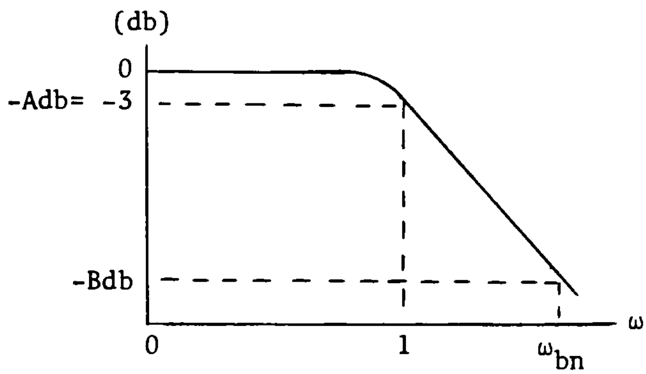
where $\varepsilon$ is chosen as one. This equation can be rearranged to a more convenient form for representing the magnitude response. Since $H(j\omega)$ and $H(-j\omega)$ are even functions in magnitude and odd functions in phase, we have

$$H(j\omega) \cdot H(-j\omega) = \left|H(j\omega)\right|^2$$

Equation 3.13 can therefore be modified to

$$20 \text{ Log } H(j\omega) = -10 \text{ Log } \left(1 + \omega\right)^{2N} \tag{3.14}$$

Figure 3.7 illustrates the magnitude response of the normalized Butterworth low pass filter described by equation 3.14. Let us briefly discuss the magnitude response shown. For $\omega = 0$, the magnitude is 0 db. At $\omega = 1$ the magnitude is -10 Log (2) or -3 db. For $\omega > 1$, the magnitude decreases at a rate depending upon the order of the filter, N. In




Normalized Butterworth Magnitude Response

Figure 3.7

the stop band region, $\omega \geq \omega_{bn}$, the response continues to attenuate beyond the minimum stop band attenuation, -Bdb. The characteristic of the ideal normalized low pass filter, achieved by the Butterworth approximation, is maximum flatness at $\omega = 0$. Additional discussion on the development and interpretation of the Butterworth polynomial can be found in most filter design or network synthesis texts such as references [3], [10].

### 3.4.2 Determining the Filter Order, N

The value N as used in equations 3.12-3.14 is defined as the order of the normalized low pass filter. Actually, the value N is the order of the characteristic polynomial, $K(\omega) = \omega^N$. The value of N determines the rate of attenuation in the transition region. As N increases, the attenuation rate beyond $\omega = 1$ also increases. The rate of attenuation in the transition region must be sufficient to achieve the designer specified attenuation, -Bdb, within the frequency range $1 < \omega \leq \omega_{bn}$. It can be shown that the value of N is related to $\omega_{bn}$ and -Bdb as follows:

$$N \geq \frac{\mathrm{Log}_{10}\left(\dfrac{10^{Bdb/10} - 1}{10^{Adb/10} - 1}\right)}{2\ \mathrm{Log}_{10}\left(\omega_{bn}\right)} \tag{3.15}$$

where the values -Bdb, -Adb and $\omega_{bn}$ are depicted in figure 3.7. The value of N is usually rounded up to the nearest integer value. An example of finding the value of N is given in the following section, 3.4.3.

### 3.4.3 The Normalized Butterworth Poles

Having determined the method of finding the filter order, N, we can now analyze the characteristic equation to determine the poles. The normalized s-plane poles of the Butterworth low pass response can be found by determining the values of s at which the insertion loss function is 0 or by use of equation 3.11. This is shown below. For $K(\omega) = \omega^N$ and $\varepsilon = 1$ we have by equation 3.11,

$$\left(s/j\right)^{2N} = -1$$

or

$$s^{2N} = \begin{cases} +1 \text{ for N even} \\ -1 \text{ for N odd} \end{cases} \tag{3.16}$$

Solving equation 3.16 for s will yield the normalized pole values for the Butterworth low pass response. Based upon this idea, the pole values are found as follows:

for N odd,

$$s_k = e^{j\left(\frac{k}{N} \cdot \pi\right)} \qquad \text{for } k = 1 \text{ to } 2N$$

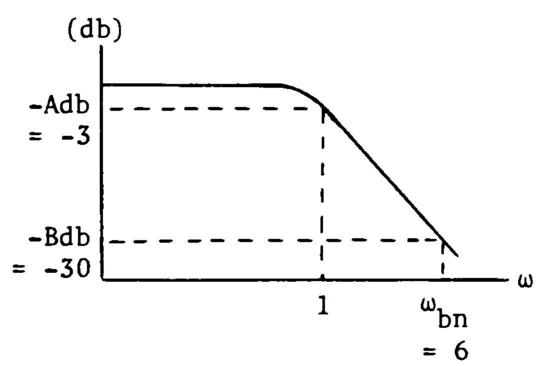
for N even, (3.17)

$$s_k = e^{j\left(\frac{2k-1}{N} \pi\right)} \qquad \text{for } k = 1 \text{ to } 2N$$

where the number of poles is 2N and $s_k$ is the kth complex pole value. Consider the following example of finding the filter order, N, and the normalized pole values for the Butterworth response. Figure 3.8 illustrates the straight line transition parameters for a desired magnitude response. The normalized stop band edge frequency is found as

$$\omega_{bn} = \frac{60}{10} = 6$$

The attenuation parameters are -Adb = -3 db and -Bdb = -30 db. Using equation 3.15, the value of N is found as




Butterworth
Low Pass Example

Figure 3.8

$$N \geq \frac{\text{Log}\left(\dfrac{10^{30/10} - 1}{10^{3/10} - 1}\right)}{2 \text{ Log } 6} = \frac{3.002}{1.556} = 1.929$$

Rounding up, N is chosen as 2. For N = 2 (even), the normalized pole values are found using equation 3.17 as

$$s_k = e^{j\left(k \frac{\pi}{2}\right)} \qquad \text{for } k = 1 \text{ to } 4$$

Using Euler's equation, $s_k$ can be expanded into real and imaginary terms as follows:

$$s_k = \sigma_k + j\omega_k = \cos\left(k \frac{\pi}{2}\right) + j \sin\left(k \frac{\pi}{2}\right)$$

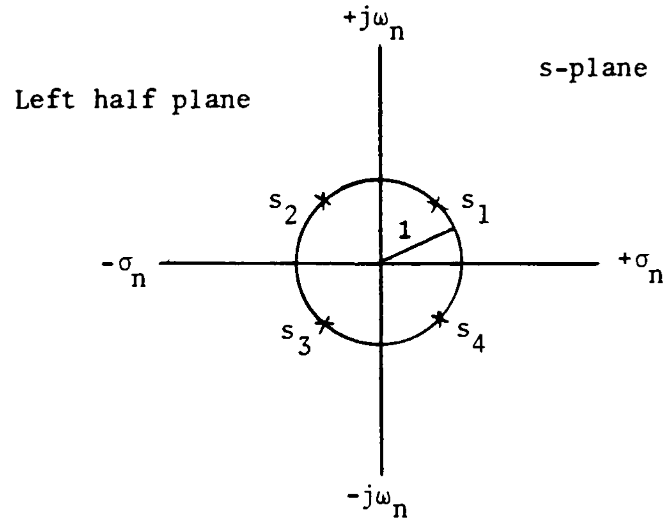
Substituting for k = 1 to 4, the normalized low pass Butterworth poles for N = 2 are as follows:

$$s_{1,4} = \ .707 \pm j\ .707$$

and

$$s_{2,3} = -.707 \pm j\ .707$$

The locations of the pole values reside on the unit circle at a rotational interval of $\pi/2$ or 45° as shown in figure 3.9.




Normalized Butterworth Poles
for N = 2

Figure 3.9

Only the left half plane poles are associated with H(s) and thus for the pole pair $s_{2,3}$ the transfer function is

$$H(s) = \frac{1}{(s-s_2)(s-s_3)} = \frac{1}{(s - (-.707 + j\ .707))(s - (-.707 - j\ .707))}$$

or

$$H(s) = \frac{1}{s^2 + \sqrt{2}\, s + 1} \tag{3.18}$$

Equation 3.18 is the characteristic form of the second order.transfer function for a Butterworth response. In general, for any value of N, there will be 2N poles residing on the unit circle. The left half plane poles are associated with H(s) and the right half plane poles are not used. Tables are available [4] which provide the values of the normalized Butterworth poles, given the value of N.

## 3.5 The Chebyshev Low Pass Approximation

We have seen one method of approximating the ideal low pass response by use of the Butterworth polynomial. In this section a second form of the characteristic function K(ω) is introduced. This function is known as the Chebyshev polynomial. As before, we will begin by examining the magnitude response characteristics followed by the method of determining the normalized pole values.

### 3.5.1 The Chebyshev Polynomial

The characteristic function, K(ω), of the Chebyshev approximation to the ideal normalized low pass response is

$$K(\omega) = \cos(N \cos^{-1} \omega) \qquad \text{for } \omega \leq 1$$

and (3.19)

$$K(\omega) = \cosh(N \cosh^{-1} \omega) \qquad \text{for } \omega \geq 1$$

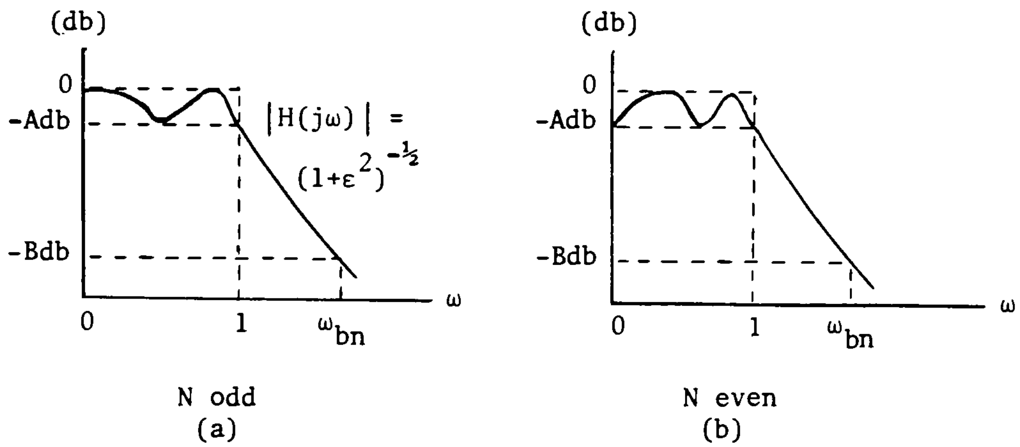
where N is the filter order and ω is a normalized low pass frequency value. The normalized low pass response is described by

$$H(j\omega) \cdot H(-j\omega) = \frac{1}{1 + \varepsilon^2 K^2(\omega)}$$

where $K(\omega)$ is the polynomial of equation 3.19 and $\varepsilon$ is a constant described by the following:

$$\varepsilon = \left(10^{Adb/10} - 1\right)^{\frac{1}{2}} \tag{3.20}$$

Figure 3.10 illustrates the magnitude response of the Chebyshev approximation for N odd and N even.




Normalized Chebyshev Magnitude Response

Figure 3.10

We will briefly discuss the magnitude response characteristics. The salient feature of the Chebyshev response is equiripple magnitude in the pass band ($|\omega| \leq 1$). In the transition region the magnitude decreases in a monotonic fashion and continues to attenuate beyond the stop band requirement, -Bdb for $\omega > \omega_{bn}$. The Chebyshev response approximates the ideal filter characteristics by minimizing the maximum variation in the pass band. Thus it attempts to approach the constant property of the ideal pass band. Additional discussions on the development of the Chebyshev approximation can be found in references [3], [12].

### 3.5.2 Determining the Filter Order, N

The value of the filter order, N, is dependent upon the attenuation requirements (-Adb, -Bdb) specified for the transition region ($1 < \omega \leq \omega_{bn}$). The value, N, describes the order of the polynomial as well as the number of normalized low pass poles (2N). The value of N is found by the following:

$$N \geq \frac{\cosh^{-1} \dfrac{\sqrt{10^{Bdb/10} - 1}}{\varepsilon}}{\cosh^{-1} (\omega_{bn})} \tag{3.21}$$

where the values -Bdb and $\omega_{bn}$ are depicted in figure 3.10 and $\varepsilon$ is described by equation 3.20. As previously mentioned, the value of N is usually rounded up to the nearest integer value.

### 3.5.3 The Normalized Chebyshev Poles

The method of determining the normalized Chebyshev poles is similar to the method used to find the Butterworth poles. We simply use equation 3.11 letting $K(\omega)$ equal the Chebyshev polynomial (3.19) and substitute $\omega = s/j$. Then solving for s yields the complex s-plane pole values. The development of the solutions of $K(\omega)$ will not be performed here (see references [10], [12]). Rather the equations describing the pole values are simply given as follows. The real ($\sigma_n$) and imaginary ($j\omega_n$) values of the Chebyshev normalized low pass poles are found from

$$s_k = \sigma_k + j\omega_k \qquad (3.22)$$

for

$$\sigma_k = \sinh\left(\frac{1}{N} \cdot \sinh^{-1}(1/\varepsilon)\right) \cdot \sin\left(\frac{2k-1}{N} \cdot \frac{\pi}{2}\right)$$

$$\omega_k = \cosh\left(\frac{1}{N} \cdot \sinh^{-1}(1/\varepsilon)\right) \cdot \cos\left(\frac{2k-1}{N} \cdot \frac{\pi}{2}\right)$$

where

N is the filter order given by equation 3.21,

k represents the kth pole out of 2N poles (k = 1 to 2N),

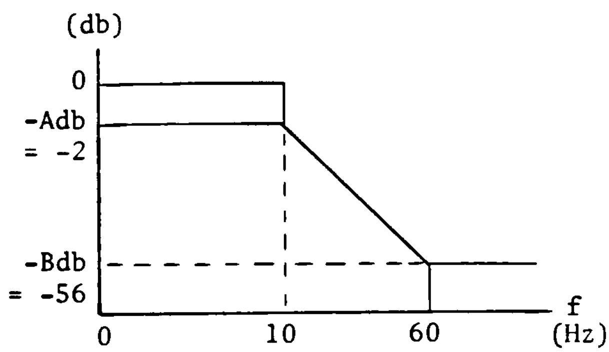
and

$$\varepsilon \text{ is } \sqrt{10^{Adb/10} - 1}$$

As an example of finding the filter order, N, and the normalized pole values, consider the following. Figure 3.11 illustrates the parameters for the desired low pass Chebyshev response. The normalized low pass value of the stop band is found using equation 3.1 as

$$\omega_{bn} = \frac{60}{10} = 6$$

The attenuation parameters shown are -Adb = -2 and -Bdb = -56. Using equation 3.20, the value of the constant $\varepsilon$ is




Requirements for a Chebyshev Low Pass Filter (example)

Figure 3.11

$$\varepsilon = \sqrt{10^{2/10} - 1} = .765$$

The value of the filter order N is then found using equation 3.21 as

$$N \geq \frac{\cosh^{-1}\left(\frac{10^{56/10} - 1}{.765}\right)}{\cosh^{-1}(6)} = 2.96$$

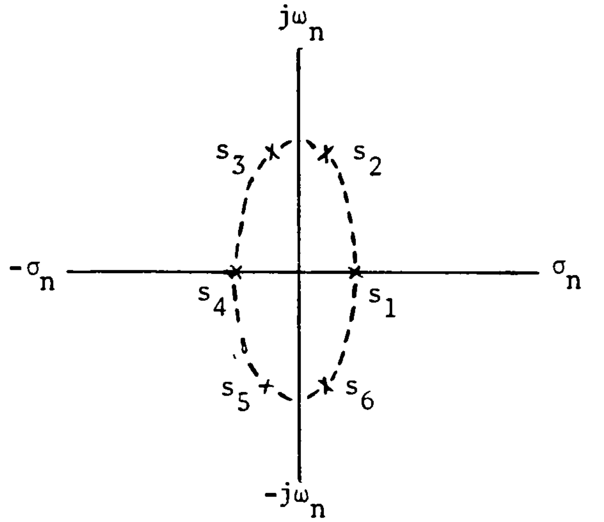
Rounding up, the value of N is chosen as 3. The pole values for N = 3 are found using equation 3.22 and are as follows for the left hand plane:

$$s_{3,5} = -.184 \pm j\ .983$$

$$s_4 = -.368$$

The locations of these pole values reside on an ellipse whose major and minor axes are governed by the constant $\varepsilon$ (equation 3.20). Figure 3.12 illustrates the normalized poles and the elliptic pattern. Tables [4], [6] are also available which provide the normalized low pass pole values given the filter order, N.

The Chebyshev magnitude response just discussed had the characteristics of an equiripple pass band and beyond the pass band the magnitude decreased monotonically. There is a second type of Chebyshev response known for its equiripple in the stop band. This is the inverse Chebyshev approximation. It turns out that $K(\omega)$ for this approximation is a rational function. It is not the intention to examine this approximation as we have done with the others. Rather its existence is




Normalized Chebyshev
Poles for N = 3

Figure 3.12

merely pointed out for those who might be interested. Reference [10] describes the inverse Chebyshev response and its design criterion. In the next section we do however examine an approximating function which has ripple in both the pass and stop bands. This is the elliptic function approximation to the ideal normalized low pass filter.

### 3.6 The Elliptic Function Low Pass Approximation

Up to this point we have seen two types of approximations to the ideal normalized low pass response, the Butterworth and Chebyshev. Their magnitude characteristics vary in the pass band but both have monotonic attenuation in the transition and stop band regions. In this section we examine a third approximation which yields the characteristics of equiripple response in the pass and stop bands. This response is achieved by the elliptic function (Cauer) filter.

### 3.6.1 The Elliptic Rational Function

The characteristic elliptic rational function is described as follows:

for N odd,

$$K(\omega) = C \cdot \frac{\omega\,(\omega_1^2 - \omega^2)(\omega_2 - \omega^2)\ \ldots\ (\omega_N^2 - \omega^2)}{\left(\frac{\omega_c^2}{\omega_1^2} - \omega^2\right)\left(\frac{\omega_c^2}{\omega_2^2} - \omega^2\right)\ \ldots\ \left(\frac{\omega_c^2}{\omega_N^2} - \omega^2\right)}$$

for N even, (3.23)

$$K(\omega) = C \cdot \frac{(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)\ \ldots\ (\omega_N^2 - \omega^2)}{\left(\frac{\omega_c^2}{\omega_1^2} - \omega^2\right)\left(\frac{\omega_c^2}{\omega_2^2} - \omega^2\right)\ \ldots\ \left(\frac{\omega_c^2}{\omega_N^2} - \omega^2\right)}$$

The terms are shown in figure 3.13 and described as follows:

$\omega_c$ is the normalized center frequency of the low pass transition region ($\sqrt{\omega_{bn}}$),

$\omega_{bn}$ is the normalized stop band edge frequency,

$\omega_1 \ldots \omega_N$ are the normalized pass band frequencies where the magnitude is 0 db,

N is the filter order,

and

C is a constant.

The characteristic equation for the normalized elliptic low pass response is the same form as previously described.

That is,

$$H(j\omega)\ H(-j\omega) = \frac{1}{1 + \varepsilon^2 K^2 (\omega)}$$

where $K(\omega)$ is the rational function just described in equation 3.23 and $\varepsilon$ is given as

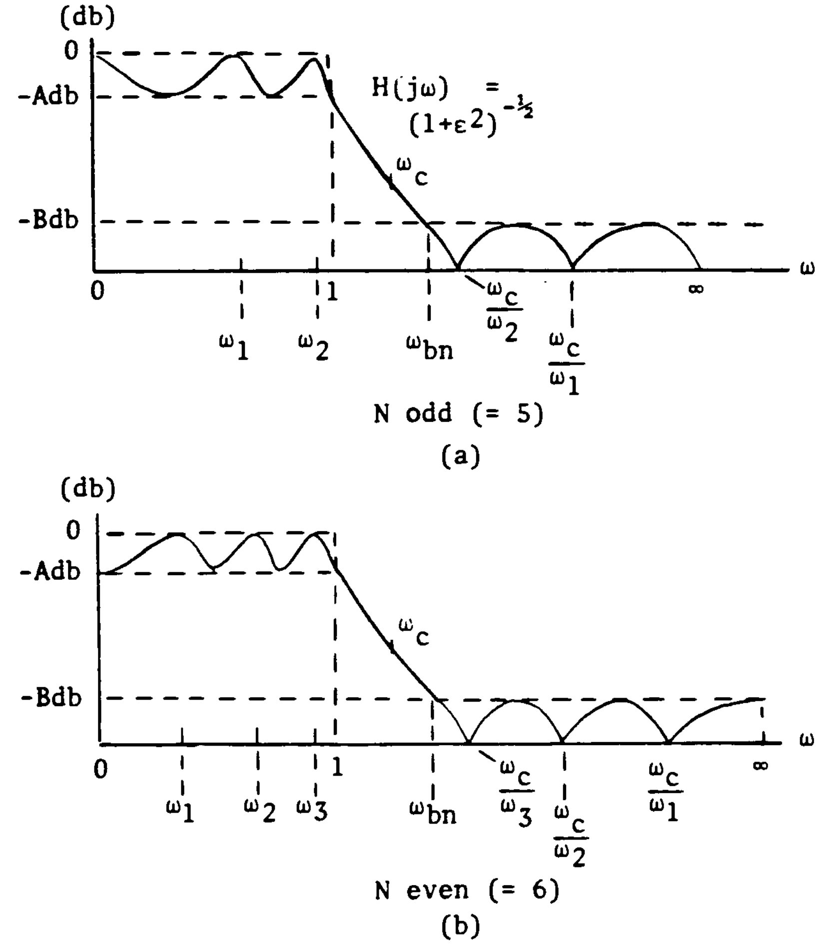
$$\varepsilon = \left(10^{Adb/10} - 1\right)^{\frac{1}{2}} \qquad (3.24)$$

Note that this constant is the same as that for the Chebyshev approximation. Let us now examine the magnitude characteristics of the elliptic response.

3.6.2 Magnitude Response of the Elliptic Function Filter

Figure 3.13 illustrates the normalized low pass magnitude response of the elliptic function filter for N odd and N even. We will discuss the magnitude characteristics as they relate to the frequency values in equation 3.23. Some new terminology will be introduced as we go along.

We start off by examing the numerator of equation 3.23 (for N odd) and its relationship to the pass band as shown in figure 3.13(a). At the normalized frequencies $\omega_1$, $\omega_2$ .... $\omega_N$ the value of $K(\omega)$ is 0.




Magnitude Response of the
Normalized Low Pass Elliptic Filter

Figure 3.13

Recall that the magnitude of the characteristic equation is described as

$$20 \text{ Log } H(j\omega) = 10 \text{ Log } \frac{1}{(1 + \varepsilon^2 K^2 (\omega))}$$

Therefore, for $K(\omega) = 0$, the magnitude of $H(j\omega)$ is 0 db. Since the magnitude is 0 db at $\omega_1$, $\omega_2$ .... $\omega_N$, we give these frequencies the name

ZEROS of ATTENUATION. Notice the difference in magnitude at $\omega = 0$ between N odd and N even in figure 3.13. The magnitude is 0 db at $\omega = 0$ for N odd due to the singular term, $\omega$, in the numerator of equation 3.23. Notice also that at $\omega = 1$ the magnitude of the response is -Adb which is derived from $20 \text{ Log } (1 + \varepsilon^2)^{-\frac{1}{2}}$. From the form of the characteristic equation above we find that $K(\omega)$ must equal 1 at the pass band frequencies where the magnitude is -Adb.

Now we focus our attention on the stop band region. By examining the denominator of equation 3.23 we see that at the frequencies $\omega_c/\omega_1$, $\omega_c/\omega_2$ .... $\omega_c/\omega_N$, the value of $K(\omega)$ is $\infty$. Then the magnitude of the response described by the characteristic function is zero or $-\infty$ db. The frequencies $\omega_c/\omega_1$ .... $\omega_c/\omega_N$ are shown in figure 3.13 at the minimum points in the stop band region. We give these frequencies the term FREQUENCIES of INFINITE LOSS.

At this point we observe that there is a relationship between the pass band and stop band. Since the magnitude is 0 db at $\omega_1$, $\omega_2$ .... $\omega_N$ in the pass band and $-\infty$ db at $\omega_c/\omega_1$ .... $\omega_c/\omega_N$ in the stop band, it follows that the characteristic function $K(\omega)$ has the following reciprocal property:

$$K(\omega_N) = \frac{L}{K(\omega_c/\omega_N)} \tag{3.25}$$

where $L$ is a constant given by

$$L = \left( \frac{10^{Bdb/10} - 1}{10^{Adb/10} - 1} \right)^{\frac{1}{2}} \tag{3.26}$$

This explains the salient features of the magnitude response shapes for N odd and N even. Now we wish to address the problem of finding the order of the filter, N, and the normalized pole/zero values.

### 3.6.3 The Normalized Poles and Zeros of the Elliptic Function Filter

The methods of determining the value N and subsequently the normalized poles and zeros is complex and involves an analysis of the characteristic rational function using elliptic integrals and elliptic trigonometric functions. The reader is referred to the appendix for a discussion of this analysis and an example. We can however gain some insight to what the pole/zero values are by examining equation 3.23.

In section 3.3.2 it was pointed out that for a rational function such as

$$K(\omega) = \frac{N(\omega)}{D(\omega)}$$

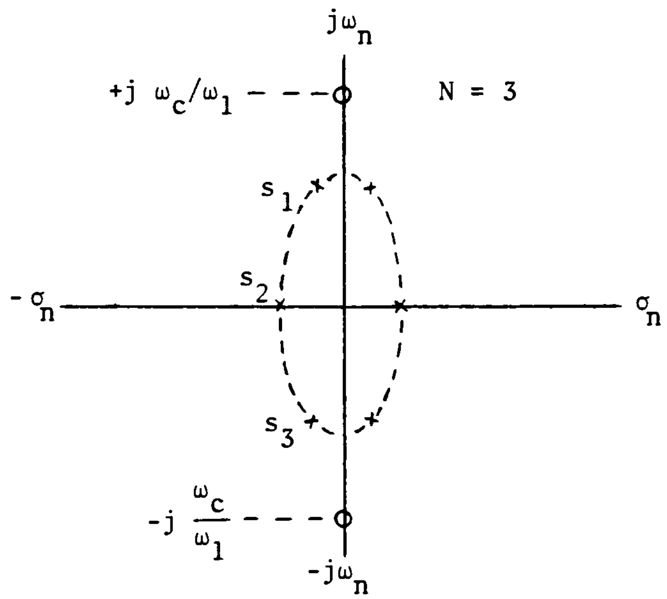
the values at which $D(\omega)$ is zero correspond to the s-plane zeros of the transfer function, $H(s)$, when $\omega = s/j$. Therefore by equation 3.23 the zeros of $H(s)$ correspond to the complex values at which the denominator of $K(\omega)$ is 0.

That is, for

$$s = \pm\, j\, \omega_c/\omega_1,\ \pm\, j\, \omega_c/\omega_2\ \ldots.\ \pm\, j\, \omega_c/\omega_N$$

we have the normalized s-plane zeros.

In regards to the pole values, we know from the examination of the Chebyshev response that equiripple in the pass band corresponds to an s-plane locus of poles in the shape of an ellipse. Combining these concepts regarding the poles and zeros we can surmise that the pole/zero pattern is of the form shown in figure 3.14. Since the left half plane




Normalized Pole/Zero Pattern
for the Elliptic Function Filter

Figure 3.14

has 3 poles and the $j\omega_n$ axis has two complex zeros, the form of H(s) describing figure 3.14 is as follows:

$$H(s) = C \cdot \frac{(s^2 + \omega_c^2/\omega_1^2)}{(s-s_1)(s-s_2)(s-s_3)}$$

where C is a constant. The appendix describes the general form of H(s) for N even and N odd.

There are many texts which discuss the elliptic function filter [5], [10]. It is difficult however to acquire a source which clearly defines a procedure for determining the filter order, N, and the normalized pole/zero values. The reason for this relates, perhaps, to the many ways of solving the elliptic functions (elliptic sine, elliptic cosine, elliptic integral, etc.). These functions inherently require greater effort to solve and usually require great accuracy. One can easily get lost in the solutions to these functions while trying to resolve the value of N and the normalized poles and zeros. The method in the appendix will hopefully aid in this regard.

A simple method of finding N and the pole/zero values is by means of tables and graphs. Reference [6] is an excellent source for this purpose. Given the normalized values -Adb, -Bdb, $\omega_{bn}$ and $\omega_c$, one can quickly determine the value of N and locate the normalized pole/zero values from the tables. It is the intent of the appendix to provide some insight into the elliptic function solutions upon which such tables and graphs are based.

In the next section a brief comparison is made among the three classical approximations.

## 3.7 Comparison of the Classical Approximations

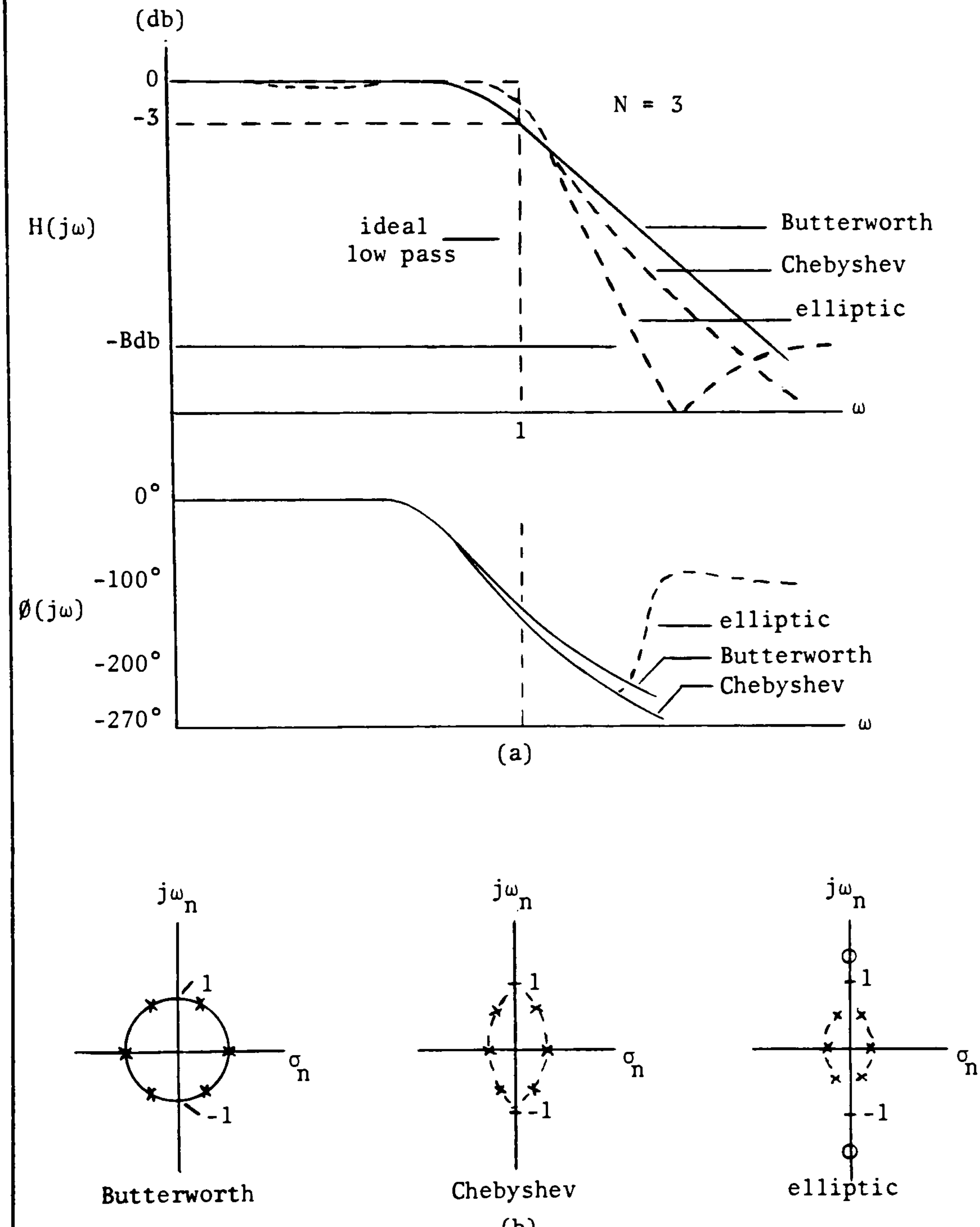
In the preceding sections three classical approximations to the magnitude response of the ideal normalized low pass filter were discussed. These approximations are the Butterworth, Chebyshev and elliptic. For each of these a discussion was presented which described the magnitude response and the s-plane characteristics. In this section a composite view of the magnitude and phase characteristics is given in order to obtain a comparison among the three classical responses.

Figure 3.15a illustrates the normalized low pass magnitude and phase response for the Butterworth, Chebyshev and elliptic filters. As a comparison basis the value of the filter order is chosen as $N = 3$. By examining the transition region we can see that the greatest attenuation rate is achieved by the elliptic response. Next to this the Chebyshev response has a faster attenuation rate than that of the Butterworth. In the pass band region the minimum variation is obtained by the elliptic and Chebyshev responses whereas the Butterworth deviates by -3 db at the cutoff. In the stop band region both the Chebyshev and Butterworth continue to attenuate in a monotonic fashion whereas the elliptic function response presents an equiripple variation.

When comparing the phase response we see that the Butterworth response provides the best linearity and the flattest group delay ($d\phi/d\omega$) among the three kinds of filter responses. It appears then that the price paid for a greater attenuation rate is greater phase non-linearity as shown for the elliptic phase response.

Shown in figure 3.15b are the s-plane diagrams corresponding to the three filter approximations. Notice the relative distance of the poles from the origin. The elliptic poles are the closest implying high Q values, greater phase non-linearity and faster attenuation. Also notice the complex zeros on the $j\omega_n$ axis for the elliptic s-plane.

The choice of which response is to be used in a design depends upon many factors. Aside from simply considering the attenuation rate, the designer must also consider the impact of phase non-linearity, design complexity, method of implementation, sensitivity, input/output impedance characteristics, etc.




Comparison of the Normalized
Butterworth, Chebyshev and Elliptic Responses

Figure 3.15

## 3.8 Chapter Summary

In this chapter the concept of normalization was introduced as a means of converting the parameters of the straight line transition diagram for any filter type into a normalized low pass form. The normalized low pass filter became the basis for analysis of the filter order N and the pole/zero values of the classical approximations.

Associated with the normalized low pass response was the characteristic equation. This equation was dependent upon a particular characteristic function, $K(\omega)$. $K(\omega)$ was then defined as a polynomial for the Butterworth and Chebyshev responses and a rational function for the elliptic response. We found that each of these had unique properties in attempting to approximate the ideal low pass response characteristics.

For each of these classical responses, we briefly examined the s-plane characteristics. It was shown that a properly selected elliptic pattern of poles resulted in an equiripple magnitude response in the pass band. In addition we found that the elliptic function filter had the unique property of complex conjugate zeros on the $j\omega$ axis which resulted in an equiripple stop band.

So at this point we have traversed from the original design parameters of the straight line transition diagram to the normalized s-plane pole/zero values. The remaining questions then pertain to how these pole/zero values are denormalized back to a form representing the original denormalized filter response. The next chapter addresses this topic of denormalizing the pole/zero values.

# CHAPTER IV

# DENORMALIZATION

## 4.1 Introduction

In the previous chapter we discussed the methods of obtaining the poles and zeros for the normalized low pass transfer function. In this chapter we examine the transformation equations which produce denormalized low pass, high pass, band pass and band stop filters from the normalized low pass filter. By substituting the transformation equations into the normalized low pass transfer function, the denormalized response is obtained. Throughout the chapter we will discuss the denormalization process as it applies to the poles of the transfer function. The same denormalization equations, however, apply to the normalized zeros of a transfer function as well.

## 4.2 Low Pass Normalized to Low Pass Denormalized

The transformation equation relating the denormalized low pass filter to the normalized low pass filter is

$$H_{LPD}(s) = H_{LPN}(\bar{s}) \tag{4.1}$$

where

$$\bar{s} = s/\omega_a$$

and the subscripts LPD, LPN are acronyms for low pass denormalized and low pass normalized respectively. Equation 4.1 states that if the transformed variable, $\bar{s}$, is substituted into the normalized low pass transfer function, then we will obtain the denormalized low pass response. Consider the case of an Nth order normalized low pass filter which has the following transfer function:

$$H_{LPN}(s) = \frac{K}{(s-s_1)(s-s_2)\ \ldots\ (s-s_N)} \tag{4.2}$$

where $s_1$, $s_2$ .... $s_N$ are the normalized low pass poles and K is a constant. The denormalized transfer function is then found for an arbitrary cutoff frequency, $\omega_a$, using equation 4.1 as follows:

$$H_{LPN}(\bar{s}) = \frac{K}{\left(\frac{s}{\omega_a} - s_1\right)\left(\frac{s}{\omega_a} - s_2\right)\ \cdots\ \left(\frac{s}{\omega_a} - s_N\right)}$$

or

$$H_{LPD}(s) = \frac{K'}{(s-s_1\cdot\omega_a)(s-s_2\cdot\omega_a)\ \ldots\ (s-s_N\cdot\omega_a)} \tag{4.3}$$

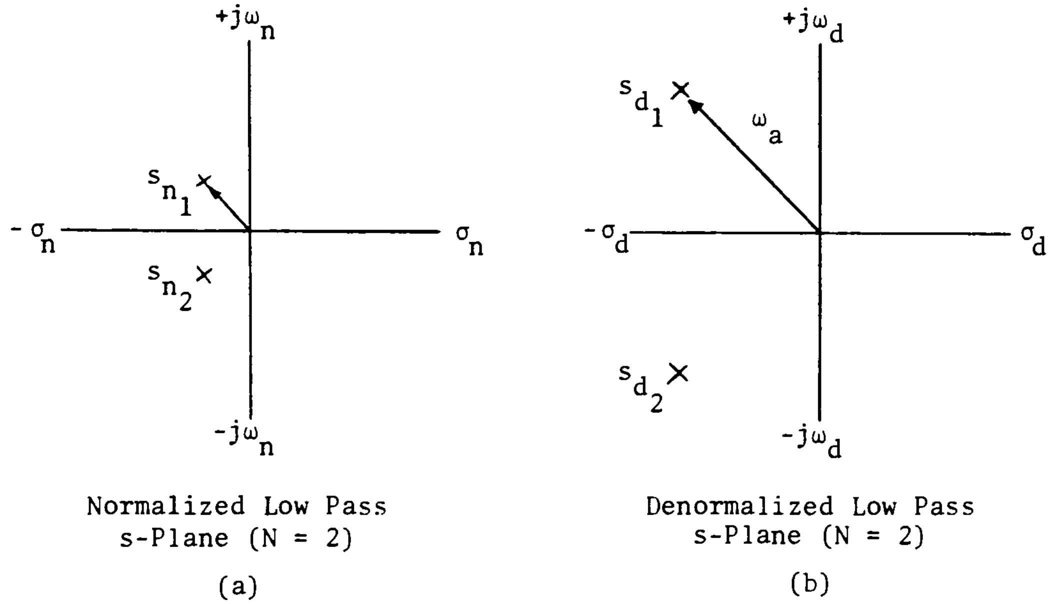
Equation 4.3 is the denormalized low pass transfer function. Notice that the values of the denormalized poles are found by multiplying the normalized poles ($s_1$, $s_2$ .... $s_N$) by the cutoff frequency, $\omega_a$.

That is,

$$s_d = \omega_a \cdot s_n \tag{4.4}$$

where the subscripts d and n refer to denormalized and normalized respectively.

Figure 4.1 illustrates the effect of denormalization. In comparing the denormalized s-plane (Figure 4.1b) with the normalized (Figure 4.1a) we see that the denormalized poles have been moved by a factor of $\omega_a$ along the same radial line of the normalized pole. The pattern of poles on the denormalized s-plane has been preserved. Therefore the low pass characteristics of the approximation chosen (Butterworth, elliptic, etc.) are retained. The same process of denormalization would be applied for the s-plane zeros such as those derived for the transfer function of the elliptic filter.




Normalized Low Pass
s-Plane (N = 2)

(a)

Denormalized Low Pass
s-Plane (N = 2)

(b)

Transformation of the Low Pass Normalized
to the Low Pass Denormalized s-Plane

Figure 4.1

## 4.3 Low Pass Normalized to High Pass Denormalized

The transformation equation relating the low pass normalized filter to the high pass denormalized filter is

$$H_{HPD}(s) = H_{LPN}(\bar{s})$$

where (4.5)

$$\bar{s} = \frac{\omega_a}{s}$$

The notation used here is similar to that of eq. 4.1. By substituting $\bar{s}$ for s in the normalized low pass transfer function (eq. 4.2), the denormalized high pass transfer function can be found as shown below.

$$H_{LPN}(\bar{s}) = \frac{K}{\left(\frac{\omega_a}{s} - s_1\right)\left(\frac{\omega_a}{s} - s_2\right) \cdots\cdots \left(\frac{\omega_a}{s} - s_N\right)}$$

$$= \frac{K\ s^N}{(\omega_a - ss_1)(\omega_a - ss_2) \cdots\cdots (\omega_a - ss_N)}$$

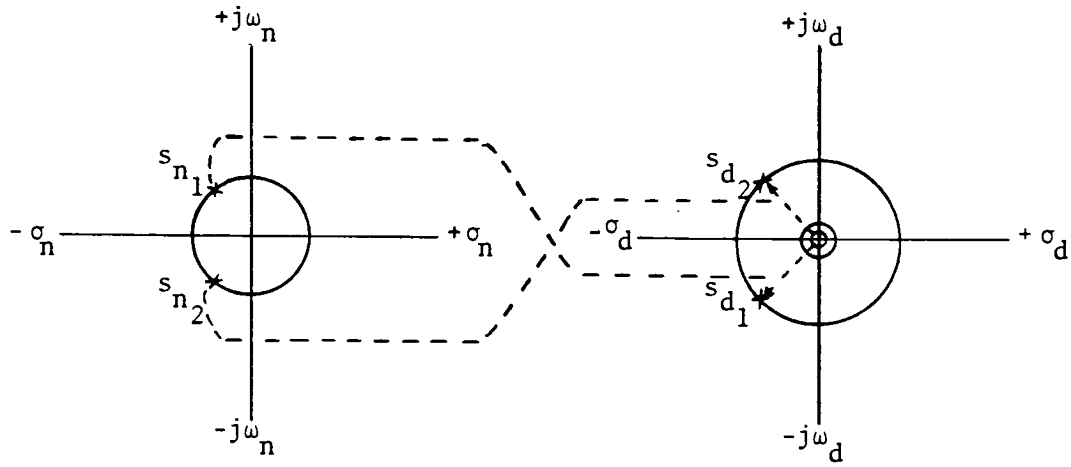
or

$$H_{HPD}(s) = \frac{K'\ s^N}{\left(s - \frac{\omega_a}{s_1}\right)\left(s - \frac{\omega_a}{s_2}\right) \cdots\cdots \left(s - \frac{\omega_a}{s_N}\right)} \qquad (4.6)$$

Equation 4.6 is the denormalized high pass transfer function for the low pass filter order, N. The denormalized poles shown in the denominator are related to the normalized poles ($s_1$, $s_2$ .... $s_N$) as follows:

$$s_d = \frac{\omega_a}{s_n} \qquad (4.7)$$

Equation 4.7 states that the denormalized high pass pole, $s_d$, can be found by inverting the normalized low pass pole, $s_n$, (generally a complex value) and then multiplying by the arbitrary denormalized high pass cutoff frequency, $\omega_a$. Figure 4.2 illustrates the effect of the denormalization. The reciprocal relationship in s is denoted by the crossing lines. Multiplying by $\omega_a$ results in a displacement along the radial line of the inverted normalized pole as shown in figure 4.2b.




Normalized Low Pass
s-Plane (N = 2)
(a)

Denormalized High Pass
s-Plane (N = 2)
(b)

Transformation of the Low Pass Normalized
to the High Pass Denormalized s-Plane

Figure 4.2

Notice the s-plane zeros at the origin in figure 4.2b. These zeros are evident in the denormalized transfer function (eq. 4.6) by the $s^N$ in the numerator and account for a magnitude equal to 0 at dc.

The magnitude and phase of each denormalized high pass pole, $s_d$, can be found using equation 4.7. We will represent the complex value, $s_n$, by its magnitude, $|s_n|$, and phase, $\emptyset_n$, as follows:

$$s_n = |s_n| \, e^{j\emptyset_n}$$

The methods for determining the magnitude, $|s_n|$, and phase, $\emptyset_n$, were presented in section 2.5.1. The denormalized high pass pole, $s_d$, is found from

$$s_d = \frac{\omega_a}{s_n} = \frac{\omega_a}{|s_n|} \cdot e^{j(-\phi_n)}$$

Notice the inverse magnitude and negative phase relationship to the normalized values in the above equation for $s_d$. In terms of the denormalized real ($\sigma_d$) and imaginary ($j\omega_d$) parts we have the following

$$s_d = \sigma_d + j\omega_d \tag{4.8}$$

where

$$\sigma_d = \frac{\sigma_n}{|s_n|^2} \cdot \omega_a$$

$$\pm j\omega_d = \frac{\mp j\omega_n}{|s_n|^2} \cdot \omega_a$$

## 4.4 Low Pass Normalized to Band Pass Denormalized

The transformation equation relating the band pass denormalized filter to the low pass normalized filter is

$$H_{BPD}(s) = H_{LPN}(\bar{s}) \tag{4.9}$$

where

$$\bar{s} = \frac{s^2 + \omega_c^2}{Bs},$$

B is the bandwidth,

and

$\omega_c$ is equal to $2\pi$ times the center frequency, $f_c$

The objective is to determine the denormalized band pass transfer function and subsequently the equations for calculating the denormalized poles. The procedure follows as before. We start by substituting $\bar{s}$ for s into the low pass normalized transfer function, $H_{LPN}(s)$, given by equation 4.2.

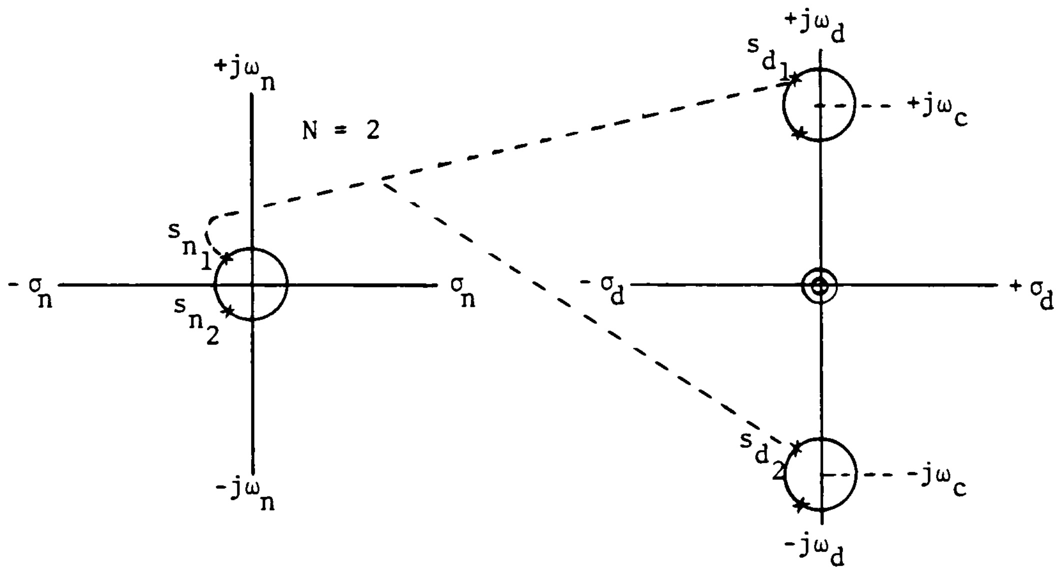
That is,

$$H_{LPN}(\bar{s}) = \frac{K}{\left(\frac{s^2 + \omega_c^2}{Bs} - s_1\right)\left(\frac{s^2 + \omega_c^2}{Bs} - s_2\right) \cdots \left(\frac{s^2 + \omega_c^2}{Bs} - s_N\right)}$$

where $s_1$, $s_2$ .... $s_N$ are the normalized poles for an Nth order low pass filter. The above equation can be rearranged to arrive at the form for $H_{BPD}(s)$ as shown below in equation 4.10.

$$H_{BPD}(s) = \frac{K\,B^N s^N}{(s^2 - Bss_1 + \omega_c^2)(s^2 - Bss_2 + \omega_c^2) \cdots (s^2 - Bss_N + \omega_c^2)} \tag{4.10}$$

Notice that each low pass pole ($s_1$, $s_2$ .... $s_N$) produces a quadratic term in the denominator of equation 4.10. Each quadratic term produces 2 denormalized band pass pole locations when factored. Thus there are 2N denormalized band pass poles derived from the Nth order normalized low pass filter. Figure 4.3 illustrates the denormalized band pass s-plane for N = 2. The dashed lines denote the production of 2 denormalized poles from each normalized low pass pole. The denormalized




Normalized Low Pass
s-Plane

(a)

Denormalized Band Pass
s-Plane

(b)

Transformation of the Low Pass Normalized
to the Band Pass Denormalized s-Plane

Figure 4.3

poles are clustered around $s = \pm j\omega_c$, where $\omega_c$ is the center frequency of the pass band. Aside from the poles, the $s^N$ term in the numerator of eq. 4.10 indicates that there are N zeros at the origin on the denormalized band pass s-plane. This can be seen for N = 2 in figure 4.3b. The zeros account for the magnitude of zero at dc. The 2N poles account for a magnitude of zero at $\infty$ frequency.

Having discussed the generalities pertaining to the denormalized s-plane, we turn our attention to determining the values of the denormalized band pass poles. The band pass poles are derived by factoring each of the quadratic terms in the denominator of equation 4.10.

That is,

$$\left(s^2 - Bss_{n_k} + \omega_c^2\right) = \left(s - s_{d_k}\right)\left(s - s_{d_{k+1}}\right)$$

where the denormalized poles are represented as $s_{d_{k,k+1}}$ and are derived for k = 1 to N as follows:

$$s_{d_{k,k+1}} = \frac{1}{2}\left[Bs_{n_k} \pm \sqrt{B^2\ s_{n_k}^2 - 4\ \omega_c^2}\right] \tag{4.11}$$

The solution for the 2N denormalized poles is complicated by the fact that the normalized poles, $s_{n_k}$, are complex $\left(\sigma_{n_k} + j\omega_{n_k}\right)$. The treatment of these solutions for the poles will not be presented here. However the results are stated in the following. Given the normalized kth pole value,

$$s_{n_k} = \sigma_{n_k} + j\omega_{n_k}$$

and the denormalized parameters,

$\omega_c$ (center frequency)

B (bandwidth),

the 2 denormalized poles $s_{d_k}$, $s_{d_{k+1}}$ are found using the following equations:

$$s_{d_k} = \sigma_{d_k} + j\omega_{d_k} \tag{4.12}$$

$$= \left(\frac{B}{2} \cdot \sigma_{n_k} + \frac{B}{2} \sqrt{R} \cos \frac{\phi}{2}\right) + j \left(\frac{B}{2} \cdot \omega_{n_k} + \frac{B}{2} \sqrt{R} \sin \frac{\phi}{2}\right)$$

and

$$s_{d_{k+1}} = \sigma_{d_{k+1}} + j\omega_{d_{k+1}} \tag{4.13}$$

$$= \left(\frac{B}{2} \cdot \sigma_{n_k} - \frac{B}{2} \sqrt{R} \cos \frac{\phi}{2}\right) + j \left(\frac{B}{2} \cdot \omega_{n_k} - \frac{B}{2} \sqrt{R} \sin \frac{\phi}{2}\right)$$

where

$$R = \left\{\left[\sigma_{n_k}^2 - \omega_{n_k}^2 - \left(\frac{2\,\omega_c}{B}\right)^2\right]^2 + 4\,\sigma_{n_k}^2\,\omega_{n_k}^2\right\}^{1/2}$$

$$\phi_k = \tan^{-1}\left[\frac{2 \cdot \sigma_{n_k} \cdot \omega_{n_k}}{\sigma_{n_k}^2 - \omega_{n_k}^2 - \left(\frac{2\,\omega_c}{B}\right)^2}\right]$$

$k$ = 1 to $N$

$N$ = order of the normalized low pass filter

The above equations can be somewhat tedious to solve given any reasonable value of N but are necessary for determining the exact denormalized pole values. For most cases, approximations can be used in place of the above equations. For the case where $\omega_c$ is much greater than $\sigma_{n_k}$, $\omega_{n_k}$ and B/2, equations 4.14 and 4.15 as shown below can be used to determine the denormalized band pass pole values.

$$s_{d_k} = \left(\sigma_{d_k} + j\omega_{d_k}\right) \approx \frac{B}{2}\,\sigma_{n_k} + j\left(\omega_c + \frac{B}{2}\,\omega_{n_k}\right) \tag{4.14}$$

$$s_{d_{k+1}} = \left(\sigma_{d_{k+1}} + j\omega_{d_{k+1}}\right) \approx \frac{B}{2}\,\sigma_{n_k} + j\left(-\,\omega_c + \frac{B}{2}\,\omega_{n_k}\right) \tag{4.15}$$

This completes the discussion of the derivation of the denormalized band pass response. An example, using the equations shown, is given in section 5.5. The next section discusses the denormalization process as it pertains to the band stop filter.

## 4.5 Low Pass Normalized to Band Stop Denormalized

Since the band stop filter is the same as the band pass filter with the bands interchanged, we can perform the denormalization by using an inverted band pass transformation (equation 4.9). The transformation equation relating the band stop denormalized filter to the low pass normalized filter is

$$H_{BSD}\,(s) = H_{LPN}\,(\bar{s}) \tag{4.16}$$

where

$$\bar{s} = \left(\frac{s^2 + \omega_c^2}{Bs}\right)^{-1}$$

and the terms B, $\omega_c$ have been previously defined in eq. 4.9. The transformation variable, $\bar{s}$, can be substituted into the normalized low pass transfer function (eq. 4.2) to derive the denormalized band stop response.

That is,

$$H_{LPN}\,(\bar{s}) = \frac{K}{\left(\frac{Bs}{s^2 + \omega_c^2} - s_1\right)\left(\frac{Bs}{s^2 + \omega_c^2} - s_2\right)\cdots\cdot\left(\frac{Bs}{s^2 + \omega_c^2} - s_N\right)}$$

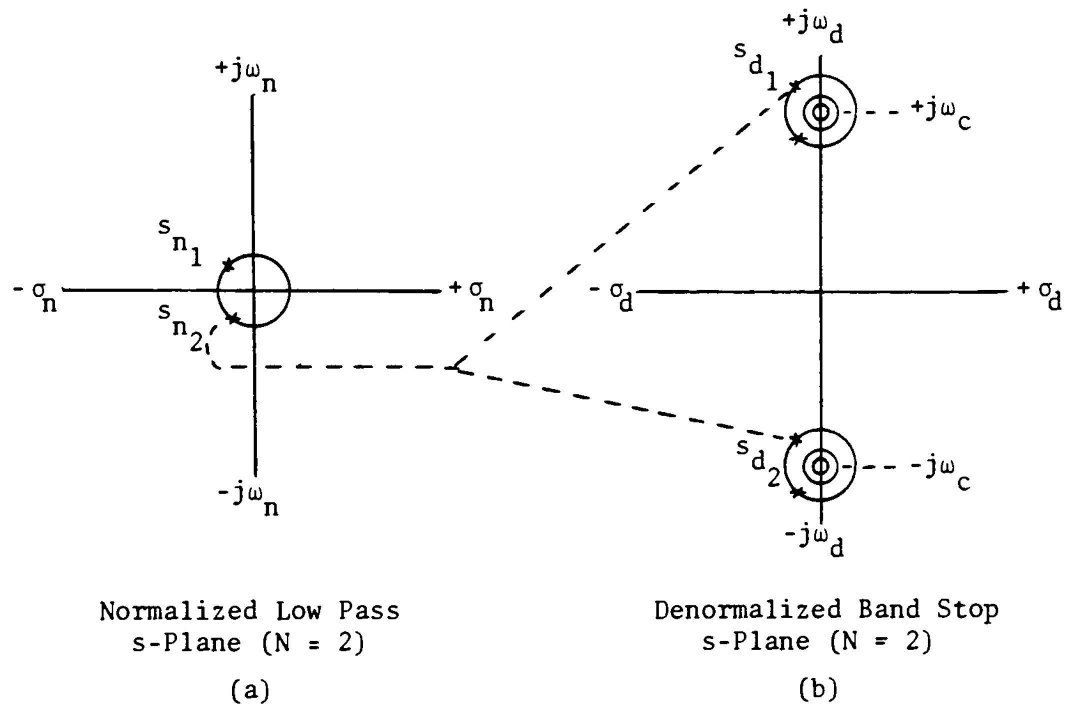
where $s_1, s_2 \ldots\ldots s_N$ are the normalized poles for the Nth order low pass filter. Rearranging this equation in terms of s, we arrive at the denormalized band stop response as follows:

$$H_{BSD}(s) = \frac{K' \left(s^2 + \omega_c^2\right)^N}{\left(s^2 - Bss_1^{-1} + \omega_c^2\right)\left(s^2 - Bss_2^{-1} + \omega_c^2\right) \ldots\ldots \left(s^2 - Bss_N^{-1} + \omega_c^2\right)} \tag{4.17}$$

The denominator of equation 4.17 is very similar to equation 4.10 for the band pass response. The quadratic terms produce 2 denormalized poles from each normalized low pass pole ($s_1, s_2 \ldots\ldots s_N$) and therefore there are 2N band stop poles. We would expect the poles to be clustered about $s = \pm j\omega_c$, in a similar fashion to the band pass response as shown in figure 4.4b. Aside from the poles, there are complex zeros indicated by the numerator of equation 4.17. For the band stop filter there are N zeros at $s = j\omega_c$ and N zeros at $-j\omega_c$. These zeros lie in the center of the cluster of poles along the $j\omega$ axis. Figure 4.4b shows the four zeros of a band stop response for N = 2 (low pass filter order). The zeros on the $j\omega$ axis account for the attenuation in the stop band defined within the region of the poles. We turn our attention now to establishing the denormalization equations for the poles and zeros for the band stop response.

The denormalized zeros are found simply by using the stop band center frequency as follows:

$$\pm z_{d_k} = \pm j\omega_{c_k} \qquad k = 1 \text{ to } N \tag{4.18}$$




Transformation of the Low Pass Normalized to the Band Stop Denormalized s-Plane

Figure 4.4

The denormalized poles are found using the same equations as those for the band pass denormalization except that the normalized poles are inverted. The equations will not be repeated here. Instead the results are summarized in the following. Given the low pass normalized kth pole value,

$$s_{n_k} = \sigma_{n_k} + j\omega_{n_k}$$

we derive the inverted normalized pole from:

$$s'_{n_k} = \frac{1}{s_{n_k}} = \frac{\sigma_{n_k}}{\left|s_{n_k}\right|^2} + j \frac{\omega_{n_k}}{\left|s_{n_k}\right|^2} \tag{4.19}$$

$$= \sigma'_{n_k} + j\, \omega'_{n_k}$$

Having found $\sigma'_{n_k}$ and $\omega'_{n_k}$ we proceed to find the denormalized band stop poles using the same equations given for the band pass (equations 4.12 to 4.15). In these band pass equations, the values $\sigma'_{n_k}$ and $\omega'_{n_k}$ (eq. 4.14) are simply substituted for $\sigma_{n_k}$ and $\omega_{n_k}$ respectively.

4.6 Chapter Summary

In this chapter we found that the denormalized transfer functions could be produced by substituting the appropriate transformation equations into the normalized low pass transfer function. The denormalized transfer function represents the response of the filter as we would expect if it were built. Therefore we are now in a position to consider an active filter design to implement the denormalized response. The next chapter presents a design methodology for active filters. The steps taken in the design process are essentially a summary of the results derived in chapters II-IV. Examples are given to fortify the design concepts.

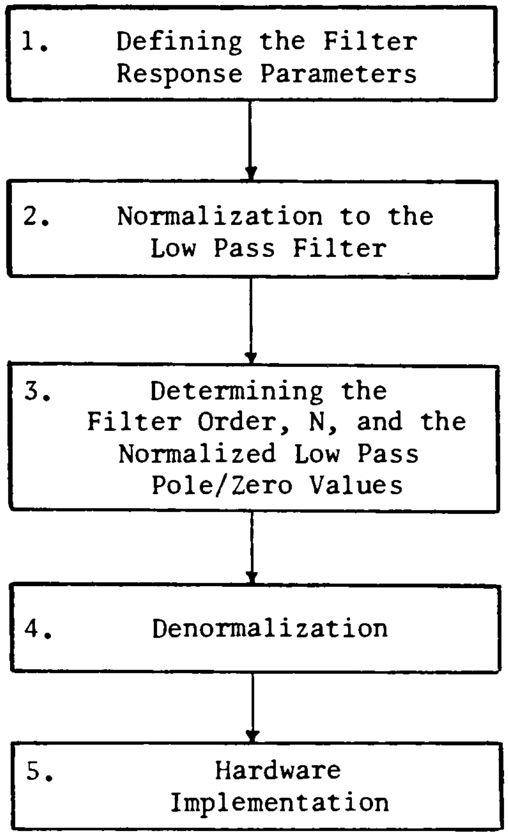
CHAPTER V

# ACTIVE FILTER DESIGN

## 5.1 Introduction

In the previous chapter we derived the denormalized transfer functions. If we were to summarize the steps taken to arrive at the denormalized transfer functions, as presented in the first four chapters, we would have a list of the basic analytical steps needed to perform a filter design. These steps are illustrated in figure 5.1. We started by defining the magnitude response parameters of the desired filter using a straight line transition diagram (step 1). Then the normalized low pass filter was derived (step 2). Using the parameters of the normalized low pass filter, the filter order, N, and the normalized pole/zero values were determined for a selected approximation (step 3). Then the normalized poles and zeros were transformed into the denormalized poles and zeros which constitute the desired filter response (step 4). What remains is the final step of implementing the denormalized transfer function (step 5).

In this chapter we examine the techniques which implement the denormalized transfer functions using active filters. Among the various classes of active filter designs, we will use the infinite-gain multiple-feedback (IGMF) configuration. The term infinite-gain applies to the ideal nature of the open loop gain, A, of an operational amplifier. The term multiple feedback refers to the use of multiple feedback paths to achieve a given transfer function. We will examine the design equations used and the active filter networks needed to implement the denormalized transfer functions.




Basic Filter Design Steps

Figure 5.1

## 5.2 Low Pass Filter Design

The general form of the low pass denormalized transfer function was described in Chapter IV and is shown below.

$$H_{LPD}(s) = \frac{K}{\left(s-s_{d_1}\right)\left(s-s_{d_2}\right) \;\cdots\; \left(s-s_{d_N}\right)}$$

where N is the filter order and $s_{d_1}$, $s_{d_2}$ .... $s_{d_N}$ are the denormalized pole values. Since the poles usually occur in conjugate pairs, we can separate the denormalized transfer function into two pole sections with the addition of a single pole section if N is odd. This is shown as follows:

$$H_{LPD}(s) = \frac{K_1}{\left(s-s_{d_1}\right)\left(s-s_{d_1}^*\right)} \cdot \frac{K_2}{\left(s-s_{d_2}\right)\left(s-s_{d_2}^*\right)} \cdots\cdots \frac{K_{N/2}}{\left(s-s_{d_N}\right)\left(s-s_{d_N}^*\right)} \cdot \frac{K'}{\left(s-\sigma_d\right)} \tag{5.1}$$

The last section shows that for N odd, one pole lies on the real axis, $\sigma_d$, of the s-plane. As a method for implementing the denormalized transfer function, we can use a cascade of two-pole filters and include a single pole active filter stage if needed. We will examine the design procedures for the two pole and single pole active filters; then a design example will be given.

### 5.2.1 Two Pole Low Pass Active Filter Design

The objective in this section is to design an active filter which will implement a two pole transfer function as shown below.

$$H(s) = \frac{K}{(s-s_d)(s-s_d^*)} \tag{5.2}$$

Given that the poles are complex conjugates $(\sigma_d \pm j\omega_d)$ we can rewrite equation 5.2 as follows:

$$H(s) = \frac{K}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2} \tag{5.3}$$

where

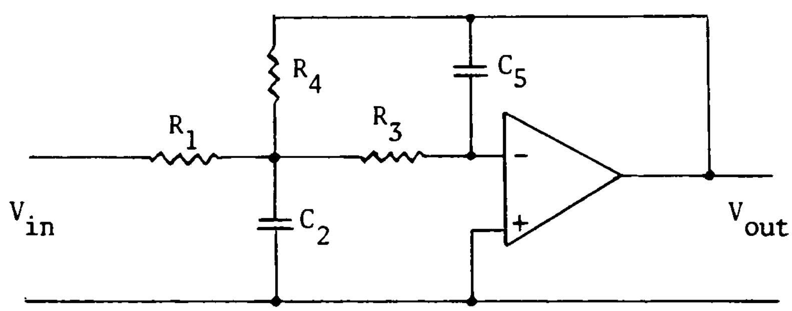
$$\omega_0 = \left(\sigma_d^2 + \omega_d^2\right)^{\frac{1}{2}}$$

and

$$Q = \omega_0 / |2\sigma_d|$$

Figure 5.2 illustrates an active filter which will implement the transfer function of equation (5.3). The components shown are related to the transfer function as shown in equation (5.4).

$$H(s) = \frac{-1/(R_1 R_3 C_2 C_5)}{s^2 + s\left(\frac{1}{R_1 C_2} + \frac{1}{R_3 C_2} + \frac{1}{R_4 C_2}\right) + \frac{1}{R_3 R_4 C_2 C_5}} \tag{5.4}$$




Low Pass IGMF Active Filter
(two-pole)

Figure 5.2

The procedure for determining the component values of figure 5.2 is as follows.

1. Specify the desired dc gain, $H_0$ (not in db)
2. Determine $\omega_0$, Q as shown in equation 5.3
3. Calculate the constant $\alpha$ as follows:

$$\alpha = Q^2 \cdot \frac{(1 + 2H_0)^2}{H_0}$$

4. Select a reasonable value for $C_5$ ($10^{-8}$ is a good starting value)

5. Find $C_2$ from $C_2 = \alpha \cdot C_5$

6. Find $R_1$ and $R_2$ as follows:

$$R_1 = R_2 = \left(\omega_0 \cdot C_5 \cdot \sqrt{\alpha \cdot H_0}\right)^{-1}$$

7. Determine $R_4$ from

$$R_4 = H_0 \cdot R_1$$

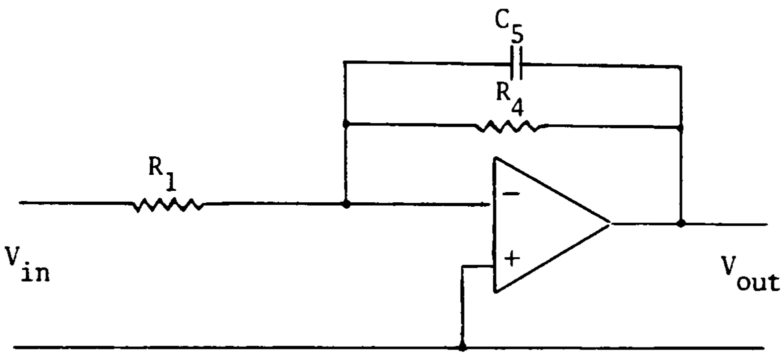
If the first iteration of the above procedure does not yield reasonable values, $C_5$ is chosen differently and steps 5-7 are repeated. The designer should consider other factors when choosing component values such as input impedance, op-amp bias currents, component values at high frequencies, Johnson noise for low level applications, etc.

### 5.2.2 Single Pole Low Pass Active Filter

For the case where the filter order, N, is odd, the transfer function will require a single pole section which has the following form

$$H(s) = \frac{K'}{s - \sigma_d} \tag{5.5}$$

where the pole value, $\sigma_d$, is real. This was previously shown in equation 5.1. Figure 5.3 illustrates the single order filter configuration used to implement the transfer function of equation 5.5. The component values shown are related to the transfer function, equation 5.5, as follows:




Low Pass Active Filter
(single pole)

Figure 5.3

$$H(s) = -\frac{R_4}{R_1} \cdot \frac{1/(C_5 \cdot R_4)}{s + 1/(C_5 \cdot R_4)} \tag{5.6}$$

The procedure for determining the component values of figure 5.3 is as follows:

1. Specify the dc gain, $H_0$ (not in db)
2. Select a reasonable value for $C_5$ ($10^{-8}$ is a good choice)
3. Find $R_1$ and $R_4$:

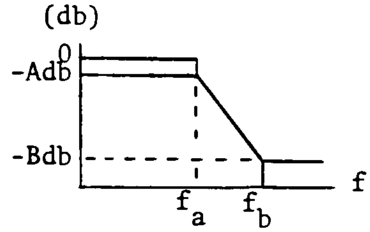
$$R_4 = \frac{1}{|\sigma_d| \cdot C_5} \quad ; \quad R_1 = R_4/H_0$$

where $\sigma_d$ is the real pole in equation 5.5.

This completes the procedures necessary for finding the component values of the two pole and single pole IGMF active filters. It should be noted that other procedures could have been developed by selecting a different

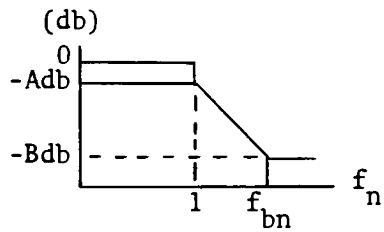
1. Define the filter parameters:

$-Adb,\ f_a,\ -Bdb,\ f_b$



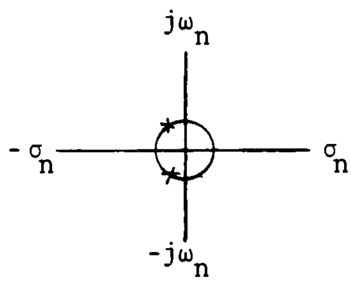

2. Find the normalized low pass filter parameters

$$f_{bn} = \frac{f_b}{f_a}$$




3. Find the filter order, N, and the normalized pole values, $s_k$

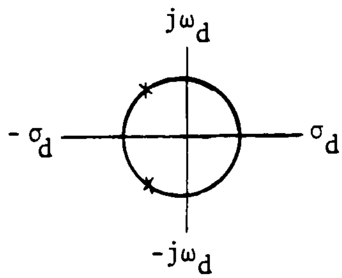
Equations for N : 3.15, 3.21
Equations for $s_k$: 3.17, 3.22




4. Denormalize the poles

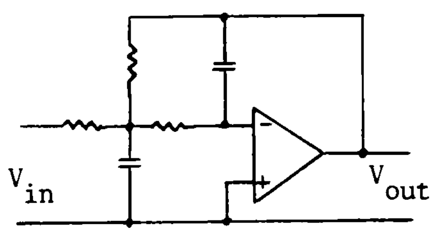
$$s_{d_k} = \omega_a \cdot s_{n_k}$$

$$\omega_a = 2\pi f_a$$




5. Implement the filter design

Two-pole, see p 110
Single pole, see p 112




Low Pass Filter Design Summary

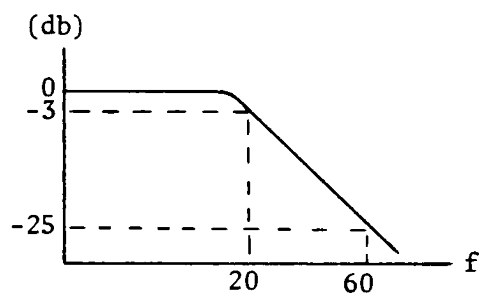
Figure 5.4

component relationship such as $C_2 = C_5$ and solving for the remaining components using equations 5.4 and 5.6.

A summary of the basic steps for designing the low pass active filter is shown in figure 5.4. The steps taken have been discussed in the previous sections. The pertinent equations for each step are given. The next section gives a complete design example.

### 5.2.3 Low Pass Design Example

In the following example, we wish to design a Butterworth active filter which has the magnitude and frequency parameters shown in figure 5.5. The pass band is specified with a maximum deviation, -Adb, of -3 db and a cutoff frequency, $f_a$, at 20 Hz. The stop band is specified with a minimum attenuation -Bdb of -25 db with a stop band edge frequency equal to 60 Hz. The requirements are summarized as shown below.




Low Pass Filter Requirements (example)

Figure 5.5

<u>Response Parameters:</u>

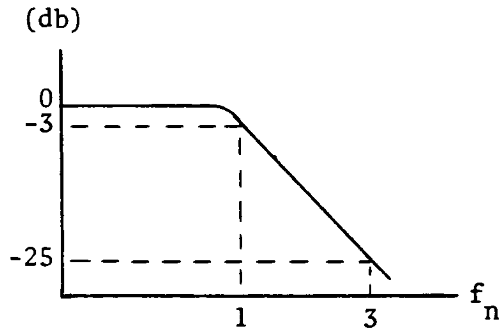
$f_a$ = 20 Hz    -Adb = -3 db

$f_b$ = 60 Hz    -Bdb = -60 db

The next step is to normalize the parameters to obtain the normalized low pass filter as shown in figure 5.6. Using the normalization equation, 3.1, we find the normalized stop band frequency as

$$f_{bn} = \frac{60}{20} = 3$$

The attenuation parameters remain the same as shown.



Normalized Low Pass Filter (derived from Figure 5.5)

Figure 5.6

Having normalized the requirements, the filter order N is obtained for the Butterworth approximation using equation 3.15 as follows:

$$N \geq \frac{\text{Log}\left(\dfrac{10^{Bdb/10} - 1}{10^{Adb/10} - 1}\right)}{2\ \text{Log}\ f_{bn}} = \frac{\text{Log}\left(\dfrac{10^{25/10} - 1}{10^{3/10} - 1}\right)}{2\ \text{Log}\ 3} = 2.6$$

Rounding up, we choose N = 3. Given the filter order, we can now obtain the normalized poles. To do this we can use the tables [4] or equation 3.17 as shown below:

$$s_k = e^{j\left(\frac{K}{3} \cdot \pi\right)} \qquad \text{for } k = 1 \text{ to } 6$$

The resulting left hand plane poles are shown in the following table.

| Butterworth Poles, N = 3 | | |
|---|---|---|
| $s_k$ | $\sigma_n$ | $j\omega_n$ |
| $s_2$ | - .5 | + j .866 |
| $s_3$ | -1.0 | 0 |
| $s_4$ | - .5 | - j .866 |

Having obtained the Butterworth normalized poles, we now need to denormalize the pole values given the cutoff frequency, $f_a$ = 20 Hz. By equation 4.4, the denormalized poles are found by multiplying by the cutoff frequency, $2\pi f_a$. The denormalized poles are shown below.

| $s_d$ | $= \sigma_d \pm j\omega_d$ |
|---|---|
| $s_{d_{2,4}}$ | -62.8 ± j 108.8 |
| $s_{d_3}$ | -125.66 |

The denormalized transfer function is found by substituting the above poles into equation 4.3.

That is,

$$H(s) = \frac{K}{(s^2 - 125.66s + 15791.4)(s + 125.66)}$$

To implement this transfer function we require a two-pole filter plus a single pole filter.

For the two pole section the filter values are found using the procedure outlined on p 110 as follows:

1. Desired gain, $H_0$ is 10.

2. Finding $\omega_0$, Q per equation 5.3 for $s_{d_{2,4}}$:

$$\omega_0 = \left(62.8^2 + 108.8^2\right)^{\frac{1}{2}} = 125.62$$

$$Q = 125.62/2(62.8) \approx 1$$

3. The design constant $\alpha$ is

$$\alpha = \frac{1^2 \left(1 + 2 \cdot 10\right)^2}{10} = 44.1$$

4. Choose $C_5 = 10^{-8}$

5. Find $C_2$:

$$C_2 = (44.1)(10^{-8})$$
$$= .44\ \mu f$$

6. Find $R_1$ and $R_3$:

$$R_1 = R_3 = \left((125.62)\ 10^{-8} \cdot \sqrt{(44.1)(10)}\right)^{-1}$$
$$= 37.9K\Omega$$

7. Find $R_4$:

$$R_4 = 10 \cdot (37.9K\Omega)$$
$$= 379K\Omega$$

Now for the single pole we have a choice between a passive R-C network design or use of the active filter of figure 5.3. If an R-C network is chosen, the load impedance is a determining factor for the resistance value selected and the gain of the previous stage.

We will choose the active filter. For the pole value $\sigma_{d_3}$ = -125.66, and for a gain of 1, the procedure on p 112 yields the following:

1. Gain, $H_0 = 1$

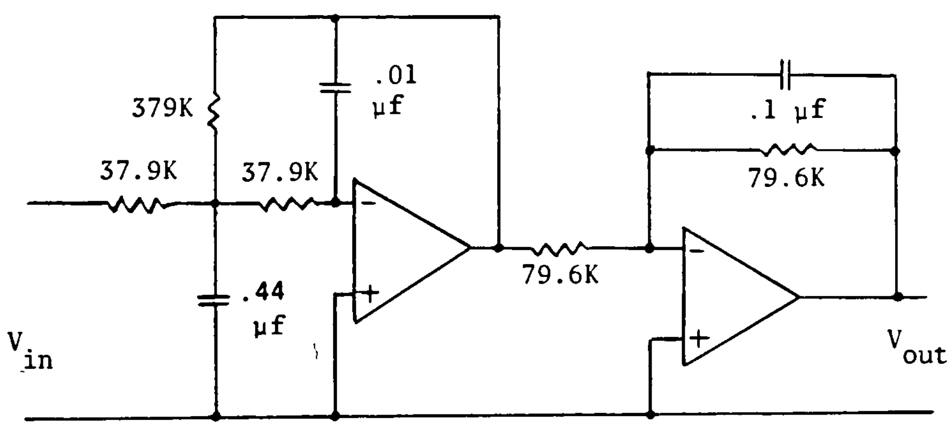
2. Choose $C_5 = 10^{-7}$

3. Finding $R_1$ and $R_4$:

$$R_4 = \frac{1}{(125.66)(10^{-7})} = 79.6\text{K}\Omega$$

$$R_1 = 79.6\text{K}\Omega/1 = 79.6\text{K}\Omega$$

The resulting three pole Butterworth filter is shown below in figure 5.7.




Low Pass Butterworth
IGMF Active Filter
(N = 3, cutoff frequency = 20 Hz)

Figure 5.7

## 5.3 High Pass Filter Design

In this section we will proceed in a similar fashion to the low pass filter design. The denormalized transfer function, eq. 4.6, is partitioned into the basic two-pole high pass sections having the following transfer function:

$$H(s) = \frac{Ks^2}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2} \tag{5.7}$$

where

$$\omega_0 = \left(\sigma_d^2 + \omega_d^2\right)^{\frac{1}{2}}$$

and

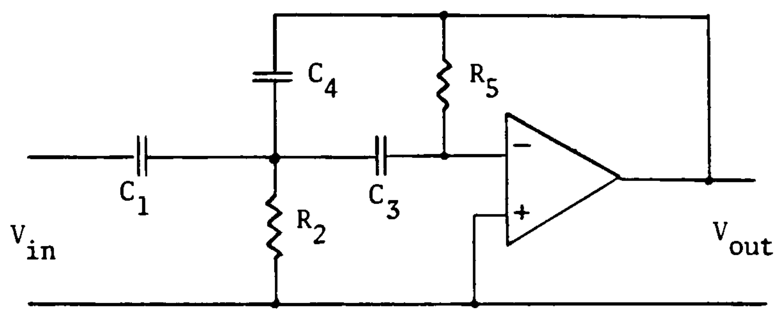
$$Q = \omega_0/|2\sigma_d|$$

It should be emphasized that the denormalized pole values, $\sigma_d \pm j\omega_d$, are different from those used for the low pass. Figure 5.8 shows the active filter used for the two pole transfer function of equation 5.7. The components are related to the transfer function as follows:

$$H(s) = \frac{-s^2 (C_1/C_4)}{s^2 + s\left(\frac{C_1}{R_5C_3C_4} + \frac{1}{R_5C_3} + \frac{1}{R_5C_4}\right) + \frac{1}{R_2R_5C_3C_4}}$$

The procedure for determining the component values shown in figure 5.8 is as follows:

1. Specify the desired gain, $H_\infty$ (not in db) at the frequency, $\omega = \infty$.
2. Determine $\omega_0$, Q for the pole pair using equation 5.7.




High Pass IGMF Active Filter
(two-pole)

Figure 5.8

3. Calculate the constant $\alpha$:

$$\alpha = Q\ (2 + H_\infty)$$

4. Select a reasonable value for C; $C_3$ and $C_4$ are equal to C.

5. Find $C_1$ from $C_1 = H_\infty \cdot C$.

6. Find $R_2$ and $R_5$ from:

$$R_2 = 1/\alpha \cdot \omega_0 \cdot C; \quad R_5 = \alpha^2 \cdot R_2$$

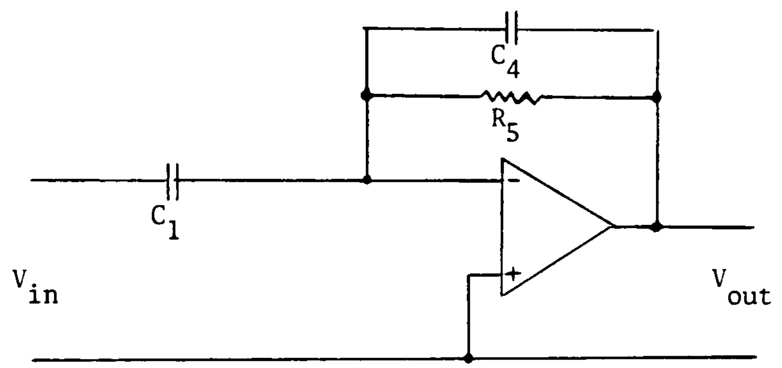
### 5.3.1 Single Pole High Pass Active Filter

Previously we saw that a single pole filter is required when the order of the filter, N, is odd. The form of the single pole high pass transfer function is

$$H(s) = \frac{K's}{s - \sigma_d} \tag{5.8}$$

where the value, $\sigma_d$, is a denormalized pole on the real s-plane axis. To implement this transfer function we use the active filter configuration shown in figure 5.9. The component values in figure 5.9 are related to the transfer function, equation 5.8 as follows:

$$H(s) = \frac{-\,s\,(C_1/C_4)}{s + 1/R_5C_4}$$




High Pass Active Filter
(single pole)

Figure 5.9

To determine the component values of figure 5.9 the following procedure can be used:

1. Specify the high frequency gain, $H_\infty$ (not db)
2. Select a reasonable value for $C_4$ ($10^{-8}$ is a good choice)
3. Find $C_1$ from

$$C_1 = H_\infty \cdot C_4$$

4. Find $R_5$ as follows:

$$R_5 = \frac{1}{|\sigma_d \cdot C_4|}$$

where $\sigma_d$ is the denormalized single pole value represented in equation 5.8.

This completes the presentation of the two pole and single pole active filter design procedures for the high pass filter

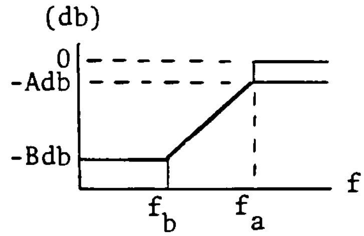
Since the design of the high pass filter is very similar in method to the low pass filter, we will not go through another complete example. However, we will briefly examine the general design process and point out the differences. Figure 5.10 illustrates the basic steps taken for the design of the high pass filter. As we can see, the general procedure used is the same in that the processes of normalization, denormalization, etc. are required. However, the equations used are different due to the transformations which convert the high pass parameters to the normalized low pass filter, etc. It should be recalled that in deriving the denormalized poles, the normalized poles are inverted. The only step which is the same between the high and low pass design procedures is the manner of finding the order, N, and the pole values for the normalized low pass filter.

## 5.4 Band Pass Filter Design

In section 4.4 we found that the denormalized band pass transfer function (equation 4.10) contained 2N poles where N is the order of the normalized low pass filter. For N odd or even, there will always be an even number of denormalized band pass poles. We can then partition the denormalized transfer function into N two-pole sections each of which has the following form:

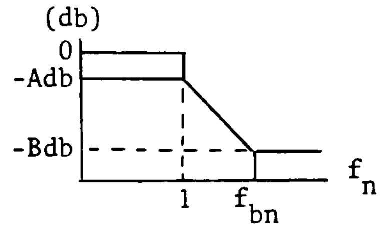
1. Define the filter parameters:

| Magnitude | Freq. |
|---|---|
| -Adb | $f_a$ |
| -Bdb | $f_b$ |



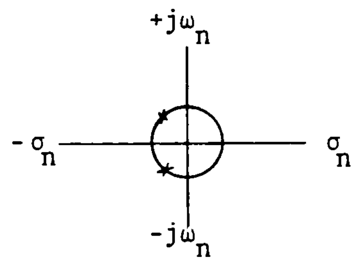

2. Normalize the filter parameters

$$f_{bn} = \left(\frac{f_b}{f_a}\right)^{-1}$$




3. Find the filter order, N, and the normalized poles, $s_{n_k}$

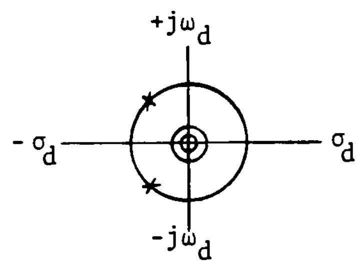
Equations for N : 3.15, 3.21
Equations for $s_{n_k}$ : 3.17, 3.22




4. Denormalize the poles

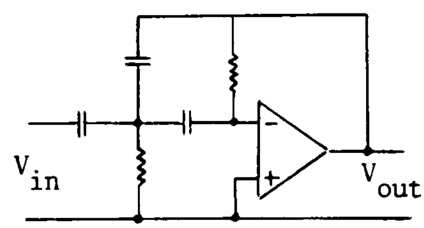
$$s_{d_k} = \omega_a \cdot \frac{1}{s_{n_k}}$$

$$\omega_a = 2\pi f_a$$




5. Implement the design

Two-pole, see p 119
Single pole, see p 121




High Pass Filter Design Summary

Figure 5.10

$$H(s) = \frac{Ks}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2} \tag{5.9}$$

where

$$\omega_0 = \left(\sigma_d^2 + \omega_d^2\right)^{\frac{1}{2}}$$

$$Q = \omega_0/|2\sigma_d|$$

and the values $\sigma_d$, $\omega_d$ are the real and imaginary components of the denormalized pole pair. Figure 5.11 illustrates the active filter used to implement the transfer function of equation 5.9. The components are related to the transfer function as follows:

$$H(s) = \frac{s(-1/R_1C_4)}{s^2 + s\left(\frac{C_3+C_4}{R_5C_3C_4}\right) + \left(\frac{R_1+R_2}{R_1R_2R_5C_3C_4}\right)}$$

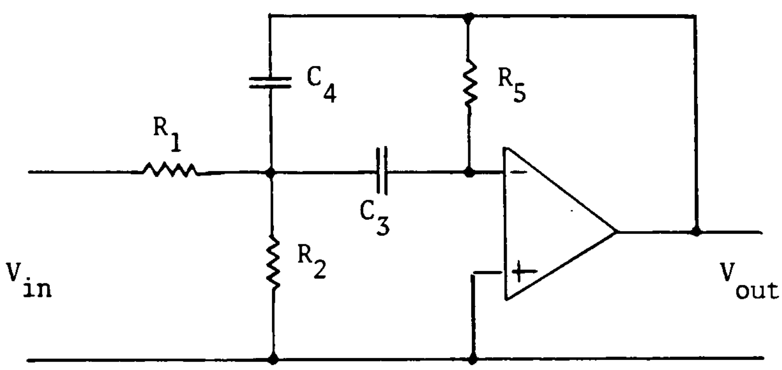
The procedure for determing the component values shown in figure 5.11 is as follows:

1. Specify the desired pass band gain, $H_c$ (not in db)
2. Determine $\omega_0$, Q for the pole pair using equation 5.9
3. Select a reasonable value for C ($10^{-8}$ is a good choice). Set $C_3 = C_4 = C$.
4. Find $R_5$ from

$$R_5 = \frac{2Q}{C\omega_0}$$

5. Find $R_1$ from

$$R_1 = \frac{R_5}{2H_c}.$$




Band Pass IGMF Active Filter
(two-pole)

Figure 5.11

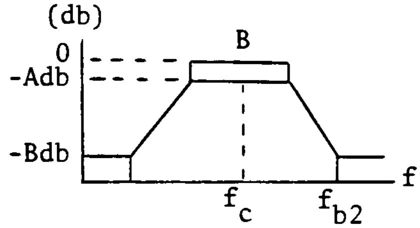
6. Determine $R_2$ from

$$R_2 = \frac{(\omega_0/Q) \cdot C}{\left(2\omega_0^2 - H_c \cdot (\omega_0/Q)^2\right)}$$

Figure 5.12 illustrates the steps required to build a band pass filter starting with the magnitude parameters given by the designer. The processes performed are the same as the other filter types, that is, normalization, denormalization, etc. The main difference which distinguishes the band pass procedure from that of the high and low pass is the generation of 2N poles from the N normalized poles. This, of course, is a result of the different transformation equations used in getting to and from the normalized parameters.

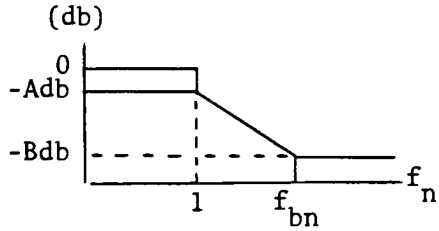
1. Define the filter parameters:

| Magnitude | Freq. |
|---|---|
| -Adb | $f_c$ |
| -Bdb | $B$ |
| | $f_{b2}$ |




2. Normalize the filter parameters:

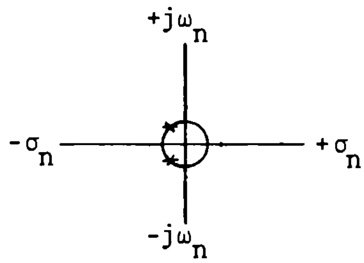
$$f_{bn} = \frac{f_b^2 - f_c^2}{B\, f_b}$$




3. Find the filter order, N, and the normalized poles, $s_{n_k}$

Equations for N : 3.15, 3.21

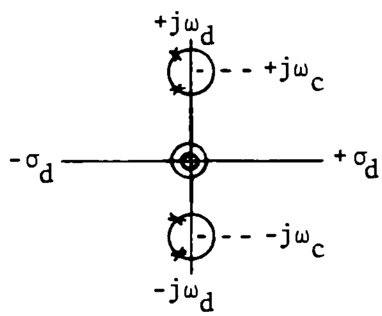
Equations for $s_{n_k}$ : 3.17, 3.22




4. Denormalize the poles

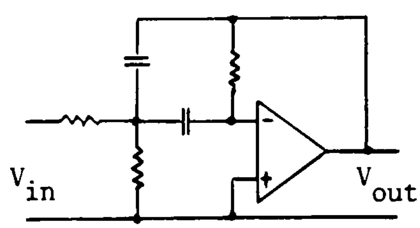
Equations for $s_{d_k}$ : 4.12, 4.13

If $\omega_0 >> \frac{B}{2}$, use equations 4.14, 4.15




5. Implement the design

Two-pole, see p 124




Band Pass Filter Design Summary

Figure 5.12

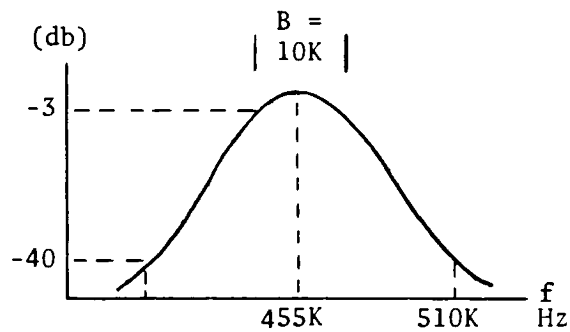
## 5.5 Band Pass Design Example

In this design, we wish to build a Butterworth band pass filter to meet the magnitude requirements depicted in figure 5.13. The center frequency, $f_c$, is 455 KHz. The -3 db bandwidth, B, is 10 KHz. The stop band attenuation requirement, -Bdb, is a minimum of 40 db down at 510 KHz. These requirements are summarized below. We start off by

| Design Parameters |
|---|
| $f_c$ = 455 KHz; B = 10 KHz |
| $f_b$ = 510 KHz; -Bdb = -40 db |

deriving the normalized low pass filter parameter $f_{bn}$ by using equation 3.5 as follows:

$$f_{bn} = \frac{510K^2 - 455K^2}{(10K)(510K)}$$
$$= 10.41$$




Band Pass Filter
Requirements (example)

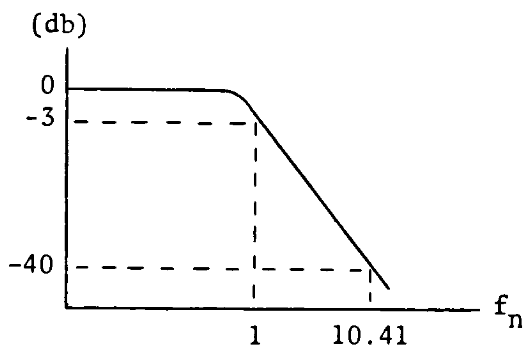
Figure 5.13

The normalized cutoff frequency will be 1 and the magnitude parameters do not change. The resulting normalized low pass filter is shown in figure 5.14. The order of the normalized low pass filter, N, is found using equation 3.15 for the Butterworth response as follows:

$$N \geq \frac{\mathrm{Log}\left(\dfrac{10^{40/10} - 1}{10^{3/10} - 1}\right)}{2\ \mathrm{Log}\ 10.41} = 1.97$$

N is rounded up to 2. The next step is to find the normalized poles for a Butterworth response where N = 2. These were previously shown in section 3.4.3 as

$$s = -.707 \pm j\ .707$$




Normalized Low Pass Filter

Figure 5.14

To find the denormalized poles we can use equations 4.12 and 4.13 which give the exact values. For the case at hand, however, the center frequency $\left(\omega_c = 2\pi \cdot 455K\right)$ is much greater than the bandwidth $(B = 2\pi \cdot 10K)$ and the normalized pole values $(-.707 \pm j\ .707)$. Therefore, equations 4.14 and 4.15 can be used as a close approximation to the exact pole values. Proceeding along these lines, we find 2 denormalized poles from each pole value using the following equations:

$$s_{d_k} = \sigma_{d_k} + j\omega_{d_k} = \frac{B}{2}\sigma_{n_k} + j\left(\omega_c + \frac{B}{2}\omega_{n_k}\right) \tag{4.14}$$

and

$$s_{d_{k+1}} = \sigma_{d_{k+1}} + j\omega_{d_{k+1}} \approx \frac{B}{2}\sigma_{n_k} + j\left(-\omega_c + \frac{B}{2}\omega_{n_k}\right) \tag{4.15}$$

Given that $\qquad B = 2\pi\ (10K) = 62.83 \times 10^3$

and $\qquad \omega_c = 2\pi\ (455K) = 2.859 \times 10^6$

the first two denormalized poles $s_{d_1}$, $s_{d_2}$ are found from $s_{n_1}$ as follows:

$$s_{n_1} = \sigma_{n_1} + j\omega_{n_1} = -.707 + j\ .707$$

$$s_{d_1} = \frac{62.83 \times 10^3}{2}(-.707) + j\left(2.859 \times 10^6 + \frac{62.83 \times 10^3}{2}(+.707)\right)$$

$$s_{d_1} = -44 \times 10^3 + j\ 2.9 \times 10^6$$

$$s_{d_2} = \frac{62.83 \times 10^3}{2}(-.707) + j\left(-2.859 \times 10^6 + \frac{62.83 \times 10^3}{2}(+.707)\right)$$

$$s_{d_2} = -44 \times 10^3 - j\ 2.815 \times 10^6$$

If we proceed to find $s_{d_3}$, $s_{d_4}$ from the second normalized pole, $s_{n_2} = -.707 - j\ .707$, we would find the following values:

$$s_{d_3} = -44 \times 10^{-3} + j\ 2.815 \times 10^6$$

$$s_{d_4} = -44 \times 10^{-3} - j\ 2.9 \times 10^6$$

The normalized and denormalized poles are summarized in the table below.

| $N = 2;\ B = 2\pi\ (10\ \text{KHz});\ \omega_c = 2\pi\ (455\ \text{KHz})$ | | | |
|---|---|---|---|
| $s_{n_1}$ | $-.707 + j\ .707$ | $s_{n_2}$ | $-.707 - j\ .707$ |
| $s_{d_1}$ | $-44 \times 10^3 + j\ 2.9 \times 10^6$ | $s_{d_3}$ | $-44 \times 10^3 + j\ 2.815 \times 10^6$ |
| $s_{d_2}$ | $-44 \times 10^3 - j\ 2.815 \times 10^6$ | $s_{d_4}$ | $-44 \times 10^3 - j\ 2.9 \times 10^6$ |

As a result we have the following pairs:

$$s_{d_{1,4}} = -44 \times 10^3 \pm j\ 2.9 \times 10^6$$

$$s_{d_{2,3}} = -44 \times 10^3 \pm j\ 2.815 \times 10^6$$

Having derived the denormalized poles, we are now in a position to perform the active filter design.

We will start by determining the component values for figure 5.11 given the denormalized pole pair $s_{d_{1,4}}$ as shown above. Proceeding through the design steps on p 124, we have the following:

1. The desired gain is selected as $H_c = 1$
2. For $s_{d_{1,4}}$ we have,

$$\sigma_d = -44 \times 10^3,\ \omega_d = 2.9 \times 10^6$$

$$\omega_{01} = \left(\sigma_d^2 + \omega_d^2\right)^{\frac{1}{2}} = 2.9 \times 10^6$$

$$Q = \omega_0 / \left|2\sigma_d\right| = 33$$

3. $C_3$ and $C_4$ are selected as $10^{-8}$

$$C = C_3 = C_4 = 10^{-10}$$

4. $R_5$ is found as

$$R_5 = \frac{2\ (33)}{10^{-10}\ (2.9 \times 10^6)} = 228\ \text{K}\Omega$$

5. $R_1$ is found as

$$R_1 = \frac{228 \times 10^3}{2 \cdot 1} = 114\ \text{K}\Omega$$

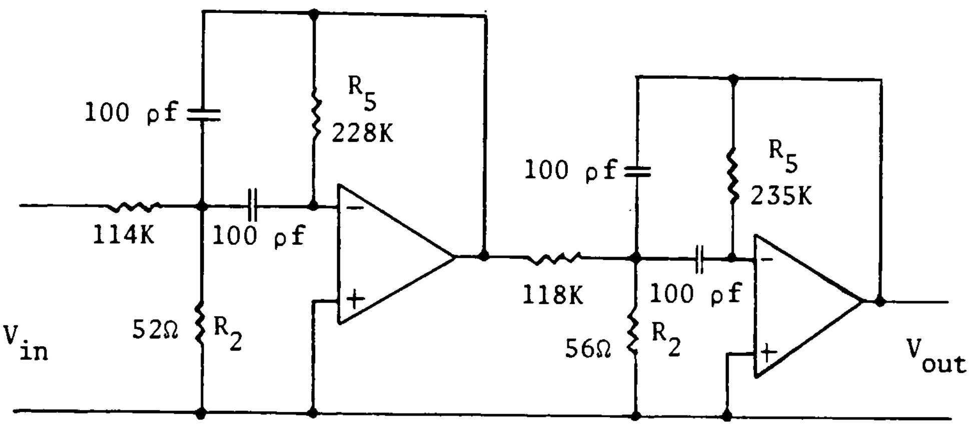
6. $R_2$ is determined as

$$R_2 = \frac{2.9 \times 10^6\ (33 \cdot 10^{-10})}{\left(2 \cdot (2.9 \times 10^6)^2 - 1 \cdot (2.9 \times 10^6\ 33)^2\right)}$$

$$= 52\Omega$$

The resulting two-pole active filter is shown in figure 5.15 on the left as the first stage. The component values for the second stage were found using the same procedure for $s_{d_{3,4}}$ with $Q = 32$ and $\omega_{02} = 2.815 \times 10^6$.

Several points are worth mentioning. Each of the two stages in figure 5.15 is tuned for the center frequency calculated in step 2 of the procedure $\left(\frac{1}{2\pi}\ \omega_{01}\ \text{or}\ \frac{1}{2\pi} \cdot \omega_{02}\right)$ by trimming the value of $R_2$. This does not affect the bandwidth because the sensitivity of B to $R_2$ is 0. Since the bandwidth sensitivity to $R_5$ is -1, we can adjust $R_5$ to acquire




Butterworth Band Pass IGMF Active Filter
($f_c$ = 455 KHz, B = 10 KHz, N = 2)

Figure 5.15

the bandwidth[1] of each stage ($B_1$ or $B_2$) and then adjust $R_2$ to tune each stage to center frequency ($\omega_{01}$ or $\omega_{02}$).

## 5.6 Band Stop Filter Design

In this section we will examine the method of implementing a denormalized transfer function having a band stop response. The general equation for the denormalized transfer function was presented in section 4.5. There it was shown that H(s) has 2N denormalized poles or N two-pole sections. Each section has the following form:

$$H(s) = \frac{K\,(s^2 + z^2)}{s^2 + \frac{\omega_0}{Q}\,s + \omega_0^2} \tag{5.10}$$

NOTE 1: bandwidth of each stage ≈ $2\sigma_d = \omega_0/Q$

where

z is the center frequency of the filter response $(2\pi f_c)$,

$$\omega_0 = \left(\sigma_d^2 + \omega_d^2\right)^{\frac{1}{2}}$$

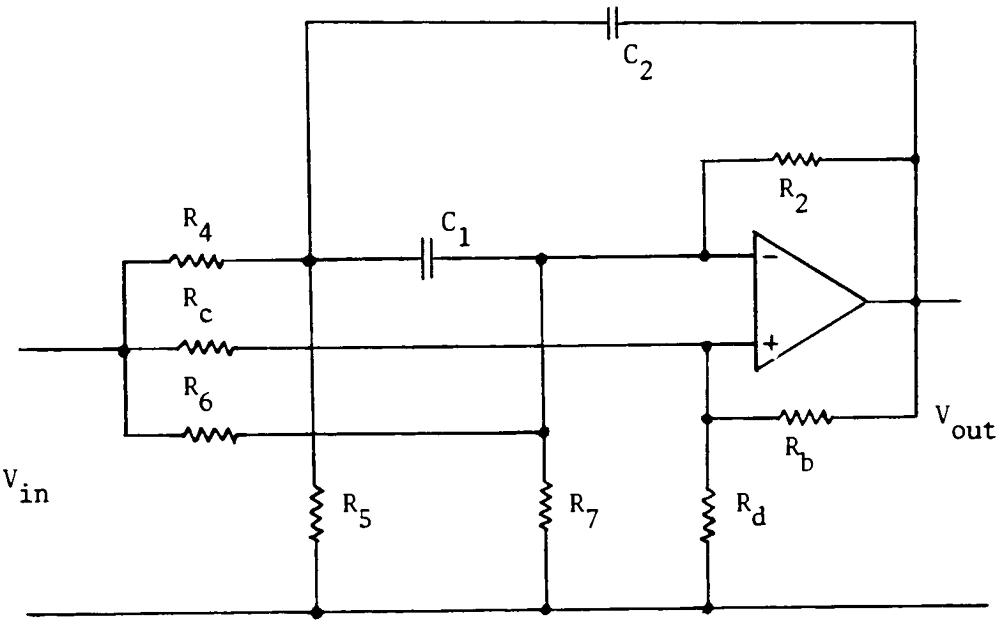
for the given pole pair and

$$Q = \omega_0 / |2\sigma_d|$$

In this section a two-pole IGMF active filter is presented which can implement the transfer function of equation 5.10.

The unique characteristic of the band stop response is that it contains complex zeros in the numerator. The previous designs did not have complex transfer function zeros. To accommodate the zeros as well as an IGMF design, we will use the active filter configuration as shown in figure 5.16. This circuit concept was presented by T. Deliyannis [13] and later expanded by J. J. Friend of Bell Laboratories [14]. The intention was to provide superior low sensitivity compared to other single amplifier circuits as well as a minimal number of components. As an added point, this circuit can be used for the elliptic transfer functions which contain s-plane zeros.

The procedure for determining the component values of figure 5.16 follows and it is rather lengthy. An example will be given after discussing the procedure.




Band Stop IGMF Active Filter
(two-pole)
Figure 5.16

Having determined the values z, Q, and $\omega_0$ for the two-pole section as described by equation 5.10, we proceed as follows:

1. Determine the frequency, $\omega_m$, at which the gain of the transfer function is at a maximum. This is found from

$$\omega_m = \omega_0 \left[ \frac{1 - \left(z/\omega_0\right)^2 \left(1 - \frac{1}{2Q^2}\right)}{\left(1 - \frac{1}{2Q^2}\right) - \left(z/\omega_0\right)^2} \right]^{\frac{1}{2}}$$

2. Determine the gain, H, at $\omega_m$ as follows:

$$H = \frac{\left| z^2 - \omega_m^2 \right|}{\left[ \left( \omega_0^2 - \omega_m^2 \right)^2 + \left( 2\omega_m \ \sigma_d \right)^2 \right]^{\frac{1}{2}}}$$

3. Find the gain normalizing constant, K, as

$$K = 1/H$$

4. Determine the constants A, B, D as follows:

$$A = \omega_0/Q$$

$$B = \omega_0^2$$

$$D = z^2 \cdot K$$

5. Select reasonable starting values for $C_1$, $C_2$, $R_c$, $R_b$. The following values are typically used:

$$C_1 = C_2 = 10^{-8}$$

$$R_c = R_b = 10^4$$

6. Find the admittance values $G_a$, $G_b$, $G_c$, $G_d$ as follows:

$$G_a = G_c/K; \ G_b = 1/R_b$$

$$G_c = 1/R_c; \ G_d = \left( G_a - G_c \right)$$

7. Calculate the constants $\gamma$, $\rho$ as follows:

$$\gamma = 1 + C_1/C_2$$

$$\rho = G_b/G_a$$

8. Calculate $G_1$ as follows:

$$G_1 = \frac{C_2}{2} \cdot \frac{1}{\rho} \cdot \left(- A + \sqrt{A^2 + 4\gamma \cdot B\rho}\right)$$

9. Find the constant K' as follows:

$$K' = \frac{K + \gamma \cdot D \cdot \left(\frac{C_2}{G_1}\right)^2}{1 + \rho}$$

K' must be between 0 and 1 if the circuit is to be realized.

10. Determine $G_4$ and $G_5$ from

$$G_4 = K' G_1$$

$$G_5 = G_1 - G_4$$

11. Find $G_3$ as follows:

$$G_3 = \frac{C_1 C_2 G_a \cdot B \cdot \left(\frac{D}{B} - K\right)}{G_1 (G_a + G_b)(K - K_2)}$$

where $K_2$ is chosen as 0 or 1 in order to derive a minimum positive value for $G_3$.

12. Determine $G_6$ and $G_7$ from

$$G_6 = K_2 \cdot G_3$$

$$G_7 = G_3 - G_6$$

13. Find $G_2$ as follows:

$$G_2 = \frac{C_1 \cdot C_2 \cdot B}{G_1} + \frac{G_b \cdot G_3}{G_a}$$

14. The resulting components are as follows:

$$C_1 = 10^{-8} \quad , \quad C_2 = 10^{-8}$$

$$R_c = 10^4 \quad , \quad R_b = 10^4$$

$$R_2 = 1/G_2 \quad , \quad R_4 = 1/G_4$$

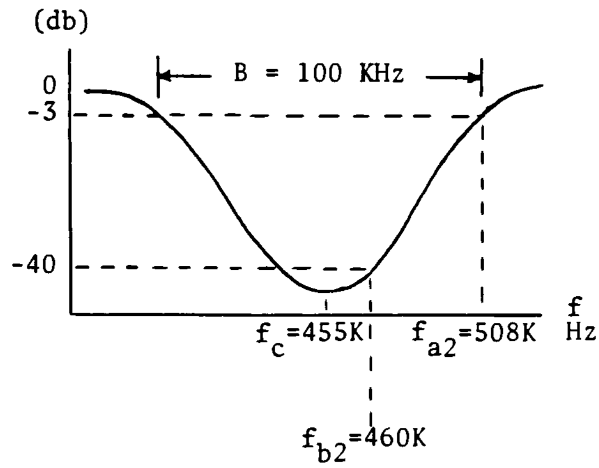
$$R_5 = 1/G_5 \quad , \quad R_6 = 1/G_6$$

$$R_7 = 1/G_7 \quad , \quad R_d = 1/G_d$$

The first attempt at calculating these values may prove unsuccessful. Various combinations of $C_1$, $C_2$, $R_c$ and $R_b$ can be tried until reasonable values are found. However, if the pole/zero values are such that K' of step 9 is not between 0 and 1, then this circuit configuration cannot be used. The next section will illustrate an example of the design procedure.

## 5.7 Band Stop Design Example

In this example we would like to design a Butterworth band stop filter having the response characteristics shown in figure 5.17. The center frequency, $f_c$, is 455 KHz. The -3 db bandwidth, B, is 100 KHz. The stop band is specified as 40 db down at $f_{b2}$ = 460 KHz. We proceed to follow the basic steps as we have done previously and as shown in figure 5.1. We start by finding the normalized low pass filter parameter, $f_{bn}$, using equation 3.7 as follows:




Butterworth Band Stop Filter
Requirements (example)

Figure 5.17

$$f_{bn} = \left(\frac{f_{b2}^2 - f_c^2}{B \cdot f_{b2}}\right)^{-1} = \left(\frac{460K^2 - 455K^2}{(100K)(460K)}\right)^{-1}$$

or

$$f_{bn} = 10.05$$

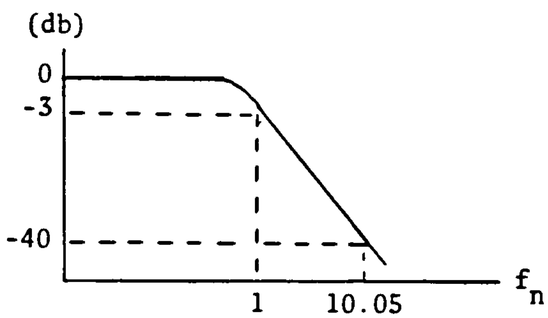
The normalized upper pass band edge is,

$$f_{an} = \left(\frac{f_{a2}^2 - f_c^2}{B \cdot f_{a2}}\right)^{-1} = \left(\frac{508K^2 - 455K^2}{(100K)(508K)}\right)^{-1}$$

or

$$f_{an} = 1$$

The magnitude requirements for the normalized response remain the same. Figure 5.18 illustrates the resulting normalized low pass filter parameters. The next step is to find the order of the filter, N.




Normalized Low Pass Filter
(derived from figure 5.17)

Figure 5.18

Using equation 3.15 for the Butterworth filter, the value of N is found as

$$N \geq \frac{\text{Log}\left(\frac{10^{40/10} - 1}{10^{3/10} - 1}\right)}{2 \text{ Log } 10.05} = 2$$

Given the filter order, N = 2, we can determine the normalized Butterworth poles. This was previously done for the band pass example (section 5.5) which also had N = 2. The normalized left hand plane poles are

$$s_{1,4} = -.707 \pm j\ .707$$

Having derived the filter order and the normalized poles, we can proceed to find the four (2N) denormalized pole values. The denormalization parameters are:

$$B = 2\pi\ (100K) = 628.32 \times 10^3$$

$$\omega_0 = 2\pi\ (455K) = 2.859 \times 10^3$$

$$s_{n_1} = \sigma_{n_1} + j\omega_{n_1} = -.707 + j\ .707$$

$$s_{n_2} = \sigma_{n_2} + j\omega_{n_2} = -.707 - j\ .707$$

Given the parameters above and using equations 4.14 and 4.15, the two denormalized poles are found from $s_{n_1}$ as follows:

$$s_{d_1} = \frac{B}{2}\left(\sigma_{n_1}\right) + j\left(\omega_0 + \frac{B}{2} \cdot \omega_{n_1}\right)$$

$$= \frac{628 \times 10^3}{2}(-.707) + j\left(2.859 \times 10^3 + \frac{628 \times 10^3}{2}(+.707)\right)$$

$$= -222 \times 10^3 + j\ 3.08 \times 10^6$$

$$s_{d_2} = \frac{B}{2}\left(\sigma_{n_1}\right) + j\left(-\omega_0 + \frac{B}{2} \cdot \omega_{n_1}\right)$$

$$= \frac{628 \times 10^3}{2}(-.707) + j\left(-2.859 \times 10^3 + \frac{628 \times 10^3}{2}(+.707)\right)$$

$$= -222 \times 10^3 + j\ 2.637 \times 10^6$$

In a similar fashion the denormalized poles $s_{d_3}$, $s_{d_4}$ are found from $s_{n_2}$. Their values are

$$s_{d_3} = -222K - j\ 2.637 \times 10^6$$

$$s_{d_4} = -222K + j\ 3.08 \times 10^6$$

At this point we select a pole pair and proceed through the IGMF active filter design using the procedure on page 134.

The pole pair $s_{d_{1,4}}$ is selected for the design example and shown below.

$$s_{d_{1,4}} = -222 \times 10^3 \pm j\ 3.08 \times 10^6$$

We start by establishing the parameters z, $\omega_0$, Q of the transfer function shown in equation 5.10.

$$\omega_0 = \left(\sigma_d^2 + \omega_d^2\right)^{1/2}$$

$$= \left((222 \times 10^3)^2 + (3.08 \times 10^6)^2\right)^{1/2}$$

$$\omega_0 = 3.09 \times 10^6$$

$$z = 2\pi\left(f_c\right) = 2\pi\ (455 \times 10^3)$$

$$= 2.859 \times 10^6$$

and

$$Q = \omega_0/\left|2\sigma_d\right| = 3.09 \times 10^6\ 2 \cdot (222 \times 10^3)$$

$$= 6.96$$

Having found $\omega_0$, Q, z we follow through the steps on page 134 to determine the component values for figure 5.16. The results of each step are summarized in the table below.

| | |
|---|---|
| 1. $\omega_m = 3.305 \times 10^6$ | 7. $\rho = .735$ |
| 2. $H = 1.36$ | 8. $G_1 = .048$ |
| 3. $K = .735$ | 9. $K' = .723$ |
| 4. $A = 444 \times 10^3$<br>$B = 9.548 \times 10^{12}$<br>$D = 6 \times 10^{12}$ | 10. $G_4 = .0347$, $G_5 = .0133$<br>11. $G_3 = 4.61 \times 10^{-3}$ |
| 5. $C_1 = C_2 = 10^{-8}$, $R_c = R_b = 10^4$ | 12. $G_6 = 4.61 \times 10^{-3}$, $G_7 = 0$ |
| 6. $G_c = 10^{-4}$, $G_b = 10^{-4}$<br>$G_a = 1.361 \times 10^{-4}$<br>$G_d = 3.61 \times 10^{-5}$ | 13. $G_2 = .023$ |

The component values determined in step 14 are:

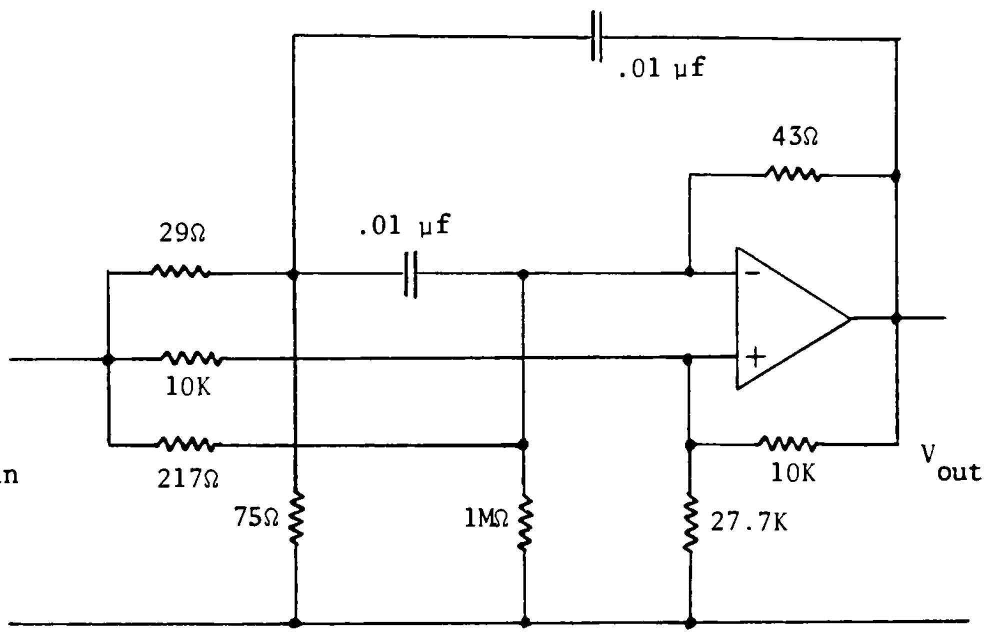
$$C_1 = 10^{-8},\ C_2 = 10^{-8},\ R_c = 10\text{K}\Omega$$

$$R_b = 10\text{K}\Omega,\ R_2 = 43\Omega,\ R_4 = 29\Omega$$

$$R_5 = 75\Omega,\ R_6 = 217\Omega,\ R_d = 27.7\text{K}\Omega$$

$$R_7 \text{ (use } 1\text{M}\Omega)$$

The two-pole IGMF active filter is shown in figure 5.19 with the resulting component values. We will not go through the calculations for the second stage since this procedure is rather tedious.




Butterworth IGMF Band Stop Filter
(two-pole example)

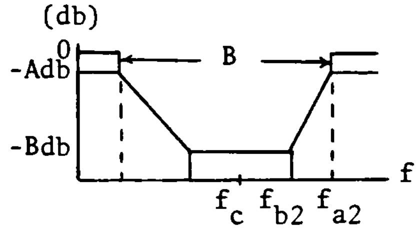
Figure 5.19

This completes the discussion of the IGMF band stop filter design. The IGMF method represents only one of several. Others such as a voltage-controlled voltage source [ 7 ], biquadratic integrated circuit and simple high/low pass combinations can be used. It is recommended that the reader investigate these alternatives.

Figure 5.20 illustrates a summary of the basic steps taken for the band stop filter design. The equations for steps 2-4 are quite similar to the band pass procedure as summarized in figure 5.12. Anyone designing the band stop filter should remember to invert the normalized poles (step 4 of figure 5.20) prior to using the denormalization equations (equations 4.14 and 4.15).

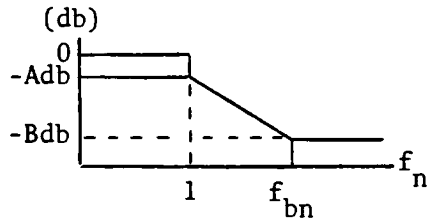
1. Define the filter parameters:

| Magnitude | Freq. |
|---|---|
| -Adb | $f_c$ |
| -Bdb | B |
| | $f_{b2}$ |



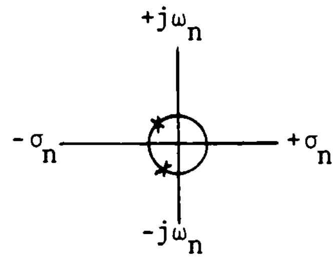

2. Normalize the filter parameters:

$$f_{bn} = \left(\frac{f_b^2 - f_c^2}{B\ f_b}\right)^{-1}$$




3. Find the filter order, N, and the normalized poles, $s_{n_k}$

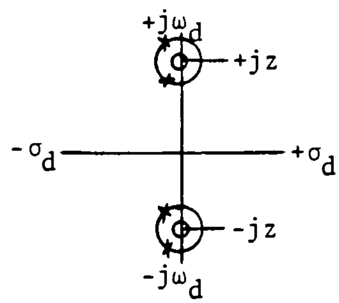
Equations for N : 3.15, 3.21
Equations for $s_{n_k}$ : 3.17, 3.22




4. Denormalize the poles

a) invert the normalized poles (equation 4.14)

b) find the denormalized poles, $s_{d_k}$, using equations 4.12, 4.13; if $\omega_0 >> \frac{B}{2}$, use equations 4.14, 4.15




5. Implement the design

Use the two-pole procedure beginning on p 134

Active Filter Circuit: see figure 5.16

Band Stop Filter Design Summary

Figure 5.20

## 5.8 Chapter Summary

In this chapter we examined the methods of implementing the denormalized transfer functions using infinite-gain multiple feedback (IGMF) active filters. We started by examining the basic steps of filter design. These steps essentially outlined the discussions of the previous chapters. The end objective was to derive the denormalized poles which were used as the starting design parameters for the active filters. The procedures were then presented for determining the component values of the IGMF active filter circuits.

In comparing the four design procedures, the band stop filter appeared unique because the circuit provided transfer function zeros. This circuit configuration is applicable to elliptic function filters which also have transmission zeros.

We have at this point completed the analysis of parameters, transformations and design procedures required for the four basic filter types (low pass through band stop). The next chapter deals with the development of a computer program called FILTER which performs the total design process for each filter type and provides the IGMF component values.

CHAPTER VI

# THE FILTER PROGRAM

## 6.1 Introduction

In this chapter we will discuss a computer program which performs the analysis and design of filters in the frequency domain. The name of the program is FILTER.

The program was developed as part of this thesis with the primary intention of acting as an aid to the filter design engineer. To accomplish this objective, all of the analytical steps required for various filter designs were assimilated into one comprehensive interactive algorithm. Of primary significance is the inclusion of a complete elliptic filter design ability. The program provides the engineer with the ability to design and evaluate various filters without the burdens of performing complex mathematics, referring to a variety of normalized tables or being limited to a familiar and perhaps non-optimal design.

The discussions in the previous chapters have provided an overview of the fundamental concepts upon which the FILTER program is based. The program is actually much broader in scope and more complex in its analytical abilities. The user interaction and guidance by the program are extensive and therefore it is unnecessary to fully comprehend the intricacies in order to put it to full use. The following sections provide a brief overview of the capabilities of the FILTER program and its software architecture.

## 6.2 Capabilities of the FILTER Program

The FILTER program has the capability of designing low pass, high pass, band pass and band stop filters using any of three classical approximations: Butterworth, Chebyshev and elliptic. Provisions have been made to include the Bessel function approximation at a later date. The following four items provide a summary of other major program capabilities.

1. The program can accept a wide variety of input parameter combinations (magnitude and frequency) for specifying the desired response. As an example, for a low pass filter, the user can specify $f_a$, $f_b$, -Adb and -Bdb* arbitrarily, regardless of the filter approximation used. This allows the designer to specify the parameters according to the application requirements. A Butterworth filter, for example, can be specified with a cutoff frequency attenuation of -1 db instead of the typical -3 db. The program determines if the parameters are appropriate and returns a message to the user if they are not. A simple example would be an entry error where the user reverses the cutoff and stop band attenuation parameters. Many other parameter checks are included. In addition, other secondary parameters are derived from those given by the designer. An example of this is the calculation of the cutoff frequency given the filter order, N, and an arbitrary attenuation value for a specific frequency.

2. Another capability of the program is to determine the normalized filter order, N, along with the normalized and denormalized poles/zeros. The maximum filter order is N = 100 for the low and high pass and N = 50 for the band pass and band

*See section 2.4 for definitions.

stop. The capabilities above are the most significant features of the program since they encompass the following analytical processes:

a) normalization of the input parameters,

b) evaluation of N and the normalized poles/zeros for a desired classical approximation response,

c) denormalization

The special feature of item (b) above is that an elliptic function algorithm has been developed to determine the Nth order poles and zeros. The elliptic function design algorithm follows the analytical methods described in the Appendix.

3. Aside from the analytical processes above, the program provides the ability to evaluate the denormalized transfer function, giving both magnitude and phase as a function of the frequency range of interest. This enables the user to determine if the selected approximation (Butterworth, Chebyshev, etc.) yields a reasonable filter order as well as sufficient attenuation and phase characteristics. If the result is unsatisfactory, the user can alter some or all of the input parameters (magnitude, frequency, filter type or selected approximation) and obtain a new response.

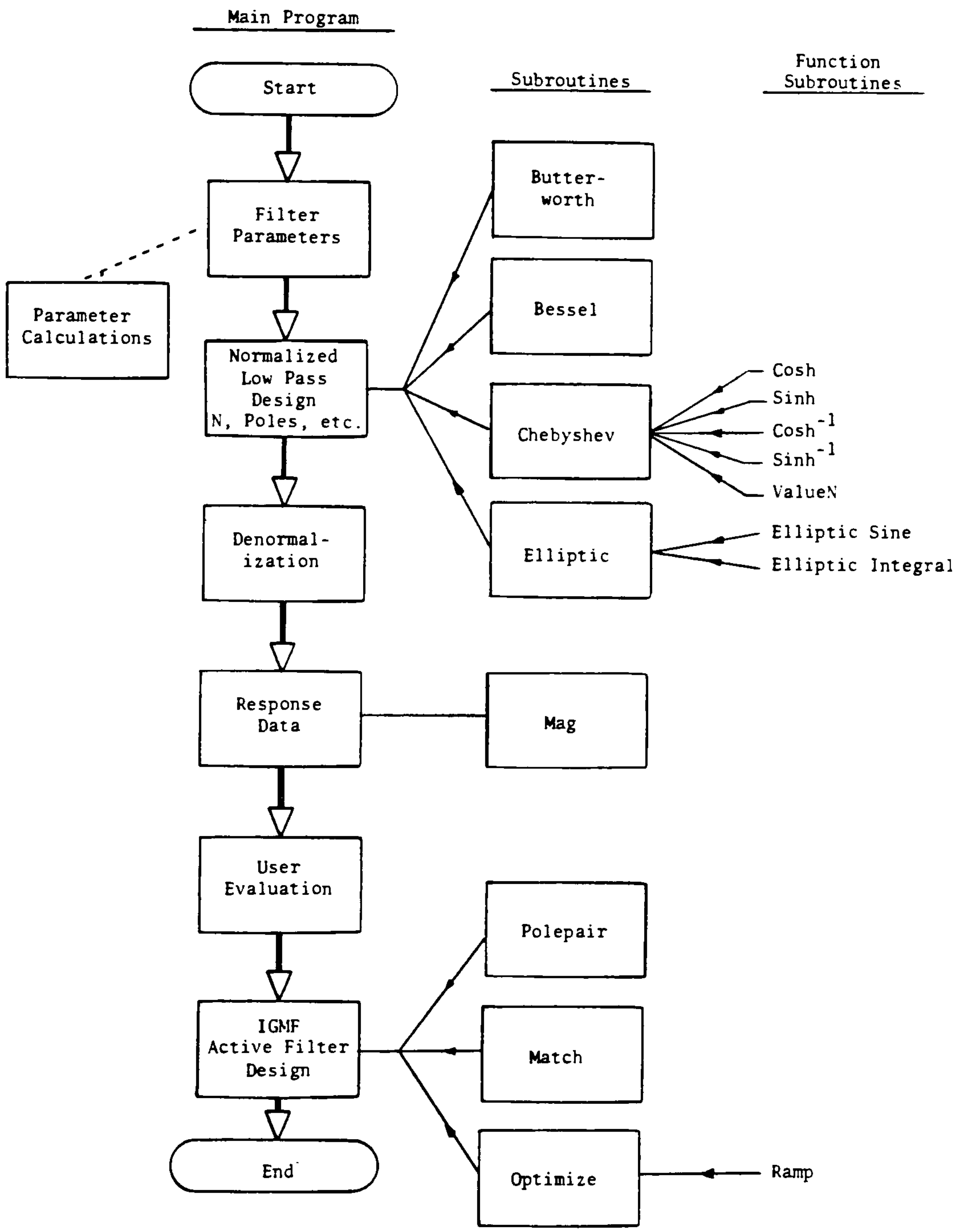
4. Another capability of the program is to design an active filter circuit using the calculated denormalized poles and zeros. The component values can be determined for multiple stages of IGMF active filter circuits. These values are displayed according to stage number; their designations are referenced to the user's

manual. When determining the active filter circuit components, multiple iterations are attempted at the design in order to obtain reasonable values. If these are not within the acceptable range of values after a prescribed number of iterations, the program stops the calculations and displays the last results. Of special significance is the use of an IGMF single-stage biquadratic active filter circuit. This circuit provides a second order numerator and denominator transfer characteristic and is employed as a canonical form for the elliptic and band stop active filter circuit stages. A special algorithm was developed specifically for the task of determining the component values for this filter circuit. Aside from these IGMF active filter circuit configurations, provisions have been made to include the universal active filter integrated circuit (FLT-U2 or equivalent) as the active element at some later date.

The four main items discussed above have given an overview of the major capabilities of the FILTER program. Following each of the processes involved within the program, the resulting information is displayed so the user can monitor and evaluate the progression of the design. Throughout the session questions are asked of the user to determine the course of action or acquire other important parameters. The next section is a brief overview of the major algorithms which constitute the program and how they are organized.

## 6.3 Program Architecture

Figure 6.1 illustrates the software architecture of the FILTER program. Each block shown represents a considerable section of software




Program FILTER Architecture

Figure 6.1

algorithms. The program is organized in three tiers: the main program, main subroutines and function subroutines.

The first tier, the main program, is denoted by the column of blocks with heavy lines as shown in figure 6.1. The main program follows the basic steps of filter design as discussed in Chapter V, figure 5.1.

The second tier consists of subroutines which support the main program by performing most of the analytical tasks. These routines are particular to the kind of filter design requested. As an example, the Butterworth subroutine, shown in figure 6.1 as one block, determines the filter order, N, as well as the normalized poles. Included at this second tier level is the elliptic function subroutine as shown.

Aside from the subroutines handling the classical approximations (Butterworth, Chebyshev and elliptic) there are four others: Mag, Polepair, Match and Optimize. The Mag subroutine finds the transfer function magnitude, $|H(j\omega)|$, for a given frequency, $\omega$. This routine supports the main program block which directs the user evaluation. The remaining three subroutines support the IGMF filter stage design. Polepair performs the function of determining a pair of poles for the second order stage transfer function. The routine, Match, scans the denormalized zeros to select a pair to minimize stage overshoot. The selection criterion is the nearest available zeros for the poles closest to the $j\omega$ axis. The routine, Optimize, determines the peak frequency and magnitude for the stage transfer function. It then calculates a stage gain to minimize overshoot. All the routine parameters are passed to and from the main program using a Fortran COMMON statement.

The third tier consists of subroutines designed to perform mathematical functions which are complex and therefore rather cumbersome to include in the second tier subroutines. The third tier also includes less complicated functions which were otherwise not available from the computer library. Examples of these function subroutines are Cosh, Inverse Cosh, Elliptic Sine, Elliptic Integral, Value N and Ramp. The latter two routines determine the filter order, N, and the peak stage amplitude respectively. Parameters are passed to the function subroutines in most cases as a simple argument value such as x in sin(x). This makes the task of analysis much easier at the second tier subroutine level.

## 6.4 Chapter Summary

This chapter presented the salient features of the FILTER program regarding its capabilities, user interaction and software structure. This overview has not by any means given a description of all the user alternatives or analytical capabilities within the program. The next chapter, the user's manual, provides more detail on the use of the program along with examples of typical interactive sessions. The next chapter is meant to be removed from the thesis as a guide when using the program.

This chapter concludes the presentation of the thesis aside from the Appendix. It is hoped that the discussions have built a sufficient foundation for the reader to understand and appreciate the design of filters and their contribution in the engineering field.

# CHAPTER VII

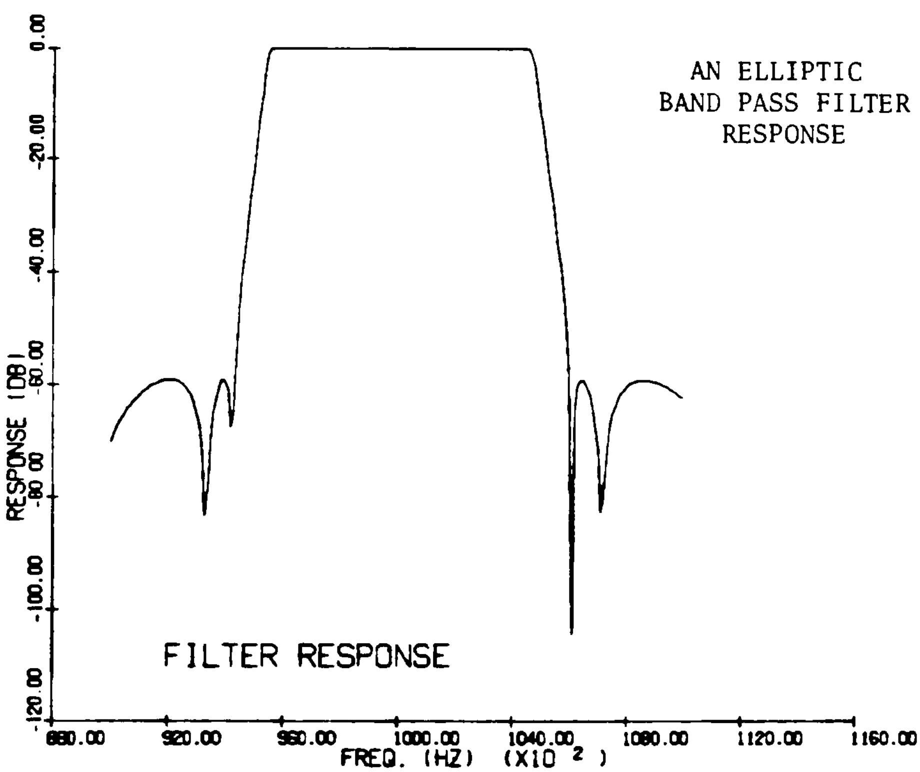
## USER'S MANUAL

The following chapter is a user's manual for the FILTER program. Specifications of the program, input parameter formats and examples of user sessions are given. The chapter is intended as an independent section to be removed for use with the program.

USER'S MANUAL

for the

FILTER PROGRAM



No responsibility is assumed for inaccuracies or use of the information within this document or information acquired from the FILTER program.

The information herein has been checked and is believed to be accurate.

FIRST EDITION

# SPECIFICATIONS

1. PROGRAM: FILTER-V1., Version 1

2. COMPUTER INFORMATION

   XEROX SIGMA 9:

| ORG | GRAN | REC | NAME | PURPOSE |
|---|---|---|---|---|
| C | 30 | 543 | B:FILTER | ROM |
| K15 | 44 | 8 | FILTER V1 | EXECUTABLE |
| K3 | 73 | 2495 | S:FILTER | SOURCE LIST |

   Average user session time: 10 minutes

3. PROGRAM OPERATIONAL SPECIFICATIONS

   Filter Kinds:

   Butterworth (also default)
   Chebyshev
   Elliptic

   Filter Types:

   low pass
   high pass
   band pass
   band stop

   Highest Filter Order, N:

   low pass, high pass: N = 100
   band pass, band stop: N = 50

   Hardware Implementation

   active filter design,
   infinite gain multiple feedback configuration

   Maximum frequency of gain and phase calculations: 100 MHz

# TABLE OF CONTENTS

## LIST OF ILLUSTRATIONS

# SECTION I

# INTRODUCTION

The purpose of the computer program, FILTER, is to provide the designer with a tool for the analysis and design of the classical filters.

The basic types of filters (low pass, high pass, band pass and band stop) can be implemented for any of three classical approximations: Butterworth, Chebyshev and elliptic (Cauer). Provisions have been made to include the Bessel function response at a later date.

The program guides the user from the initial step of defining the filter response parameters to the last step where active filter components are determined. Analysis data is provided during the program session in the form of the s-plane poles and zeros (both normalized and denormalized). In addition, the user has the opportunity to evaluate magnitude and phase response. The sections which follow present the major portions of the program, FILTER, in sequence. In reading these sections, the user will gain familiarity with how the program session progresses along with the program capabilities and interaction.

The FILTER program was developed as a part of a thesis presented to the Rochester Institute of Technology, entitled "AN ACTIVE FILTER DESIGN PROGRAM" (theory and application).

# SECTION II

## SYSTEM COMMANDS FOR THE FILTER PROGRAM

### 2.1 Introduction

The following presents a list of commands which are commonly used during a session with the FILTER program. The symbol, (CR), is used to indicate that the user should hit the RETURN key. The assumption is made that the program is operating on the Sigma 9 (XEROX) computer.

### 2.2 Starting the Program

After logging on and receiving the !, enter the program name as shown below:

FILTER-V1. (CR)

Example:

!FILTER-V1.

### 2.3 Correcting a Data Entry Error

To correct an entry error, the DELETE key can be used as shown in the following example where the user wishes to enter the cutoff frequency, FA, as 100 Hz.

Question from the program:

ENTER FA:

User entry:

ENTER FA: 1ØB\Ø (CR)

incorrect entry

DELETE key is hit

Ø is entered as the correct value

## 2.4 Stopping the Program Execution

In the event that the user recognizes that the program is executing with erroneous entry data, the execution can be stopped. The BREAK key is used for this purpose as shown in the following example.

User entry:

(BREAK key hit)

Computer response:

!

User Entry:

!FILTER-V1. (CR)

Computer response:

QUIT?

User response:

Hit (CR) and the program will restart.

## 2.5 Ending the Session

To end a session after the program is completed or interrupted, enter OFF as shown below:

Last computer response:

!

User entry:

!OFF (CR)

Figure 2.1 illustrates an example of starting and ending the FILTER program.

```
!FILTER-V1.


                        ACTIVE FILTER DESIGN PROGRAM
-****************************************************************************
                     COPYRIGHT BY LOUIS R. GABELLO,1981

            NO RESPONSIBILITY IS ASSUMED FOR THE USE OR
            ACCURACY OF THE INFORMATION PRESENTED IN
            THIS PROGRAM.SUBJECT TO CHANGE WITHOUT NOTICE.

            VERSION 1                LAST UPDATE:5-7-81



ENTER THE KIND OF FILTER DESIRED:
                        BW FOR BUTTERWORTH,
                        CB FOR CHEBYSHEV,
                        EL FOR ELLIPTIC,
                        BS FOR BESSEL,
                        00(ZERO) FOR DONT KNOW.

ENTER KIND:



!OFF
```

Starting and Ending the FILTER Program

Figure 2.1

# SECTION III

# FILTER PARAMETERS
(entry data)

## 3.1 Introduction

After the user logs on, the FILTER program will request the design parameters of the filter. These parameters include the selection of classical responses (Butterworth, Chebyshev, elliptic), type of response (low pass, high pass, etc.) along with the attenuation and frequency parameters. The FILTER program accommodates many combinations of input parameters (including defaults). The user should try various combinations to become familiar with the various possibilities. The program however will guide the user in any event.

## 3.2 Defining the Filter Parameters

Prior to running the program, it is advisable to have a list of the response parameters which you will use to define the desired filter. Figures 3.1 and 3.2 illustrate the straight line transition diagrams for the four filter types. The parameters used to define the filter responses are shown. The symbols and their definitions are those used by the FILTER program.

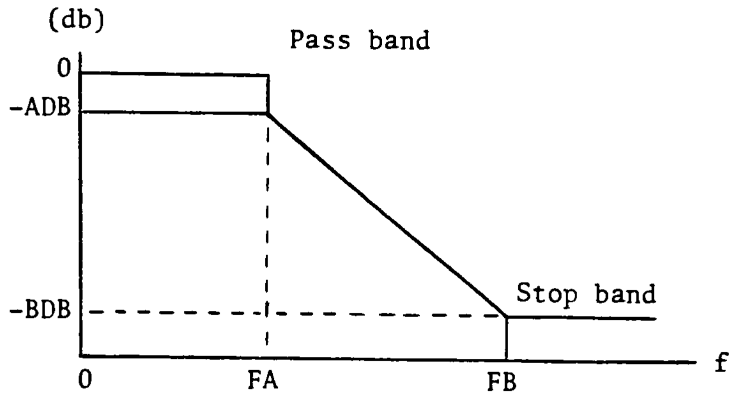
## 3.3 Parameter Entry Examples

Figure 3.3 illustrates the parameter entry section for a Butterworth low pass filter. The desired filter has a cutoff frequency. FA = 100 Hz with a maximum pass band deviation, -ADB = -3 db. The stop band edge frequency, FB, is 860 Hz. The minimum stop band attenuation, -BDB, is -40 db.

Figure 3.4 illustrates an example of the input response parameters for a Butterworth band pass filter. The desired filter has a center frequency, FC = 455 KHz and a bandwidth, BD = 10 KHz. The maximum pass band deviation -ADB, is -0.1 db. The stop band edge frequency, FB2, is 500 KHz with a minimum attenuation, -BDB = -40 db. All frequency entries are in hertz.

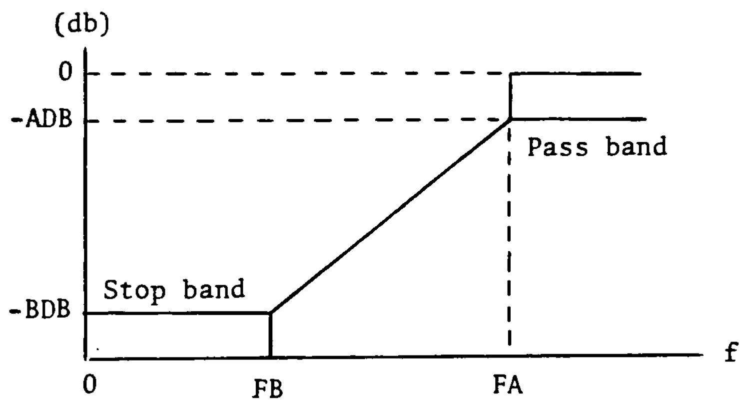
As part of the parameter entry section of the program, various internal validation checks are made on the user data; when necessary the program notifies the user and asks for a correction or verification on the entered parameter(s). An example of this is shown in Figure 3.4 where the user defines the bandwidth for the Butterworth filter as 10 KHz but the pass band edge attenuation, -ADB, is -1 db. The proper bandwidth is defined for a Butterworth filter by the -3 db frequencies. As shown, the program accepts the data, calculates the proper -3 db bandwidth and notifies the user.

LOW PASS




FA: pass band edge frequency
FB: stop band edge frequency
-ADB: maximum pass band attenuation
-BDB: minimum stop band attenuation

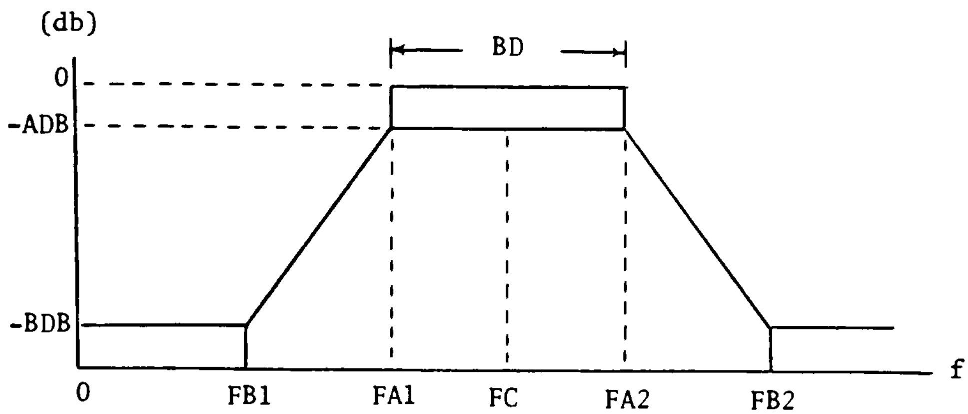
HIGH PASS




(Parameter definitions are the same as the LOW PASS above)

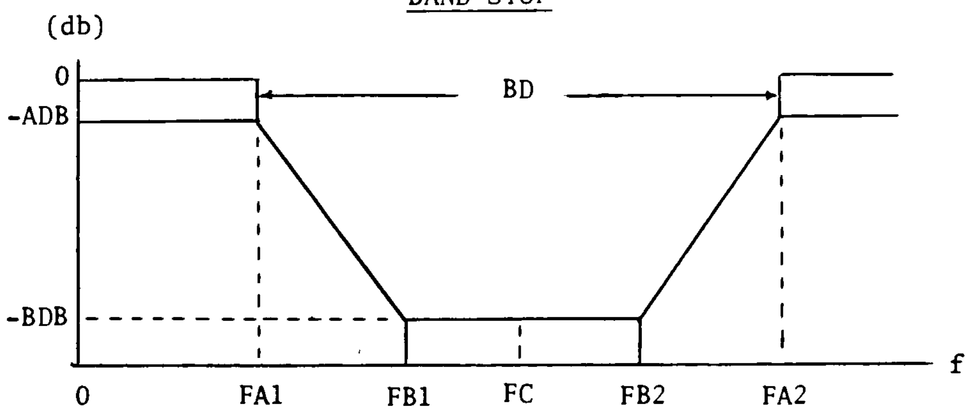
Filter Parameters
(low pass and high pass)
Figure 3.1

BAND PASS




| | |
|---|---|
| FC: | center frequency ( $\sqrt{FA1 \cdot FA2}$) |
| FA1, FA2: | pass band edge frequencies |
| FB1, FB2: | stop band edge frequencies |
| BD: | bandwidth defined by FA1 and FA2 at -ABD (not necessarily -3 db) |
| -ADB: | maximum pass band attenuation |
| -BDB: | minimum stop band attenuation |

BAND STOP




(Parameter definitions are the same as the BAND PASS above)

Filter Parameters
(band pass and band stop)
Figure 3.2

```
                    ACTIVE FILTER DESIGN PROGRAM
*******************************************************************

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          VERSION 1                LAST UPDATE:5-7-81


ENTER THE KIND OF FILTER DESIRED:
                    BW FOR BUTTERWORTH,
                    CB FOR CHEBYSHEV,
                    EL FOR ELLIPTIC,
                    BS FOR BESSEL,
                    00(ZERO) FOR DONT KNOW.

ENTER KIND:
?BW

-

ENTER THE TYPE OF FILTER DESIRED:
                        LP FOR LOW PASS ,
                        HP FOR HIGH PASS,
                        BP FOR BANDPASS,
                        BS FOR BANDSTOP.

ENTER TYPE:
?LP
```

```
                    INPUT RESPONSE PARAMETERS
             --------------------------------------

DO YOU KNOW THE FILTER ORDER,N?
ENTER Y OR N:
?N


PLEASE ENTER THE RESPONSE PARAMETERS
                    PASSBAND EDGE FREQUENCY,FA,
                    PASSBAND -DB AT FA,-ADB,
                    STOPBAND EDGE FREQUENCY,FB,
                    STOPBAND -DB AT FB,-BDB.
ENTER FA:
?100
ENTER ADB(MINUS SIGN NOT REQUIRED):
?3
ENTER FB:
?860
ENTER BDB(MINUS SIGN NOT REQUIRED):
?40


                    SUMMARY OF INPUT PARAMETERS
----------------------------------------------------------------

           (ZERO INDICATES THAT NO VALUE WAS GIVEN)
                KIND OF FILTER:BW
                TYPE OF FILTER:LP
                CENTER FREQUENCY:  .000
                LOWER PASSBAND EDGE FREQUENCY:  .000
                HIGHER PASSBAND EDGE FREQUENCY:  .000

                BANDWIDTH:  .000
          ------------------------------------------
                PASSBAND EDGE FREQUENCY:  100.
                PASSBAND EDGE DB DOWN:  3.00
                STOPBAND EDGE FREQUENCY:  860.
                STOPBAND EDGE DB DOWN:  40.0
          ------------------------------------------
                FILTER ORDER:    .000
```

band pass and band stop only.

low pass and high pass only.

Parameter Entry Section for a
Butterworth Low Pass Filter

Figure 3.3

```
ENTER THE TYPE OF FILTER DESIRED:
                              LP FOR LOW PASS .
                              HP FOR HIGH PASS.
                              BP FOR BANDPASS.
                              BS FOR BANDSTOP.


ENTER TYPE:
?BP


                         INPUT RESPONSE PARAMETERS
                    -----------------------------------


DO YOU KNOW THE FILTER ORDER,N?
ENTER Y OR N:
?N


ENTER THE REQUESTED RESPONSE PARAMETERS
DEFINED AS FOLLOWS:
                         CENTER FREQUENCY(FC),
                         PASSBAND EDGE FREQUENCIES(FA1,FA2),
                         PASSBAND EDGE -DB AT FA1 OR FA2(-ADB).
                         UPPER STOPBAND EDGE FREQUENCY(FB2),
                         STOPBAND -DB AT FB2(-BDB)
                         BANDWIDTH,BD.


ENTER FC(0(ZERO) FOR DONT KNOW):
?455000

ENTER BANDWIDTH(0(ZERO) FOR DONT KNOWN.):
?10000
```

```
ENTER ADB(MINUS SIGN NOT REQUIRED):
?1
ENTER FB2:
?500000
ENTER BDB(MINUS SIGN NOT REQUIRED):
?40


                     SUMMARY OF INPUT PARAMETERS
------------------------------------------------------------------

          (ZERO INDICATES THAT NO VALUE WAS GIVEN)
               KIND OF FILTER:BW
               TYPE OF FILTER:BP
               CENTER FREQUENCY:  .455E+06
               LOWER PASSBAND EDGE FREQUENCY:  .450E+06
               HIGHER PASSBAND EDGE FREQUENCY:  .460E+06
               BANDWIDTH:  .100E+05
     ----------------------------------------
               PASSBAND EDGE FREQUENCY:  .000
               PASSBAND EDGE DB DOWN:  1.00
               STOPBAND EDGE FREQUENCY:  .500E+06
               STOPBAND EDGE DB DOWN:  40.0
     ----------------------------------------
               FILTER ORDER:   .000
------------------------------------------------------------------

                ACTUAL BANDWIDTH IS MEASURED AT -3DB
               AND NOT AT -  1.00   DB.THE PROPER BANDWIDTH IS
               AS FOLLOWS:
                -3DB BANDWIDTH=  .132E+05
------------------------------------------------------------------


               THE PROPER -3DB FREQUENCIES ARE:
                 .448E+06 AND   .462E+06
------------------------------------------------------------------

------------------------------------------------------------------


               FILTER ORDER,N,EQUALS:    3.00
------------------------------------------------------------------
```

calculated by the program

Input Response Parameters for a
Butterworth Band Pass Filter

Figure 3.4

# SECTION IV

## TRANSFER FUNCTION DATA
(the poles, zeros and filter order, N)

### 4.1 Introduction

After entering the input parameters, the program evaluates the filter order, N, and the transfer function poles (and zeros). Both the normalized and denormalized values are provided to the user. Other information such as elliptic integral parameters, bandwidth, $\omega_0$ and Q (for the pole pair), etc. are provided as secondary parameters calculated by the program.

### 4.2 Example of the Transfer Function Data

Figure 4.1 illustrates an example of acquiring the transfer function poles and zeros for an elliptic low pass filter. The filter has a -0.1 db attenuation at the cutoff frequency, FA = 100 Hz. The stop band is defined by FB = 860 Hz with a minimum attenuation, -BDB = -40 db. The resulting filter order, N, is three.

Figure 4.1 also illustrates the elliptic filter parameters derived from the data entered by the user. These secondary parameters (small Kl, Kl, VO, etc.) are significant only to those familiar with the formulation of the normalized poles and zeros.

```
                INPUT RESPONSE PARAMETERS
        -----------------------------------

 DO YOU KNOW THE FILTER ORDER,N?
 ENTER Y OR N:
?N

 PLEASE ENTER THE RESPONSE PARAMETERS
                     PASSBAND EDGE FREQUENCY,FA,
                     PASSBAND -DB AT FA,-ADB,
                     STOPBAND EDGE FREQUENCY,FB,
                     STOPBAND -DB AT FB,-BDB:
 ENTER FA:
?100
 ENTER ADB(MINUS SIGN NOT REQUIRED):
?.1
 ENTER FB:
?860
 ENTER BDB(MINUS SIGN NOT REQUIRED):
?40



                SUMMARY OF INPUT PARAMETERS
 ----------------------------------------------------------------

         (ZERO INDICATES THAT NO VALUE WAS GIVEN)
              KIND OF FILTER:EL
              TYPE OF FILTER:LP
              CENTER FREQUENCY:   .000
              LOWER PASSBAND EDGE FREQUENCY:  .000
              HIGHER PASSBAND EDGE FREQUENCY:  .000
              BANDWIDTH:  .000
         ------------------------------------
              PASSBAND EDGE FREQUENCY:  100.
              PASSBAND EDGE DB DOWN:  .100
              STOPBAND EDGE FREQUENCY:  860.
              STOPBAND EDGE DB DOWN:  40.0
         ------------------------------------
              FILTER ORDER:    .000
 ----------------------------------------------------------------


            THE ELLIPTIC PARAMETERS HAVE BEEN
            DETERMINED AND ARE SHOWN BELOW:
   SMALL K=.1162791          SMALL K PRIME=.9932166

   SMALL K1= .00152627       SMALL K1 PRIME= .99999884

 ----------------------------------------------------------------
```

```
                ELLIPTIC INTEGRAL VALUES
                ------------------------
      K=  1.5761                      K PRIME=  3.5467

      K1=  1.5708                     K1 PRIME=  7.8710
 ----------------------------------------------------------------


      ZEROS OF ATTEN.                FREQ. OF INF. LOSS
  NORM.           DENORM.         NORM.           DENORM.
   .000           .293E-05        .100E+09        .293E+11
   .236           86.7            3.38            992.

 ----------------------------------------------------------------
                THE VALUE OF VO IS:    .862

 ----------------------------------------------------------------


                FILTER ORDER,N,EQUALS:    3.00
 ----------------------------------------------------------------


                NORMALIZED POLE VALUES
 ****************************************************************

       NO.     REAL PLE.     IMAG.PLE.       WO          Q
        1      -.33275        .00000        .333        .500
        2      -.16353        .41163        .443        1.35
        3      -.16353       -.41163        .443        1.35

                DENORMALIZED POLE VALUES
 ****************************************************************

       NO.     REAL PLE.     IMAG.PLE.       WO          Q
        1      -613.12        .00000        613.        .500
        2      -301.31        758.47        816.        1.35
        3      -301.31       -758.47        816.        1.35

                      ZERO VALUES
 ****************************************************************

                NORMALIZED VALUES          DENORMALIZED VALUES
       NO.        REAL        IMAG.          REAL        IMAG.
        1       .00000       3.3834          .000       .623E+04
        2       .00000      -3.3834          .000      -.623E+04
```

elliptic
pole
parameter

Transfer Function Data for an
Elliptic Low Pass Filter
Figure 4.1

# SECTION V

## RESPONSE EVALUATION
(magnitude and phase vs. frequency)

### 5.1 Introduction

After the FILTER program has provided the transfer function poles and zeros, the user has the opportunity to request magnitude and phase data for the denormalized filter response. The start and stop frequencies specify the range of data desired. The granularity is specified by the frequency increment.

### 5.2 Example of Evaluating the Filter Response

Figures 5.1 and 5.2 illustrate an example of evaluating the filter response. The filter being considered is a normalized low pass Butterworth filter having a -3 db cutoff frequency, FA = 1; the stop band is defined by the frequency, FB = 10 with a minimum attenuation, -BDB = -40 db. As shown, the user requests the magnitude and phase response data over the frequency range from 0 to 12.

As part of the filter response evaluation, the user can also alter any of the original filter parameter entries during this portion of the program. This is prompted by the question, "Is the response satisfactory? Enter Y or N." If the user enters N, choices are given for redefining some or all of the filter parameters. Figure 5.2 illustrates an example where the user has decided to alter the cutoff frequency, FA, from 1 to 2.15. As a result, the normalized filter order, N, changes from 2 to 3. The stop band requirements have been allowed to remain fixed.

```
                    INPUT RESPONSE PARAMETERS
               ------------------------------------

DO YOU KNOW THE FILTER ORDER,N?
ENTER Y OR N:
?N

PLEASE ENTER THE RESPONSE PARAMETERS
                    PASSBAND EDGE FREQUENCY,FA,
                    PASSBAND -DB AT FA,-ADB,
                    STOPBAND EDGE FREQUENCY,FB,
                    STOPBAND -DB AT FB,-BDB:
ENTER FA:
?1
ENTER ADB(MINUS SIGN NOT REQUIRED):
?3.01
ENTER FB:
?10
ENTER BDB(MINUS SIGN NOT REQUIRED):
?40

                    SUMMARY OF INPUT PARAMETERS
---------------------------------------------------------------

               (ZERO INDICATES THAT NO VALUE WAS GIVEN)
                    KIND OF FILTER:BW

-

                    TYPE OF FILTER:LP
                    CENTER FREQUENCY:  .000
                    LOWER PASSBAND EDGE FREQUENCY:  .000
                    HIGHER PASSBAND EDGE FREQUENCY:  .000
                    BANDWIDTH:  .000
               ---------------------------------------
                    PASSBAND EDGE FREQUENCY:  1.00
                    PASSBAND EDGE DB DOWN:  3.01
                    STOPBAND EDGE FREQUENCY:  10.0
                    STOPBAND EDGE DB DOWN:  40.0
               ---------------------------------------
                    FILTER ORDER:    .000
---------------------------------------------------------------
```

```
          ---------------------------------------

                    FILTER ORDER,N,EQUALS:    2.00
------------------------------------------------------------------

                    NORMALIZED POLE VALUES
******************************************************************
     NO.     REAL PLE.      IMAG.PLE.        WO          0
      1      -.70711         .70711         1.00       .707
      2      -.70711        -.70711         1.00       .707

                    DENORMALIZED POLE VALUES
******************************************************************
     NO.     REAL PLE.      IMAG.PLE.        WO          0
      1      -4.4430         4.4430         6.28       .707
      2      -4.4430        -4.4430         6.28       .707

------------------------------------------------------------------

WOULD YOU LIKE DATA ON THE GAIN AND PHASE RESPONSE?

ENTER Y OR N:
?Y

ENTER THE FOLLOWING DATA RESPONSE PARAMETERS:
                         STARTING FREQUENCY(HERTZ),
                         STOP FREQUENCY(HERTZ),
                         FREQUENCY INCREMENT(HERTZ).
ENTER STARTING FREQUENCY:
?0
 ENTER STOP FREQUENCY:
?12
 ENTER THE FREQUENCY INCREMENT:
?1
```

Input Parameters and Response Evaluation
for a Butterworth Low Pass Filter

Figure 5.1

```
                 LP FREQUENCY RESPONSE DATA
                     FOR THE BW FILTER
-------------------------------------------------------------------
   NO.      FREQ.          MAG        GAIN(DB)      PHASE(DEG)
    1      .00000        1.0000         .000        .137E-04
    2      1.0000        .70713        -3.01       -90.0
    3      2.0000        .24255       -12.3       -137.
    4      3.0000        .11044       -19.1       -152.
    5      4.0000        .62383E-01   -24.1       -159.
    6      5.0000        .39971E-01   -28.0       -164.
    7      6.0000        .27769E-01   -31.1       -166.
    8      7.0000        .20406E-01   -33.8       -168.
    9      8.0000        .15624E-01   -36.1       -170.
   10      9.0000        .12346E-01   -38.2       -171.
   11      9.9999        .10000E-01   -40.0       -172.
   12      11.000        .82649E-02   -41.7       -173.
   13      12.000        .69448E-02   -43.2       -173.
-------------------------------------------------------------------
   NO.      FREQ.          MAG        GAIN(DB)      PHASE(DEG)


DO YOU WANT MORE FREQUENCY RESPONSE DATA?
ENTER Y OR N:
?N

IS THE RESPONSE SATISFACTORY,
ENTER Y OR N:
?N

                YOU HAVE THE FOLLOWING OPTIONS:
                     A)CHANGE THE INPUT RESPONSE PARAMETERS,
                     B)CHANGE THE FILTER KIND(BW.CHEBY,ELL,BESSEL),
                     C)CHANGE THE CUTOFF OR CENTER FREQUENCY,
                     D)START OVER,
                     E)CONTINUE AS IS.

ENTER A,B,C,D OR E:
?D
ENTER FA:
?2.15
-------------------------------------------------------------------
-
                     FILTER ORDER.N.EQUALS:     3.00
-------------------------------------------------------------------
```

```
                    NORMALIZED POLE VALUES
*******************************************************************
   NO.    REAL PLE.     IMAG.PLE.       WO          Q
    1     -.50000        .86603        1.00       1.00
    2     -1.0000        .00000        1.00       .500
    3     -.50000       -.86603        1.00       1.00

                   DENORMALIZED POLE VALUES
*******************************************************************
   NO.    REAL PLE.     IMAG.PLE.       WO          Q
    1     -6.7546        11.699        13.5       1.00
    2     -13.509        .00000        13.5       .500
    3     -6.7546       -11.699        13.5       1.00
-------------------------------------------------------------------


WOULD YOU LIKE DATA ON THE GAIN AND PHASE RESPONSE?

ENTER Y OR N:
?N

DO YOU WISH TO CONTINUE ONTO THE FILTER
DESIGN IMPLEMENTATION SECTION,

ENTER Y OR N:
?N

                         DO YOU WISH TO RERUN THE PROGRAM?
ENTER Y OR N:
?N

END OF PROGRAM
*STOP*
```

Response Data; Alteration of the
Cutoff Frequency (cont. of Figure 5.1)
Figure 5.2

# SECTION VI

# ACTIVE FILTER DESIGN

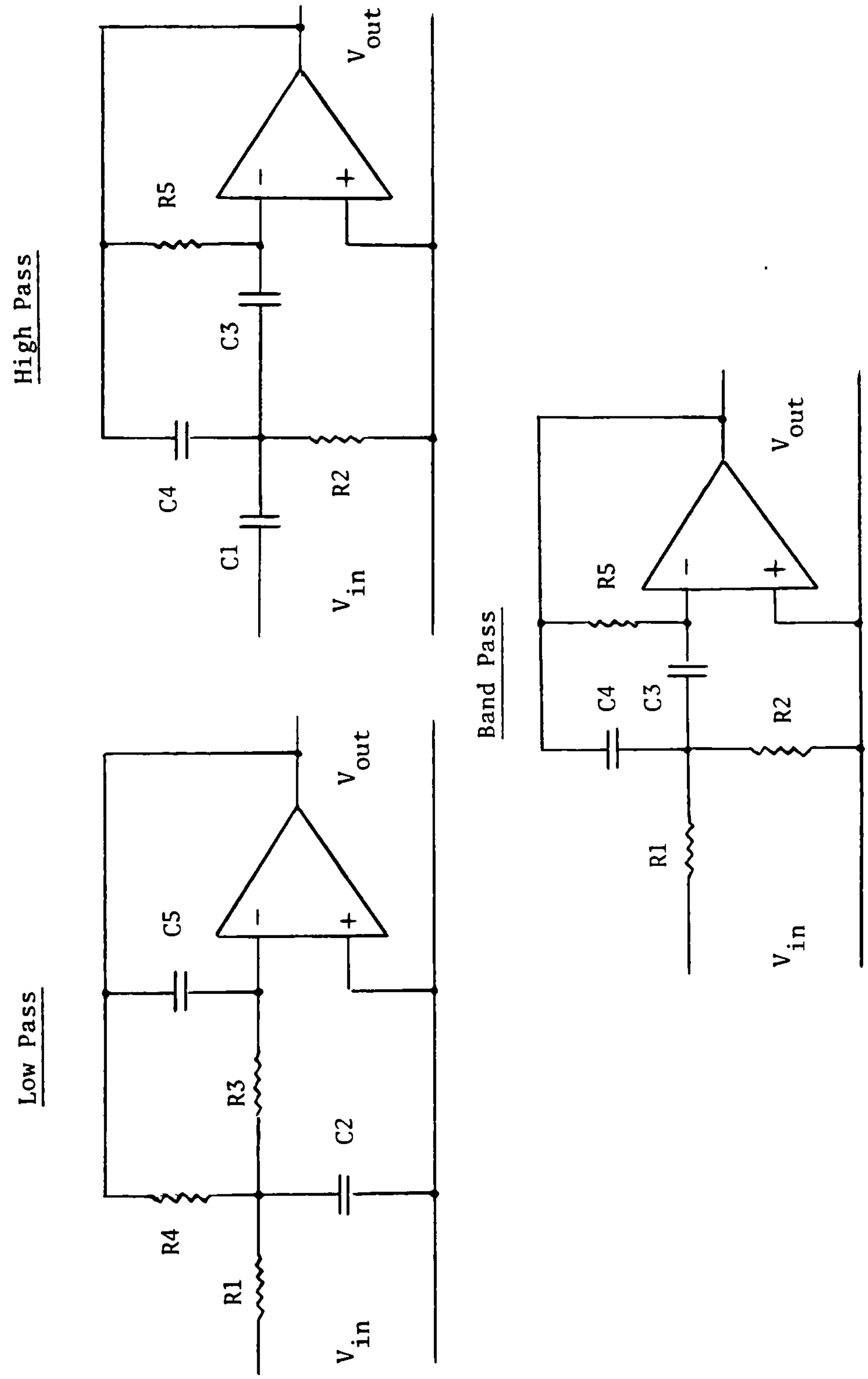
## 6.1 Introduction

The last major portion of the FILTER program provides for the hardware implementation of the denormalized transfer function (poles and zeros) using active filters. The user enters the desired gain of the filter in db. The program then identifies the circuit and component values using the nomenclature in Figures 6.1 and 6.2.

## 6.2 Active Filter Circuits

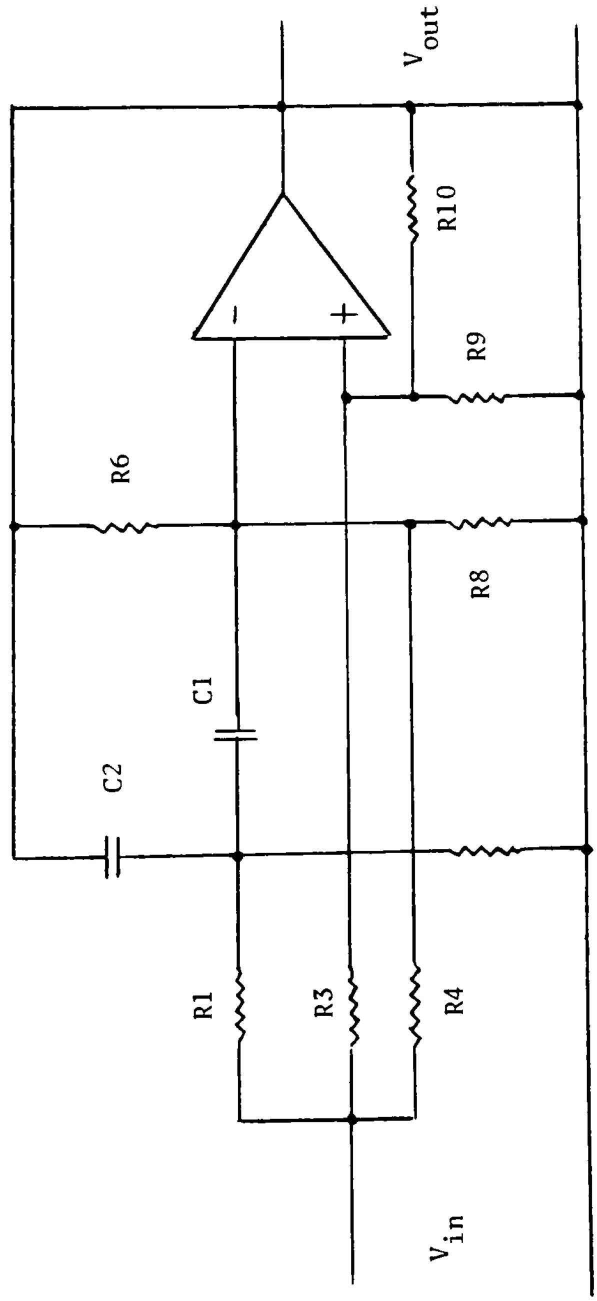
Each of the active filter circuits shown in Figures 6.1 and 6.2 will implement one pole pair (second order transfer function). The design of the overall active filter is then accomplished by cascading stages of these second order active filters. The parameters identified for each stage are the pole pair, $\omega_0$ and Q. When N is odd and the filter is a low or high pass, there will be one stage which implements a single order function or one pole.

For the case where the filter is non-elliptic, the program provides component values using the circuits in Figure 6.1. For the elliptic filters and band stop filters, the program uses the circuit configuration shown in Figure 6.2. This circuit provides for the complex numerator zeros in the denormalized transfer function. There are other ways of implementing a transfer function with complex zeros (such as the dual integrator or the voltage controlled voltage source methods). However, this concept is unique in that it uses an infinite gain multiple feedback (IGMF) design approach. This is consistent with the IGMF design philosophy of the circuits in Figure 6.1.




Active Filter Circuit Configurations

Figure 6.1




Elliptic Active Filter Configuration
(low pass, high pass, band stop and band pass)
Figure 6.2

# SECTION VII

# USER INTERACTION
(examples)

## 7.1 Introduction

This section illustrates two complete examples of sessions with the FILTER program. The first example being considered is a Chebyshev band pass filter design. The second example is an elliptic low pass filter design.

## 7.2 A Chebyshev Band Pass Filter Design (example one)

The following example illustrates a session with the FILTER program where the user wishes to design a Chebyshev band pass filter (Figures 7.1 to 7.4). The desired filter response has a center frequency, FC = 455 KHz and a bandwidth, BD = 10 KHz. The maximum pass band deviation is -ADB = -0.1 db. The desired stop band attenuation is a minimum of -40 db at the upper stop band edge frequency, FB2 = 500 KHz. The filter order, N, is unknown. The resulting input parameters are as follows:

Pass band:

FC = 455 KHz
BD = 10 KHz
-ADB = -0.1 db

Stop band:

FB2 = 500 KHz
-BDB = -40 db

The session progresses through the filter parameters, response evaluation and the active filter design.

```
                    ACTIVE FILTER DESIGN PROGRAM
**********************************************************************
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 ENTER THE KIND OF FILTER DESIRED:
                        BW FOR BUTTERWORTH,
                        CB FOR CHEBYSHEV,
                        EL FOR ELLIPTIC,
                        BS FOR BESSEL,
                        00(ZERO) FOR DONT KNOW.

 ENTER KIND:
?CB

 ENTER THE TYPE OF FILTER DESIRED:
                             LP FOR LOW PASS ,
                             HP FOR HIGH PASS,
                             BP FOR BANDPASS,
                             BS FOR BANDSTOP.


 ENTER TYPE:
?BP
                      INPUT RESPONSE PARAMETERS
               -----------------------------------

 DO YOU KNOW THE FILTER ORDER,N?
 ENTER Y OR N:
?N
```

```
 ENTER THE REQUESTED RESPONSE PARAMETERS
 DEFINED AS FOLLOWS:
                     CENTER FREQUENCY(FC),
                     PASSBAND EDGE FREQUENCIES(FA1,FA2),
                     PASSBAND EDGE -DB AT FA1 OR FA2(-ADB),

                     UPPER STOPBAND EDGE FREQUENCY(FB2),
                     STOPBAND -DB AT FB2(-BDB)
                     BANDWIDTH,BD.


 ENTER FC(0(ZERO) FOR DONT KNOW):
?455E3

 ENTER BANDWIDTH(0(ZERO) FOR DONT KNOWN.):
?10E3
 ENTER ADB(MINUS SIGN NOT REQUIRED):
?.1
 ENTER FB2:
?500E3
 ENTER BDB(MINUS SIGN NOT REQUIRED):
?40

                SUMMARY OF INPUT PARAMETERS
---------------------------------------------------------

          (ZERO INDICATES THAT NO VALUE WAS GIVEN)
              KIND OF FILTER:CB
              TYPE OF FILTER:BP
              CENTER FREQUENCY:  .455E+06
              LOWER PASSBAND EDGE FREQUENCY:  .450E+06
              HIGHER PASSBAND EDGE FREQUENCY:  .460E+06
              BANDWIDTH:  .100E+05
       --------------------------------------
              PASSBAND EDGE FREQUENCY:  .000
              PASSBAND EDGE DB DOWN:  .100
              STOPBAND EDGE FREQUENCY:  .500E+06
              STOPBAND EDGE DB DOWN:  40.0
       --------------------------------------
              FILTER ORDER:    .000
---------------------------------------------------------


              BANDWIDTH=  .100E+05
---------------------------------------------------------
```

Chebyshev Band Pass Filter Design (example one)
Input Parameters
Figure 7.1

```
FILTER ORDER,N.EQUALS:    3.00
------------------------------------------------------------

NORMALIZED POLE VALUES
************************************************************

NO.   REAL PLE.    IMAG.PLE.     WO        Q
 1    -.48497      -1.2065      1.30      1.34
 2    -.96994       .00000      .970      .500
 3    -.48498      1.2065       1.30      1.34

DENORMALIZED POLE VALUES
************************************************************

NO.   REAL.PLE.    IMAG.PLE.     WO         Q         BW AT-ADB
1    -.150E+05    .282E+07    .282E+07    93.8322   .469E+04
2    -.305E+05   -.286E+07    .286E+07    46.9111   .935E+04
3    -.150E+05   -.282E+07    .282E+07    93.8313   .469E+04
4    -.154E+05   -.290E+07    .290E+07    93.8241   .486E+04
5    -.305E+05    .286E+07    .286E+07    46.9086   .935E+04
6    -.154E+05    .290E+07    .290E+07    93.8233   .486E+04
------------------------------------------------------------

WOULD YOU LIKE DATA ON THE GAIN AND PHASE RESPONSE?

ENTER Y OR N:
?Y

ENTER THE FOLLOWING DATA RESPONSE PARAMETERS:
                         STARTING FREQUENCY(HERTZ),
                         STOP FREQUENCY(HERTZ),
                         FREQUENCY INCREMENT(HERTZ).
ENTER STARTING FREQUENCY:
?0
 ENTER STOP FREQUENCY:
?50\\700000
 ENTER THE FREQUENCY INCREMENT:
?50000
```

```
          BP FREQUENCY RESPONSE DATA
          FOR THE C8 FILTER
------------------------------------------------------------

NO.    FREQ.          MAG           GAIN(DB)     PHASE(DEG)
 1    .00000         .74517E-24     -483.         270.
 2    50000.         .23961E-07     -152.         270.
 3    .10000E+06     .21442E-06     -133.         269.
 4    .15000E+06     .88098E-06     -121.         269.
 5    .20000E+06     .28159E-05     -111.         269.
 6    .25000E+06     .84901E-05     -101.         268.
 7    .30000E+06     .27642E-04     -91.2         267.
 8    .35000E+06     .11658E-03     -78.7         265.
 9    .40000E+06     .10148E-02     -59.9         261.
10    .45000E+06     .98733         -.111         101.
11    .50000E+06     .26095E-02     -51.7        -257.
12    .55000E+06     .31431E-03     -70.1        -264.
13    .60000E+06     .99070E-04     -80.1        -266.
14    .65000E+06     .45051E-04     -86.9        -267.
15    .69999E+06     .24838E-04     -92.1        -267.

------------------------------------------------------------

NO.    FREQ.          MAG           GAIN(DB)     PHASE(DEG)

------------------------------------------------------------
DO YOU WANT MORE FREQUENCY RESPONSE DATA?
ENTER Y OR N:
?Y

ENTER THE FOLLOWING DATA RESPONSE PARAMETERS:
                         STARTING FREQUENCY(HERTZ),
                         STOP FREQUENCY(HERTZ),
                         FREQUENCY INCREMENT(HERTZ).
ENTER STARTING FREQUENCY:
?447000
 ENTER STOP FREQUENCY:
?464000
 ENTER THE FREQUENCY INCREMENT:
?1000
```

Transfer Function and Response Evaluation Data
(example one cont.)
Figure 7.2

```
BP FREQUENCY RESPONSE DATA
FOR THE CB FILTER
--------------------------------------------------------------------------------
NO.      FREQ.          MAG        GAIN(DB)       PHASE(DEG)
 1     .44700E+06     .48024      -6.37           179.
 2     .44800E+06     .68371      -3.30           158.
 3     .44900E+06     .88628      -1.05           130.
 4     .45000E+06     .98740      -.110           101.
 5     .45100E+06     .99867      -.116E-01       76.4
 6     .45200E+06     .99000      -.873E-01       55.6
 7     .45300E+06     .98972      -.897E-01       36.8
 8     .45400E+06     .99625      -.327E-01       18.4
 9     .45500E+06     1.0000       .000           .622E-01
10     .45600E+06     .99628      -.323E-01      -18.3
11     .45699E+06     .98979      -.891E-01      -36.5
12     .45799E+06     .98990      -.882E-01      -55.1
13     .45899E+06     .99835      -.143E-01      -75.5
14     .45999E+06     .98979      -.891E-01      -99.5
15     .46099E+06     .89961      -.919          -128.
16     .46199E+06     .70853      -2.99          -155.
17     .46299E+06     .50655      -5.91          -177.
18     .46399E+06     .35536      -8.99          -192.
--------------------------------------------------------------------------------
NO.      FREQ.          MAG        GAIN(DB)       PHASE(DEG)


--------------------------------------------------------------------------------

DO YOU WANT MORE FREQUENCY RESPONSE DATA?
ENTER Y OR N:


-


?N

IS THE RESPONSE SATISFACTORY.
ENTER Y OR N:
?Y

DO YOU WISH TO CONTINUE ONTO THE FILTER
DESIGN IMPLEMENTATION SECTION.

ENTER Y OR N:
?Y
```

Response Evaluation Data
(example one cont.)
Figure 7.3

```
                              FILTER DESIGN IMPLEMENTATION SECTION
**********************************************************************************

 ENTER THE TYPE OF DESIGN DESIRED:
    A)INFINITE GAIN MULTIPLE FEEDBACK(IGMF).
    B)DUAL INTEGRATOR

ENTER A OR B:
?A

                              IGMF DESIGN
**********************************************************************************


INDICATE THE GAIN DESIRED IN THE PASSBAND
OR AT THE CENTER FREQUENCY.

ENTER DB (SIGN IS REQUIRED):
?0


                              BP FILTER COMPONENT VALUES
----------------------------------------------------------------------------------

                    (REFER TO FIGURE  6.1  IN THE USER MANUAL)


                    RESISTOR VALUES WITHIN THE RANGE OF
                    100 OHMS TO 1 MEGOHM COULD NOT BE SELECTED FOR THIS
                    STAGE.THE FOLLOWING REPRESENTS THE CLOSEST AVAILABLE
                    VALUES NEAREST THESE LIMITS FOR STAGE NO. 1:
STAGE       R1              R2              R5              C3              C4
         (KOHMS)         (KOHMS)         (KOHMS)          (UF)            (UF)
------   -------         -------         -------         -------         -------
 1    :  880.         :  .500E-01     :  .176E+04     :  .378E-04     :  .378E-04
----------------------------------------------------------------------------------


STAGE       R1              R2              R5              C3              C4
         (KOHMS)         (KOHMS)         (KOHMS)          (UF)            (UF)
------   -------         -------         -------         -------         -----
 2    :  440.         :  .100         :  880.         :  .373E-04     :  .373E-04
----------------------------------------------------------------------------------


                    RESISTOR VALUES WITHIN THE RANGE OF
                    100 OHMS TO 1 MEGOHM COULD NOT BE SELECTED FOR THIS
                    STAGE.THE FOLLOWING REPRESENTS THE CLOSEST AVAILABLE
                    VALUES NEAREST THESE LIMITS FOR STAGE NO. 3:
STAGE       R1              R2              R5              C3              C4
         (KOHMS)         (KOHMS)         (KOHMS)          (UF).           (UF)
------   -------         -------         -------         -------         -------
 3    :  880.         :  .500E-01     :  .176E+04     :  .368E-04     :  .368E-04
----------------------------------------------------------------------------------


                         DO YOU WISH TO RERUN THE PROGRAM?
 ENTER Y OR N:
?N

 END OF PROGRAM
 *STOP*

!OFF
```

Active Filter Design Section
(example one completion)
Figure 7.4

## 7.3 An Elliptic Low Pass Filter Design (example two)

This second example (Figures 7.5 to 7.8) illustrates a session where the user wishes to design an elliptic low pass filter. The desired cutoff frequency, FA, is 100 Hz with a maximum pass band deviation, -ADB = -0.1 db. The desired stop band is defined by the frequency, FB = 860 Hz, with a minimum attenuation, -BDB = -40 db. The example illustrates the user inputs, response evaluation and the active filter design.

As part of the elliptic active filter design, the FILTER program performs an optimization algorithm which matches the zeros to the poles to minimize stage overshoot (Figure 7.7). The desired overall filter gain is then divided among the filter stages to equalize the maximum magnitude of each stage (Figure 7.8). Having performed the optimization process, the filter component values are then calculated (Figure 7.8).

```
                    ACTIVE FILTER DESIGN PRDGRAM
*****************************************************************


               CDPYRIGHT BY LDUIS R. GABELLD,19B1

          ND RESPDNSIBILITY IS ASSUMED FDR THE USE DR
          ACCURACY DF THE INFDRMATIDN PRESENTED IN
          THIS PRDGRAM.SUBJECT TD CHANGE WITHDUT NDTICE.

          VERSIDN 1                LAST UPDATE:5-7-B1



 ENTER THE KIND DF FILTER DESIRED:
                         BW FDR BUTTERWDRTH,
                         CB FDR CHEBYSHEV,
                         EL FDR ELLIPTIC,
                         BS FDR BESSEL,
                         00(ZERD) FDR DDNT KNDW.


 ENTER KIND:
?EL



 ENTER THE TYPE DF FILTER DESIRED:
                              LP FDR LDW PASS ,
                              HP FDR HIGH PASS,
                              BP FDR BANDPASS.
                              BS FDR BANDSTDP.


 ENTER TYPE:
?LP
```

```
                    INPUT RESPDNSE PARAMETERS
                -------------------------------------

 DD YDU KNDW THE FILTER DRDER,N?
 ENTER Y DR N:
?N



 PLEASE ENTER THE RESPDNSE PARAMETERS
                         PASSBAND EDGE FREQUENCY,FA,
                         PASSBAND -DB AT FA,-ADB,
                         STDPBAND EDGE FREQUENCY,FB,
                         STDPBAND -DB AT FB,-BDB.
 ENTER FA:
?100
 ENTER ADB(MINUS SIGN NDT REQUIRED):
?.1
 ENTER FB:
?B60
 ENTER BDB(MINUS SIGN NDT REQUIRED):
?40

                    SUMMARY DF INPUT PARAMETERS
 ----------------------------------------------------------------

            (ZERD INDICATES THAT ND VALUE WAS GIVEN)
                 KIND DF FILTER:EL
                 TYPE DF FILTER:LP
                 CENTER FREQUENCY:  .000
                 LDWER PASSBAND EDGE FREQUENCY:  .000
                 HIGHER PASSBAND EDGE FREQUENCY:  .000
                 BANDWIDTH:  .000
          ----------------------------------------
                 PASSBAND EDGE FREQUENCY:  100.
                 PASSBAND EDGE DB DDWN:  .100
                 STDPBAND EDGE FREQUENCY:  B60.
                 STDPBAND EDGE DB DDWN:  40.0
          ----------------------------------------
                 FILTER DRDER:    .000
 ----------------------------------------------------------------
```

Elliptic Low Pass Filter Design (example two)
Input Parameter Section
Figure 7.5

```
            THE ELLIPTIC PARAMETERS HAVE BEEN
            DETERMINED AND ARE SHOWN BELOW:
   SMALL K=.1162791            SMALL K PRIME=.9932166

   SMALL K1= .00152627         SMALL K1 PRIME= .99999884
----------------------------------------------------------------

            ELLIPTIC INTEGRAL VALUES
            ------------------------
   K=  1.5761                      K PRIME=  3.5467

   K1=  1.5708                     K1 PRIME=  7.8710
----------------------------------------------------------------


   ZEROS OF ATTEN.                FREQ. OF INF. LOSS
 NORM.         DENORM.          NORM.          DENORM.
  .000         .293E-05         .100E+09       .293E+11
  .296         86.7             3.38           992.




----------------------------------------------------------------

            THE VALUE OF V0 IS:     .862
----------------------------------------------------------------

            FILTER ORDER,N.EQUALS:    3.00
----------------------------------------------------------------
```

```
                      NORMALIZED POLE VALUES
*****************************************************************

   NO.     REAL PLE.      IMAG.PLE.        W0          O
    1      -.33275         .00000         .333        .500
    2      -.16353         .41163         .443        1.35
    3      -.16353        -.41163         .443        1.35

                     DENORMALIZED POLE VALUES
*****************************************************************

   NO.     REAL PLE.      IMAG.PLE.        W0          O
    1      -613.12         .00000         613.        .500
    2      -301.31         758.47         816.        1.35
    3      -301.31        -758.47         816.        1.35

                          ZERO VALUES
*****************************************************************

              NORMALIZED VALUES          DENORMALIZED VALUES
   NO.         REAL          IMAG.          REAL        IMAG.
    1         .00000        3.3834          .000        .623E+04
    2         .00000       -3.3834          .000       -.623E+04

-----------------------------------------------------------------

  WOULD YOU LIKE DATA ON THE GAIN AND PHASE RESPONSE?


  ENTER Y OR N:
 ?Y


  ENTER THE FOLLOWING DATA RESPONSE PARAMETERS:
                        STARTING FREQUENCY(HERTZ),
                        STOP FREQUENCY(HERTZ),
                        FREQUENCY INCREMENT(HERTZ).
  ENTER STARTING FREQUENCY:
 ?0
   ENTER STOP FREQUENCY:
 ?1000
   ENTER THE FREQUENCY INCREMENT:
 ?50
```

Transfer Function Data
(example two cont.)
Figure 7.6

```
                LP FREQUENCY RESPONSE DATA
                FOR THE EL FILTER
------------------------------------------------------------------
  NO.     FREQ.          MAG         GAIN(DB)      PHASE(DEG)
   1     .00000        1.0000          .000          .000
   2     50.000        .98855         -.100         -45.6
   3     100.00        .98855         -.100         -100.
   4     150.00        .58195         -4.70         -168.
   5     200.00        .23621         -12.5         -204.
   6     250.00        .11146         -19.1         -221.
   7     300.00        .60338E-01     -24.4         -231.
   8     350.00        .35795E-01     -28.9         -237.
   9     400.00        .22600E-01     -32.9         -241.
  10     450.00        .14902E-01     -36.5         -245.
  11     500.00        .10130E-01     -39.9         -247.
  12     550.00        .70295E-02     -43.1         -249.
  13     600.00        .49379E-02     -46.1         -251.
  14     650.00        .34842E-02     -49.2         -253.
  15     699.99        .24485E-02     -52.2         -254.
  16     749.99        .16954E-02     -55.4         -255.
  17     799.99        .11386E-02     -58.9         -256.
  18     849.99        .72092E-03     -62.8         -257.
  19     899.99        .40402E-03     -67.9         -258.
  20     949.99        .16126E-03     -75.8         -258.
  21     999.99        .26147E-04     -91.7         -78.9
------------------------------------------------------------------
  NO.     FREQ.          MAG         GAIN(DB)      PHASE(DEG)

------------------------------------------------------------------
DO YOU WANT MORE FREQUENCY RESPONSE DATA?
ENTER Y OR N:
?N

IS THE RESPONSE SATISFACTORY,
ENTER Y OR N:
?Y

DO YOU WISH TO CONTINUE ONTO THE FILTER
DESIGN IMPLEMENTATION SECTION,

ENTER Y OR N:
?Y
```

```
                    FILTER DESIGN IMPLEMENTATION SECTION
****************************************************************

-

 ENTER THE TYPE OF DESIGN DESIRED:
    A)INFINITE GAIN MULTIPLE FEEDBACK(IGMF),
    B)DUAL INTEGRATOR

ENTER A OR B:
?A

                    IGMF DESIGN
****************************************************************

INDICATE THE GAIN DESIRED IN THE PASSBAND
OR AT THE CENTER FREQUENCY.

ENTER DB (SIGN IS REQUIRED):
?0

****************************************************************
```

stage optimization:

```
              MATCHED POLES/ZEROS
```

$\omega_0$

```
----------------------------------------------------------------
REAL PLE.     IMAG.PLE.     RADIAN FREQ.      Q          ZERO
-----------:------------:-------------:-----------:-------------

-301.     :   758.      :   816.      :   1.35    :   .623E+04
-613.     :   .000      :   613.      :   .500    :   .000
```

no zero
for stage two

Response Evaluation and Active Filter Design
(example two cont.)
Figure 7.7

```
                    LP FILTER COMPONENT VALUES
----------------------------------------------------------------------
              (REFER TO FIGURE  6.2  IN THE USER MANUAL)



                    STAGE GAIN PARAMETERS
----------------------------------------------------------------------
STAGE     MAX.GAIN FREQ.        MAG.         CONST.      RESULTING MAG.
----      --------------     ----------    ----------    --------------
  1            110.            84.0         .143E-01         1.20
  2            .000           .163E-02      736.             1.20
```

{frequency where magnitude is a maximum} {maximum magnitude} {normalizing constant} {new maximum magnitude}

```
STAGE      R1            R3           R4           C2           C5
         (KOHMS)      (KOHMS)      (KOHMS)        (UF)         (UF)
------   -------      -------      -------      -------      -------
 1   :   600.      :  10.0      :  OPEN      :  .100      :  .100E-01
STAGE      R6            R7           R8           R9           R10
         (KOHMS)      (KOHMS)      (KOHMS)      (KOHMS)      (KOHMS)
------   -------      -------      -------      -------      -------
 1   :   98.2      :  8.58      :  3.14      :  .145      :  10.0
----------------------------------------------------------------------
```

Refer to Figure 6.1:

```
STAGE      R1            R3           R4           C2           C5
         (KOHMS)      (KOHMS)      (KOHMS)        (UF)         (UF)
------   -------      -------      -------      -------      -------
 2   :   .222      :  SHORT     :  163.      :  OPEN      :  .100E-01

                 DO YOU WISH TO RERUN THE PROGRAM?
ENTER Y OR N:
?N

END OF PROGRAM
*STOP*
!OFF
```

{Stage 2 has no zeros. Therefore, use a non-elliptic low pass active filter.}

Active Filter Design Section
(example two completion)
Figure 7.8

# SECTION VIII

## A NOTE TO THE USER
(concerning evaluation and program limitations)

The program FILTER attempts to assist the designer in the analysis of various filter kinds and responses. Although much of the analysis and component evaluation is assumed on the part of the program, the user should be cognizant of its limitations and evaluate the resulting data and component values according to his or her own criterion.

The following should be considered:

1. Not all filter responses can be implemented with the hardware configurations shown. The component values are determined based upon a particular method and certain starting component values. In the event the first criterion yields unreasonable values, multiple combinations of component values are attempted according to a sensitivity order and formulation. After a predetermined number of iterations, the calculations are interrupted and a notice is issued to the user indicating that reasonable component values could not be reached. In this case the user must judge the values to determine if an alternate design should be used.
2. The design stages, component values, and filter configurations are suggestions and not necessarily the best design. The user should consider the merits of the suggested design against his or her own experiences and preferences.
3. This program is written in ANS Fortran. Minor changes are required, however, for use on computers other than the Xerox

Sigma 9. The program has been used on the Digital Equipment Corporation VAX Computer without difficulty and with a few minor alterations in syntax.

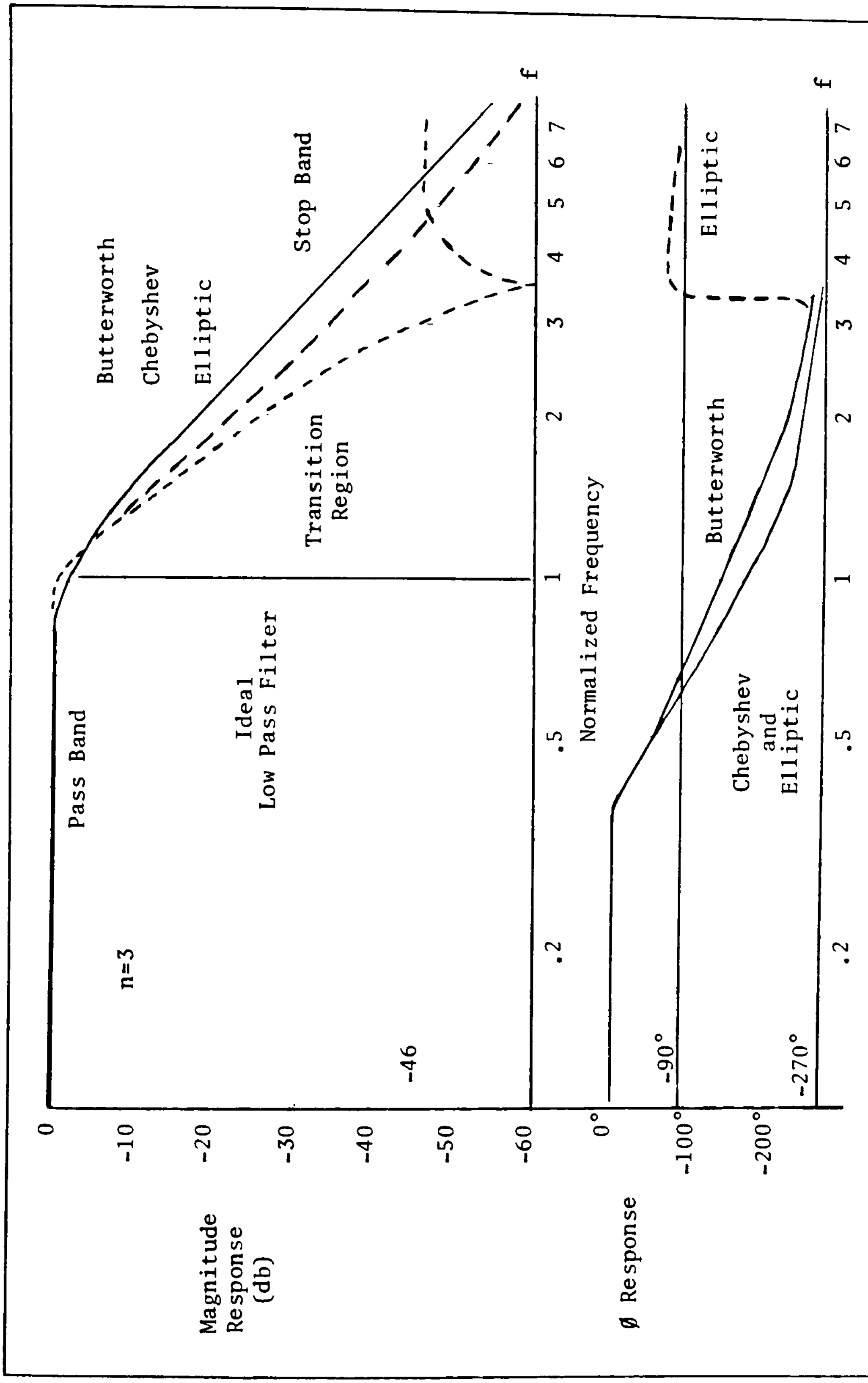
APPENDIX

# APPENDIX A

## QUICK COMPARISON of the CLASSICAL FILTER RESPONSES

Figure A.1 illustrates a comparison among three classical normalized filter responses (Butterworth, Chebyshev and elliptic). The classical filter responses are characterized by the manner in which they approximate the ideal filter response. The Butterworth filter approximates the constant (ideal) pass band characteristic by having maximum flatness at f = 0. The Chebyshev filter approximates the constant pass band characteristic by minimizing the maximum deviation throughout the pass band; this results in an equiripple response in the pass band. The elliptic function filter approximates both the constant pass band characteristic and ideal cutoff rate by allowing an equiripple magnitude response in both the pass and stop bands.

As a comparison among the three responses shown, the elliptic filter provides the greatest rate of attenuation from the pass band to the stop band for the given filter order. However, the elliptic filter also has the greatest phase non-linearity. In contrast, the Butterworth filter provides the least rate of attenuation. In this example a higher filter order, N, would be required to achieve the stop band edge attenuation provided by the Chebyshev and elliptic filters.



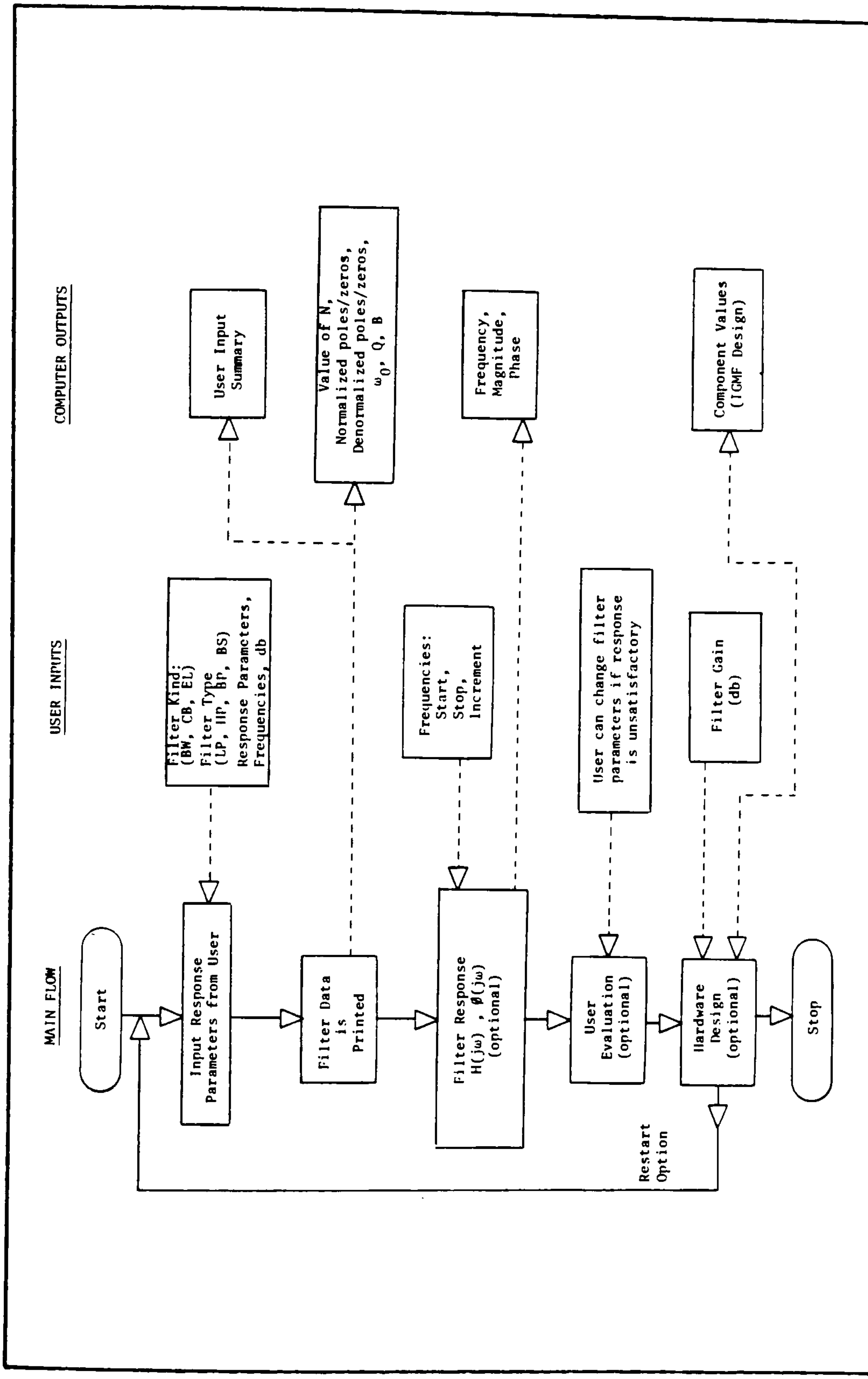

Comparison of the Classical Filter Responses

Figure A.1

# APPENDIX B

## PROGRAM FLOW AND I/O SUMMARY

Figure B.1 illustrates a simplified flow chart for the FILTER program. The diagram shows the main sections of the program along with the user inputs and computer outputs. The intention here is to acquaint the user with the overall format and expectations of a user session.




Program Flow and I/O Summary

Figure B.1

APPENDIX A

# DEVELOPMENT OF THE SOLUTIONS FOR THE ELLIPTIC FUNCTION POLES AND ZEROS

## A.1 Introduction

In this appendix we will examine the background and development of a method for determining the poles and zeros for the Nth order normalized elliptic low pass filter. This method was developed by expanding upon the theory presented by A. J. Grossman [11] for the case of an odd order filter. The following provides an overview of the sections in this appendix.

In section A.2 we will become familiar with the terminology and the parameters of the elliptic filter magnitude response. These parameters describe an insertion loss characteristic which has an equiripple pass band and stop band magnitude response.

The next section (A.3) describes the characteristics of a rational function, $R_N(\omega)$, which will yield the prescribed insertion loss characteristics defined in section A.2.

Section A.4 then develops the relationship between the prescribed form of the rational function and the required frequencies which yield the equiripple response. This relationship turns out to be the elliptic sine function.

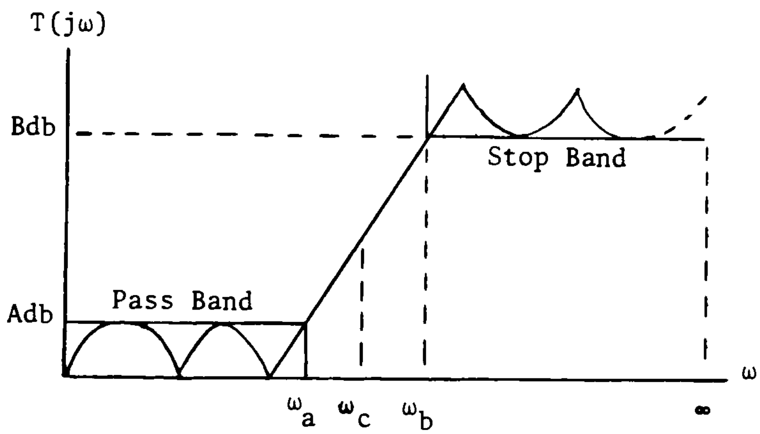
The next section provides a more intuitive examination of the elliptic integral and elliptic sine functions as they apply towards describing the elliptic rational function and its solutions.

The last section (A.6) utilizes the concepts presented in sections A.2 to A.6 to formulate the solution of the s-plane poles and zeros.

## A.2 The Normalized Elliptic Parameters

In this section we examine the parameter definitions which will commonly be used in evaluating the normalized elliptic poles and zeros. Figure A.1 illustrates the desired insertion loss characteristics of a denormalized low pass elliptic filter for $\omega \geq 0$. We use the inverted form of $|H(j\omega)|$ to maintain convention with reference [11]. That is,

$$|T(j\omega)| = |H(j\omega)|^{-1}$$




Denormalized Elliptic Filter Response

Figure A.1

The elliptic filter has equiripple attenuation characteristics in both the pass band ($0 \leq \omega \leq \omega_a$) and in the stop band ($\omega \geq \omega_b$). The equiripple maximum in the pass band is Adb. The equiripple minimum in the stop band is Bdb.

Using the parameters given in figure A.1 we now obtain the normalized low pass filter (see chapter III for the normalization process). We will choose the geometric center of the transition region ($\omega_a < \omega < \omega_b$) as the normalizing frequency, $\omega_c$. That is,

$$\omega_c = \left(\omega_a \cdot \omega_b\right)^{\frac{1}{2}} \tag{A.1}$$

where

$\omega_a$ is the denormalized pass band edge frequency

and

$\omega_b$ is the denormalized stop band edge frequency.

We next define a value, k, known as the selectivity parameter or modulus. This parameter is a measure of the transition rate from the pass band to the stop band and is defined as

$$k = \left(\omega_a/\omega_b\right) \qquad k < 1 \tag{A.2}$$

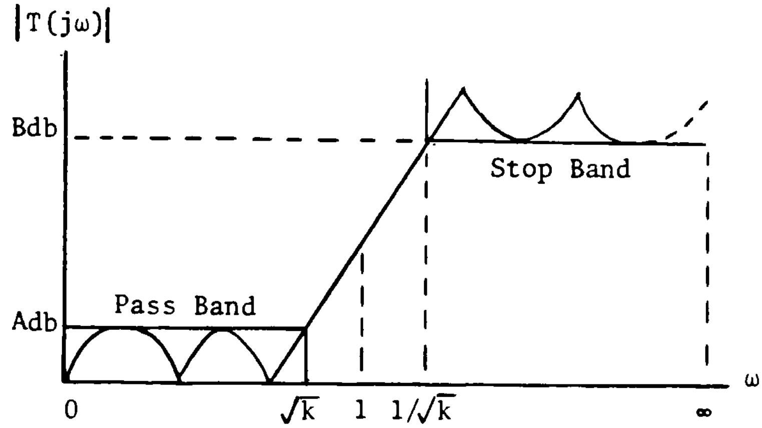
If we normalize the frequency values $\omega_a$ and $\omega_b$ with respect to $\omega_c$ we have

$$\omega_{an} = \left(\omega_a/\omega_c\right) = \sqrt{k}$$

and (A.3)

$$\omega_{bn} = \left(\omega_b/\omega_c\right) = 1/\sqrt{k}$$

As a result of the normalization process we have the normalized insertion loss parameters shown below in figure A.2.




Normalized Elliptic Insertion
Loss Parameters

Figure A.2

## A.3 Magnitude Characteristics of the Elliptic Rational Function, $R_N(\omega)$

Having normalized the insertion loss parameters, we now make some observations regarding the equiripple characteristics in order to derive a proper form of the elliptic approximation function, $R_N(\omega)$. (We assume that the function, $R_N(\omega)$, is unknown at the present.)

Recall from chapter III that the general transfer function, $H(j\omega)$, is characterized using an approximation function $K(\omega)$ as follows:

$$H(j\omega) \cdot H(-j\omega) = \frac{1}{1 + \varepsilon^2 K^2(\omega)} = \left| H(j\omega) \right|^2$$

where

$\varepsilon$ is a constant.

Since we defined $|T(j\omega)|$ as

$$|T(j\omega)| = |H(j\omega)|^{-1}$$

we can rewrite the characteristic equation as

$$T(j\omega) \cdot T(-j\omega) = 1 + \varepsilon^2 R_N^2(\omega) \qquad (A.4)$$

where

> $\varepsilon$ is a constant (which will be determined shortly),
>
> $R_N(\omega)$ is the elliptic approximation function substituted for $K(\omega)$

and

> the magnitude, $|T(j\omega)|$, has equiripple characteristics as shown in figure A.2

We would like to determine the function $R_N(\omega)$ which results as a consequence of the prescribed attenuation parameters as shown in figure A.2. In particular, we are interested in the values of $R_N(\omega)$ at the known frequencies of equal maxima and equal minima ($\sqrt{k}$ and $1/\sqrt{k}$ are two such frequencies respectively).

The pass band edge frequency, $\sqrt{k}$ , is an equimaximum point having a prescribed attenuation, Adb. This is related to $R_N(\omega)$ by

$$Adb = 10 \text{ Log} \left(1 + \varepsilon^2 R_N^2 (\sqrt{k})\right)$$

The required value of $R_N^2 (\sqrt{k})$ is then found as

$$R_N^2 (\omega = \sqrt{k}) = \frac{\left(10^{Adb/10} - 1\right)}{\varepsilon^2}$$

At this point it becomes convenient to define the value of the constant, $\varepsilon$. The value $\varepsilon$ is chosen such that $R_N^2(\omega)$ is equal to unity at the cutoff frequency ($\omega = \sqrt{k}$). This results in the half power point for $|T(j\omega)|^2$. Then for

$$R_N^2\ (\omega = \sqrt{k}) = 1$$

the required value for $\varepsilon$ is

$$\varepsilon = \sqrt{10^{\ A/10} - 1} \tag{A.5}$$

Notice that $R_N^2(\omega)$ equals one at all the equimaximum frequencies in the pass band as shown in figure A.3. ($\omega = \omega_x,\ \omega_y,\ \sqrt{k}$)

The attenuation in the stop band at the equiminima frequencies is Bdb. The equiminimum value for $R_N^2(\omega)$ can be found at $\omega = 1/\sqrt{k}$ as follows:

$$\text{Bdb} = 10\ \text{Log}\left(1 + \varepsilon^2\ R_N^2\ (1/\sqrt{k})\right)$$

from which we find that,

$$R_N^2\ (1/\sqrt{k}) = \left(\frac{10^{\ \text{Bdb}/10} - 1}{10^{\ \text{Adb}/10} - 1}\right) \tag{A.6}$$

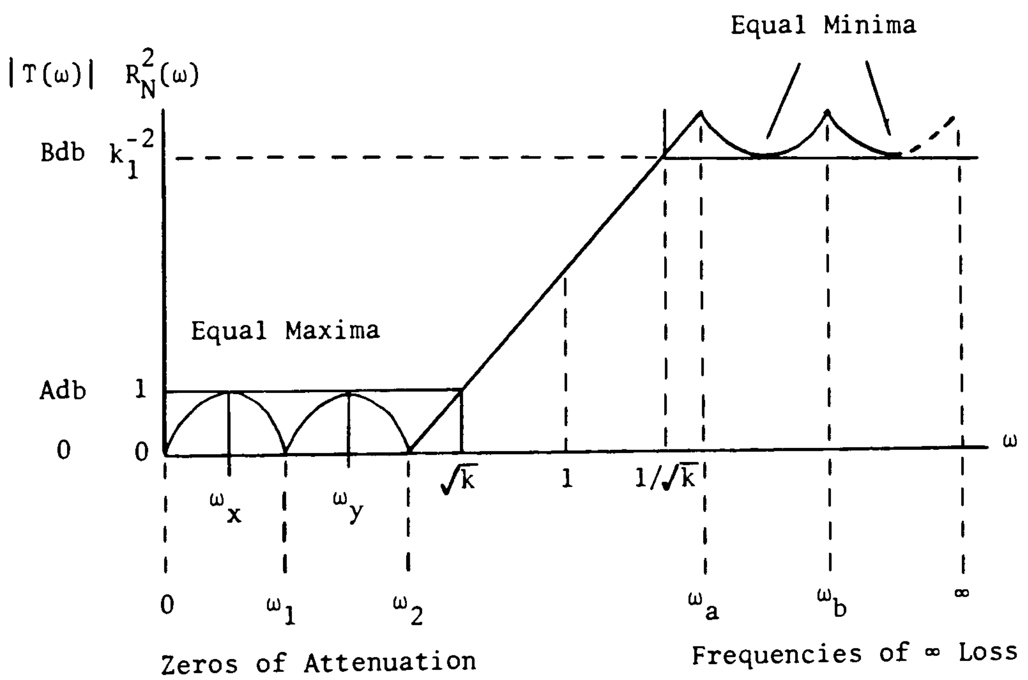
For convenience we define the value $k_1$ (known as the selectivity parameter) as

$$k_1 = \left(\frac{10^{\ A/10} - 1}{10^{\ B/10} - 1}\right)^{1/2}$$

Equation A.6 then becomes

$$R_N^2\,(1/\sqrt{k}) = 1/k_1^2$$

In addition to these values we observe that $R_N^2(\omega)$ equals zero at the pass band frequencies ($\omega = 0, \omega_1, \omega_2$) which are defined as the zeros of attenuation (see figure A.3). In the stop band we have $R_N^2(\omega) = \infty$ at frequencies defined as the frequencies of infinite loss ($\omega = \omega_a, \omega_b, \infty$).




Prescribed Characteristics of the Normalized Elliptic Function, $R_N^2(\omega)$

Figure A.3

The prescribed values of $R_N^2(\omega)$ are therefore summarized as follows:

$$R_N^2(\omega) = 0 \qquad \text{for } \omega = 0,\ \omega_1,\ \omega_2$$

$$= \infty \qquad \text{for } \omega = \omega_a,\ \omega_b,\ \infty$$

$$= 1 \qquad \text{at the equimaxima frequencies: } \omega = \omega_x,\ \omega_y,\ \sqrt{k}$$

$$= 1/k_1^2 \qquad \text{at the equiminima frequencies}$$

A rational function which satisfies these prescribed parameters shown in figure A.3 is as follows:

$$R_N(\omega) = \frac{1}{\sqrt{k_1}} \cdot \frac{\omega\left(\omega_1^2 - \omega^2\right)\left(\omega_2^2 - \omega^2\right) \cdots\cdots \left(\omega_m^2 - \omega^2\right)}{\left(\omega_a^2 - \omega^2\right)\left(\omega_b^2 - \omega^2\right) \cdots\cdots \left(\omega_N^2 - \omega^2\right)}$$

As a result of the equiripple characteristics in the pass band and stop band, it can be shown (see A. J. Grossman, reference 11) that the normalized poles of $R_N(\omega)$ ($\omega_a$, $\omega_b$ .... $\omega_N$) are the inverse values of the zeros of $R_N(\omega)$. That is, $\omega_a = 1/\omega_2$, $\omega_b = 1/\omega_1$, etc. As a consequence, the elliptic rational function has the following reciprocal property:

$$R_N(\omega) = \frac{1}{k_1} \cdot \frac{1}{R_N(1/\omega)} \tag{A.7}$$

Applying this reciprocal property to the normalized frequency values, the elliptic rational function can be rewritten as follows:

for N odd, (A.8)

$$R_N(\omega) = \frac{1}{\sqrt{k_1}} \cdot \frac{\omega \cdot \left(\omega_1^2 - \omega^2\right) \cdot \left(\omega_2^2 - \omega^2\right) \cdots\cdots \left(\omega_N^2 - \omega^2\right)}{\left(1/\omega_1^2 - \omega^2\right)\left(1/\omega_2^2 - \omega^2\right) \cdots\cdots \left(1/\omega_N^2 - \omega^2\right)}$$

and for N even,

$$R_N(\omega) = \frac{1}{\sqrt{k_1}} \cdot \frac{\left(\omega_1^2 - \omega^2\right)\cdot\left(\omega_2^2 - \omega^2\right) \ldots\ldots \left(\omega_N^2 - \omega^2\right)}{\left(1/\omega_1^2 - \omega^2\right)\left(1/\omega_2^2 - \omega^2\right) \ldots\ldots \left(1/\omega_N^2 - \omega^2\right)}$$

where

$\omega_1, \omega_2 \ldots\ldots \omega_N$ are the normalized zeros of attenuation,

and

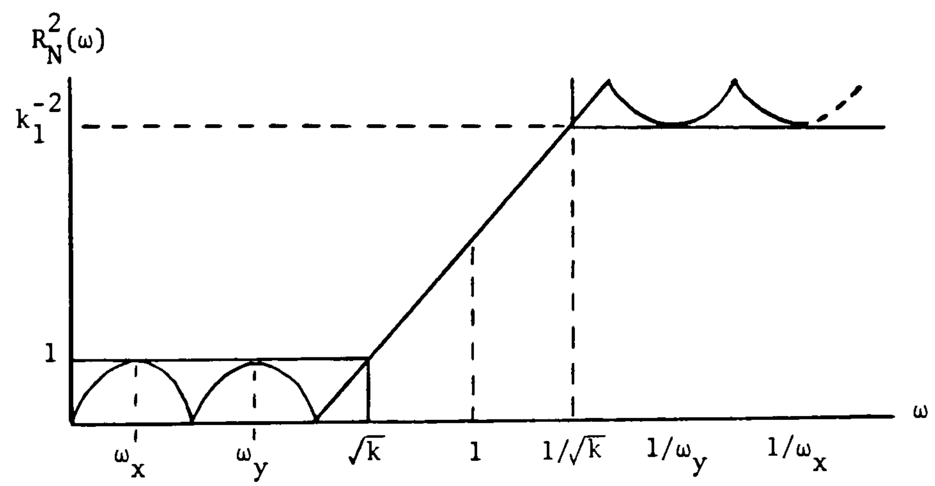
$1/\omega_1, 1/\omega_2 \ldots\ldots 1/\omega_N$ are the normalized frequencies of infinite loss.

This form of $R_N(\omega)$ allows a clear interpretation of the frequencies where $R_N(\omega) = 0$ (zeros of $R_N(\omega)$) and where $R_N(\omega) = \infty$ (poles of $R_N(\omega)$). We have not however defined what these frequencies shall be in order to achieve the prescribed equiripple magnitude characteristics in the pass and stop bands. We shall see in the next section that these frequencies are found using the elliptic sine function.

## A.4 The Poles and Zeros of $R_N(\omega)$

In the previous section we determined the equimaxima and equiminima values of the normalized rational function, $R_N(\omega)$. These values were a consequence of the prescribed insertion loss parameters and the desired equiripple characteristics. We would now like to determine the frequency values of $R_N(\omega)$ which yield the equiripple response, that is, the zeros of attenuation (zeros of $R_N(\omega)$) and the frequencies of infinite loss (poles of $R_N(\omega)$). The analysis will initially follow the approach presented by A. J. Grossman [11] for the case of N odd. (N = 5 will typically be shown.) This will then be expanded for the general Nth order elliptic filter.

We will start by developing a general equation based upon the known values of $R_N^2(\omega)$ $\left(1 \text{ and } k_1^{-2}\right)$ and the corresponding frequencies of equal maxima and minima ($\omega_x$, $\omega_y$, $\sqrt{k}$ and $1/\omega_x$, $1/\omega_y$, $1/\sqrt{k}$ respectively as shown in figure A.4). Since we know the frequency values $\sqrt{k}$ and $1/\sqrt{k}$




Characteristics of the Normalized Elliptic Rational Function

Figure A.4

(from our normalization process), it is desirable to combine certain related functions of $R_N^2(\omega)$ to eliminate the unknown equal maximum/minimum frequencies $\omega_x$, $\omega_y$, $1/\omega_x$, $1/\omega_y$ as shown in figure A.4. We therefore consider three functions of $R_N^2(\omega)$:

$$1 - R_N^2(\omega),$$

$$\left(1 - k_1^2\, R_N^2(\omega)\right)$$

$$dR_N(\omega)/d\omega$$

The function, $\left(1 - R_N^2(\omega)\right)$, has a single known root at $\omega = \sqrt{k}$ and double roots at the unknown equimaxima frequencies in the pass band $(\omega = \omega_x, \omega_y)$. The function, $\left(1 - k_1^2\, R_N^2(\omega)\right)$, has a single known root at $\omega = 1/\sqrt{k}$ and double roots at the unknown equiminima frequencies in the stop band $\omega = 1/\omega_x, 1/\omega_y$ . The third function, $dR_N(\omega)/d\omega$, has single roots at the unknown equimaxima and equiminima frequencies ($\omega = \omega_x, \omega_y$ and $\omega = 1/\omega_x, 1/\omega_y$ respectively). By dividing the derivative, $dR_N(\omega)/d\omega$, by the square root of the product of the other two functions, the unknown poles and zeros of $dR_N(\omega)/d\omega$ cancel those of $\left(1 - R_N^2(\omega)\right)$ and $\left(1 - k_1^2\, R_N^2(\omega)\right)$. The result relates the three functions of $R_N^2(\omega)$ to the known frequency values $\sqrt{k}$ and $1/\sqrt{k}$ as follows:

$$\frac{dR_N(\omega)/d\omega}{\sqrt{\left(1 - R_N^2(\omega)\right)\left(1 - k_1^2\, R_N^2(\omega)\right)}} = \frac{\pm\, M_0}{\sqrt{\left(\omega^2 - (\sqrt{k})^2\right)\left(\omega^2 - (1/\sqrt{k})^2\right)}} \tag{A.9}$$

where

$M_0$ is a constant,

$R_N(\omega)$ is the elliptic rational function (eq. A.8),

and

$\sqrt{k}$, $1/\sqrt{k}$ are the normalized pass band and stop band edge frequencies (eq. A.3)

Now we will solve equation A.9 for the relationship between $R_N(\omega)$ and $\omega$.

Equation A.9 can be expressed in terms of a definite integral. Using the variable transformations,

$$y = \omega/\sqrt{k}$$

and

$$x = R_N(\omega)$$

equation A.9 becomes

$$\int_0^{R_N(\omega)} \frac{dx}{\sqrt{(1 - x^2)(1 - k_1^2\, x^2)}} = \pm M_0 \cdot \sqrt{k} \int_0^{\omega/\sqrt{k}} \frac{dy}{\sqrt{(1 - y^2)(1 - k^2\, y^2)}} + C \qquad \text{(A.10)}$$

where

x and y are the variables of integration

and

C is a constant

We then perform a transformation on the above equation to obtain a familiar integral form. We let

$$x = \sin \gamma_1$$

$$R_N(\omega) = \sin \phi_1$$

$$y = \sin \gamma$$

and

$$\omega/\sqrt{k} = \sin \phi$$

Equation A.10 is then transformed into the following:

$$\int_0^{\emptyset_1} \frac{d\gamma_1}{\sqrt{1 - k_1^2 \sin^2 \gamma_1}} = \pm M_0 \cdot \sqrt{k} \int_0^{\emptyset} \frac{d\gamma}{\sqrt{1 \quad k^2 \sin^2 \gamma}} \tag{A.11}$$

Equation A.11 expresses the relationship between $R_N(\omega)$ and $\omega$ by way of elliptic integrals (see appendix E for a method of solving for the elliptic integral value). We define an elliptic integral as

$$u = F\,(\emptyset,\, k) = \int_0^{\emptyset} \frac{d\gamma}{\sqrt{1 - k^2 \sin^2 \gamma}}$$

where

k is the modulus determined as a normalized frequency parameter

and

$\emptyset$ is referred to as the amplitude of the integral.

$R_N$ and $\omega$ are related by the elliptic integral value, u, in the following manner:

$$\omega(u) = \sqrt{k}\ \mathrm{sn}\ (u,\, k) = \sin \emptyset \tag{A.12}$$

and

$$R_N(u) = \mathrm{sn}\ (\pm\, Mu + C,\, k_1) = \sin \emptyset_1 \tag{A.13}$$

where

sn represents the elliptic sine trigonometric function (see appendix E for a method of evaluating the elliptic sine).

These two equations indicate that if we can provide the elliptic integral values, u, for the given modulus values k and $k_1$, we can then solve for the frequencies (eq. A.12) which yield the equiripple characteristics. In addition, the value of the rational function, $R_N(\omega)$ can be found by equation A.13 for the corresponding values of u associated with $\omega$. (Appendix B shows an example of how these values of u are derived and applied.)

Equations A.12 and A.13 provide the primary tools by which we analyze the elliptic response. They provide the basis for deriving the poles and zeros of the normalized low pass elliptic transfer function, H(s).

### A.5 Interpreting the Rational Function by way of the Elliptic Integral and Elliptic Sine

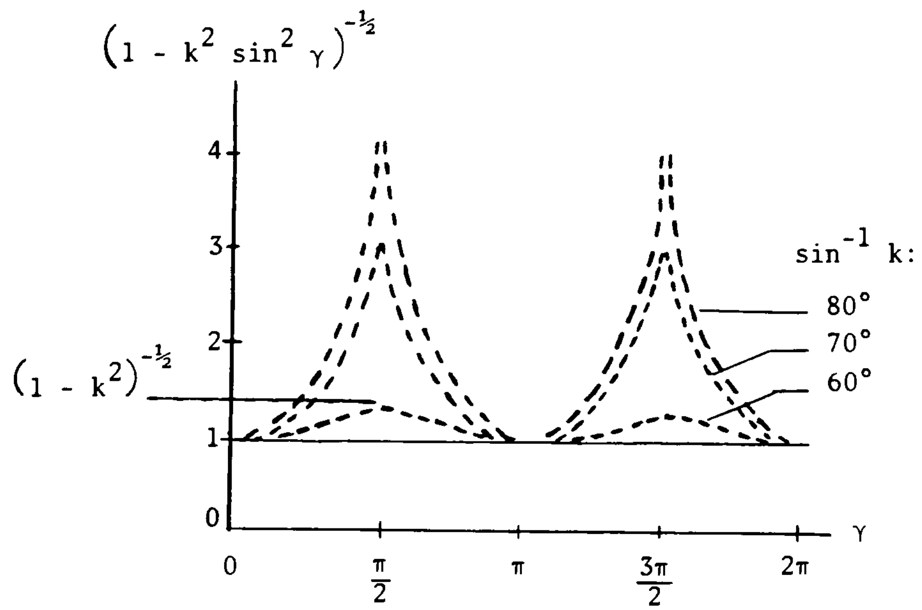
In this section we would like to gain some insight into the concept of the elliptic integral and elliptic sine functions. Then we will examine how we determine the integral values, $u = F(\emptyset, k)$, which are used in the solutions of the equiripple frequencies. As an end result, we will have two sets of equations, in terms of the elliptic sine, which describe the rational function and the proper normalized equimaxima/equiminima frequencies.

#### A.5.1 The Elliptic Integral and Elliptic Sine Functions

Suppose we are given the normalized modulus value, k (see equation A.2). Using k we can determine various elliptic integral values, u, using the expression:

$$u = F\,(\emptyset,\, k) = \int_0^{\emptyset} \frac{d\gamma}{\sqrt{1 - k^2 \sin^2 \gamma}} \tag{A.14}$$

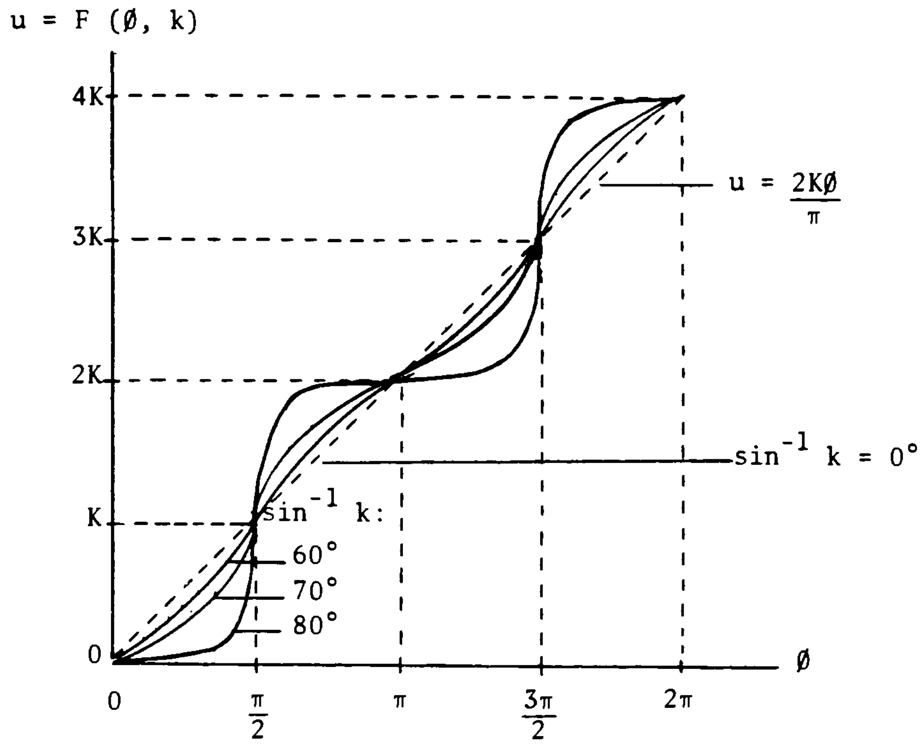
Figure A.5 illustrates the integrand of u. We notice that this function, $\left(1 - k^2 \sin^2 \gamma\right)^{-\frac{1}{2}}$, varies in amplitude between 1 and $\left(1 - k^2\right)^{-\frac{1}{2}}$. As




Integrand Function of the Elliptic Integral

Figure A.5

the modulus value k increases, the peak amplitude grows large (which explains the difficulty in evaluating the integral value given $\sin^{-1} k \simeq \pi/2$). The elliptic integral equation (eq. A.14) directs us to integrate the function $\left(1 - k^2 \sin^2 \gamma\right)^{-\frac{1}{2}}$ over the range 0 to $\emptyset$. (We will see in appendix B how the required values of $\emptyset$ are defined.) If we allow $\emptyset$ to vary, we find that the integral value, u, will vary as shown in figure A.6.




Elliptic Integral Values (Real) as a
Function of the Amplitude, $\phi$

Figure A.6

The value of the elliptic integral, u, at $\phi = \pi/2$ is, by convention, designated as K, that is

$$u = F(\pi/2, k) = K \tag{A.15}$$

Because of the periodicity of the integrand of u, we find that for every $\Delta\phi = \pi/2$, the increment of u is $\Delta u = K$. That is,

$$u = F\left(\phi = N\frac{\pi}{2}, k\right) = NK \tag{A.16}$$

The value, u = K, is therefore designated as the COMPLETE elliptic integral value. We can also say that for any known elliptic integral value u where

$$u = F(\phi, k) \qquad (\phi < \pi/2)$$

the following relationship holds:

$$u' = F(n\pi + \phi, k) = 2nK + u \qquad \begin{pmatrix}\text{n is an} \\ \text{integer}\end{pmatrix}$$

Let us assume that we have determined a value of $\phi$ which corresponds to a given integral value, u. The elliptic sine is simply the sin ($\phi$). We represent this as follows:

$$\mathrm{sn}(u, k) = \sin(\phi) \tag{A.17}$$

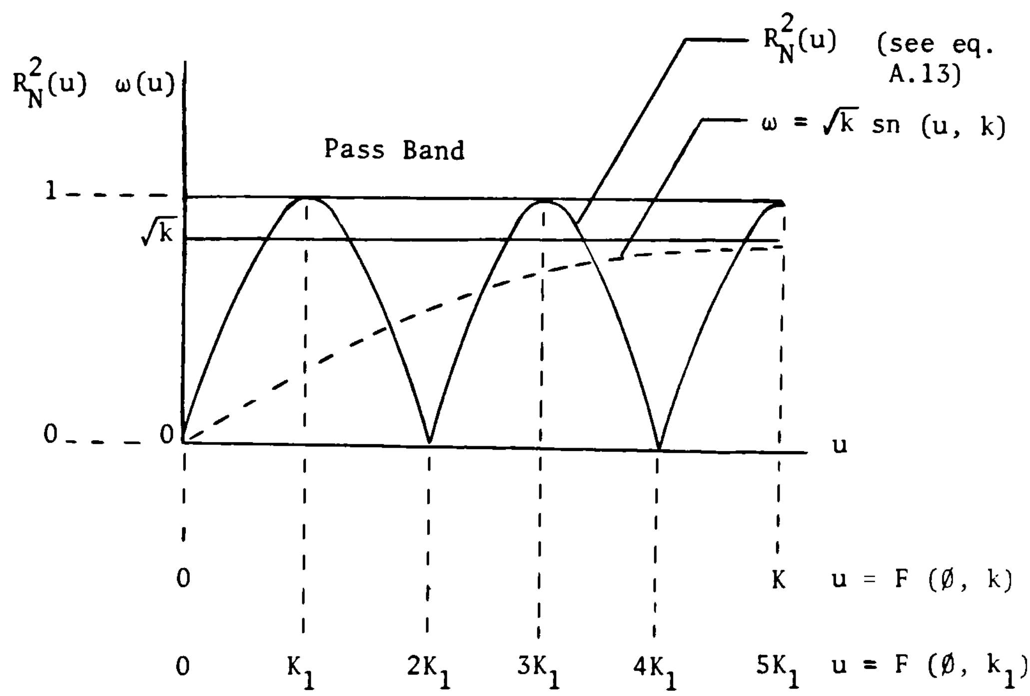
For the complete elliptic integral value just discussed, we had $u = K$ and $\phi = \pi/2$ for the given modulus, k. The elliptic sine value for this case is

$$\begin{aligned} \mathrm{sn}(u, k) &= \mathrm{sn}(K, k) \\ &= \sin(\pi/2) = 1 \end{aligned}$$

Other examples of finding the elliptic sine values are given in appendix E.

### A.5.2 Finding the Poles and Zeros of $R_N(\omega)$ in Terms of the Elliptic Sine

In section A.4 we found that $R_N(\omega)$ and $\omega$ could be described using elliptic sine functions governed by the modulus values k and $k_1$. In the equation describing $\omega$ (eq. A.12), we find that when $\mathrm{sn}(u, k) = 1$, the value of $\omega = \sqrt{k}$ (the pass band edge). This occurs when u has the amplitude, $\phi = \pi/2$. The value of u at the pass band edge is therefore the complete elliptic integral, K. This is shown in figure A.7 for u = k




The Pass Band of the Elliptic Filter
Transformed by the Elliptic Integral
(N = Odd = 5)

Figure A.7

and $\omega = \sqrt{k}$. The value of $R_N^2(u)$ at $\omega = \sqrt{k}$ is one which also corresponds to the integral value u = K. We also note that $R_N(u)$ by equation A.13 is governed by the modulus $k_1$. The corresponding complete elliptic integral value is designated by $K_1$ as shown in figure A.7. We know from our examination of the insertion loss characteristics (section A.3) that when $R_N^2(\omega)$ is equal to unity we are at an equimaximum frequency. In addition, since $R_N(u)$ is described by the elliptic sine function, the angle $\emptyset$ must be some multiple of $\pi/2$ for $R_N$ to equal unity. Therefore, the value of u corresponding to the equimaxima frequencies must be some multiple of the complete integral, $K_1$. As shown in figure A.7 for the case of N = 5, the complete elliptic integral, $K_1$, is traversed 5 times in the pass band. The zeros of attenuation occur at u = 0, $2K_1$ and $5K_1$. We have therefore defined the integral values, u, as needed by equation A.12 to determine the zeros of $R_N(\omega)$. The zeros of $R_N(\omega)$ are found as follows for N odd:

$$\omega = \sqrt{k}\ \mathrm{sn}\ (u, k) \tag{A.18}$$

where

$$u = \frac{2\ell K}{N} \qquad \text{for } \ell = 0 \text{ to } \frac{N-1}{2}$$

and

N is the order of the normalized low pass elliptic filter

We can perform a similar analysis for the case when N is even. Figure A.8 illustrates $\omega(u)$ and $R_N^2(u)$ for the case where N = 6. As shown, we have three zeros of attenuation corresponding to the integral values $K_1$, $3K_1$ and $5K_1$. The zeros of attenuation (zeros of $R_N(\omega)$) are found as follows for N even:

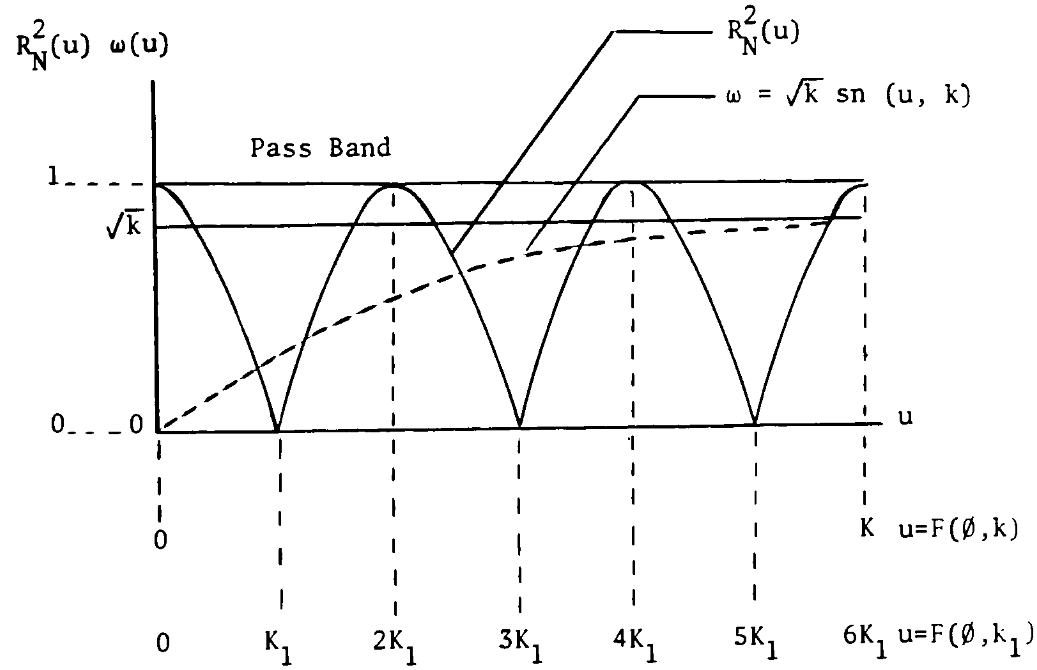
$$\omega = \sqrt{k}\ \mathrm{sn}\ (u, k) \tag{A.19}$$

where

$$u = \frac{(2\ell - 1)}{N} \cdot K \qquad \text{for } \ell = 1 \text{ to } N/2$$

and

N is the order of the normalized low pass elliptic filter.




The Pass Band of the Elliptic Filter Transformed by the Elliptic Integral (N = Even = 6)

Figure A.8

The value of the function $R_N(u)$ in the pass band is found as follows for N odd or even:

$$R_N(u) = \text{sn}\left(\frac{NK_1}{K}\cdot u, k_1\right) \tag{A.20}$$

where the constants M and $C_1$ of equation A.13 have now been set equal to $NK_1/K$ and 0 respectively.

At this point we have two methods of evaluating the function $R_N$. We can use equation A.18 or A.19 to determine the poles and zeros of $R_N(\omega)$ as required by equation A.8. The second approach is to use equation A.20 to evaluate $R_N(u)$. As we will find out shortly, equation A.20 requires complex integral values in the stop band which makes the task of evaluating $R_N(u)$ a little more difficult than evaluating $R_N(\omega)$.

## A.6 Solution of the Poles and Zeros for the Normalized Elliptic Transfer Function, H(s)

In this section we will examine a method of finding the s-plane poles and zeros for the general Nth order normalized elliptic low pass filter. We will start by finding the normalized zeros.

### A.6.1 Normalized Elliptic Transfer Function Zeros

In chapter III of the main text we found that the zeros of H(s) are derived from the poles of the rational function, $R_N(\omega)$. Given the normalized rational function,

$$R_N(\omega) = \frac{1}{\sqrt{k_1}} \cdot \frac{\left(\omega_1^2 - \omega^2\right)\cdot\left(\omega_2^2 - \omega^2\right) \cdots\cdots \left(\omega_N^2 - \omega^2\right)}{\left(1/\omega_1^2 - \omega^2\right)\left(1/\omega_2^2 - \omega^2\right) \cdots\cdots \left(1/\omega_N^2 - \omega^2\right)}$$

the poles of $R_N(\omega)$ are $\omega = \pm\, 1/\omega_1, \pm\, 1/\omega_2 \cdots \pm\, 1/\omega_N$. The s-plane zeros are found by letting $\omega = s/j$. The zeros of H(s) are therefore

$$\pm\, j\, 1/\omega_1, \pm\, j\, 1/\omega_2 \cdots \pm\, j\, 1/\omega_N$$

or in general

$$z_i = \pm\, j\, 1/\omega_i \qquad \text{for } i = 1 \text{ to } N \tag{A.21}$$

where $\omega_1$, $\omega_2$ .... $\omega_N$ have previously been defined by equation A.19 as the zeros of $R_N(\omega)$ or zeros of attenuation.

### A.6.2 Normalized Elliptic Transfer Function Poles

Having solved for the normalized zeros of H(s), we now pursue the normalized s-plane poles. This is the final step required in our effort to define the elliptic transfer function, H(s).

The determination of the complex s-plane poles for the elliptic transfer function is based upon the understanding of the complex properties of the elliptic trigonometric functions. We will not get too involved with the application of these functions for the solution of the poles (see reference [11]). Rather, a simplified argument will be presented. First we will discuss why complex integral values are required; then the solution of the poles will be presented.

We start by considering the basic equation for the frequencies defined as the zeros of attenuation. Recall from equation A.19 that

$$\omega = \sqrt{k}\ \mathrm{sn}\ (u, k)$$

We would like to use this equation to transform the normalized frequency axis. In doing so we first realize that the maximum value of sn (u, k) is one. Therefore, in the stop band where $\omega$ is greater than one, we require a complex integral value which we will designate as jv.

That is,

$$\omega = \sqrt{k}\ \mathrm{sn}\ (jv, k) \qquad \text{for } \omega > \sqrt{k} \qquad \text{(A.22)}$$

Next we attempt to define the complex integral value, v.

We set the insertion loss function equal to zero and solve for $R_N(\omega)$.

That is,

$$1 + \varepsilon^2 R_N^2(\omega) = 0$$

from which we find $R_N(\omega)$ as

$$R_N(\omega) = \pm j/\varepsilon$$

Substituting equation A.20 for $R_N(\omega)$ we have

$$\mathrm{sn}\left(\frac{NK_1}{K}\cdot u, k_1\right) = \pm j/\varepsilon \qquad (A.23)$$

We are interested in solving for an integral value which will yield the imaginary value $1/\varepsilon$ in equation A.23. If we allow the following relationship,

$$\frac{NK_1}{K}\cdot u = jv \qquad (A.24)$$

then equation A.23 becomes

$$\mathrm{sn}\ (jv,\ k_1) = \pm j/\varepsilon \qquad (A.25)$$

As shown in reference [11], one elliptic property is

$$\mathrm{sn}\ (jv,\ k_1) = j\,\mathrm{tn}\ (v,\ k_1')$$

where

tn is the elliptic tangent

and

$$k_1' = \left(1 - k_1^2\right)^{\frac{1}{2}}$$

Using this property, $\pm j/\varepsilon$ can be related to the elliptic tangent as follows:

$$\pm j/\varepsilon = \pm j\,\mathrm{tn}\ (v,\ k_1')$$

or

$$v = F\ (\phi,\ k_1') \tag{A.26}$$

where

$$\phi = \tan^{-1}\ (1/\varepsilon)$$

and

$$\varepsilon = \sqrt{10^{Adb/10}-1}$$

Equation A.26 states that v is an elliptic integral value governed by the modulus $k_1'$ and whose amplitude (upper limit of integration) is defined by $\phi$; both $1/\varepsilon$ and $k_1'$ are derived directly from the given normalized parameters, $k_1$ and Adb. If we now relate v to u by equation A.24, we have the complex value,

$$u = jv \cdot \frac{K}{K_1 N} = jv_0 \tag{A.27}$$

It can be shown [11] that this complex value is the imaginary part of the integral value u + jv as required in equation A.22. Equation A.22 then becomes

$$\omega = \sqrt{k}\ \mathrm{sn}\ (u + jv_0,\ k) \tag{A.28}$$

where the values of u are found by equations A.18 and A.19; $v_0$ is found by equation A.27.

Equation A.28 defines the complex s-plane poles when we allow $s = \omega/j$.

That is

$$s_\ell = (-1)^\ell \cdot j\sqrt{k}\ \mathrm{sn}\ (jv_0 \pm u_\ell,\ k) \tag{A.29}$$

where for N odd

$$u = \frac{2\ell K}{N} \qquad \text{for } \ell = 0 \text{ to } (N - 1)$$

and for N even

$$u = \frac{(2\ell - 1)}{N} K \qquad \text{for } \ell = 1 \text{ to } N$$

Equation A.29, by way of other complex elliptic properties [11], is translated into the final form which we desire for the normalized elliptic poles.

That is,

$$\sigma_\ell \pm j\ \beta_\ell = \frac{(a_0) \cdot P(\omega_\ell) \pm j\ \omega_\ell \cdot C}{\left(1 + a_0^2\ \omega_\ell^2\right)} \tag{A.30}$$

where

$$P(\omega_\ell) = \sqrt{\left(1 - k \cdot \omega_\ell^2\right)\left(1 - \omega_\ell^2/k\right)}$$

$$C = \text{constant} = \sqrt{\left(1 + k \cdot a_0^2\right)\left(1 + a_0^2/k\right)}$$

$$a_0 = j\sqrt{k}\ \mathrm{sn}\ (jv_0,\ k) = -\sqrt{k}\ \mathrm{tn}\ (v_0,\ k)$$

$$k = (\omega_a/\omega_b) \text{ (see eq. A.2)}$$

and

$$\omega_\ell = \sqrt{k}\ \mathrm{sn}\ (u_\ell, k)$$

For N odd,

$$u_\ell = \frac{2\ell K}{N} \qquad\qquad \ell = 0 \text{ to } (N - 1)$$

and for N even,

$$u_\ell = \frac{(2\ell - 1)}{N} \cdot K \qquad \ell = 1 \text{ to } N$$

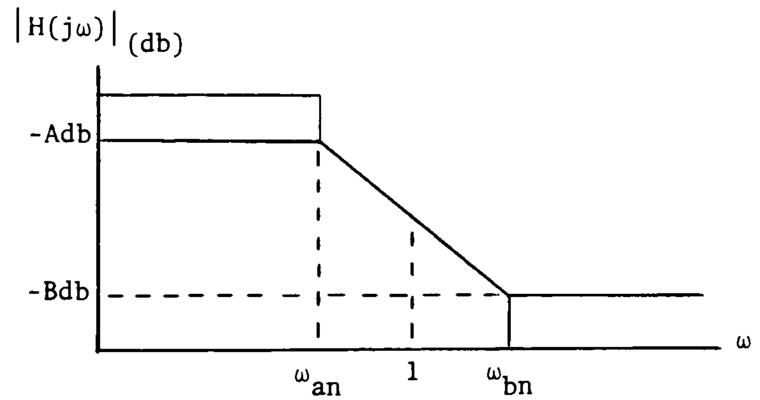
We see that the analysis of the elliptic poles is quite involved and somewhat difficult to determine. However, a summary of this analysis is provided as a step-by-step procedure in appendix B. A comprehensive example is given there also.

## APPENDIX B

## PROCEDURE FOR DETERMINING THE NORMALIZED ELLIPTIC POLES AND ZEROS

### B.1 Introduction

This appendix provides a step-by-step method of finding the s-plane poles and zeros of an Nth order normalized low pass filter. The method is based upon the concepts and formulae presented in appendix A. This procedure assumes that the reader can determine an elliptic integral value, F (Ø, k) (see appendices C and D). The assumption is made that the user has derived the four normalized low pass filter parameters defined in figure B.1 (refer to appendix A).




$\omega_{an}$: the normalized pass band edge frequency
$\omega_{bn}$: the normalized stop band edge frequency
-Bdb: the minimum stop band attenuation
-Adb: the maximum pass band attenuation

Normalized Low Pass Filter Parameters
(parameter basis for an elliptic filter design)

Figure B.1

Throughout the procedure, work tables are provided as a means for organizing and recording the calculated values. At the end of the procedure a summary work sheet is provided. A complete example of using the procedure is provided in section B.3.

## B.2 Procedure for Finding the Poles and Zeros

1. Using the normalized low pass parameters $\omega_{an}$, $\omega_{bn}$, -Adb, -Bdb, determine the elliptic parameters k, k', $k_1$, $k_1'$ where

$$k = \omega_{an}/\omega_{bn} \qquad k_1 = \sqrt{\frac{10^{Adb/10}-1}{10^{Bdb/10}-1}}$$

$$k' = \sqrt{1 - k^2} \qquad k_1' = \sqrt{1 - k_1^2}$$

elliptic parameters

| k | k' | $k_1$ | $k_1'$ |
|---|---|---|---|
| | | | |

2. Using the elliptic parameters from step 1, find the complete elliptic integral values K, K', $K_1$, $K_1'$ (see appendices C and D) where

$$K = F\ (90°,\ k), \quad K_1 = F\ (90°,\ k_1)$$

$$K' = F\ (90°,\ k'), \quad K_1' = F\ (90°,\ k_1')$$

The work table below can be used to record the data for step 2.

| | | | |
|---|---|---|---|
| $\theta_k = \sin^{-1} k$ | $\theta_{k'} = \sin^{-1} k'$ | $\theta_{k_1} = \sin^{-1} k_1$ | $\theta_{k_1'} = \sin^{-1} k_1'$ |
| $\theta_k =$ | $\theta_{k'} =$ | $\theta_{k_1} =$ | $\theta_{k_1'} =$ |
| $K = F\,(90°, \theta_k)$ | $K' = F\,(90°, \theta_{k'})$ | $K_1 = F\,(90°, \theta_{k_1})$ | $K_1' = F\,(90°, \theta_{k_1'})$ |
| $K =$ | $K' =$ | $K_1 =$ | $K_1' =$ |

3. Determine the filter order N using the results of step 2 as follows:

$$N \geq \left(\frac{K \cdot K_1'}{K' \cdot K_1}\right)$$

Round up the N value.

4. Having found the filter order N, we use the elliptic sine function (see appendix E) to determine the zero loss frequencies, $\omega_\ell$ where

$$\omega_\ell = \sqrt{k}\ \mathrm{sn}\,(u_\ell, k)$$

For N even, the values of $u_\ell$ are found from

$$u_\ell = \frac{(2\ell - 1)}{N} \cdot K \qquad \text{for } \ell = 1 \text{ to } \frac{N}{2}$$

and for N odd we have

$$u_\ell = \frac{2\ell K}{N} \qquad \text{for } \ell = 0 \text{ to } \frac{N - 1}{2}$$

To calculate the values of $\omega_\ell$, we perform the following steps for each value of $\ell$. We already have the values K, $\theta_k$ and k from steps 1 and 2.

a) Find $u_\ell$.

b) Find $\phi_\ell$ from the integral tables (appendix D) given the values $u_\ell$ and $\theta_k$.

c) Find $\omega_\ell = \sqrt{k} \sin \phi_\ell$

The work table below might assist in these calculations.

| $k =$ | $\sqrt{k} =$ | $\theta_k =$ | $K =$ | |
|---|---|---|---|---|
| $\ell$ | $u_\ell$ | $\phi_\ell$ | $\sin \phi_\ell$ | $\omega_\ell = \sqrt{k} \sin \phi_\ell$ |
| | | | | |

5. Next we determine the constant v. This constant is an elliptic integral value needed in the step 6 calculations for $a_0$. We find v from

$$v = F(\phi_\varepsilon, \theta_{k'}) \cdot \left(\frac{K}{NK_1}\right)$$

where

$$\varepsilon = \left(10^{Adb/10} - 1\right)^{\frac{1}{2}},$$

$$\phi_{\varepsilon} = \tan^{-1}(1/\varepsilon)$$

and

$$\theta_{k'} = \sin^{-1}(k')$$

6. Having found the integral value v, we next determine the constant $a_0$. The value, $a_0$, is used in the computation of the poles. We determine $a_0$ from:

$$a_0 = -\sqrt{k}\ \mathrm{tn}\ (v, k')$$

This is found as follows:

a) given the integral value v and the angle $\theta_{k'} = \sin^{-1}(k')$ we examine the integral tables to find $\phi_v$.

b) the value $a_0$ is then

$$a_0 = -\sqrt{k}\tan(\phi_v)$$

(the value $a_0$ is also the first pole value (real) for the case where N is odd).

7. The remaining normalized pole values $(\sigma_\ell \pm j\ \beta_\ell)$ are found using each value of $\omega_\ell$ as follows:

$$\sigma_\ell \pm j\ \beta_\ell = \frac{a_0 \cdot P(\omega_\ell) \pm j\ \omega_\ell \cdot C}{\left(1 + a_0^2\ \omega_\ell^2\right)}$$

where

the values, $\omega_\ell$, are the zeros of attenuation from step 4,

C is a constant $= \sqrt{\left(1 + k \cdot a_0^2\right)\left(1 + a_0^2/k\right)}$

and

$$P(\omega_\ell) = \sqrt{\left(1 - k \cdot \omega_\ell^2\right)\left(1 - \omega_\ell^2/k\right)}$$

The work table below will assist in these calculations.

| $k =$ | | $a_0 =$ | | $C =$ | |
|---|---|---|---|---|---|
| $\ell$ | $\omega_\ell$ | $\left(1 + a_0^2\ \omega_\ell^2\right)$ | $P(\omega_\ell)$ | $\sigma_\ell$ | $\pm\ \beta_\ell$ |
| | | | | | |

8. The s-plane zeros, $z_\ell$, are found using the zero loss frequencies, $\omega_\ell$, from step 7 as follows:

$$\pm z_\ell = \pm j/\omega_\ell$$

| $\ell$ | $\pm z_\ell = \pm j/\omega_\ell$ |
|---|---|
| | |

This completes the procedure for finding the normalized poles and zeros for the elliptic low pass filter. Table B.1 provides a work sheet for summarizing the calculated data.

Step 1.

ELLIPTIC PARAMETERS

| $k$ | $k'$ | $k_1$ | $k_1'$ |
|---|---|---|---|
| | | | |

Step 2.

ELLIPTIC INTEGRAL VALUES

| | | | |
|---|---|---|---|
| $\theta_k = \sin^{-1} k$ | $\theta_{k'} = \sin^{-1} k'$ | $\theta_{k_1} = \sin^{-1} k_1$ | $\theta_{k_1'} \quad \sin^{-1} k_1'$ |
| $\theta_k =$ | $\theta_{k'} =$ | $\theta_{k_1} =$ | $\theta_{k_1'}$ |
| $K = F\ (90°, \theta_k)$ | $K' = F\ (90°, \theta_{k'})$ | $K_1 = F\ (90°, k_1)$ | $K_1' \quad F\ (90°, k_1')$ |
| $K =$ | $K' =$ | $K_1 -$ | $K_1'$ |

Step 3.

$$N \geq \frac{K \cdot K_1'}{K' \cdot K_1} -$$

Step 4.

ZEROS OF ATTENUATION

| $k =$ | $\sqrt{k} =$ | $\theta_k -$ | $K$ | |
|---|---|---|---|---|
| $\ell$ | $u_\ell$ | $\phi_\ell$ | $\sin \phi_\ell$ | $\omega_\ell = \sqrt{k} \ \sin \phi_\ell$ |
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| | | | | |

Elliptic Design Summary Work Sheet

Table B.1

Step 5.

| $\varepsilon = \sqrt{10^{Adb/10}-1}$ | $\phi_\varepsilon = \tan^{-1}\left(\frac{1}{\varepsilon}\right)$ | $\theta_{k'}$ | $\frac{K}{NK_1}$ |
|---|---|---|---|
| | | | |
| $v = F(\theta_{k'}, \phi_\varepsilon) \cdot \frac{K}{NK_1} =$ | | | |

Step 6.

| $\theta_{k'}$ | $\phi_v$ for $\left\{F(\phi_v, \theta_{k'}) \quad v\right\}$ | $\tan \phi_v$ |
|---|---|---|
| | | |
| $a_0 \quad - \sqrt{k} \tan \phi_v$ | | |

Step 7.

POLES

| k = | | $a_0$ | | C = | |
|---|---|---|---|---|---|
| $\ell$ | $\omega_\ell$ | $\left(1 + a_0^2\, \omega_\ell^2\right)$ | $P(\omega_\ell)$ | $\sigma_\ell$ | $\pm \beta_\ell$ |
| | | | | | |
| | | | | | |
| | | | | | |
| | | | | | |
| | | | | | |
| | | | | | |

Table B.1 (Cont'd.)

Equations for step 7:

$$C = \left[\left(1 + k \cdot a_0^2\right)\left(1 + a_0^2/k\right)\right]^{1/2} \qquad P(\omega_\ell) \quad \left[\left(1 + k \cdot \omega_\ell^2\right)\left(1 - \omega_\ell^2/k\right)\right]^{1/2}$$

$$\sigma_\ell = \frac{a_0 \cdot P(\omega_\ell)}{\left(1 + a_0^2\ \omega_\ell\right)} \quad ; \qquad \pm j\ \beta_\ell \quad \frac{\pm j\ \omega_\ell \cdot C}{\left(1 + a_0^2\ \omega_\ell\right)}$$

Step 8.

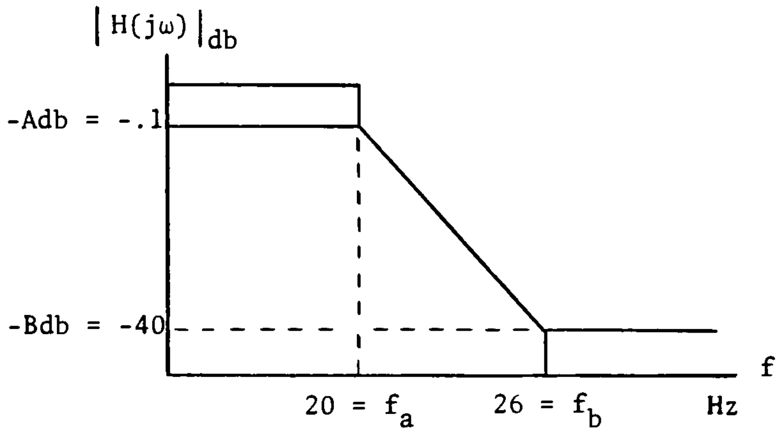
ZEROS

| $\ell$ | $\pm z \qquad = \qquad \pm j/\omega_\ell$ |
|---|---|
| | |
| | |
| | |
| | |

Table B.1 (Cont'd.)

## B.3 Example of Finding the Poles and Zeros

In this example we determine the normalized poles and zeros for a low pass elliptic filter having the insertion loss characteristics shown below in figure B.2.

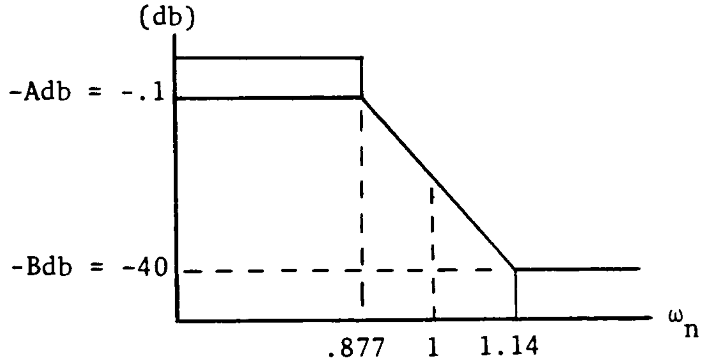



Elliptic Low Pass Filter Requirements

Figure B.2

We will follow through the steps given in the procedure just discussed and record the calculated values on the summary work sheet. In the context of our example, when the term TABLE is used, it refers to the Elliptic Integral Tables of the First Kind (see appendix D).

Our initial step is to determine the normalized low pass filter. For the low pass filter we choose to normalize the frequency axis by dividing all values by the center frequency, $f_c = \sqrt{20.26} = 22.8$ Hz. The figure below illustrates the normalized low pass filter. We will now perform each step in the procedure.




Normalized Low Pass Elliptic Filter (example)

Figure B.3

1. For $\omega_{an} = .877$, $\omega_{bn} = 1.14$, $-Adb = -.1$ and $-Bdb = -40$, we find:

$$k = .769,\ k' = .639,\ k_1 = 1.526 \times 10^{-3}$$

$$k_1' = .99999884$$

(Precision values are required especially for $k_1$ and $k_1'$ in order to accurately find $K_1$, $K_1'$ and N in steps 2 and 3.)

2. Taking the inverse sine of the k parameters we have

$$\theta_k = 50.26°,\ \theta_{k'} = 39.72°,\ \theta_{k_1} = .087°,\ \theta_{k_1'} = 89.913°$$

By interpolating the table values we have the following complete elliptic integral values:

$$K = F\ (90°,\ 50.26°) = 1.9411$$

$$K' = F\ (90°,\ 39.72°) = 1.7837$$

$$K_1 = F\ (90°,\ .087°) = 1.5708$$

and

$$K_1' = F\ (90°,\ 89.913°) = 7.8462$$

3. Using the integral values of step 2, the filter order N is calculated as 5.436.

   N is then rounded up to the value 6.

4. Since N = 6 (even) we will have N/2 or 3 zeros of attenuation. The integral values $u_\ell$ for the zeros of attenuation are found from

$$u_\ell = \frac{(2\ell - 1)}{N} \cdot K \qquad \text{for } \ell = 1 \text{ to } N/2$$

   Therefore,

$$u_1 = K/6 = 1.9411/6 = .324$$

$$u_2 = (3/6) \cdot K = .971$$

$$u_3 = (5/6) \cdot K = 1.6176$$

   Given $\theta_k = 50.26°$ we find $\theta_\ell$ from

$$u_\ell = F\ (\emptyset_\ell,\ 50.26°)$$

   Using the tables we find the three values of $\emptyset_\ell$ corresponding to $u_\ell$. That is,

$$\emptyset_1 = 18.35°,\ \emptyset_2 = 51.36°,\ \emptyset_3 = 78.04°$$

   Then $\omega_\ell$ is found from

$$\omega_\ell = \sqrt{k}\ \sin\ \emptyset_\ell$$

We then have

$$\omega_1 = .276,\ \omega_2 = .685,\ \omega_3 = .858$$

By inverting these values of $\omega_\ell$ we find the frequencies of infinite loss corresponding to the infinite loss points in the stop band. These values are also the zeros of H(s). Thus

$$\pm j\ z_1 = \pm j\ 1/\omega_1 = \pm j\ 3.623$$

$$\pm j\ z_2 = \pm j\ 1/\omega_2 = \pm j\ 1.46$$

$$\pm j\ z_3 = \pm j\ 1/\omega_3 = \pm j\ 1.166$$

5. The respective values of $\varepsilon$, $\phi_\varepsilon$, $\theta_{k'}$, and $F(\phi_\varepsilon, \theta_{k'})$ are:

$$\varepsilon = \left(10^{0.1/10}-1\right)^{\frac{1}{2}} = .153$$

$$\phi_\varepsilon = \tan^{-1}\left(\frac{1}{\varepsilon}\right) = 81.3°$$

$$\theta_{k'} = 89.913°,$$

and

$$F\ (81.3°,\ 89.913°) = 2.566$$

The value v is then found from:

$$v = F\ (81.3°,\ 89.913°) \cdot \frac{K}{NK_1}$$

$$= 2.566\ (.206)$$

or

$$v = .5284$$

6. To find the constant $a_0$, we first find $\phi_v$ given

$$v = .5284 = F\left(\phi_v, \theta_{k'}\right) = F\left(\phi_v, 39.72°\right).$$

By interpolating the table values we find

$$\phi_v = 29.73°$$

and

$$a_0 = -\sqrt{k}\tan\phi_v$$

$$= -\sqrt{.769}\tan(29.73°)$$

$$= -.5$$

7. The final step is to calculate the normalized pole values given the values $a_0$, $\omega_\ell$ and k from our work sheet. Since this is rather lengthy, only one example will be shown. Given

$$\omega_1 = .276,\ a_0 = -.5 \text{ and } k = .769$$

we have,

$$C = \left[\left(1 + k\cdot a_0^2\right)\left(1 + a_0^2/k\right)\right]^{\frac{1}{2}} = 1.257$$

$$P(\omega_\ell) = \left[\left(1 - k\cdot\omega_\ell^2\right)\left(1 - \omega_\ell^2/k\right)\right]^{\frac{1}{2}} = .92$$

and

$$\left(1 + a_0^2\ \omega_\ell\right) = 1.019$$

Then

$$\sigma_1 \pm j\ \beta_1 = \frac{a_0\ \ P(\omega_\ell) \pm j\ \omega_\ell \cdot C}{\left(1 + a_0^2\ \omega_\ell\right)}$$

$$= \frac{(-.5)(.92) \pm j\ (.276)(1.257)}{1.019}$$

or

$$\sigma_1 \pm j\ \beta_1 = (-.451 \pm j\ .34)$$

This completes the example of finding the normalized poles and zeros. The results are summarized on the work sheet (Table B.2) which follows. In addition, the primary results (zeros of attenuation, frequencies of infinite loss and the pole/zero values) are illustrated in figure B.4.

Step 1. ELLIPTIC PARAMETERS

| $k$ | $k'$ | $k_1$ | $k_1'$ |
|---|---|---|---|
| .769 | .639 | $1.53 \times 10^{-3}$ | .99999884 |

Step 2. ELLIPTIC INTEGRAL VALUES

| | | | |
|---|---|---|---|
| $\theta_k = \sin^{-1} k$ | $\theta_{k'} = \sin^{-1} k'$ | $\theta_{k_1} = \sin^{-1} k_1$ | $\theta_{k_1'} = \sin^{-1} k_1'$ |
| $\theta_k = 50.26°$ | $\theta_{k'} = 39.72°$ | $\theta_{k_1} = .087°$ | $\theta_{k_1'} = 89.913°$ |
| $K = F(90°, \theta_k)$ | $K' = F(90°, \theta_{k'})$ | $K_1 = F(90°, k_1)$ | $K_1' \quad F(90°, k_1')$ |
| $K = 1.9411$ | $K' = 1.7837$ | $K_1 = 1.5708$ | $K_1' = 7.8462$ |

Step 3.

$$N \geq \frac{K \cdot K_1'}{K' \cdot K_1} = 5.4$$

CHOOSE
N = 6

Step 4. ZEROS OF ATTENUATION

| $k = .769$ | $\sqrt{k} = .877$ | $\theta_k = 50.26$ | $K = 1.9411$ | |
|---|---|---|---|---|
| $\ell$ | $u_\ell$ | $\phi_\ell$ | $\sin \phi_\ell$ | $\omega_\ell = \sqrt{k} \sin \phi_\ell$ |
| 1 | .324 | 18.35° | .3146 | $\omega_1 = .276$ |
| 2 | .971 | 51.36° | .7811 | $\omega_2 = .685$ |
| 3 | 1.6176 | 78.04° | .9783 | $\omega_3 = .858$ |
| | | | | |
| | | | | |
| | | | | |

Elliptic Design Summary Work Sheet (Example)

Table B.2

Step 5.

| $\varepsilon = \sqrt{10^{Adb/10}-1}$ | $\phi_\varepsilon = \tan^{-1}\left(\frac{1}{\varepsilon}\right)$ | $\theta_{k'}$ | $\frac{K}{NK_1}$ |
|---|---|---|---|
| .153 | 81.3° | 89.913° | .206 |
| $v = F\left(\theta_{k'}, \phi_\varepsilon\right) \cdot \frac{K}{NK_1} = .528$ | | | |

Step 6.

| $\theta_{k'}$ | $\phi_v$ for $\left\{F\left(\phi_v, \theta_{k'}\right) = v\right\}$ | $\tan \phi_v$ |
|---|---|---|
| 39.72° | 29.73° | .57108 |
| $a_0 = -\sqrt{k}\tan \phi_v = -.5$ | | |

Step 7.

POLES

| $k = .769$ | | | $a_0 = -0.5$ | | $C = 1.257$ |
|---|---|---|---|---|---|
| $\ell$ | $\omega_\ell$ | $\left(1 + a_0^2\,\omega_\ell^2\right)$ | $P\left(\omega_\ell\right)$ | $\sigma_\ell$ | $\pm\,\beta_\ell$ |
| 1 | .276 | 1.019 | .92 | -.45 | ± j .34 |
| 2 | .685 | 1.1173 | .5 | -.22 | ± j .771 |
| 3 | .858 | 1.184 | .136 | -.058 | ± j .911 |
| | | | | | |
| | | | | | |
| | | | | | |

Table B.2 (Cont'd.)

Equations for step 7:

$$C = \left[\left(1 + k\cdot a_0^2\right)\left(1 + a_0^2/k\right)\right]^{\frac{1}{2}} \qquad P(\omega_\ell) = \left[\left(1 - k\cdot\omega_\ell^2\right)\left(1 - \omega_\ell^2/k\right)\right]^{\frac{1}{2}}$$

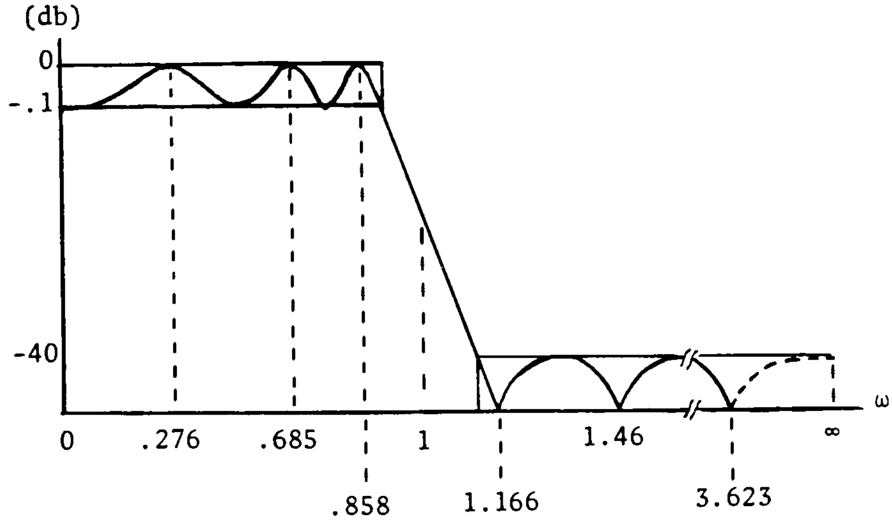
$$\sigma_\ell = \frac{a_0\ P(\omega_\ell)}{\left(1 + a_0^2\ \omega_\ell\right)} \quad ; \quad \pm j\ \beta_\ell = \frac{\pm j\ \omega_\ell \cdot C}{\left(1 + a_0^2\ \omega_\ell\right)}$$

Step 8.

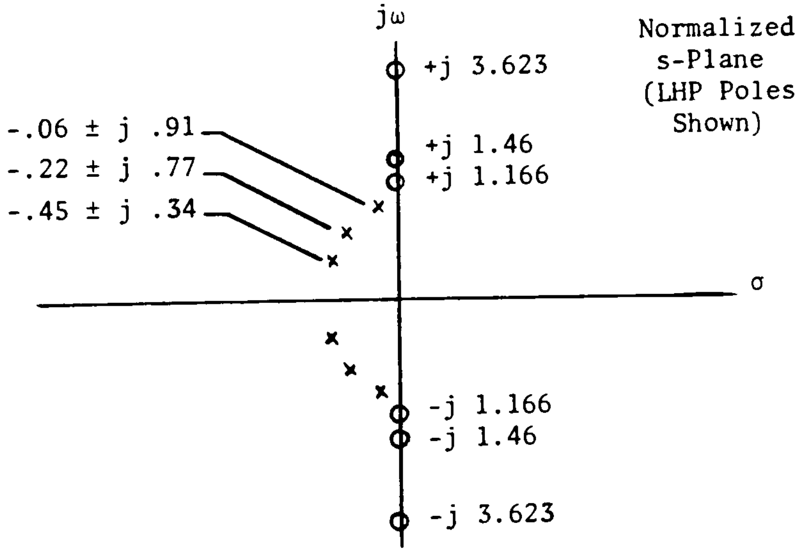
ZEROS

| $\ell$ | $\pm z$ | $= \pm j/\omega_\ell$ |
|---|---|---|
| 1 | $\pm j\ z_1$ | $\pm j\ 3.623$ |
| 2 | $\pm j\ z_2$ | $\pm j\ 1.46$ |
| 3 | $\pm j\ z_3$ | $\pm j\ 1.166$ |
| | | |

Table B.2 (Cont'd.)




Normalized Low Pass
Elliptic Filter (N = 6)
(a)

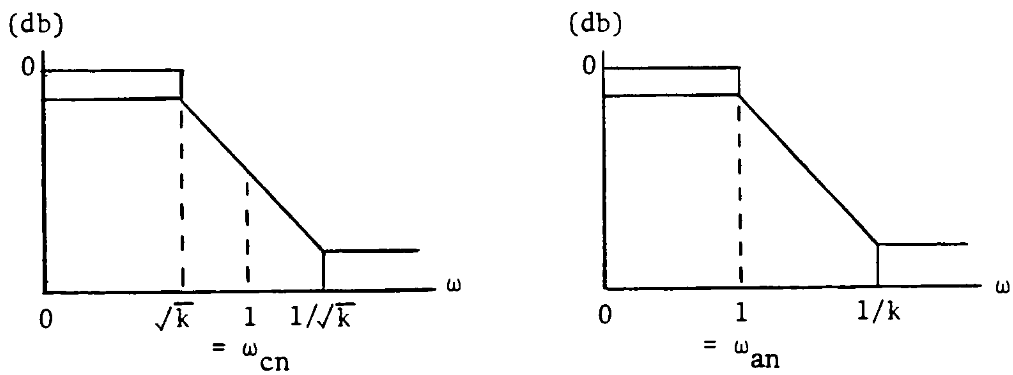



Normalized Poles/Zeros
for the Low Pass Elliptic
Filter (N = 6)
(b)

Figure B.4

## B.4 Translation of the Normalized Values to the Pass Band Edge

In the procedure for finding the poles and zeros for the elliptic function filter we used the center frequency of the transition region to obtain the normalized low pass filter. The resulting s-plane poles and zeros are therefore normalized with respect to the center $\left(\omega_{cn} = 1\right)$ of the transition region (figure B.5a). It is often desirable to obtain the normalized s-plane poles and zeros with respect to the pass band edge $\left(\omega_{an} = 1\right)$ as shown in figure B.5b.




Center Frequency Normalization (a)

Cutoff Frequency Normalization (b)

Normalization with Respect to the Center Frequency vs. the Cutoff Frequency

Figure B.5

The normalized poles and zeros for the filter of figure B.5b can be found from

$$\bar{s}_i = s_i / \sqrt{k}$$

where $s_i$ represents a pole or zero value normalized to the geometric center frequency. Consider a pole and a zero value for the example presented in section B.3:

$$s_1 = -.45 + j\ .34 \quad \text{(pole)}$$

$$j\ z_1 = j\ 3.623$$

The pass band edge is

$$\omega_{an} = \sqrt{k} = .877$$

The corresponding pole $\bar{s}_1$ and zero $j\ \bar{z}_1$ normalized with respect to the pass band edge $\left(\omega_{an} = 1\right)$ are therefore:

$$\bar{s}_1 = \frac{1}{.877} \cdot (-.45 + j\ .34)$$

$$= -.513 + j\ .388$$

and

$$j\ \bar{z}_1 = j\ 3.623/.877 = 4.131$$

Regardless of the normalized filter used (figure B.5a or B.5b), the resulting denormalized poles and zeros will be the same.

## APPENDIX C

## METHOD FOR FINDING THE VALUE OF THE ELLIPTIC INTEGRAL

### C.1 Introduction

In this appendix we will examine one method of finding the value of the elliptic integral. The method is based upon Landen's Ascending Approximation [15].

In our efforts to solve for the normalized elliptic poles and zeros (appendices A and B), we are required to find the values of various elliptic integrals. One elliptic integral is described by

$$F(\emptyset, k) = \int_0^{\emptyset} \frac{d\gamma}{\sqrt{1 - k^2 \sin^2 \gamma}} \tag{C.1}$$

where the given parameters are the modulus value, k, and the upper limit of integration $\emptyset$. When $\emptyset = \pi/2$, we have the complete elliptic integral.

When finding the elliptic poles and zeros, four modulus values are defined ($k$, $k'$, $k_1$, $k_1'$) using the normalized frequency values and the attenuation parameters (see B.1). The associated complete elliptic integral values were designated as $K$, $K'$, $K_1$, $K_1'$.

The method presented in this appendix resolves the issue of an unobtainable closed form solution to the elliptic integral. This method is directly applicable for determining the values $K$, $K'$, $K_1$, $K_1'$ and the elliptic sine values. An example will be given.

### C.2 Landen's Ascending Transformation

Landen's ascending transformation is based upon the following principle. Suppose we have two angles $\theta_n$, $\theta_{n+1}$ (related to the $\sin^{-1}$ of modulus values) which have the relationships:

$$\left(1 + \sin \theta_n\right)\left(1 + \cos \theta_{n+1}\right) = 2 \tag{C.2}$$

and

$$\theta_{n+1} > \theta_n$$

where

n is an iterative value

We then let $\phi_n$, $\phi_{n+1}$ be two iterative angles which define the upper limit of integration in the elliptic integral. We assume that

$$\sin\left(2\, \phi_{n+1} - \phi_n\right) = \sin \theta_n \cdot \sin \phi_n$$

and (C.3)

$$\phi_{n+1} < \phi_n$$

As the iterative value, n, increases, the corresponding value of $\theta_n$ decreases while the modular angle, $\phi_n$, increases. We start with the given value, $\theta$, as $\theta_{n=0}$ and determine a $\theta_{n+1} = \theta_1$ from equation C.2. Given $\phi$ as $\phi_0$, and the values $\theta_0$, $\theta_1$, we can use equation C.3 to find $\phi_{n+1}$. We now start over using the new values of $\theta$ and $\phi$. This process continues until we find

$$\left(\phi_{n+1} - \phi_n\right) \simeq 0$$

or the number of decimal places required.

The value F ($\phi$, $\theta$) is then found from

$$F(\emptyset, \theta) = \csc\theta \left[ \prod_{i=0}^{n} \sin\theta_i \right]^{\frac{1}{2}} \cdot \ln\tan\left(\frac{\pi}{4} + \frac{1}{2}\Phi\right) \tag{C.4}$$

where

$$\Phi = \lim_{n \to \infty} \emptyset_n$$

To summarize, the following procedure is presented based upon this transformation just discussed. An example is given in section C.4.

## C.3 Procedure for Finding the Elliptic Integral Value, F (Ø, θ)

The following procedure assumes that the elliptic parameters θ and Ø are given and that we wish to find the value of F (Ø, θ). (Recall for example that $\theta_k = \sin^{-1} k$ and that k is related to the normalized frequency axis. See B.2.)

1. We start with

$$n = 0,\ \theta_0 = \theta \text{ and } \emptyset_0 = \emptyset$$

   where θ and Ø are the given parameters and n is an iterative value.

2. We find $\theta_{n+1}$ by using the relationship (from equation C.2) that

$$1 + \cos\theta_{n+1} = \frac{2}{1 + \sin\theta_n}$$

   or

$$\theta_{n+1} = \cos^{-1}\left[\frac{2}{1 + \sin\theta_n} - 1\right]$$

3. We next solve for $\phi_{n+1}$ from equation C.3. Rearranging equation C.3 we have

$$\phi_{n+1} = \frac{\sin^{-1}\left(\sin \theta_n \cdot \sin \phi_n\right) + \phi_n}{2}$$

4. We then determine the difference

$$\Delta\phi = \left|\phi_n - \phi_{n+1}\right|$$

If $\Delta\phi$ is less than the desired decimal accuracy we continue. Otherwise the value n is incremented and we return to step 2.

5. Having found the desired accuracy of $\phi_{n+1}$, the value of the elliptic integral is found from equation C.4. That is,

$$F(\phi, \theta) = F(\phi_0, \theta_0) =$$

$$\csc \theta_0 \left[\prod_{i=0}^{n} \sin \theta_i\right]^{\frac{1}{2}} \cdot \ln \tan\left(\frac{\pi}{4} + \frac{1}{2}\phi_{n+1}\right)$$

## C.4 Example of Finding the Elliptic Integral Value

Assume that we have found the elliptic parameter k = 1/2 from the normalized low pass filter. We wish to find the complete elliptic integral value K where

$$K = F(\phi, \theta) = \int_0^{\pi/2} \frac{d\gamma}{\sqrt{1 - (.5)^2 \sin^2 \gamma}}$$

and

$$\phi = 90°, \ \theta = \sin^{-1}(k) = 30°$$

We desire an accuracy out to five decimal places for $\phi$. The following steps directly follow the procedure in section C.3.

1. From our given parameters we have

$$n = 0, \ \phi = 90°, \ \theta_0 = 30°$$

2. For $n = 0$ and $\theta_n = \theta_0 = 30°$, we have

$$\theta_{n+1} = \theta_1 = \cos^{-1}\left[\frac{2}{1 + \sin 30°} - 1\right]$$

$$= 70.53°$$

3. For $n = 0$, $\phi_n = 90°$ and $\theta_n = 30°$ we have

$$\phi_{n+1} = \phi_1 = \frac{\sin^{-1}(\sin 30° \cdot \sin 90°) + 90°}{2}$$

$$= 60°$$

4. For $\phi_1 = 60°$ and $\phi_0 = 90°$ we have

$$\Delta\phi = 90° - 60° = 30°$$

   This is much greater than the $\Delta\phi$ required ($\Delta\phi = 10^{-5}$ for five decimal place accuracy). Therefore, n is incremented from 0 to 1 and the process is repeated (return to step 2).

The following table summarizes the results of each step. The iterations continued until for $n = 4$ we have

$$\Delta\phi = \left|\phi_4 - \phi_3\right|$$

$$= 57.348426 - 57.348425$$

$$= 10^{-6}$$

| $n$ | $\theta_n$ | $\sin\theta_n$ | $\phi_n$ |
|---|---|---|---|
| 0 | 30° | .5 | 90.0° |
| 1 | 70.53° | .94280 | 60.0° |
| 2 | 88.313° | .999566 | 57.3678° |
| 3 | 89.9876° | .99999998 | 57.348426° |
| 4 | 90° | 1.0 | 57.348425° |

5. Continuing now with the procedure and using the values from the above table we have,

$$n = 4, \ \phi_{n+1} = 57.348425°$$

$$\csc\theta_0 = \left(\sin\theta_0\right)^{-1} = 2$$

and

$$\left[\prod_{i=0}^{n=4} \left(\sin\theta_i\right)\right]^{\frac{1}{2}} = .685094$$

Using equation C.4 we then have

$$F\ (90°,\ 30°) = 1.6857417$$

Tables [16] are available which provide the elliptic integral values given the parameters $\phi$ and $\theta$ (these tables are also shown in appendix D). For the example given the table value is

$$F\ (\phi,\ \theta) = F\ (90°,\ 30°) = 1.68575$$

As a final point we note that the elliptic sine value is equal to sine $\phi$ where $\phi$ is that angle producing the given integral value. Using the previous example, we are given

$$F\ (\emptyset,\ 30°) = 1.68575 = K$$

We know that the required value for $\emptyset$ is $\pi/2$ to obtain the complete elliptic integral (K).

Then the elliptic sine value is

$$\begin{aligned} sn\ (u,\ \theta) &= sn\ (1.68575,\ 30°) \\ &= \sin \emptyset = 1 \end{aligned}$$

Appendix E gives a fuller explanation of using the integral tables to find the elliptic sine.

# APPENDIX D

# THE COMPLETE ELLIPTIC INTEGRAL TABLES

## D.1 Using the Tables

The elliptic integral tables (section D.2) provide values for the elliptic integral function F (ϕ, k) given the parameters k and ϕ. The equation which relates F (ϕ, k) to ϕ, k is

$$u = F(\phi, k) = \int_0^{\phi} \frac{d\gamma}{\sqrt{1 - k^2 \sin^2 \gamma}} \tag{D.1}$$

where u is the specific value of F (k, ϕ), having the modulus k and whose amplitude is ϕ.

The table value $\theta_k$ is related to k by

$$\theta = \sin^{-1} k$$

The elliptic integral value, u, could then be represented as

$$u = F(\phi, k)$$

In the procedure for finding the normalized poles and zeros (appendix B), we will often be required to find the integral value, u, given ϕ, k. We will also be required to find the value, ϕ given u and k. As an example, consider finding the value u given

$$\phi = \pi/2 \text{ and } k = .766$$

The corresponding angles are

$$\phi = 90°$$

and

$$\theta_k = \sin^{-1} .766 = 50°$$

The elliptic integral value, u = F (90°, 50°) is found from the tables at the intersection of the 90° row for $\phi$ and the 50° column for $\theta$. The value is 1.9356. We then have

$$u = F(\phi, \theta) = F(90°, 50°) = 1.9356$$

Whenever $\phi = \pi/2$, we refer to u as the complete elliptic integral of the first kind.

As a second example, consider finding the value $\phi$ given that

$$u = .4446 \text{ and } k = .766$$

To determine $\phi$ we scan the $\theta = 50°$ column until we locate the value 0.4446. The corresponding value for $\phi$ is 25°. For the case where exact values are not shown in the tables, interpolation is sufficient (for $\theta < 90°$). For cases where $\theta$ is greater than 85°, a separate table is provided to supplement the first.

The elliptic trigonometric functions have the argument $\phi$ as derived from F ($\phi$, $\theta$) and are defined as follows:

$$\begin{aligned} \text{elliptic sine} &= \sin(\phi) \\ \text{elliptic cosine} &= \cos(\phi) \\ \text{elliptic tangent} &= \tan(\phi) \end{aligned}$$

Since the value $\phi$ is determined from a given integral value u and the modulus k, the elliptic trigonometric functions are often represented as follows:

$$\text{elliptic sine} = \text{sn}\ (u, k) = \sin \phi$$

$$\text{elliptic cos} = \text{cn}\ (u, k) = \cos \phi$$

$$\text{elliptic tan} = \text{tn}\ (u, k) = \tan \phi$$

where we use the letter u to represent the elliptic integral value $F(\phi, k)$.

## D.2 The Elliptic Integral Tables

The following tables (D.1 and D.2) provide the values of the elliptic integral function $F(\phi, k)$. The values k (modulus) and $\phi$ (amplitude) are the variables for which we desire the integral value. Note that the table value $\theta$ corresponds to

$$\theta = \sin^{-1}(k)$$

With respect to other terminology, the following abbreviation is often used

$$u = F(\phi, k)$$

or

$$u = F(\phi, \theta)$$

The tables were taken from the C.R.C. Standard Mathematical Tables [16]. For values of $\theta \geq 85°$, a separate table is provided.

## *ELLIPTIC INTEGRALS OF THE FIRST KIND: $F(k, \phi)$

$$F(k, \phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \qquad \theta = \sin^{-1} k$$

| $\phi$ \ $\theta$ | 5° | 10° | 15° | 20° | 25° | 30° | 35° | 40° | 45° |
|---|---|---|---|---|---|---|---|---|---|
| 1° | 0.0175 | 0.0175 | 0.0175 | 0.0175 | 0.0175 | 0.0175 | 0.0175 | 0.0175 | 0.0175 |
| 2° | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 |
| 3° | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 |
| 4° | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0698 |
| 5° | 0.0873 | 0.0873 | 0.0873 | 0.0873 | 0.0873 | 0.0873 | 0.0873 | 0.0873 | 0.0873 |
| 6° | 0.1047 | 0.1047 | 0.1047 | 0.1047 | 0.1048 | 0.1048 | 0.1048 | 0.1048 | 0.1048 |
| 7° | 0.1222 | 0.1222 | 0.1222 | 0.1222 | 0.1222 | 0.1222 | 0.1223 | 0.1223 | 0.1223 |
| 8° | 0.1396 | 0.1396 | 0.1397 | 0.1397 | 0.1397 | 0.1397 | 0.1398 | 0.1398 | 0.1399 |
| 9° | 0.1571 | 0.1571 | 0.1571 | 0.1572 | 0.1572 | 0.1572 | 0.1573 | 0.1573 | 0.1574 |
| 10° | 0.1745 | 0.1746 | 0.1746 | 0.1746 | 0.1747 | 0.1748 | 0.1748 | 0.1749 | 0.1750 |
| 11° | 0.1920 | 0.1920 | 0.1921 | 0.1921 | 0.1922 | 0.1923 | 0.1924 | 0.1925 | 0.1926 |
| 12° | 0.2095 | 0.2095 | 0.2095 | 0.2096 | 0.2097 | 0.2098 | 0.2099 | 0.2101 | 0.2102 |
| 13° | 0.2269 | 0.2270 | 0.2270 | 0.2271 | 0.2272 | 0.2274 | 0.2275 | 0.2277 | 0.2279 |
| 14° | 0.2444 | 0.2444 | 0.2445 | 0.2446 | 0.2448 | 0.2450 | 0.2451 | 0.2453 | 0.2456 |
| 15° | 0.2618 | 0.2619 | 0.2620 | 0.2621 | 0.2623 | 0.2625 | 0.2628 | 0.2630 | 0.2633 |
| 16° | 0.2793 | 0.2794 | 0.2795 | 0.2797 | 0.2799 | 0.2802 | 0.2804 | 0.2808 | 0.2811 |
| 17° | 0.2967 | 0.2968 | 0.2970 | 0.2972 | 0.2975 | 0.2978 | 0.2981 | 0.2985 | 0.2989 |
| 18° | 0.3142 | 0.3143 | 0.3145 | 0.3148 | 0.3151 | 0.3154 | 0.3159 | 0.3163 | 0.3167 |
| 19° | 0.3317 | 0.3318 | 0.3320 | 0.3324 | 0.3327 | 0.3331 | 0.3336 | 0.3341 | 0.3347 |
| 20° | 0.3491 | 0.3493 | 0.3495 | 0.3499 | 0.3503 | 0.3508 | 0.3514 | 0.3520 | 0.3526 |
| 21° | 0.3666 | 0.3668 | 0.3671 | 0.3675 | 0.3680 | 0.3685 | 0.3692 | 0.3699 | 0.3706 |
| 22° | 0.3840 | 0.3842 | 0.3846 | 0.3851 | 0.3856 | 0.3864 | 0.3871 | 0.3879 | 0.3887 |
| 23° | 0.4015 | 0.4017 | 0.4021 | 0.4027 | 0.4033 | 0.4041 | 0.4049 | 0.4059 | 0.4068 |
| 24° | 0.4190 | 0.4192 | 0.4197 | 0.4203 | 0.4210 | 0.4219 | 0.4229 | 0.4239 | 0.4250 |
| 25° | 0.4364 | 0.4367 | 0.4372 | 0.4379 | 0.4387 | 0.4397 | 0.4408 | 0.4420 | 0.4433 |
| 26° | 0.4539 | 0.4542 | 0.4548 | 0.4556 | 0.4565 | 0.4576 | 0.4588 | 0.4602 | 0.4616 |
| 27° | 0.4714 | 0.4717 | 0.4724 | 0.4732 | 0.4743 | 0.4755 | 0.4769 | 0.4784 | 0.4800 |
| 28° | 0.4888 | 0.4893 | 0.4899 | 0.4908 | 0.4921 | 0.4934 | 0.4950 | 0.4967 | 0.4985 |
| 29° | 0.5063 | 0.5068 | 0.5075 | 0.5086 | 0.5099 | 0.5114 | 0.5132 | 0.5150 | 0.5170 |
| 30° | 0.5238 | 0.5243 | 0.5251 | 0.5263 | 0.5277 | 0.5294 | 0.5313 | 0.5334 | 0.5356 |
| 31° | 0.5412 | 0.5418 | 0.5427 | 0.5440 | 0.5456 | 0.5475 | 0.5496 | 0.5519 | 0.5543 |
| 32° | 0.5587 | 0.5593 | 0.5603 | 0.5617 | 0.5635 | 0.5656 | 0.5679 | 0.5704 | 0.5731 |
| 33° | 0.5762 | 0.5769 | 0.5780 | 0.5795 | 0.5814 | 0.5837 | 0.5862 | 0.5890 | 0.5920 |
| 34° | 0.5937 | 0.5944 | 0.5956 | 0.5973 | 0.5994 | 0.6018 | 0.6046 | 0.6077 | 0.6109 |
| 35° | 0.6111 | 0.6119 | 0.6133 | 0.6151 | 0.6173 | 0.6200 | 0.6231 | 0.6264 | 0.6300 |
| 36° | 0.6286 | 0.6295 | 0.6309 | 0.6329 | 0.6353 | 0.6383 | 0.6416 | 0.6452 | 0.6491 |
| 37° | 0.6461 | 0.6470 | 0.6486 | 0.6507 | 0.6534 | 0.6565 | 0.6602 | 0.6641 | 0.6684 |
| 38° | 0.6636 | 0.6646 | 0.6662 | 0.6685 | 0.6714 | 0.6749 | 0.6788 | 0.6831 | 0.6877 |
| 39° | 0.6810 | 0.6821 | 0.6839 | 0.6864 | 0.6895 | 0.6932 | 0.6975 | 0.7021 | 0.7071 |
| 40° | 0.6985 | 0.6997 | 0.7016 | 0.7043 | 0.7076 | 0.7116 | 0.7162 | 0.7213 | 0.7267 |
| 41° | 0.7160 | 0.7173 | 0.7193 | 0.7222 | 0.7258 | 0.7301 | 0.7350 | 0.7405 | 0.7463 |
| 42° | 0.7335 | 0.7348 | 0.7370 | 0.7401 | 0.7440 | 0.7486 | 0.7539 | 0.7598 | 0.7661 |
| 43° | 0.7510 | 0.7524 | 0.7548 | 0.7580 | 0.7622 | 0.7671 | 0.7728 | 0.7791 | 0.7859 |
| 44° | 0.7685 | 0.7700 | 0.7725 | 0.7760 | 0.7804 | 0.7857 | 0.7918 | 0.7986 | 0.8059 |
| 45° | 0.7859 | 0.7876 | 0.7903 | 0.7940 | 0.7987 | 0.8044 | 0.8108 | 0.8181 | 0.8260 |

## ELLIPTIC INTEGRALS OF THE FIRST KIND: $F(k, \phi)$

$$F(k, \phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \qquad \theta = \sin^{-1} k$$

| $\phi$ \ $\theta$ | 50° | 55° | 60° | 65° | 70° | 75° | 80° | 85° | 90° |
|---|---|---|---|---|---|---|---|---|---|
| 1° | 0.0175 | 0.0175 | 0.0175 | 0.0175 | 0.0175 | 0.0175 | 0.0175 | 0.0175 | 0.0175 |
| 2° | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 |
| 3° | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 |
| 4° | 0.0698 | 0.0699 | 0.0699 | 0.0699 | 0.0699 | 0.0699 | 0.0699 | 0.0699 | 0.0699 |
| 5° | 0.0873 | 0.0873 | 0.0873 | 0.0874 | 0.0874 | 0.0874 | 0.0874 | 0.0874 | 0.0874 |
| 6° | 0.1048 | 0.1048 | 0.1049 | 0.1049 | 0.1049 | 0.1049 | 0.1049 | 0.1049 | 0.1049 |
| 7° | 0.1224 | 0.1224 | 0.1224 | 0.1224 | 0.1224 | 0.1225 | 0.1225 | 0.1225 | 0.1225 |
| 8° | 0.1399 | 0.1399 | 0.1400 | 0.1400 | 0.1400 | 0.1401 | 0.1401 | 0.1401 | 0.1401 |
| 9° | 0.1575 | 0.1576 | 0.1576 | 0.1576 | 0.1577 | 0.1577 | 0.1577 | 0.1577 | 0.1577 |
| 10° | 0.1751 | 0.1751 | 0.1752 | 0.1753 | 0.1753 | 0.1754 | 0.1754 | 0.1754 | 0.1754 |
| 11° | 0.1927 | 0.1928 | 0.1929 | 0.1930 | 0.1930 | 0.1931 | 0.1931 | 0.1932 | 0.1932 |
| 12° | 0.2103 | 0.2105 | 0.2106 | 0.2107 | 0.2108 | 0.2109 | 0.2109 | 0.2110 | 0.2110 |
| 13° | 0.2280 | 0.2282 | 0.2284 | 0.2285 | 0.2286 | 0.2287 | 0.2288 | 0.2288 | 0.2289 |
| 14° | 0.2458 | 0.2460 | 0.2462 | 0.2464 | 0.2465 | 0.2466 | 0.2467 | 0.2468 | 0.2468 |
| 15° | 0.2636 | 0.2638 | 0.2641 | 0.2643 | 0.2645 | 0.2646 | 0.2647 | 0.2648 | 0.2648 |
| 16° | 0.2814 | 0.2817 | 0.2820 | 0.2823 | 0.2825 | 0.2827 | 0.2828 | 0.2829 | 0.2830 |
| 17° | 0.2993 | 0.2997 | 0.3000 | 0.3003 | 0.3006 | 0.3008 | 0.3010 | 0.3011 | 0.3012 |
| 18° | 0.3172 | 0.3177 | 0.3181 | 0.3185 | 0.3188 | 0.3191 | 0.3193 | 0.3194 | 0.3195 |
| 19° | 0.3352 | 0.3357 | 0.3362 | 0.3367 | 0.3371 | 0.3374 | 0.3377 | 0.3378 | 0.3379 |
| 20° | 0.3533 | 0.3539 | 0.3545 | 0.3550 | 0.3555 | 0.3559 | 0.3561 | 0.3563 | 0.3564 |
| 21° | 0.3714 | 0.3721 | 0.3728 | 0.3734 | 0.3740 | 0.3744 | 0.3747 | 0.3749 | 0.3750 |
| 22° | 0.3896 | 0.3904 | 0.3912 | 0.3919 | 0.3926 | 0.3931 | 0.3935 | 0.3937 | 0.3938 |
| 23° | 0.4078 | 0.4088 | 0.4097 | 0.4106 | 0.4113 | 0.4119 | 0.4123 | 0.4126 | 0.4127 |
| 24° | 0.4261 | 0.4272 | 0.4283 | 0.4292 | 0.4301 | 0.4308 | 0.4313 | 0.4316 | 0.4317 |
| 25° | 0.4446 | 0.4458 | 0.4470 | 0.4481 | 0.4490 | 0.4498 | 0.4504 | 0.4508 | 0.4509 |
| 26° | 0.4630 | 0.4645 | 0.4658 | 0.4670 | 0.4681 | 0.4690 | 0.4697 | 0.4701 | 0.4702 |
| 27° | 0.4816 | 0.4832 | 0.4847 | 0.4861 | 0.4873 | 0.4884 | 0.4891 | 0.4896 | 0.4897 |
| 28° | 0.5003 | 0.5021 | 0.5038 | 0.5053 | 0.5067 | 0.5079 | 0.5087 | 0.5092 | 0.5094 |
| 29° | 0.5190 | 0.5210 | 0.5229 | 0.5247 | 0.5262 | 0.5275 | 0.5285 | 0.5291 | 0.5293 |
| 30° | 0.5379 | 0.5401 | 0.5422 | 0.5442 | 0.5459 | 0.5474 | 0.5484 | 0.5491 | 0.5493 |
| 31° | 0.5568 | 0.5593 | 0.5617 | 0.5639 | 0.5658 | 0.5674 | 0.5686 | 0.5693 | 0.5696 |
| 32° | 0.5760 | 0.5786 | 0.5812 | 0.5837 | 0.5858 | 0.5876 | 0.5889 | 0.5898 | 0.5900 |
| 33° | 0.5950 | 0.5980 | 0.6010 | 0.6037 | 0.6060 | 0.6080 | 0.6095 | 0.6104 | 0.6107 |
| 34° | 0.6143 | 0.6176 | 0.6208 | 0.6238 | 0.6265 | 0.6287 | 0.6303 | 0.6313 | 0.6317 |
| 35° | 0.6336 | 0.6373 | 0.6408 | 0.6441 | 0.6471 | 0.6495 | 0.6513 | 0.6526 | 0.6528 |
| 36° | 0.6531 | 0.6571 | 0.6610 | 0.6647 | 0.6679 | 0.6706 | 0.6726 | 0.6739 | 0.6743 |
| 37° | 0.6727 | 0.6771 | 0.6814 | 0.6854 | 0.6890 | 0.6919 | 0.6941 | 0.6955 | 0.6960 |
| 38° | 0.6925 | 0.6973 | 0.7019 | 0.7063 | 0.7102 | 0.7135 | 0.7159 | 0.7175 | 0.7180 |
| 39° | 0.7123 | 0.7176 | 0.7227 | 0.7275 | 0.7318 | 0.7353 | 0.7380 | 0.7397 | 0.7403 |
| 40° | 0.7323 | 0.7380 | 0.7436 | 0.7488 | 0.7535 | 0.7575 | 0.7604 | 0.7623 | 0.7629 |
| 41° | 0.7524 | 0.7586 | 0.7647 | 0.7704 | 0.7758 | 0.7799 | 0.7831 | 0.7852 | 0.7859 |
| 42° | 0.7727 | 0.7794 | 0.7858 | 0.7922 | 0.7979 | 0.8026 | 0.8062 | 0.8084 | 0.8092 |
| 43° | 0.7931 | 0.8004 | 0.8076 | 0.8143 | 0.8204 | 0.8258 | 0.8296 | 0.8320 | 0.8328 |
| 44° | 0.8139 | 0.8215 | 0.8293 | 0.8367 | 0.8433 | 0.8490 | 0.8533 | 0.8560 | 0.8569 |
| 45° | 0.8343 | 0.8428 | 0.8512 | 0.8592 | 0.8665 | 0.8727 | 0.8774 | 0.8804 | 0.8814 |

Elliptic Integral Tables (1° ≤ Ø ≤ 45°)
(taken from: C.R.C. Mathematical Tables, 1960 pp 256-269)

Table D.1

**ELLIPTIC INTEGRALS OF THE FIRST KIND: $F(k, \phi)$**

$$F(k, \phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \qquad \theta = \sin^{-1} k$$

| $\phi$ \ $\theta$ | 5° | 10° | 15° | 20° | 25° | 30° | 35° | 40° | 45° |
|---|---|---|---|---|---|---|---|---|---|
| 46° | 0.8034 | 0.8052 | 0.8080 | 0.8120 | 0.8170 | 0.8230 | 0.8300 | 0.8378 | 0.8462 |
| 47° | 0.8208 | 0.8227 | 0.8258 | 0.8300 | 0.8354 | 0.8418 | 0.8492 | 0.8575 | 0.8666 |
| 48° | 0.8381 | 0.8403 | 0.8436 | 0.8480 | 0.8537 | 0.8606 | 0.8685 | 0.8773 | 0.8870 |
| 49° | 0.8559 | 0.8579 | 0.8614 | 0.8661 | 0.8721 | 0.8794 | 0.8878 | 0.8972 | 0.9076 |
| 50° | 0.8734 | 0.8756 | 0.8792 | 0.8842 | 0.8905 | 0.8982 | 0.9072 | 0.9173 | 0.9283 |
| 51° | 0.8909 | 0.8932 | 0.8970 | 0.9023 | 0.9090 | 0.9172 | 0.9267 | 0.9374 | 0.9491 |
| 52° | 0.9084 | 0.9108 | 0.9148 | 0.9204 | 0.9275 | 0.9361 | 0.9462 | 0.9575 | 0.9700 |
| 53° | 0.9259 | 0.9284 | 0.9326 | 0.9385 | 0.9460 | 0.9551 | 0.9658 | 0.9778 | 0.9912 |
| 54° | 0.9434 | 0.9460 | 0.9505 | 0.9567 | 0.9646 | 0.9742 | 0.9855 | 0.9982 | 1.0124 |
| 55° | 0.9609 | 0.9637 | 0.9683 | 0.9748 | 0.9832 | 0.9933 | 1.0052 | 1.0187 | 1.0337 |
| 56° | 0.9784 | 0.9813 | 0.9862 | 0.9930 | 1.0018 | 1.0125 | 1.0250 | 1.0393 | 1.0552 |
| 57° | 0.9959 | 0.9989 | 1.0041 | 1.0112 | 1.0204 | 1.0317 | 1.0449 | 1.0600 | 1.0768 |
| 58° | 1.0134 | 1.0166 | 1.0219 | 1.0295 | 1.0391 | 1.0509 | 1.0648 | 1.0807 | 1.0985 |
| 59° | 1.0309 | 1.0342 | 1.0398 | 1.0477 | 1.0578 | 1.0702 | 1.0848 | 1.1016 | 1.1204 |
| 60° | 1.0484 | 1.0519 | 1.0577 | 1.0660 | 1.0766 | 1.0896 | 1.1049 | 1.1226 | 1.1424 |
| 61° | 1.0659 | 1.0695 | 1.0757 | 1.0843 | 1.0953 | 1.1089 | 1.1250 | 1.1436 | 1.1646 |
| 62° | 1.0834 | 1.0872 | 1.0936 | 1.1026 | 1.1141 | 1.1284 | 1.1452 | 1.1648 | 1.1868 |
| 63° | 1.1009 | 1.1049 | 1.1115 | 1.1209 | 1.1330 | 1.1478 | 1.1655 | 1.1860 | 1.2091 |
| 64° | 1.1184 | 1.1225 | 1.1295 | 1.1392 | 1.1518 | 1.1674 | 1.1859 | 1.2073 | 1.2318 |
| 65° | 1.1359 | 1.1402 | 1.1474 | 1.1575 | 1.1707 | 1.1869 | 1.2063 | 1.2288 | 1.2545 |
| 66° | 1.1534 | 1.1579 | 1.1654 | 1.1759 | 1.1896 | 1.2065 | 1.2267 | 1.2503 | 1.2773 |
| 67° | 1.1709 | 1.1756 | 1.1833 | 1.1943 | 1.2085 | 1.2262 | 1.2472 | 1.2719 | 1.3002 |
| 68° | 1.1884 | 1.1932 | 1.2013 | 1.2127 | 1.2275 | 1.2458 | 1.2678 | 1.2936 | 1.3232 |
| 69° | 1.2059 | 1.2109 | 1.2193 | 1.2311 | 1.2465 | 1.2655 | 1.2885 | 1.3154 | 1.3464 |
| 70° | 1.2234 | 1.2286 | 1.2373 | 1.2495 | 1.2655 | 1.2853 | 1.3092 | 1.3372 | 1.3697 |
| 71° | 1.2410 | 1.2463 | 1.2553 | 1.2680 | 1.2845 | 1.3051 | 1.3299 | 1.3592 | 1.3931 |
| 72° | 1.2585 | 1.2640 | 1.2733 | 1.2864 | 1.3036 | 1.3249 | 1.3507 | 1.3812 | 1.4167 |
| 73° | 1.2760 | 1.2817 | 1.2913 | 1.3049 | 1.3226 | 1.3448 | 1.3715 | 1.4033 | 1.4403 |
| 74° | 1.2935 | 1.2994 | 1.3093 | 1.3234 | 1.3417 | 1.3647 | 1.3924 | 1.4254 | 1.4640 |
| 75° | 1.3110 | 1.3171 | 1.3273 | 1.3418 | 1.3608 | 1.3846 | 1.4134 | 1.4477 | 1.4879 |
| 76° | 1.3285 | 1.3348 | 1.3454 | 1.3603 | 1.3800 | 1.4045 | 1.4344 | 1.4700 | 1.5118 |
| 77° | 1.3460 | 1.3525 | 1.3634 | 1.3788 | 1.3991 | 1.4245 | 1.4554 | 1.4923 | 1.5359 |
| 78° | 1.3636 | 1.3702 | 1.3814 | 1.3974 | 1.4183 | 1.4445 | 1.4765 | 1.5147 | 1.5600 |
| 79° | 1.3811 | 1.3879 | 1.3995 | 1.4159 | 1.4374 | 1.4645 | 1.4976 | 1.5372 | 1.5842 |
| 80° | 1.3986 | 1.4056 | 1.4175 | 1.4344 | 1.4566 | 1.4846 | 1.5187 | 1.5597 | 1.6085 |
| 81° | 1.4161 | 1.4234 | 1.4356 | 1.4530 | 1.4758 | 1.5046 | 1.5399 | 1.5823 | 1.6328 |
| 82° | 1.4336 | 1.4411 | 1.4536 | 1.4715 | 1.4950 | 1.5247 | 1.5611 | 1.6049 | 1.6572 |
| 83° | 1.4512 | 1.4588 | 1.4717 | 1.4901 | 1.5143 | 1.5448 | 1.5823 | 1.6276 | 1.6817 |
| 84° | 1.4687 | 1.4765 | 1.4897 | 1.5086 | 1.5335 | 1.5649 | 1.6035 | 1.6502 | 1.7062 |
| 85° | 1.4862 | 1.4942 | 1.5078 | 1.5272 | 1.5527 | 1.5850 | 1.6248 | 1.6730 | 1.7308 |
| 86° | 1.5037 | 1.5120 | 1.5259 | 1.5457 | 1.5720 | 1.6052 | 1.6461 | 1.6957 | 1.7554 |
| 87° | 1.5212 | 1.5297 | 1.5439 | 1.5643 | 1.5912 | 1.6253 | 1.6673 | 1.7184 | 1.7801 |
| 88° | 1.5388 | 1.5474 | 1.5620 | 1.5829 | 1.6105 | 1.6454 | 1.6886 | 1.7412 | 1.8047 |
| 89° | 1.5563 | 1.5651 | 1.5801 | 1.6015 | 1.6297 | 1.6656 | 1.7099 | 1.7640 | 1.8294 |
| 90° | 1.5738 | 1.5828 | 1.5981 | 1.6200 | 1.6490 | 1.6858 | 1.7312 | 1.7868 | 1.8541 |

**ELLIPTIC INTEGRALS OF THE FIRST KIND: $F(k, \phi)$**

$$F(k, \phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \qquad \theta = \sin^{-1} k$$

| $\phi$ \ $\theta$ | 50° | 55° | 60° | 65° | 70° | 75° | 80° | 85° | 90° |
|---|---|---|---|---|---|---|---|---|---|
| 46° | 0.8552 | 0.8644 | 0.8734 | 0.8821 | 0.8900 | 0.8968 | 0.9019 | 0.9052 | 0.9063 |
| 47° | 0.8741 | 0.8840 | 0.8958 | 0.9053 | 0.9139 | 0.9212 | 0.9268 | 0.9304 | 0.9316 |
| 48° | 0.8974 | 0.9079 | 0.9185 | 0.9287 | 0.9381 | 0.9461 | 0.9523 | 0.9561 | 0.9575 |
| 49° | 0.9186 | 0.9300 | 0.9415 | 0.9525 | 0.9627 | 0.9714 | 0.9781 | 0.9824 | 0.9838 |
| 50° | 0.9401 | 0.9523 | 0.9647 | 0.9766 | 0.9876 | 0.9971 | 1.0044 | 1.0091 | 1.0107 |
| 51° | 0.9617 | 0.9748 | 0.9881 | 1.0010 | 1.0130 | 1.0233 | 1.0313 | 1.0364 | 1.0381 |
| 52° | 0.9835 | 0.9976 | 1.0118 | 1.0258 | 1.0387 | 1.0499 | 1.0587 | 1.0642 | 1.0662 |
| 53° | 1.0055 | 1.0205 | 1.0359 | 1.0509 | 1.0649 | 1.0771 | 1.0866 | 1.0927 | 1.0948 |
| 54° | 1.0277 | 1.0437 | 1.0602 | 1.0764 | 1.0915 | 1.1048 | 1.1152 | 1.1219 | 1.1242 |
| 55° | 1.0500 | 1.0672 | 1.0848 | 1.1022 | 1.1186 | 1.1331 | 1.1444 | 1.1517 | 1.1542 |
| 56° | 1.0725 | 1.0908 | 1.1097 | 1.1285 | 1.1462 | 1.1619 | 1.1743 | 1.1823 | 1.1851 |
| 57° | 1.0952 | 1.1147 | 1.1349 | 1.1551 | 1.1743 | 1.1914 | 1.2049 | 1.2136 | 1.2167 |
| 58° | 1.1180 | 1.1389 | 1.1605 | 1.1822 | 1.2030 | 1.2215 | 1.2362 | 1.2458 | 1.2492 |
| 59° | 1.1411 | 1.1632 | 1.1864 | 1.2097 | 1.2321 | 1.2522 | 1.2684 | 1.2789 | 1.2826 |
| 60° | 1.1643 | 1.1879 | 1.2126 | 1.2376 | 1.2619 | 1.2837 | 1.3014 | 1.3129 | 1.3170 |
| 61° | 1.1877 | 1.2128 | 1.2392 | 1.2660 | 1.2922 | 1.3159 | 1.3352 | 1.3480 | 1.3524 |
| 62° | 1.2113 | 1.2379 | 1.2661 | 1.2949 | 1.3231 | 1.3490 | 1.3701 | 1.3841 | 1.3890 |
| 63° | 1.2351 | 1.2634 | 1.2933 | 1.3242 | 1.3547 | 1.3828 | 1.4059 | 1.4214 | 1.4268 |
| 64° | 1.2591 | 1.2890 | 1.3209 | 1.3541 | 1.3870 | 1.4175 | 1.4429 | 1.4599 | 1.4659 |
| 65° | 1.2833 | 1.3149 | 1.3489 | 1.3844 | 1.4199 | 1.4532 | 1.4810 | 1.4998 | 1.5065 |
| 66° | 1.3076 | 1.3411 | 1.3773 | 1.4153 | 1.4536 | 1.4898 | 1.5203 | 1.5411 | 1.5485 |
| 67° | 1.3321 | 1.3675 | 1.4060 | 1.4467 | 1.4880 | 1.5274 | 1.5609 | 1.5840 | 1.5923 |
| 68° | 1.3568 | 1.3942 | 1.4351 | 1.4786 | 1.5232 | 1.5661 | 1.6030 | 1.6287 | 1.6379 |
| 69° | 1.3817 | 1.4212 | 1.4646 | 1.5111 | 1.5591 | 1.6059 | 1.6466 | 1.6752 | 1.6856 |
| 70° | 1.4068 | 1.4484 | 1.4944 | 1.5441 | 1.5959 | 1.6468 | 1.6918 | 1.7237 | 1.7354 |
| 71° | 1.4320 | 1.4759 | 1.5246 | 1.5777 | 1.6335 | 1.6890 | 1.7388 | 1.7745 | 1.7877 |
| 72° | 1.4574 | 1.5036 | 1.5552 | 1.6118 | 1.6720 | 1.7324 | 1.7876 | 1.8277 | 1.8427 |
| 73° | 1.4830 | 1.5315 | 1.5862 | 1.6465 | 1.7113 | 1.7774 | 1.8384 | 1.8837 | 1.9008 |
| 74° | 1.5087 | 1.5597 | 1.6175 | 1.6818 | 1.7516 | 1.8237 | 1.8915 | 1.9427 | 1.9623 |
| 75° | 1.5345 | 1.5882 | 1.6492 | 1.7176 | 1.7927 | 1.8715 | 1.9468 | 2.0050 | 2.0276 |
| 76° | 1.5606 | 1.6171 | 1.6812 | 1.7540 | 1.8347 | 1.9207 | 2.0047 | 2.0711 | 2.0973 |
| 77° | 1.5867 | 1.6457 | 1.7136 | 1.7909 | 1.8777 | 1.9719 | 2.0658 | 2.1414 | 2.1721 |
| 78° | 1.6130 | 1.6748 | 1.7462 | 1.8284 | 1.9215 | 2.0240 | 2.1288 | 2.2164 | 2.2528 |
| 79° | 1.6394 | 1.7040 | 1.7792 | 1.8664 | 1.9663 | 2.0781 | 2.1954 | 2.2969 | 2.3404 |
| 80° | 1.6660 | 1.7335 | 1.8125 | 1.9048 | 2.0119 | 2.1339 | 2.2653 | 2.3836 | 2.4362 |
| 81° | 1.6926 | 1.7631 | 1.8461 | 1.9438 | 2.0584 | 2.1913 | 2.3387 | 2.4775 | 2.5421 |
| 82° | 1.7193 | 1.7929 | 1.8799 | 1.9831 | 2.1057 | 2.2504 | 2.4157 | 2.5705 | 2.6603 |
| 83° | 1.7462 | 1.8228 | 1.9140 | 2.0228 | 2.1537 | 2.3110 | 2.4965 | 2.6911 | 2.7942 |
| 84° | 1.7731 | 1.8528 | 1.9482 | 2.0630 | 2.2024 | 2.3731 | 2.5811 | 2.8136 | 2.9487 |
| 85° | 1.8001 | 1.8830 | 1.9826 | 2.1035 | 2.2518 | 2.4366 | 2.6694 | 2.9487 | 3.1313 |
| 86° | 1.8271 | 1.9132 | 2.0172 | 2.1442 | 2.3017 | 2.5013 | 2.7612 | 3.0978 | 3.3547 |
| 87° | 1.8542 | 1.9435 | 2.0519 | 2.1852 | 2.3520 | 2.5670 | 2.8561 | 3.2620 | 3.6425 |
| 88° | 1.8813 | 1.9739 | 2.0867 | 2.2263 | 2.4026 | 2.6336 | 2.9537 | 3.4412 | 4.0481 |
| 89° | 1.9084 | 2.0043 | 2.1216 | 2.2675 | 2.4535 | 2.7007 | 3.0530 | 3.6328 | 4.7413 |
| 90° | 1.9356 | 2.0347 | 2.1565 | 2.3088 | 2.5046 | 2.7681 | 3.1534 | 3.8317 | — |

Elliptic Integral Tables ($46° \leq \phi \leq 90°$)
(taken from: C.R.C. Mathematical Tables, 1960 pp 256-269)

Table D.1 (cont'd.)

COMPLETE ELLIPTIC INTEGRALS

| $\sin^{-1} k$ | $K$ | $\log K$ | $\sin^{-1} k$ | $K$ | $\log K$ |
|---|---|---|---|---|---|
| 80° | 3.1534 | 0.498777 | 85° | 3.8317 | 0.583396 |
| 81 | 3.2553 | 0.512591 | 86 | 4.0528 | 0.607751 |
| 82 | 3.3699 | 0.527613 | 87 | 4.3387 | 0.637355 |
| 83 | 3.5004 | 0.544120 | 88 | 4.7427 | 0.676027 |
| 84 | 3.6519 | 0.562514 | 89 | 5.4349 | 0.735192 |
| 85 | 3.8317 | 0.583396 | 90 | $\infty$ | $\infty$ |

Values of $K$ for $\sin^{-1} k$ = 85° to 89° by 0.1° and 89° to 90° by minutes

| $\sin^{-1} k$ | $K$ | $\log K$ | $\sin^{-1} k$ | $K$ | $\log K$ |
|---|---|---|---|---|---|
| 85.0° | 3.832 | 0.58343 | 89° 0′ | 5.435 | 0.73520 |
| 85.1 | 3.852 | 0.58569 | 89 2 | 5.469 | 0.73791 |
| 85.2 | 3.872 | 0.58794 | 89 4 | 5.504 | 0.74068 |
| 85.3 | 3.893 | 0.59028 | 89 6 | 5.540 | 0.74351 |
| 85.4 | 3.914 | 0.59262 | 89 8 | 5.578 | 0.74648 |
| 85.5 | 3.936 | 0.59506 | 89 10 | 5.617 | 0.74950 |
| 85.6 | 3.958 | 0.59748 | 89 12 | 5.658 | 0.75266 |
| 85.7 | 3.981 | 0.59999 | 89 14 | 5.700 | 0.75587 |
| 85.8 | 4.004 | 0.60249 | 89 16 | 5.745 | 0.75929 |
| 85.9 | 4.028 | 0.60509 | 89 18 | 5.791 | 0.76275 |
| 86.0 | 4.053 | 0.60778 | 89 20 | 5.840 | 0.76641 |
| 86.1 | 4.078 | 0.61045 | 89 22 | 5.891 | 0.77019 |
| 86.2 | 4.104 | 0.61321 | 89 24 | 5.946 | 0.77422 |
| 86.3 | 4.130 | 0.61595 | 89 26 | 6.003 | 0.77837 |
| 86.4 | 4.157 | 0.61878 | 89 28 | 6.063 | 0.78269 |
| 86.5 | 4.185 | 0.62170 | 89 30 | 6.128 | 0.78732 |
| 86.6 | 4.214 | 0.62469 | 89 32 | 6.197 | 0.79218 |
| 86.7 | 4.244 | 0.62778 | 89 34 | 6.271 | 0.79734 |
| 86.8 | 4.274 | 0.63083 | 89 36 | 6.351 | 0.80284 |
| 86.9 | 4.306 | 0.63407 | 89 38 | 6.438 | 0.80875 |
| 87.0 | 4.339 | 0.63739 | 89 40 | 6.533 | 0.81511 |
| 87.1 | 4.372 | 0.64068 | 89 41 | 6.584 | 0.81849 |
| 87.2 | 4.407 | 0.64414 | 89 42 | 6.635 | 0.82210 |
| 87.3 | 4.444 | 0.64777 | 89 43 | 6.696 | 0.82582 |
| 87.4 | 4.481 | 0.65137 | 89 44 | 6.756 | 0.82969 |
| 87.5 | 4.520 | 0.65514 | 89 45 | 6.821 | 0.83385 |
| 87.6 | 4.562 | 0.65916 | 89 46 | 6.890 | 0.83822 |
| 87.7 | 4.603 | 0.66304 | 89 47 | 6.964 | 0.84286 |
| 87.8 | 4.648 | 0.66727 | 89 48 | 7.044 | 0.84782 |
| 87.9 | 4.694 | 0.67154 | 89 49 | 7.131 | 0.85315 |
| 88.0 | 4.743 | 0.67605 | 89 50 | 7.226 | 0.85891 |
| 88.1 | 4.794 | 0.68070 | 89 51 | 7.332 | 0.86522 |
| 88.2 | 4.818 | 0.68556 | 89 52 | 7.449 | 0.87210 |
| 88.3 | 4.905 | 0.69064 | 89 53 | 7.583 | 0.87984 |
| 88.4 | 4.965 | 0.69592 | 89 54 | 7.737 | 0.88857 |
| 88.5 | 5.030 | 0.70157 | 89 55 | 7.919 | 0.89867 |
| 88.6 | 5.099 | 0.70749 | 89 56 | 8.143 | 0.91078 |
| 88.7 | 5.173 | 0.71374 | 89 57 | 8.430 | 0.92583 |
| 88.8 | 5.253 | 0.72041 | 89 58 | 8.836 | 0.94626 |
| 88.9 | 5.340 | 0.72754 | 88 59 | 9.529 | 0.97905 |
| 89.0 | 5.435 | 0.73520 | 90 0 | $\infty$ | $\infty$ |

Complete Elliptic Integral Table (85° $\leq \theta \leq$ 90°)
(taken from: C.R.C. Math Tables, 1960 pp 256-269)

Table D.2

## APPENDIX E

## PROCEDURE FOR FINDING THE ELLIPTIC SINE VALUE

### E.1 Introduction

In appendix B it was necessary to determine various values of the elliptic sine function in order to derive the poles and zeros of the elliptic transfer function. This appendix provides a method of determining the elliptic sine value by using the elliptic integral tables in appendix D.

### E.2 Development of the Procedure

We start with the requirement that we are to find the elliptic sine value represented as

$$\mathrm{sn}\ (u,\ k) \tag{E.1}$$

where

k is the modulus value (see eq A.2)

and

u is the elliptic integral value

The integral value, u, is found by

$$u = F\ (\emptyset,\ k) = \int_0^{\emptyset} \frac{d\gamma}{\sqrt{1 - k^2 \sin^2 \gamma}} \tag{E.2}$$

The elliptic sine value is related to u by

$$\mathrm{sn}\ (u,\ k) = \sin\ (\emptyset) \tag{E.3}$$

Having provided the mathematical basis, let us interpret these equations.

We are given the parameters u and k from which we are to find the elliptic sine. Equation E.2 states that u is a function of the angle $\emptyset$ (called the amplitude) and governed by the modulus k. The elliptic integral tables provide the values of u given $\emptyset$ and $\theta = \sin^{-1} k$. Therefore, we can first find the value $\emptyset$ given the parameters u and $\theta$. Then taking the trigonometric sine of the found value $\emptyset$ yields the elliptic sine value as expressed by equation E.3. Let us summarize this into a procedure and then look at an example.

## E.3 Procedure

We are given the following two parameters:

u (elliptic integral value)

and

k (modulus)

We wish to find

$$\rho = \text{sn}\ (u, k)$$

The procedure is as follows:

1. Find

$$\theta_k = \sin^{-1} k$$

2. Using the elliptic integral tables (see appendix D), determine the two integral values in the column for $\theta_k$ which are closest to the given value u.

3. Read the two values of $\emptyset$ which correspond to the rows which locate the two values of u from step 2.

4. Interpolate between the two values of $\emptyset$ to find the required value of $\emptyset$ corresponding to the given value, u.

5. Determine the elliptic sine, $\rho$, from

$$\rho = \sin \emptyset$$

where $\emptyset$ is the interpolated value found in step 4.

Example

Suppose we are given the values

$$u = \frac{K}{6} = .3245 \quad (1)$$

and

$$k = \frac{1}{1.305} = .766$$

We wish to find

$$\rho = \mathrm{sn}\ (u,\ k)$$

Using the procedure we have:

1. $\theta_k = \sin^{-1} k = 50°$

2. For u = .3245, the two closest integral values under the column $\theta = 50°$ are

$$u_1 = .3172$$

$$u_2 = .3352$$

NOTE 1: K = 1.9407 and is the complete elliptic integral value for the modulus value k = 1/1.3

3. The corresponding values of $\phi$ are

$$\phi_1 = 18°, \; \phi_2 = 19°$$

4. Interpolating u to find $\phi$ we have

$$\phi = \frac{.3245 - .3172}{.3352 - .3172} (19° - 18°) + 18°$$

$$= (.4056)\ (1°) + 18° = 18.4056°$$

5. The elliptic sine value is then

$$\rho = \mathrm{sn}\ (u, k) = \sin\ (18.4056°) = .3157$$

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