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## an actuator-dISC Analysis of helicopter wake GEOMETRY AND THE CORRESPONDING BLADE RESPOnSE

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## ABSTRACT

Assuming that a helicopter rotar in forward flight can be represented by c flat circular disc with an appropriate steady pressure discontinuity across it, expressions were developed for relating the pressure field (1) to the total aerodynamic thrust and the total steady pitching and rolling moments experienced by the rotor or (2) to the time-dependent aerodynamic flapping moments at the roots of the blades as they rotate. The selected pressure field (1) satisfies the requirement of a loss of lift at the center and periphery of the disc, (2) accounts for first and second azimuthal harmonic variations in the lift density, and (3) satisfies Laplace's equation, which is the governing equation for the linearized steady flow of an incompressible fluid.

A digital computing program was written for calculating the velocity field from the selected pressure field and for calculating the positions of streamlines emanating from points above the disc. Two sample cases were clusen, corresponding to the UH-l rotor at advance ratios, $\mu$, of 0.26 and 0.08 respectively. The computed streamlines for both cases show the tip-vortex phenomenon, indicating Prandtl-Lanchester traili:2g vortices emanating from the tip areas of the circular wing.

The geometry of the vortex wake that is shed and trailed from the blades was predicted from the above-menticned streamline calculations. The computed deviations from a helical wake were greater for the $\mu=0.08$ case than for the $\mu=0.26$ case.

In order to examine the sensitivity of blade-load calculations to wake geometry, an existing computer program for calculating blade airloads was modified to accept the computed wake geometry, as contrasted to the rigid helix usually assumed. The blade-ioads predictions of this program using the computed wake geometry input were compared with the predictions resulting from a helical wakegeometry input and with experimentally measured airloads for the two cases mentioned above. The comparison indicates that for the UH-1 two-bladed teetering rotor system, the computed airloads are more sensitive to the wake geometry for the slower flight ( $\mu=0.08$ ) than for the faster one $(\mu=0.26)$. This weaker dependence at higher forward speeds may be due to (1) the wake being blown farther behind the rotor and/or (2) the relative predominance of cyclic pitch input, tangential velocity variation, and blade flapping motion in the total time-varying environment of the blades at a higher speed.

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In preparing parts of the report, material was drawn freely from the work of Mr. R. F. Piziali, listed as Reference 2 in this report. The authors are grateful for the cooperation of cornell Aeronatical Laboratory, and particularly to Mr. R. A. Piziali for making available the source decks for their blade-loads program BLP2.

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## LIST OF SYMBOLS

| $\bar{A}_{o_{k}}, \bar{A}_{n_{k}}$ | cosine coefficients of Fourier expansion of $G_{k}$ |
| :---: | :---: |
| $a_{o k}, a_{n k}$ | cosine coefficients of Fourier expansion of $q_{k}$ |
| $A_{0_{k}}, A_{n_{k}}$ | ```coefficients in trigonometric expansion for //k``` |
| $\frac{b}{B}$ | blade semi-chord <br> vector magnetic induction field |
| BLP2 | blade airload digital computer program described in Reference 2 |
| $\bar{B}_{n k}$ | sine coefficients of nth harmonic in Fourier Expansion of $\mathbf{G}_{\mathbf{k}}$ |
| $b_{n k}$ | sine coefficients of nth harmonic in Fourier Expansion of $q_{k}$ |
| $C_{n}^{m}$ | "arbitrary" constant in the solution of Legendre's equation, multiplying the cosine term |
| $C_{T}$ | $\text { thrust coefficient }=\frac{T}{\pi \rho R^{4} \Omega^{2}}$ |
| $C_{M_{r}}$ | $\text { coefficient of rolling moment }=\frac{M_{r}}{T \cdot R}$ |
| $C_{M_{p}}$ | $\text { coefficient of pitching moment }=\frac{M_{p}}{T \cdot R}$ |
| $C_{1}$ | Mangler thrust coefficient $=\frac{T}{\pi \rho / 2 V_{F B}^{e} R^{2}}$ |


| $D_{n}^{m}$ | "arbitrary" constant in the solution of Legendre's equation, multiplying the sine term |
| :---: | :---: |
| dA | elemental area of rotor disc |
| $\left(\frac{\partial C_{h}}{\partial a}\right)$ | measured lift curve slope |
| $e_{k}$ | $\sqrt{g_{k k}}$ |
| $F(r, t)$ | blade aerodynamic lift per unit span |
| $g_{k}$ | structural damping coefficient in the $k^{t h}$ natural mode |
| $G_{k}$ | generalized force in the $\boldsymbol{k}^{\text {th }}$ natural mode |
| i | imaginary number $=\sqrt{-i}$ |
| $I_{K}$ | circulation associated with the part of the bound vortex due to all parts of the airfoil angle of attack except that resulting from wake vorticity |
| $\vec{J}$ | vector current distribution |
| $k_{a}$ | azimuthal position of blade when shedding vorticity |
| $L$ | implies the total of line filaments of vortex strength |
| $\ell$ | length along vortex filament |
| $M_{k}$ | generalized mass in the $\boldsymbol{k}^{\text {th }}$ natural mode |
| m | "constant of separation" in solution of a partial differential equation |
| $M_{r}$ | rolling moment (about the $\boldsymbol{x}^{\prime}$ axis) |
| $M_{p}$ | pitching moment (about the $y$ axis) xiv |


| $M(\psi)$ | moment applied by rotor blade to shaft at angle |
| :---: | :---: |
| $M_{0}, M_{c 1}, M_{31}$ | harmonic components of $M(\psi)$ |
| $N_{m}$ | total number of normal modes chosen to represent total blade response |
| $N_{h}$ | total number of harmonics retained in Fourier expansion |
| $\hat{n}$ | unit vector, normal to a surface |
| $n$ | "constant separation" in solution of <br> a partial differential equation |
| $N_{b}$ | total number of blades in a rotor |
| $\mathrm{N}_{\mathbf{a}}$ | total number of azimuthal stations on rotor disc |
| $N_{r}$ | total number of radial stations on rotor disc |
| $P(r, t)$ | blade aerodynamic pitching moment per unit span |
| $p$ | disturbance pressure |
| Po | static pressure |
| $P_{n}^{m}(\nu)$ | associated Legendre function of the first kind |
| $P$ | nondimensionalized disturbance <br> pressure $\qquad$ |
| $Q(u, v, w)$ | nondimensional induced velocity $q(u, v, w) / V_{f_{s}}$ |
| $Q_{n}^{m}$ | associated Legendre functions of the second kind |
| $g_{i j}$ | metric tensor |

$q_{k}(t)$
$\vec{q}$
$q_{i}$
$r$
$\vec{r}$
$R$
$R_{i j k_{a}}$
$S_{m_{k j}}$
$t$
$T$
$T_{m_{k i}}$
$u$
$u$
$v_{k}$
$\bar{v}_{o_{k}}, \bar{v}_{n_{k}}$
$V_{1}$
$V_{T}$
$V$
generalized displacement (a function of time) in the $k^{\text {th }}$ normal mode
induced velocity vector
component of fluid velocity in the 1 -axis direction
radial distance from the rotor center to a point on a blade
vector distance between source point and field point
radius of actuator disc
point in the three-dimensional wake model
induced velocity coefficients in the trip expansion for $W_{k s}$
(independent) time coordinate (sec.) total thrust
induced velocity coefficients in trig expansion for $\mathbf{W}_{K_{T}}$
disturbance velocity in $x$-direction nondimensionalized disturbance velocity in the $x$-direction $=$ $\qquad$
"impressed" normal flow due to all effects except wake vorticity
coefficients in trig expansion for $\bar{W}_{k}$
"tangential" relative airflow velocity component in $x$-direction
total relative fluid velocity normal to the blade leading edge
volume of fluid

| V | nondimensionalized disturbance ve- |
| :---: | :---: |
|  | locity in the $\mathbf{y}$-direction $=\frac{V}{V_{F S}}$ |
| $v_{i}$ | induced velocity |
| $V_{\text {fs }}$ | free-stream velocity |
| $v$ | disturbance velocity in the $\boldsymbol{y}$-direction |
|  | nondimensionalized disturbance ve- |
| W | locity in the $\mathbf{z}$-direction $=\frac{\mathbf{w}}{V_{F s}}$ |
| $\bar{w}(x)$ | normal relative velocity component (positive in the plus z-direction) |
| W | disturbance velocity in the $\mathbf{z}$-direction |
| $W_{k}$ | "impressed" normal flow component due to wake vorticity |
| $\underline{w}$ | (assuned) uniform induced velocity over rotor disc |
| $x$ | $x / R$ |
| $x^{w}$ | "wind coordinates" |
| $x$, | blade chordwise coordinate, positive aft |
| $x$ | Cartesian coordinate in horizontal, longitudinal direction, positive forward |
| $x^{1}$ | longitudinal Cartesian coordinate in plane of rotor disc, positive forward |
| $\hat{\chi}$ | unit vector in direction of positive $x$-axis |
| $y$ | displacement of blade normal to the chord plane |
| $y^{\prime}$ | Cartesian coordinate in lateral direction, positive starboard |


| Yoc | built-in coning angle plus steady blade bending deflection (plus, up) |
| :---: | :---: |
| $Y^{\prime}$ | $y^{\prime} / \mathrm{R}$ |
| $z, Z^{\prime}$ | $z / R, z / R$ |
| 2 | Cartesian coordinate in vertical direction, positive down |
| $z^{\prime}$ | Cartesian coordinate normal to rotor disc, positive duwn |
| $\alpha_{\text {max }}$ | angle of attack corresponding to airfoil stall |
| $a_{s}$ | geometric angle of attack of blade (airfoil section (positive, nose up)) |
| d | incidence angle of rotor disc, (positive, nose down) |
| $\Gamma_{m_{k}}$ | maximum value of total bound, circulation, corresponding to blade stall |
| $\stackrel{\rightharpoonup}{\gamma}$ | vector distribution of vorticity |
| $\Gamma$ | total circulation around the blade section |
| $\overline{\Gamma_{i}}$ | maximum bound vorticicy on a blade at azimuth station i |
| $\gamma$ | strength of vortex sheet per unit length |
| $\Gamma$ | scalar vorticity per unit lengt:h in the wake |
| 8 | fraction of time interval which wake is advanced from blade position at which it was shed, trailed |
| $\Delta()$ | difference in ( ) between upper and lower surface, or increment in ( ) |
| $\delta_{k_{j}} \Gamma_{j}$ | circulation associated with bound vorticity resulting from wake vorticity trailed and shed when blade was at azimuth $j$ |
|  | xviii |


| (4) | torsional displacement of blade |
| :---: | :---: |
| $\theta$ | Glauert coordinate |
| $\omega_{k}$ | $\mathbf{k}^{\boldsymbol{*}}$ natural frequency ( $\mathrm{rad} / \mathrm{sec}$ ) |
| $5^{m}$ | ellipsoidal coordinates |
| $\xi$ | "dumny" variable of integration in chord or $x$-axis direction |
| $\varepsilon^{m}$ | disc coordinates |
| $\phi$ | velocity potential $=\Phi_{1}(\boldsymbol{\nu}) \cdot \Phi_{2}(\boldsymbol{\eta}) \cdot \Phi_{3}(\psi)$ |
| $\pi(r, \psi)$ | pressure difference at $\boldsymbol{r}$ ! $W$ when a blade is at that point on the disc |
| $p$ | air mass density |
| $\mu_{0}$ | electromagnetic induction coefficient defined by $\nabla \times \bar{B}=\mu_{0} J$ |
| $d \tau$ | elemental volume |
| $d \Sigma$ | ```elemental surface area (of a vortex tube)``` |
| $\nabla$ | $\text { vector operator }=\frac{\partial}{\partial x} \hat{x}+\frac{\partial}{\partial y} \hat{y}+\frac{\partial}{\partial z} \hat{z}$ |
| $2,7,3$ | curvilinear coordinates (see Figure 7), the last corresponds to angle of azimuth on the rotor disc |
| $\mu$ | tip speed ratio: $V_{\text {Fs }} / \mathbf{R R}_{R}$ |

Subscripts
c
k

i

| $U, L$ | identifies upper, lower airfoil surface, respectively |
| :---: | :---: |
| $T$ | identifies trailed vortex |
| BU | " bound vortex |
| S | " shed vortex |
| Q | " field point |
| K | particular point on the rotor disc |
| 9 | indicates partial derivative with respect to variable(s) represented by subscript(s) which follow(s) |

Superscripts
(k)
$\sim$
$\wedge$
_
identifies upper, lower airfoil surface, respectively
identifies trailed vortex to variable(s) represented by subscript(s) which follow(s)

7
$\pi$
"
\#
vector
perturbation quantity
unit vector
steady component

## CHAPTER 1

## INTRODUCTION

### 1.1 A First View of the Helicopter Wake-Geometry Problem

The desirability of an accurate determination of the wake geometry for a helicopter rotor stems primarily from the necessity of predicting the airloads experienced by the blades. These airloads, in addition to providing the lift and propulsive force, are a major source of vibratory excitation and drive the structurally important oscillatory blade bending moments. Since aerodynamic forces are determined by the flow field, the relative flow field must be known if one is to satisfactorily predict the airloads on a prescribed airfoil executing a prescribed motion.

It has been demonstrated from fixed-wing experience (see, for example, the reduced polar diagrams and reduced lift/angle of attack curves in Reference 1), that for a finite lifting body, the velocity field perturbation resulting from the distribution of the "free" vorticity in the wake must be taken into account in order to predict the aerodynamic forces (aerodynamic force is sufficiently definitive) on the body. It may be argued that this wake-induced velocity is even more important for rotary wings since (1) rotation causes the wake vorticity to remain in the vicinity of the (succeeding) blades for a much longer time and (even for the ". Thest known advance ratios) is not blown behind the wing so quickly as with fixed-wing craft, and (2) as a result of a lower forward speed in the case of a pure helicopter, the change in angle of attack for a given downwash velocity will be larger over most of the disc than in the fixed-wing case.

In addition to the determination of the rotor airloads, the wakeinduced velocity field is likely to be influential in the determination of (1) interference effects between rotors, (2) positioning and effectiveness of auxiliary surfaces, (3) interference with the fuselage and other nonlifting bodies, (4) debris entrainment, (5) projectile trajectories, etc; hence, the continuing interest in the study of helicopter wake geometry and in the resultant wake-induced velocity field.

One of the difficulties in an accurate determination of the wakeinduced velocity should be imnediately apparent: the interdependence of the wake geometry and the induced velocity field. The wake-
induced velocity field can be expressed only as an integral (over the wake) of a function of the wake geometry and the vorticity strengths, while the wake geometry is obtained by time integrations of the flow field along particle paths. As can be seen from subsequent sections, this one complication alone is sufficient to make the general wake-geometry determination problem difficult. This interdependence between wake geometry and induced velocity field does not arise in the customary treatment of the fixed-wing problem because one assumes there that the induced velocity is quite small as compared to the forward speed and hence the trailed vorticity is assumed to lie on a "rigid, horizontal" surface behind the wing.

Another complicating factor is the flexibility of rotor blades. Most helicopter blades currently have their first flapwise natural frequencies within a few percent of rotor speed. Table I summarizes the flapping natural frequencies for the $\mathrm{UH}-1$ and $\mathrm{H}-34$ rotor systems. It has been demonstrated that significant variations in downwash exist up to the 4th azimuthal harmonic. Under such circumstances, the azimuthal variations in aerodynamic loads are almost certain to excite substantial oscillatory flapping displacements as well as torsional displacements, and the resulting "plunging" velocities will affect the aerodynamic loads on the blades to the first order, while the torsional displacements (i.e., angle of attack changes due to torsion) will affect them to the zeroth order. Thus, the situation is not one of a "prescribed blade executing a prescribed motion". As a final note, it may be mentioned that the rotor control settings and attitude are themselves affected by th periodic variations in the hub loads.

Figure $I$ indicates the interrelations of these various aspects of the problem. The figure can be looked upon as an information fiow diagram*, the rectangular blocks representing various phenomena as customarily treated in isolation. Thus, for example, the airloads and wake vorticity strengths resulting from the phenomenon of blade aerodynamics can be calculated if the (1) blade structural displacements and motions, (2) control settings, (3) vehicle attitude and motion, and (4) wake-induced velocity (inflow) are all known as "inputs" to the blade aerodynamic equations. These equations are, for unstalled subsonic motion, simple algebraic relations. (The assumption that makes this possible is the customary strip theory postulation that each spanwise section of the airfoil has a twodimensional pattern of aerodynamic behavior.) We are not so fortunate in regard to some of the other phenomena; e.g., the wake-

[^0]

Quantities of interest:

1. Control settings
2. Airloads
3. Vortex strength, position due to unperturbed flow
4. Induced velocity field
5. Blade displacements and motions
6. Vehicle attitude and motion
7. Wake vorticity position (wake geometry)

Figure 1. Schematic Representation of the Helicopter Rotor Aerodynamics Problem.

| TABLE | ROTOR BLADE VIBRATION NATURAL FREQUENCIES AND COMPARISON WITH RCTOR SPEEDS, FOR UH-1 AND H-34 |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Rotor Sys:em | Nominal <br> Rotor Speed | Rotor <br> Modie No. | Nacural <br> $\mathrm{rad} / \mathrm{sec}$ | $\begin{aligned} & \frac{\text { Frequency }}{\text { Rutor }} \\ & \text { Speeds } \end{aligned}$ | Mode of Oscillation |
| Bell Uhi-1 <br> (teetering) <br> 2 blades | 315 rpm <br> (33 rad/sec) | 1 | 33.8 | 0.97 | rigid flapping |
|  |  | 2 | 38.8 | 1. .8 | first symmetric flap-bending |
|  |  | 3 | 96.4 | 2.92 | first antisymmetric flap-bendirg |
|  |  | 4 | 112.8 | 3.42 | second symmetric flap-bending |
|  |  | 5 | 171.6 | 5.20 | second antisymmetric flap-bending |
| $\left\lvert\, \begin{aligned} & \text { Sikorsky } \\ & \mathrm{H}-34 \\ & \text { (fully } \\ & \text { articulated) } \end{aligned}\right.$ | 210 rpm <br> (22 rad/sec) | 1 | 23.65 | 1.08 | rigid flapping |
|  |  | 2 | 61.45 | 2.79 | first flap-bending |
|  |  | 3 | 171.84 | 5.08 | second flap-bending |
|  |  | 4 | 177.19 | 8.05 | third flap-bending |
| 4 blades |  | 5 | 297.19 | 13.51 | fourth flap-Eending |

induced velocity relations (i.e., Biot-Savart law integrated over the wake) and the wake-transport relations. These are integral expressions, as mentioned above.

### 1.2 Some Relevant Physical Phenomena

Equations representing some of the phenomena shown in Figure 1 will be stated in this section. More specific description of these phenomena as represented in the computer program appears in Chapter 3.

### 1.21 Blade Dynamics

The oscillatory response of a rotor blade to variations in airloads can usually be satisfactorily represented in terms of its response in each of a judiciously chosen finite number, $N_{m}$, of its normal modes of displacement (rigid as well as elastic). If $q_{k}(t)$ is the generalized displacement in the $k^{\text {th }}$ normal mode, the flapwise and torsional displacements at a point radius $r$ can then be written as*

$$
\begin{align*}
& y(r, t)=\sum_{k=1}^{N_{m}} q_{k}(t) \cdot y^{(k)}(r)  \tag{I}\\
& \theta(r, t)=\sum_{k=1}^{N_{m}} q_{k}(t) \cdot \Theta^{(k)}(r) \tag{2}
\end{align*}
$$

where $y^{(k)}(r), \Theta^{(k)}(r)$ is the mode shape in the $k^{t h}$ normal mode.

[^1]The differential equation for determining the generalized displacements $q_{k}(t)$ is of the form

$$
\begin{equation*}
M_{k} \ddot{q}_{k}+i g_{k} M_{k} \omega_{k}^{2} q_{k}+M_{k} \omega_{k}^{2} q_{k}=G_{k}(t) \tag{3}
\end{equation*}
$$

$$
\cdots k=1, N_{m}
$$

$M_{k}, g_{k}, \omega_{k}$ and $G_{k}$ are the generalized mass, structural damping, natural frequency, and generalized forcing function respectively in the $k^{\text {th }}$ mode. The first three of these are dynamic "properties" of the rotating blade, as is the mode shape, and can be prodetermined for each mode. The forcing function depends on the airloads experienced by the blade:

$$
G_{k}(t)=\int_{r_{1}}^{r_{2}}\left[F(r, t) \cdot y^{(k)}(r)+P(r, t) \cdot \Theta^{(k)}(r)\right] d r \quad(4)
$$

$F(r,+)$ and $P(r, t)$ represent the time-varying aerodynamic lift and pitching moment per unit span at radius $r$.

### 1.22 Blade Aerodynamics

Consider a blade segment with its leading edge at $x=-b$ and trailing edge at $\boldsymbol{x}=+\boldsymbol{b}$ in a blade-attached coordinate system; see Figure 2(a). Let the relative airflow have a "tangential" component $V_{1}$ in the $x_{1}$ direction and an "impressed" normal flow component $\bar{W}(x)$ in the $z$ direction. $\bar{W}(x)$ is the combined effect of (l) the component of $V_{1}$ normal to the blade chord as a result of the geometrical angle of attack $\alpha_{\&}$ and from the local camber of che airfoil section, (2) the component of the forward speed normal to the blade chord as a result of shaft tilt, spanwise blade slope, etc., (3) instantaneous flapping velocity of the blade section, (4) instantaneous torsional velocity of the blade section (due to blade flexibility as well as cyclic control inputs), and (5) induced velocity due to the wake vorticity. We wish to estimate the aerodynamic forces experienced by this blade segment.

Since no through-flow can exist on the impermeable airfoil, the presence of the airfoil must give rise to such a "disturbance" that its effect will exactly cancel the "impressed" normal velocity $\bar{w}\left(x_{1}\right)$ in the region $-b \leq x_{1} \leq+b$. It can be shown (see, for example, Reference 3) that for a thin airfoil, this disturbance can be represented by a vortex sheet over the chord length, as shown in Figure 2(b). The distribution of the
strength $\mathcal{Y}\left(x_{1}\right)$ of this vortex sheet must be such that

$$
\begin{equation*}
\bar{w}\left(x_{1}\right)+\frac{1}{2 \pi} \int_{-b}^{+b} \frac{\gamma(\xi)}{\varepsilon_{0}-x_{1}} d \xi_{5}=0 \tag{5}
\end{equation*}
$$


(a) blade element and "impressed" normal velocity distribution $\bar{w}\left(x_{i}\right)$.

(c) Glauert coordinate 6 .

$$
x_{1}=-b \cos \theta
$$

Figure 2. Physical and Mathematical Blade Element Characteristics.

In order to uniquely specify the circulation around the airfoil section, it is customary to further require that the flow be smooth over the sharp trailing edge (Kutta hypothesis)*. This can occur only if

$$
\begin{equation*}
\gamma(+b)=0 \tag{6}
\end{equation*}
$$

Subject to the condition (6), equation (5) has now to be solved. The solution, by Sohngen's formula, is

$$
\begin{equation*}
\gamma\left(x_{1}\right)=\frac{2}{\pi} \cdot \sqrt{\frac{b-x_{1}}{b+x_{1}}} \cdot \int_{-b}^{b} \sqrt{\frac{b+\xi}{b-\xi}} \cdot \frac{\overline{\bar{w}}(\varepsilon)}{x_{1}-\xi} \cdot d \xi \tag{7}
\end{equation*}
$$

After thus determining the bound vorticity strength $\boldsymbol{y}\left(x_{1}\right)$ from the impressed normal velocity $\overline{\mathbf{W}}\left(x_{1}\right)$, it now remains to deduce the pressure difference between the two faces of the airfoil. Assuming that (1) the disturbance velocities are small compared to $V_{1}$, (2) the flow is irrotationel outside of the vortex sheet, and (3) the flow is incompressible, the momentum equation for unsteady flow can be reduced to

$$
\begin{equation*}
\frac{\partial}{\partial t}\left(-\frac{\partial \phi}{\partial x_{1}}\right)+V_{1} \frac{\partial}{\partial x_{1}}\left(-\frac{\partial \phi}{\partial x_{1}}\right)=-\frac{1}{\rho} \cdot \frac{\partial p}{\partial x_{1}} \tag{8}
\end{equation*}
$$

where $\boldsymbol{p}$ is the disturbance pressure and $\phi$ is a velocity

[^2]potential, such that the disturbance velocity $u$ equals $-\frac{\partial \phi}{\partial x_{1}}$.
Changing the order of differentiation in the first term and integrating from $x_{1}=-\infty$ to a point on the upper face of the vortex sheet,
\[

$$
\begin{equation*}
-\frac{\partial \phi_{4}}{\partial t}-V_{1} \quad \frac{\partial \phi_{u}}{\partial x_{1}}+\frac{1}{\rho} p_{u}=f(t) \tag{9}
\end{equation*}
$$

\]

and similarly at a point on the lower face,

$$
\begin{equation*}
-\frac{\partial \phi_{L}}{\partial t}-V_{q} \frac{\partial \phi_{L}}{\partial x_{1}}+\frac{1}{\rho} P_{L}=f(t) \tag{10}
\end{equation*}
$$

But it can be shown (by considering an elemental rectangular contour around a point $x_{1}$ ) that

$$
\begin{equation*}
\frac{\partial \phi_{L}}{\partial x_{1}}-\frac{\partial \phi_{L}}{\partial x_{1}}=u_{L}-u_{L}=\gamma\left(x_{1}, t\right) \tag{11}
\end{equation*}
$$

and, by integrating,

$$
\begin{equation*}
\phi_{L}-\phi_{L}=\int_{-b}^{x_{1}} \gamma\left(\xi_{0}, t\right) d \xi_{1} \tag{12}
\end{equation*}
$$

Using equations (9) through (12), one gets the aerodynamic pressure difference across the vortex sheet:

$$
\begin{equation*}
p_{L}\left(x_{1}\right)-p_{L}\left(x_{1}\right)=\Delta p\left(x_{1}\right)=p\left[V_{1} \gamma\left(x_{1}, t\right)+\frac{\partial}{\partial t} \int_{-b}^{x_{1}} \gamma\left(\xi_{1}, t\right) d \xi_{0}\right] \tag{13}
\end{equation*}
$$

which is the unsteady form of Zhukovski ${ }^{\prime}$ 's theorem.

The total circulation $\Gamma$ around the blade section, the aerodynamic lift $F$ per unit spar, and the aerodynamic pitching moment about mid-chord $P_{\text {e }}$ per unit span are obtained by integration over the chord:

$$
\begin{align*}
& \Gamma=\int_{-b}^{+b} \gamma\left(x_{1}\right) d x_{1} \\
& F=\int_{-b}^{+b} \Delta p\left(x_{1}\right) d x_{1}  \tag{15}\\
& P_{\frac{c}{2}}=\int_{-b}^{+b} x \cdot \Delta p\left(x_{1}\right) d x_{1} \tag{16}
\end{align*}
$$

The other "output" from the blade aerodynamics block in Figure 1 is the wake vortex strength. The strength of free vortices trailed and shed from a blade can be deduced by logical extrapolation of the fixed-wing "lifting line" theory. Thus, when the instantaneous bound vorticity varies along the blade span, trailed vortex strengths are given by

$$
\begin{equation*}
V_{T}(r)=\frac{d \Gamma_{o v}}{d r} \tag{17}
\end{equation*}
$$

where $\gamma_{T}(r)$ is the strength of the trailed vortex sheet and $\Gamma_{s v}(r)$ is the strength of the bound vortex at the spanwise location $r$. Similarly, for a blade with continuously changing (tempozal change) bound vorticity, the shed vortex strength is given by

$$
\begin{equation*}
\gamma_{S}(r, t)=\frac{d \Gamma_{w}(r, t)}{V_{T} d t} \tag{18}
\end{equation*}
$$

where $\mathcal{V}_{0}(r, t)$ is the strength of the shed vortex sheet as a function of spanwise position $r$ and time $t$, and $V$ is the total relative fluid velocity normal to the blade leading edge.

When the strength of the bound vortex varies both along the span and with time, the above expressions for $\gamma_{T}$ and $\gamma_{5}$ will represent the components of the strength of the vortex sheet in the chordwise and spanwise directions respectively.

### 1.23 Wake-Induced Velocity Relations

These relations are used for determining the "induced" velocity qa at a field point $Q$ due to a distribution of vorticity in a volume $\mathbf{V}$. This is directly analogous to the problem of determining the magnetic induction field $\mathbf{E}_{\mathbf{a}}$ at $\mathbf{a}$ due to a current distribution $J$ in $V$. The latter problem has the solution given by the Biot-Savart field law:

$$
\begin{equation*}
\vec{B}_{Q}=\frac{\mu_{0}}{4 \pi} \int_{V} \frac{\vec{J} \times \vec{r}}{r^{3}} d \tag{17}
\end{equation*}
$$

where $\bar{r}$ is the vector distance from the source point to the field point. See, for example, Reference 5.

Analogously, for our induced velocities,

$$
\vec{q}_{a}=\frac{1}{4 \pi} \int_{V} \frac{\bar{\Gamma} \times \bar{r}}{r^{3}} d \tau
$$

(The $\mu_{0}$ is a characteristic of the units used for the electromagnetic quantities such that $\nabla \times \bar{B}=\mu_{0} J \quad$.)

If the source distribution of vorticity is over line segments as in our wake model, our formula is modified to

$$
\begin{equation*}
\vec{q}_{Q}=\frac{1}{4 \pi} \int_{L} \Gamma \frac{\bar{d} l}{} \times \bar{r} \tag{20}
\end{equation*}
$$

where $\bar{r}$ is the vector from $\overline{d \ell}$ to $Q$, and $\Gamma$ is the scalar vorticity per unit length along $\overline{\boldsymbol{l}}$, an elemental length of vortex filament in the wake. The integration is over $L$, which is the aggregate of all the wake filaments.

### 1.24 Wake-Transport Relations

Kelvin's theorem of conservation of circulation (see Reference 6) states that for an ideal fluid, the substantial derivative of the circulation $\Gamma_{c} \triangleq \oint_{c} \bar{q} \cdot d \boldsymbol{d}$ around a $\left.c\right]$ used contour $c$ is zero.

$$
\begin{equation*}
\frac{D \Gamma_{c}}{D i}=0 \tag{21}
\end{equation*}
$$

Now, if $c$ lies on tie surface $\Sigma$ of a vortex tube,

$$
\begin{equation*}
\Gamma_{c}=\oint_{c} \bar{q} \cdot \overline{d l}=\iint_{\Sigma}(\nabla \times \vec{q}) \cdot \hat{n} d \Sigma=0 \tag{22}
\end{equation*}
$$

using Stoke's theorem and the fact that $\boldsymbol{\nabla} \times \bar{q}$ is at right angles to the unit normal $\hat{n}$ on the surface $\Sigma$. Therefore, using Kelvin's theorem, one conciudes that a contour on a vortex-tube surface will always remain on a vortex-tube surface. Using two such intersecting surfaces, it is easy to see that vortex lines always travel with the fluid (Helmholtz theorem; see Reference 7).

We may consider all vorticity in the wake model to have originated from the blades. Thus, if we can calculate the positions of fluid particles released into the flo l field from discrete locations on the disc*, at all subsequent (for our purposes, discrete) instants, we will be able to predict the wake geometry (i.e., time-dependent

[^3]positions of the wake-filament end points in Figure $15(\mathrm{~b})$ ).

### 1.3 Review of Various Approaches to Handling Rotary-Wing Aerodynamics

1.31 Analytical

Among existing theoretical approaches are the actuator theories, the blade element theories, and the wake theories.
a. The early actuator disc theory for a propeller assumed a uniform pressure distribution and induced velocity over the disc, and explained their interrelationship through a simple momentum transfer approach, leading to a relation like

$$
\begin{equation*}
T=2 \pi R^{2} \rho v_{i} \sqrt{V_{F_{3}}^{2}+v_{i}^{2}} \tag{23}
\end{equation*}
$$

where $T, V_{i}$, and $V_{F s}$ are the total thrust, the induced velocity at the disc, and the forward velocity respectively. Glauert modified this assumption of the uniform pressure distribution to one increasing linearly from the front to the rear of the rotor disc (see Reference 8). A modification due to Kinner (Reference 9) and Mangler (Reference 10) assumes an azimuthally symmetric lift distribution satisfying the requirement of a loss of lift both at the center and at the periphery of the disc.

All the actuator disc theories implicitly assume an infinite number of blades. The inherently unsteady nature of the aerodynamics associated with a finite number of blades is, therefore, ignored, as are all effects concerned with the shape of airfoil sections.
b. The blade element theories, on the other hand, consider the forces experienced by individual blades in their motion through the fluid; they are thus intimately concerned with airfoil section characteristics. The blade is divided into a number of radial elements, and airfoil properties and momentum balance are used for each element to yield the tinrust and induced velocity (see Reference li). An interesting combination of momentum and blade-element theories for forward flight
has been suggested by Wood and Hermes (Reference 12). They postulate that the instantaneous induced velocity at a point on the disc can be obtained by superposition of transient momentum theory induced velocity distributions corresponding to the number of blade passages over that point in space.
c. In contrast to the above, the early vortex wake theories represented the wake by a series of circular vortices. The geometry, however, was based on an assumed induced velocity field and on an infinite number of blades. The wake vortex theories then attempted to obtain the induced velocities with the aid of the Biot-Savart formula. Recent modifications of the vortex wake theories assume helical vortices trailed from near the blade tips and a "warped-pie" type of meshvortex system that is shed from the blade. These vortex wake theories may be looked upon as a modification of Prandtl's theory for a finite fixed wing.

### 1.32 Experimental

Considerable experimentation has resulted in a good deal of test data both for structural loads and aerodynamics of rotary wings. Excellent smoke-trail photographs for a model rotor in a wind tunnel were obtained by Tararine (Reference 13). Very extensive blade-load and blade-motion data were presented by Gcheimann (Reference 14) and by Burpo and others (Reference 15). Much of the experimentation has been carried out primarily to gain insight, and rigid model rotors have frequently been used (e.g., Tararine). A recent water-tunnel study of rotary-wing flow visualization was carried out by Lehman (Reference 16). An interesting electromagnetic analog was reported by Gray (Reference 17).

### 1.33 Computational

There appear to be very few numerical solutions of the simultaneous equations of wake motion resulting from blade loading, vortex strength, and vortex-induced velocity as described by the Biot-Savart law. One such calculation was accomplished by Crimi (Reference 18) with an extensive iterative calculation for the time-varying flow and the vortex geometry.

Reference 2 (and its subsequent extension, Reference 19) is an example of the elaborate blade-loads programs that have been developed in recent years. Reference 19 numerically simulates most of the phenomena shown in Figure 1. (A notable exclusion is
the wake-geometry computation.) We will c'escribe some details of the Reference 2 program in Chapter 3 of this report. Another example is the program reported in Reference 20 , which compared the blade-load predictions using the rigid wake-induced velocity with similar predictions using a uniform inflow velocity at the disc for a four-bladed, fully articulated rotor.

### 1.34 Some Complicating Factors* Encountered in the Approaches Cited

1.341 Physical Aspects

As a result of the inherently unsteady aerodynamics of the rotor in forward flight, the "shed" vortices are likely to be significant, and this requires that considerable detail be retained in the wake model (see Figure $15(\mathrm{~b})$ ). Also, due to rotation the wake geometry is more complex than for a fixed wing with a similarly varying bound vorticity.

The flexibility of the blades, as mentioned earlier, makes unprescribed blade motions inevitable.
"...For example, a typical blade of a flapping, threebladed, 25 foot radius rotor has a first bending frequency of about 2.5 times the rotational speed for operating tip speeds; by comparison, a rotating string or chain has a corresponding frequency of about 2.4 times its rotational speed. Certainly, for airloads with frequencies of third harmonic and over, the blade is quite flexible and will respond elastically. This interaction between dynamic structural deformations and airloads is the second serious physical complication in the blade airloads problem." (Reference 21)

### 1.342 Theoretical Difficulties

Even if the equations were not coupled as regards blade motion and

[^4]aerodynamics, the problem is diffisult to solve because of the interdependence of the wake geometry and the induced velocity. To avoid this complication, drastic assumptions have been made regarding vortex geometry. Momentum theories, which bypass wake effects completely, have been subject to other limitations. The Kinner-Mangler theory, for example, which is one of the more sophisticated actuator disc thecries, ignores the nonuniformity of lift with azimuthal position.

Another disturbing factor is the questionable validity of small perturbation assumptions in linearizing the basic flow equations. For such assumptions to be valid, the downward velocity induced by the lifting surface, which from the momentum point of view is the basic source of lift, must be very small compared with forward speed. This is the case for a fixed wing and makes possible the assumption that the trailed vorticity lies on a rigid cylindrical surface through the trailing edge with generators parallel to the direction of flow. It is much less obvious that this assumption will give good results when using an "actuator disc" to represent a lifting rotor at any forward speed. Although the downward flow component near the blade tip may be small compared to the total forward velocity of the tip, there are likely to be points on the rotor where this is not the case. In any event, the "actuator disc" rarnot recojnize blade rotatior, but only free-stream velocities, so that "perturbations" may not be small, and linearized flow equations could lead to large erors.

### 1.31:-1xpilmential and Computational Difficulties

M, ••|" | lmental atudy of blade airlnads has to be very elaborate, |। $\mid 1,1$ ( $\mid: 11$ l; |ifficult to design a blade which is both aero$\because, \| \operatorname{lil} \mid$ |l|y $\| m$ dynamically similar to the full scale item; ,.. .1 .1 . masurements have to be made to define streamines, (IN. I'lom. oririllatory blade pressures, drag, and power; ; |mfいい lon wf lunnel wall effect is especially difficult where - $1,1,1.1 \quad 1.1,11 \|$ don over the rotor disc and unsteady effects 13 11, 11.1111.

A wh $\cdot \| \cdots m+1$ y 'omputation such as that done in Reference 18,
 wh 'h. lime intedm equations (involving unprescribed velocities) $1,1,0,1 \cdots 1$.

### 1.4 Mutivion Ior the Present Work

The mute-trail photographs taken by Tararine (Reference 13) for a riji: model rot:s! in a wind tunnel showed significant departures
from the traditionally assumed helical tip trail, especially over the rear of the disc. Numerical results of blade-loads computations (at the Comell Aeronautical Laboratory*) reported by Piziali in Reference 2 suggested that a more accurate description of the wake geometry might lead to a better correlation between the computed and measured blade loads. A further modification of the Reference 2 work was performed by Chang at CAL (Reference 19), who incorporated (1) the adjustments in the pitch control settingsand shaft tilt for the given flight condition (thus "representing" the pilot in the steady-state computational algorithm) and (2) torsional degrees of flexibility of the blades into the computer program. Both Reference 2 and Reference 19 used a rigid helical representation for the wake, however. The results of Reference 19
"...are not significantly closer to the measured results than those obtained in Reference 1.** Therefore, it appr that, at least for these configurations, further improv ...t in the correlations will probably require the establishment of a better wake model and an even more complete representation of the blade motions...." (Reference 19)

Further evidence of the possible importance of wake geometry is reported in Reference 20 . This comparison of the results of two blade-load computations, one using a uniform inflow and the other using a variable inflow resulting from a CAL type rigid wake, showed marked differences in the two blade-load predictions.

It was concluded that a blade-load computation using a more realistic wake-geometry input for calculating induced velocities would be worthwhile. Since an accurate wake-geometry computation (such as Reference 18) from a time-varying velocity field resulting from wake interaction, fuselage effects, and blade bound circulation would be quite extensive, a modified actuator theory approach was adopted. The approximate method reported here uses a Kinner-Mangler type actuator disc with an improved lift distribution to get an approximation to the time-averaged induced velocity field. The lift distribution is choser so as to realistically represent the radial variation typical of helicopter rotors and to accommodate up to the

[^5]second harmonic components of the azimuthal variation. The method then predicts the spatial positioning of individual vortices released from a finite number of blades, without any a priori assumptions regarding the wake geometry.

CHAPTER 2

## THEOREITCAL MATERIAL

### 2.1 Theoretical Baskground

Kinner (Reference 9) noticed that Laplace's equation, which is the governing equation for the pressure field in case of linearized, steady, incompressible fluid flow, is separable in the ellipsoidal coordinates and that it is possible to choose the coordinate system and the harmonics in such a manner that the solution is discontinuous across a finite, flat, circular disc. Thus, he related the pressure field to the total lift developed by a circular wing. He also gave approximate values of the induced downwash on the disc. His analysis did not allow any disc-incidence angle, however, nor did he calculate the sideways induced velocities. In addition, the discontinuities Kinner dealt with yield a variation over the disc for certain combinations of these solutions that appears quite similar to the lift distribution on a typical rotor.

Mangler (Reference 10) modified Kinner's work to include the discincidence angle, but he restricted his solutions to azimuthally symmetric lift distributions. For such a restricted flight condition, he obtained numerical values of the induced downwash on the disc. Figures 3 and 4 show the distribution of the induced velocity over the disc for a disc incidence of $0^{\circ}$ and $15^{\circ}$ respectively. Figure 5 shows the same results as Figure 4, but they are displayed through the variation of the induced velocity along three longitudinal lines on the disc.

### 2.2 Capsule Statement of the Work Done

In order to evaluate the importance of the details of the wake geometry, it was decided to compare the blade-load predictions from an existing blade-load program, using two different wake-geometry inputs. The program used is that* reported in Reference 2. The two

[^6]

Figure 3. Induced Velocity Over Disc for Azimuthally Symmetric Lift Distribution, Expressed as Contours for $\overline{V_{F s} C_{\phi}}$. For Disc Incidence $=0^{\circ}$. (From Mangler, Reference 10)


Figure 4. Induced Velocity Over Disc For Azimuthally Symmetric Lift Distribution, Expressed as Contours for $\frac{\mathrm{V}}{\mathrm{V}} \mathrm{C}_{\uparrow}$.
For Disc Incidence $=15^{\circ}$. (From Mangler, Reference 10)


Figure 5. Induced Velocity Distribution According to Mangler and Squire. Disc Incidence $=15^{\circ}$. (From Reference 8)
wake-geometry inputs were (1) a rigid, helical wake geometry resulting from the assumption of a uniform induced velocity. (This is the type of wake geometry used in the CAL computations in Reference 2 and will be termed the CAL wake geometry.) (2) A distorted wake geometry resulting from the steady velocity field of an actuator disc with a lift distribution that includes up to the second azimuthal harmonic variations. (This wake geometry will be termed the UR wake geometry.) The steady velocity field used here includes induced velocities in all directions. Specifically, it does not ignore the sideways velocities. This is considered to be a definite advantage, since even small sideways distortions of the wake may be of importance, especially where the trailed vortex from one blade passes near the succeeding blade(s).

No changes were made in BLP2 other than to modify it to suit the IBM System $360 / 65$ at the Computing Center of the University of Rochester* and to modify it to interface with the UR wake computing program. The representation of rotor dynamics, blade aerodynamics, and wakeinduced velocity relations was retained in its entirety.

Section 2.3 describes the theoretical scheme leading to a determination of the steady induced velocity field (used for calculating the UR wake geometry) for a given flight condition.

### 2.3 Theoretical Scheme

### 2.31 Equations of Fluid Motion

Consider a stationary, flat, circular disc, radius $R$, in an infinite expanse of an incompressible fluid. Establish a Cartesian coordinate system $x y z$ such that the velocity of the fluid at an infinite distance upstream of the disc is $-V_{F B} \hat{X}$ at a discincidence angle $a$ (see Figure 6). The equations for incompressible fluid flow are
momentum:

$$
\frac{\partial q_{i}}{\partial t}+q_{j} q_{i, j}=-\frac{1}{p} p_{0,1} \quad \ldots i_{, j}=1,2,3
$$

continuity:

$$
\begin{equation*}
q_{i, i}=0 \quad \cdots i, j=1,2,3 \tag{25}
\end{equation*}
$$

[^7]
$\left|V_{\mathrm{Fs}}\right| \longrightarrow$

Wind
Direction


Figure 6. Cartesian Coordinate Systems xy'z ("Wind System") and $x^{\prime} y^{\prime} \mathbf{z}^{\prime}$ ("Disc System") in Dimensional Form.
where $q_{i}$ are the components of the fluid velocity, $p$ is the constant density of the fluid, and $p_{0}$ is the pressure. For steady flow, the momentum equations become

$$
\begin{equation*}
q_{j} q_{i, j}=-\frac{1}{p} p_{0, i} \quad \ldots i, j=1,2,3 \tag{26}
\end{equation*}
$$

Noting that the components of $\vec{q}$ are $\left(-V_{F s}+u, v, w\right)$, where $\overrightarrow{\vec{q}}(u, v, w)$ is the velocity perturbation, and assuming a lightly loaded rotor, ie., assuming

$$
\begin{equation*}
(u, v, w) \ll V_{F s} \tag{27}
\end{equation*}
$$

allows linearizing equation (26) to

$$
\begin{equation*}
-V_{F s} \tilde{q}_{i, x}=-\frac{1}{\rho} p_{0, i} \quad \ldots i=1,2,3 \tag{28}
\end{equation*}
$$

Differentiating (28) with respect to

$$
V_{F S}\left(\tilde{q}_{i, i}\right)_{3 x}=\frac{1}{p} p_{0, i i}
$$

But the left-hand side of this equation vanishes as a consequence of the continuity condition (25). Hence,

$$
\begin{align*}
\frac{1}{p} p_{0, i i} & =0  \tag{29}\\
\text { i.e., } \quad \nabla^{2} p_{0} & =0
\end{align*}
$$

which is the governing equation for the pressure field in a steady incompressible fluid flow. Hence, the disturbance pressure field $p$ must satisfy Laplace's equation.

Since we are interested in obtaining a pressure field that satisfies the boundary conditions of discontinuity across the disc, we attempt to transform Laplace's equation in 3-space to a coordinate system suitable for that purpose.

Define a Cartesian coordinate system $x^{\prime} y^{\prime} z^{\prime}$, which is obtained by rotating the $x y^{\prime} z$-system through the angle a about the $y^{\prime}$-axis (see Figure 6). For convenience, let us nondimensionalize our Cartesian systems on the disc radius $R$, ie., define new systems $X Y^{\prime} Z$ and $X^{\prime} Y^{\prime} Z^{\prime}$, such that

$$
\begin{equation*}
\left(X, Y^{\prime}, Z\right)=\frac{1}{R}\left(X, Y^{\prime}, z\right) \tag{30}
\end{equation*}
$$

and

$$
\left(X^{\prime}, y^{\prime}, Z^{\prime}\right)=\frac{1}{R}\left(x^{\prime}, y^{\prime}, z^{\prime}\right)
$$

The transformations between the wind system $X Y^{\prime} Z$ and the disc system $X^{\prime} Y^{\prime} Z^{\prime}$ are:

$$
\begin{align*}
& \left|\begin{array}{l}
X \\
Y^{\prime} \\
Z
\end{array}\right|=\left[\begin{array}{ccc}
\cos \alpha & 0 & -\sin \alpha \\
0 & 1 & 0 \\
\sin \alpha & 0 & \cos \alpha
\end{array}\right]\left|\begin{array}{l}
X^{\prime} \\
Y^{\prime} \\
Z^{\prime}
\end{array}\right|  \tag{31}\\
& \left|\begin{array}{l}
X^{\prime} \\
Y^{\prime} \\
Z^{\prime}
\end{array}\right|=\left[\begin{array}{ccc}
\cos \alpha & 0 & \sin \alpha \\
0 & 1 & 0 \\
-\sin \alpha & 0 & \cos \alpha
\end{array}\right]\left|\begin{array}{l}
X \\
Y^{\prime} \\
Z
\end{array}\right| \tag{32}
\end{align*}
$$

Define a curvilinear coordinate system $2 \boldsymbol{\%}$, such that

$$
\begin{aligned}
& X^{\prime}=\sqrt{1-\nu^{2}} \sqrt{1+\eta^{2}} \cos \psi \\
& Y^{\prime}=\sqrt{1-\nu^{2}} \sqrt{1+\eta^{2}} \quad \sin \psi \\
& Z^{\prime}=\nu \eta
\end{aligned}
$$

It may be noted that this $\boldsymbol{\nu}$, coordinate system will cover the entire 3 -space once, if we restrict $\nu, \boldsymbol{\eta}$, and $\psi$ (see Figure 6) to the ranges
$-1 \leq \nu \leq+1$
$0 \leq \eta \leq \infty$
$0 \leq \psi \leq 2 \pi$

Figure 7 shows the $\nu \geqslant \not \geqslant$ coordinate system viewed in the $X^{\prime} Z^{\prime}$ plane (pitch plane of the helicopter). The $\boldsymbol{\nu}$ : constant surfaces are hyperboloids of one sheet and the $\eta=$ constant surfaces are ellipsoids, both families of surfaces being azimuthally symmetric about the $z^{\prime}$ axis. $\mathcal{Z}$ is the azimuthal angle measured from the negative $X^{\prime}$ axis, counterclockwise looking along the plus $z^{\prime}$ axis. $\bar{y}=0$ represents the two faces of the disc. The coordinate $\nu$ changes sign as one crosses the disc.

The inverse of the transformation (33) is

$$
\begin{aligned}
& \nu=-\frac{1}{\sqrt{2}} \operatorname{sgn} Z^{\prime} \cdot \sqrt{\left[1-\left(x^{\prime^{2}}+Y^{\prime 2}+Z^{\prime 2}\right)\right]+\sqrt{\left.\left[1-x^{2}+Y^{2}+Z^{\prime 2}\right)\right]^{2}+4 Z^{1^{2}}}} \\
& \eta=-\frac{Z^{\prime}}{\nu} \\
& \psi=\tan ^{-1}\left(\frac{y^{\prime}}{-x^{\prime}}\right)
\end{aligned}
$$

The matrix of first partials of the Cartesian coordinates $X^{\prime}, Y^{\prime}, Z^{\prime}$ (represented by $\xi^{k}$ ) with respect to the curvilinear coordinates $\nu, \eta, o n \nLeftarrow$ (represented by $\zeta^{k}$ ) is



Figure 7. Ellipsoidal Coordinate Systen, Viewed in the $x^{\prime 2} \mathbf{z}^{\prime} \quad$ Plane.

The metric tensor $g_{i j}$ for the $\nu \geqslant \nLeftarrow$ system is

which shows that the $\mu \geqslant 7$ coordinates form an orthcgonal system. This makes writing out Laplace's equation especially easy, since in orthogonal coordinates (see Reference 22),

$$
\nabla^{2} \phi=\frac{1}{e_{1} e_{2} e_{3}}\left[\frac{\partial}{\partial \zeta^{2}}\left(\frac{e_{2} e_{1}}{e_{1}} \frac{\partial \phi}{\partial \zeta_{5}^{2}}\right)+\frac{\partial}{\partial \zeta^{2}}\left(\frac{e_{1} e_{1}}{e_{2}} \frac{\partial \phi}{\partial \zeta_{3}^{2}}\right)+\frac{\partial}{\partial \zeta^{3}}\left(\frac{e_{1} e_{2}}{e_{3}} \frac{\partial \phi}{\partial \zeta_{3}^{3}}\right)\right]
$$

where

$$
e_{k} \leqslant \sqrt{g_{k k}}
$$

(no summation).
For our ellipsoidal coordinate system, then, Laplace's equation $\nabla^{2} \phi=0$ takes the form
$\underset{\text { Letting }}{\frac{\partial}{\partial \mu}}\left[\left(1-\nu^{2}\right) \frac{\partial \phi}{\partial v}\right]+\frac{\partial}{\partial \eta}\left[\left(1+\eta^{2}\right) \frac{\partial \phi}{\partial \eta}\right]+\frac{\partial}{\partial \gamma^{2}}\left[\frac{\nu^{2}+\eta^{2}}{\left(1-\nu^{2}\right)\left(1+\eta^{2}\right)} \frac{\partial \phi}{\partial \gamma}\right]=0$

$$
\begin{equation*}
\phi(\nu, \eta, \psi)=\Phi_{1}(\nu) \cdot \Phi_{2}(\eta) \cdot \Phi_{3}(\psi) \tag{39}
\end{equation*}
$$

it is found that (38) separates into the three equations

$$
\begin{array}{r}
\frac{d^{2} \Phi_{3}}{d z^{2}}+m^{2} \Phi_{3}=0 \\
\frac{d}{d \mu}\left[\left(i-v^{2}\right) \frac{\left.d \Phi_{1}\right]+\left[\frac{-m^{2}}{d v}+n(n+1)\right] \Phi_{1}=0}{1-v^{2}}=0\right. \tag{41}
\end{array}
$$

$$
\begin{equation*}
\frac{d}{d \eta}\left[\left(1+\eta^{2}\right) \frac{d \xi_{2}}{d \eta}\right]+\left[\frac{m^{2}}{1+\eta^{2}}-n(n+1)\right] \Phi_{2}=0 \tag{42}
\end{equation*}
$$

where $m$ and $n$ arc the constants of separation. The singlevalued solutions of (40) are

$$
\begin{equation*}
\Phi_{3}(\psi)=[\cos m \psi, \text { in } n, \psi] \ldots m=\text { integer } \tag{43}
\end{equation*}
$$

Equations (41) and (42) are recognized as forms of Legendre's associated differential equation (see Reference 23), and their solutions are

$$
\begin{align*}
& \Phi_{1}(\nu)=\left[P_{n}^{m}(\nu), Q_{n}^{m}(\nu)\right]  \tag{44}\\
& \Phi_{2}(\eta)=\left[P_{n}^{m}(i \eta), Q_{n}^{m}(i \eta)\right] \tag{45}
\end{align*}
$$

Thus, we find that the mos+ reneral single-valued solution of Laplace's equation in our e_ipsoidal coordinate system is

$$
\phi=\left\{\begin{array}{l}
P_{n}^{m}(\nu)  \tag{46}\\
Q_{n}^{m}(\nu)
\end{array}\right\}\left\{\begin{array}{l}
P_{n}^{m}(i \eta) \\
Q_{n}^{m}(i \eta)
\end{array}\right\}\left\{\begin{array}{l}
\cos m \nsim \\
\sin m \nsim
\end{array}\right\}
$$

where $m$ is an integer.
2.33 Choice of the Pressure Field, Based on Total Thrust and Net Moments

We shald deal with the disturbance pressure in its nondimensionalized form $P$, normalized on twice the free-stream dynamic pressure, i.e.,

$$
\begin{equation*}
P \Leftrightarrow \frac{P}{\rho V_{F s}^{2}} \tag{47}
\end{equation*}
$$

The general solution of Laplace's equation is given by equation (46). Appendix $I$, which was prepared using Reference 24 , shows some of the associated functions of Legendre, $P_{n}^{m}$ and $Q_{n}^{m}$, and their properties. Before concluding that equation (46) is the general solution for the pressure field, we must examine it for any contradictions with the physical restrictions on the disturbance pressure $P$.

Thus, we notice from Appendix $I$ that the $Q_{n}^{n}(\mu)$ would be infinite at $\nu= \pm 1$, since they contain terms with denominators that vanish at $\boldsymbol{\nu}= \pm 1$. This leads to an infinite disturbance pressure on the $Z^{\prime}$ axis. Hence, we must discard the $Q_{n}^{m}(\tau)$ type of component from equation (44), thus reducing it to

$$
\begin{equation*}
\Phi_{1}(\nu)=P_{n}^{m}(\nu) \tag{48}
\end{equation*}
$$

This form of $\Phi_{1}$ eliminates disturbances of infinite magnitude along the rotor shaft axis.

Similarly, it is noted from Appendix $I$ that $P_{n}^{m}(i \eta) \longrightarrow \infty$ as $\eta \rightarrow \infty$. Since this would mean an infinite disturbance at infinity, this condition must also be ruled out on physical grounds. This reduces equation (45) to

$$
\begin{equation*}
\Phi_{2}(\eta)=Q_{n}^{m}(i \eta) \tag{49}
\end{equation*}
$$

Appendix I shows that (49) tends to zero as $\geqslant$ tends to $\infty$. Equations (39), (43), (48), and (49) are combined to obtain the general form of the disturbance pressure:

$$
\begin{equation*}
P=\sum_{m, n}^{\infty} P_{n}^{m}(\nu) \cdot Q_{n}^{m}(i y) \cdot\left[C_{n}^{m} \cos m \psi+D_{n}^{m} \sin m \psi\right] \tag{50}
\end{equation*}
$$

where $C_{n}^{\infty}$ and $D_{n}^{m}$ are arbitrary constants. From among the infinity of terms in this general form, we will retain only those terms that are necessary to simulate a reasonable lifting rotor or disc.

Since over the disc area the lift density should correspond to the difference in the pressures $p$ just below ( $\eta=0, \nu<0$ ) and above ( $n=0, \nu>0$ ) the disc, we drop all combinations of $m, n$ in (50) that result in an even ( $m+n$ ). This is because $P_{n}^{m}(\nu)$ is an even fundlion if $(m+n)$ is even.

Let us set the practical requirements that (1) an integration of the lift density over the disc area should correspond to the cotal (steady) thrust developed by the rotor, (2) the lift density should vanish at the center and at the periphery of the disc, (3) the expression for the pressure distribution should be able to represent net pitching and rolling moments supplied by the lifting disc to the craft, (4) the pressure distribution should reasonably represent the first harmonic components of lift density as experienced by lightly loaded rotor blades, and (5) there should be some provision for simulating the second harmonic components of the blade bending moments.

Total Thrust
Using equation (50), the total thrust $T$ is

$$
\begin{aligned}
& T=\int_{\text {(lowerfaes) }} P d A-\int_{\text {(upperface) }} P d A \\
& =\sum_{m, n} \rho V_{F s}^{2}\left[\int_{0}^{2 \pi}\left(C_{n}^{m} \cos m z \gamma+r^{m}, \sin m \psi\right) a \psi\right] \cdot\left[\int_{0}^{n} P_{n}^{m}(\nu) Q_{n}^{m}(i 0) r d r\right] \\
& =\rho V_{F s}^{2} \sum_{n} 2 \pi C_{n}^{0} Q_{n}^{0}(i 0) \int_{0}^{0} P_{n}^{0}(\nu) r d r
\end{aligned}
$$

From equations (33), we find $r^{2}=R^{2}\left(1-w^{2}\right)$ for a point on the disc. Hence, while integrating on the disc,

(buer.upper)

$$
\therefore T=-p V_{f s}^{2} \cdot 2 \pi R^{2} \cdot \sum_{n} c_{n}^{0} Q_{n}^{0}(i o) \int_{-1}^{+!} P_{n}^{0}(\nu) \nu d \nu
$$

Noting that $\nu \geq P_{i}^{0}(\nu)$ and using the orthogonality relations in Appendix $I$, only the ( $n=1$ ) term will have a nonvanishing integral, and we get

$$
\begin{equation*}
T=\frac{4}{3} \cdot \pi R^{2} \cdot p v_{F S}^{2} \cdot c_{1}^{0} \tag{51}
\end{equation*}
$$

Defining a coefficient of thrust $C_{T}$,

$$
\begin{equation*}
C_{T} \triangleq \frac{T}{\pi p R^{4} \Omega^{2}} \tag{52}
\end{equation*}
$$

we get

$$
\begin{equation*}
C_{1}^{0}=\frac{3}{4} \frac{C_{T}}{M^{2}} \tag{53}
\end{equation*}
$$

$\Omega$ is the rotor speed in radians per second, and $\mu$ is called the tip-speed ratio

$$
\begin{equation*}
\mu \triangleq \frac{V_{F S}}{\Omega R} \tag{54}
\end{equation*}
$$

Thus, the combination $(m=0, m=1)$ is necessary for getting a steady thrust from the pressure field.
From Figure 8, it is seen that $P_{i}^{0}(\boldsymbol{J})$ does not vanish at the center of the disc, i.e., at $\nu= \pm 1$. In order to have a nonvanishing thrust, and still to satisfy condition (2) above, we must include the combination ( $m=0, n=1$ ) along with any other combination (s) having ( $m=0$ ). Their coefficients will have to be so related that the composite lift density vanishes both at $r=0$ and at $r=R$ Let us try a combination of $(m=0, n=1)$ and $(m=0, n=3)$. Since $v=0^{\circ}$ at the periphery and $V= \pm 1$ at the center, we require that

$$
\begin{align*}
& \quad c_{1}^{0} \cdot P_{1}^{0}( \pm 1) \cdot Q_{1}^{0}(i 0)+C_{3}^{0} \cdot p_{3}^{0}( \pm 1) \cdot Q_{3}^{0}(i 0)=0 \\
& \text { and } \quad C_{1}^{0} \cdot P_{1}^{0}(0) \cdot Q_{1}(i o)+C_{3}^{0} \cdot P_{3}^{0}(0) \cdot Q_{3}^{0}(i o)=0 \\
& \therefore \quad c_{3}^{0}=\frac{3}{2} C_{1}^{0}=\frac{9}{8} \frac{c_{T}}{\mu^{\pi}} \tag{55}
\end{align*}
$$



Figure 8. $\boldsymbol{P}_{\boldsymbol{n}}^{\boldsymbol{m}}(\boldsymbol{\nu}) \begin{gathered}\text { Plotted } \\ \text { Disc. }\end{gathered}$ as a Function of Radius $\boldsymbol{\mu}$ on the

Thus, the combinations $(m=0, M=1)$ and ( $M=0, M=3$ ) can together satisfy conditions (1) and (2) on page 32 if $C_{3}^{0}=\frac{3}{2} C_{1}^{c}$.
Rolling and Piuching Moments
Using equation (50), the rolling moment about the $X^{\prime}$ axis is
$M_{r}=P V_{F S}^{2} \sum_{m, n} Q_{n}^{m}(i 0) \int_{\substack{n=0 \\(\text { Lower-UPper })}}^{R} \int_{\psi=0}^{2 \pi} P_{h}^{m}(v)\left(C_{n}^{m} \cos m \psi+D_{n}^{m} \sin m \psi\right) \cdot(-r \sin \psi) \cdot r d r d \psi$
Using orthogonality of the circular function over $0-2 \pi$

while integrating on the disc, we have
$\left.r^{2} d r\right|_{0} ^{E} \Rightarrow\left[-\left.R^{3} \nu \sqrt{1-v^{2}} d \tau\right|_{-1} ^{0}\right]-\left[-R^{3} v-\left.\sqrt{1-v^{2}} d v\right|_{+1} ^{0}\right]$
$\therefore M_{r}=\sum_{n} \rho \pi R^{3} V_{F S}^{2} Q_{n}^{1}(0 i) D_{n}^{1} \int_{-1}^{1} P_{n}^{1}(v) \cdot \sqrt{ } \sqrt{1-v^{2}} d \nu$
Noting that $P_{2}^{1}(\nu)=-3 \nu \sqrt{1-\nu^{\prime}}$, and using the orthogonality relations in Appendix I only the ( $n=2$ ) term will survive integration, and we get

$$
\begin{equation*}
M_{r}=\frac{8}{5} i \cdot \pi R^{3} \cdot \rho V_{F S}^{2} \cdot O_{2}^{1} \quad \text { DX } x^{\prime} a \times i s \tag{56}
\end{equation*}
$$

Defining a coefficient of rolling moment

$$
\begin{equation*}
C_{M_{r}} \triangleq \frac{M_{r}}{T \cdot R} \tag{57}
\end{equation*}
$$

we get

$$
\begin{equation*}
D_{2}^{1}=-\frac{5}{8} i \frac{C_{M_{r}} \cdot C_{T}}{\mu^{2}} \tag{58}
\end{equation*}
$$

A similar calculation for the pitching moment $M_{P}$ leads to

$$
\begin{align*}
& M_{p}=\frac{8}{5} i \cdot \pi R^{3} \cdot p V_{F S}^{2} \cdot C_{2}^{1}+M Y \text { axis }  \tag{59}\\
& C_{2}^{1}=-\frac{5}{8} i \frac{C_{M} \cdot C_{T}}{M^{2}}  \tag{60}\\
& C_{M_{p}}=\frac{M_{p}}{T \cdot R} \tag{61}
\end{align*}
$$

Thus, the combination ( $m=1, M=2$ ) will satisfy condition (3) on page 32.

Figure 9 shows typical plots of the steady and first two harmonic components of experimentally measured blade loads. Since the radial variation of $\boldsymbol{P}_{\mathcal{L}}(\sqrt{ })$ in Figure 8 is quite similar to the variation of the first härmonic components in Figure 9, it was decided to retain the combination $(m=1, n=4)$ in equation (50). Also retained was the combination $(m=2, n=3)$.

Table II shows the combinations of $m$ and $n$ to which the steady disturbance pressure field $\boldsymbol{P}$ was restricted.

$$
\begin{equation*}
P=\sum_{m, n} P_{n}^{m}(v) \cdot Q_{n}^{m}(i \eta) \cdot\left[c_{n}^{m} \cos m \forall+E_{n}^{m} \sin m \psi\right] \tag{62}
\end{equation*}
$$

2.34 Choice of the Pressure Field, Based on Blade Flapping Moments

In Section 2.33 the coefficients $C_{2}^{0}, C_{3}^{0}, C_{8}^{2}$, and $D_{2}^{2}$ were calculated from $C_{r}, C_{A p}$, and $C_{M m}$. The latter represent the "total", or integrated, effect of the disc as felt by the craft, in the form of the thrust, rolling moment, and pitching moment respectively. While this sort of a determination of the pressure field may seem


Figure 9. Typical Steady and First Two Harmonic Components of Experimental Rlade Loads. (From Scheimann, Reference 14. $\mathrm{H}-34$, Flight $20, \mathrm{~V}=122$ Knots.)

TABLE II. THE TERMS RETAINED IN THE EXPRESSION (62) FOR THE DISTURBANCE PRESSURE P

| m | n | Remarks |
| :---: | :---: | :---: |
| 0 | 1 | This is the only term that gives rise to a non-zero total lift (but does not have zero lift density at the center of the disc). |
| 0 | 3 | Together with the above term, will give zero lift density at the center of the disc if $c_{g}^{0}=\frac{3}{2} c_{g}^{0}$. |
| 1 | 2 | This is the only term that will give rise to net pitching and rolling moments. |
| 1 | 4 | Although the net moments over the disc are zero for this term, the radial distribution of pressure due to this term is quite similar to experimentally observed pressure distributions. |
| 2 | 3 | To account for second harmonic variations in lift density and blade-flapping moments. |

quite obvious, it certainly is not the only way to relate the pressure field to physical quantities. For a real helicopter with a finite number of blades, barring the case for which experimental measurements of blade differential pressures are available (as in case of Reference 14 and 15), it may be quite difficult to accurately determine the steady components of total thrust, rolling, and pitching moments, because of uncertainties in fuselage drag, vehicle trim, etc. It would probably be advisable, then, to be able to base the determination of the steady pressure field on some more easily (and con?idently) estimable quantity. The flapwise moment on the blade could be one such quantity. Even for an experimental case, it is probably as easy, if not easier, to get measurements of the blade-flapping moments as it is to measure the differential pressures. In any event, since lift variations higher than the first harmonic do not result in any "integrated" effect at all, it is clear that expressing the $\boldsymbol{C}_{n}^{m}, D_{n}^{m}$ in terms of the instantaneous flapping moments on the blade would be helpful in simulating higher harmonic variations in lift in any actuator disc approach. This section deals with this approach.

There are two possible ways in which the coefficients $C_{n}^{m}$ and $D_{n}^{m}$ can be expressed in texms of the flapping moments $M(\nLeftarrow)$ experienced by the blades at their roots as they rotate:
(1) Assuming that the value $[0(r, y)-p(r, y)]$ of the theoretical
steady lift density at a point $(\Gamma, \psi)$ on the disc is the time average of the lift supplied to the rotor when there is and is not a blade at that point, or
(2) Assuming that the $M(\nmid \nmid)$ is simply the flapping moment on a blade, when it is at the azimuthal position of, and subjected to the steady lift density $[p(r, \psi)$-p $p(r, \psi)\rangle$ upper $(r, \psi)$.
It may be noted that with assumption (1), the total momentum transferred from the fluid to the disc will equal the total momentum transferred from the fluid to the $N_{b}$ blades. Assumption (2) does not satisfy this condition.

Making assumption (1), let us further assume that the time history of the blade lift density at a point $(r, \psi)$ of the disc is a series of pulses as chown in Figure 10. TT(r, $\boldsymbol{F}^{2}$ )represents the pressure difference experienced by each of the blades at $\mu$ when at $\psi$. $\bar{p}(r, \psi)$ represents the assumed steady lift density at $(r, \psi)$.

$$
\bar{p}(r, \psi)=p(r, \psi)-p(r, \psi)=\sim p(r, \psi)
$$

 assuned pulsating lift
finite number of blades

-     -         -             -                 -                     - assumed steady lift density for an actuator disc

Firure 10. Tine Histories of the Lift Density at a Point $(\boldsymbol{r}, \boldsymbol{W})$ C $\pm$ the Disc.

Assumption (1) requires that the areas under the two time-history curves be equal.

$$
\therefore \pi(r, \psi)=\frac{2 \pi}{6 N_{b}} \cdot r p(r, \psi)
$$

and $M(\psi)=\int_{0}^{R} r \cdot \pi(r, \psi) \cdot 2 b \cdot d r$

$$
\begin{equation*}
=\frac{4 \pi}{N_{b}} \int_{0}^{P} r_{l o w e r}^{2}(r, \psi) d r \tag{63}
\end{equation*}
$$

Since on the disc $r^{2}=R^{2}\left(2-v^{2}\right)$, we have $r^{2} d r / \Rightarrow-\left.R^{5} v \sqrt{2-v^{3}} d v\right|_{-1} ^{0}$ lowepface

$$
\begin{equation*}
\therefore M(v)=\frac{4 \pi R^{3}}{N_{b}} \cdot \rho v_{f v}^{2} \int_{0}^{-1} v \sqrt{2-v^{2}} p(v, \psi) d v \tag{64}
\end{equation*}
$$

Using expression (62) for $\boldsymbol{P}$, we get

$$
\begin{aligned}
M(\psi)=-\frac{4 \pi R^{3}}{N_{b}} \cdot \rho v_{F s}^{2} \cdot \sum_{m, n}\left\{\left[c_{n}^{m} \cos m v+D_{n}^{m} \sin m \psi\right]\right. \\
\left.\cdot Q_{n}^{m}(i 0) \cdot \int_{n}^{\infty 1}(v) v \sqrt{1-v^{2}} d v\right\}
\end{aligned}
$$

Evaluating the integrals for the five combinations of $(m, n)$ shown in Table II, this redlines to

$$
\begin{gathered}
M(\psi)=-\frac{4 \pi R^{3}}{N_{h}} \cdot \rho \nu_{F=}^{2} \cdot\left\{-\frac{\pi}{16} c_{2}^{0}-\frac{\pi}{96} c_{3}^{0}+\frac{4}{5} i\left(c_{2}^{2} \cos \psi+\eta_{2}^{2} \sin \psi\right)\right. \\
\\
\left.+\frac{15}{4} \pi\left(\xi_{5}^{2} \cos 2 \psi+\eta_{3}^{2} \sin 2 \psi\right)\right\}
\end{gathered}
$$

Or, remembering that $C_{3}^{0}=\frac{3}{2} C_{1}^{0}$, we nave

$$
\begin{equation*}
C_{i}^{0}=\frac{16 N_{G}}{5 \pi^{2} \cdot \rho r_{F S}^{2} \cdot R^{3}} M_{0} \tag{65}
\end{equation*}
$$

$$
\begin{align*}
& C_{3}^{0}=\frac{24 N_{b}}{5 \pi^{2} \cdot \rho V_{F S}^{2} \cdot R^{3}} \cdot M_{0}  \tag{66}\\
& c_{9}^{2}=\frac{5 i N_{b}}{16 \pi^{2} \rho V_{F S}^{2} \cdot R^{3}} \cdot M_{C 1}  \tag{67}\\
& D_{2}^{1}=\frac{5 i N_{b}}{16 \pi \cdot \rho V_{F S}^{2} \cdot R^{3}} \cdot M_{s 1}  \tag{68}\\
& C_{3}^{2}=-\frac{N_{b}}{15 \pi^{2} \cdot \rho V_{F S}^{2} \cdot R^{3}} \cdot M_{C 2}  \tag{69}\\
& D_{3}^{2}=-\frac{N_{b}}{15 R^{3} \pi^{2} \cdot \rho V_{F S}^{2}} \cdot M_{S 2} \tag{70}
\end{align*}
$$

where $M_{0}, M_{C_{1}}, M_{S 1}, M_{C 2}$, and $M_{S 2}$ are the harmonic components of $\boldsymbol{M}(\boldsymbol{\psi}):$

$$
M(\psi)=M_{0}+M_{C 1} \cos \psi+M_{S 1} \sin \psi+M_{C 2} \cos 2 \psi+M_{S 2} \sin 2 K_{71)}
$$

Equations (65) through (70) relate the steady pressure field to the flapping moments resulting from the airloads on the blades, and were so used in the UR wake computations reported herein. These relations wre obtained under the assumption (1) of equal average momentum transfer on page 39 .

If we choose the simpler assumption (2) on page 39 , thus ignoring the average momentum transfer requirement, we get

$$
\begin{equation*}
M(\psi)=\int_{0}^{R} 2 p(r, \psi) \cdot r \cdot 2 b \cdot d r \tag{72}
\end{equation*}
$$

Since on the disc $r^{2}=R^{2}\left(1-v^{2}\right)$, we have $\left.\operatorname{ror} \int^{R} \rightarrow-R^{2} v d v\right)_{-1}^{0}$
lower

$$
\therefore M(\psi)=4 b R^{2} \cdot \rho v_{F s}^{2} \cdot \int_{0}^{-1} v p(r, \psi) d v
$$

Using (62) for $\boldsymbol{p}$, we get
$M(\psi)=-4 b R^{2} \cdot \rho v_{F S}^{2} \cdot \sum_{m, n}^{-}\left\{\left(c_{n}^{m} \cos m \psi+D_{n}^{m} \sin m \psi\right) \cdot Q_{n}^{m}(i o) \cdot \int_{0}^{+1} v p_{n}^{m}(v) d v\right\}$

Evaluating the five integrals,

$$
\begin{aligned}
& M(\psi)=4 b R^{2} \cdot \rho V_{F S}^{2} \cdot\left\{\frac{2}{3} c_{1}^{0}+\left(\frac{5}{24} i \pi c_{4}^{3}-\frac{3}{8} i \pi c_{2}^{2}\right) \cos \psi\right. \\
&\left.+\left(\frac{5}{24} i \pi D_{4}^{2}-\frac{3}{8} i \pi D_{2}^{2}\right) \sin \psi-16 c_{3}^{2} \cos 2 \psi-16 D_{3}^{2} \sin 2 \psi\right\}
\end{aligned}
$$

Noting that both $\boldsymbol{C}_{2}^{\mathbf{2}}$ and $\boldsymbol{C}_{4}^{\mathbf{2}}$ contribute to the first harmonic cosine flapping moment, we need an additional "constraint" to determine them. Let us choose the total pitching moment Mp from equation (59) for the purpose. Similarly, equation (56) will be used to determine $\boldsymbol{D}_{\boldsymbol{Z}}^{2}$ and $\boldsymbol{D}_{\boldsymbol{4}}^{\mathbf{2}}$. The results are:

$$
\begin{align*}
& C_{1}^{0}=\frac{3}{4 b \cdot \rho V_{r s}^{2} \cdot R^{2}} M_{0}  \tag{73}\\
& C_{3}^{0}=\frac{9}{8 N \cdot \rho V_{F S}^{2} \cdot R^{2}} M_{0}  \tag{74}\\
& C_{2}^{1}=-\frac{5 i}{8 \pi \cdot \rho V_{F S}^{2} \cdot R^{3}} M_{p}  \tag{75}\\
& D_{2}^{1}=-\frac{5 i}{8 \pi \cdot \rho V_{F S}^{2} \cdot R^{3}} M_{r}
\end{align*}
$$

$$
\begin{align*}
& C_{4}^{1}=\frac{9}{5} C_{2}^{2}-\frac{6 i}{25 \pi \cdot b \cdot \rho V_{F S}^{2} \cdot R^{2}} M_{C I}  \tag{77}\\
& D_{4}^{1}=\frac{9}{5} D_{2}^{2}-\frac{6 i}{25 \pi \cdot 6 \cdot \rho V_{F S}^{2} \cdot R^{2}} M_{G 1}  \tag{78}\\
& C_{3}^{2}=-\frac{1}{64 b \cdot \rho V_{A S}^{2} \cdot R^{2}} M_{C 2}  \tag{79}\\
& D_{3}^{2}=-\frac{1}{64 b \cdot \rho V_{F S}^{2} \cdot R^{2} M_{S 2}} \tag{80}
\end{align*}
$$

Equations (73), (74) and (77) through (80) do not satisfy the average momentum transfer condition.

### 2.35 Induced Velocity Field

We shall deal with the induced velocity in its nondinensionalized form $\overrightarrow{\boldsymbol{Q}}(\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W})$, normalized on the free-stream velocity $\boldsymbol{V}_{\text {Fs }}$, i.e.,

$$
\begin{equation*}
\overrightarrow{\widetilde{Q}}(u, v, w) \triangleq \frac{1}{v_{F s}} \overrightarrow{\tilde{q}}(u, v, w) \tag{81}
\end{equation*}
$$

Using (81) and (47), equation (28) becomes

$$
\begin{equation*}
\widetilde{Q}_{i, x}=\rho_{, i} \quad \ldots i=1,2,3 \tag{82}
\end{equation*}
$$

which is the completely dimensionless form of the linearized momentum equations. Integrating (82) and noting that all disturbances vanish at an infinite distance upstream of the disc, we get the indure: velocity field

$$
\begin{equation*}
\tilde{Q}_{i}(x, y, z)=\int_{+\infty}^{x} P_{, i}(\xi, y, z) d \xi \quad \ldots i=1,2,3 \tag{83}
\end{equation*}
$$

The subscript $i$ represents the Cartesian "wind coordinates" $x, y$, and $\boldsymbol{Z}$. In order to evaluate (83), then, for field point $\left(x, y, z\right.$ ), we must first obtain the partials $\frac{\partial p}{\partial x}$, $\frac{\partial p}{\partial y} y$, and $\frac{\partial p}{\partial z}$ of the pressure field $\boldsymbol{P}$, and then carry out the integration from $\xi=+\infty$ to $\boldsymbol{\xi}=\boldsymbol{X}$. Using the chain rule for first partials,

$$
\begin{equation*}
\frac{\partial \rho}{\partial x^{i}}=\frac{\partial \xi^{k}}{\partial x^{i}} \cdot \frac{\partial \zeta^{j}}{\partial \xi^{k}} \cdot \frac{\partial p}{\partial \zeta^{j}} \quad \cdots i, j, k=1,2,3 \tag{84}
\end{equation*}
$$

where $\boldsymbol{X}^{\boldsymbol{m}}$ represent the "wind coordinates" $(x, y, z)$

$$
\xi \xi^{\prime \prime} \text { represent the "disc coordinates" }\left(x^{\prime}, y^{\prime}, z^{\prime}\right)
$$

$$
\boldsymbol{\zeta}^{m} \text { represent the "ellipsoidal coordinates" (ひ, 壮). }
$$

$$
\begin{aligned}
& \text { The } \frac{\partial \xi^{k}}{\partial x^{i}} \text { are available from equation (32) } \\
& \text { by inverting the matrix of first partials } \frac{\partial \zeta^{j}}{\partial j} \text { are obtained }
\end{aligned}
$$ by inverting the matrix of first partials $\frac{d \zeta^{4}}{\partial \zeta J}$ in equation (36):



The $\frac{\partial P}{\partial b^{2}}$ are obtained by using the expression (62) for $\boldsymbol{P}$ and the formulas for derivatives of Legendre's associated functions in Appendix I:
$\frac{\partial p}{\partial v}=\sum_{m, n} Q_{n}^{m}(i \eta) \cdot\left[C_{n}^{m} \cos m \psi+D_{n}^{m} \sin m \geqslant\right]-\left[(n+n) \frac{P_{n-1}^{m}(v)}{1-v^{2}}-n \nu \frac{p_{n}^{m}(\nu)}{1-v^{2}}\right]$

$$
\begin{align*}
& \frac{\partial P}{\partial \eta}=\sum_{m, n} P_{n}^{m}(\nu) \cdot\left[C_{n}^{m} \cos m \psi+D_{n}^{m} \sin m \psi 7 \cdot\left[n \eta \frac{Q_{m-1}^{m}(i \eta)}{1+\eta^{2}}+i(m+n) \frac{Q_{n-1}^{m}(i \eta)}{1+\eta^{2}}\right]\right.  \tag{87}\\
& \frac{\partial p}{\partial \psi}=\sum_{m, n} m P_{n}^{m}(\nu) Q_{n-1}^{m}(i \eta)\left[\sum_{n}^{m} \cos m \psi-c_{n}^{m} \sin m \psi\right] \tag{88}
\end{align*}
$$

We now have all the necessary formulas for numerically integrating equations (83). Some simplification, however, is possible in case of the $X$-component of equation (83):

$$
\begin{equation*}
u(x, y, z)=\int_{+\infty}^{x} \frac{\partial p}{\partial \zeta} d \xi \tag{89}
\end{equation*}
$$

So long as the path of integration does not pass through points where $P$ is discontinuous, evaluation of the integral on the righthand side yields

$$
\begin{equation*}
u(x, y, z)=P(x, y, z) \tag{90}
\end{equation*}
$$

Since the onjy place where the horizontal path of integration encounters a discontinuity in $P$ is the disc surface, (90) holds for all points upstream of the disc. For a point $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ on a streamline which crosses the disc, we use the physical condition that the fluid velocity at a point on the lower face of the disc is the same as that at its image on the upper face (i.e., the actuator is permeable $)^{*}$, an* ovaluate the integral for $U$ in two parts:



[^8]or, since for our disturbance pressure $\boldsymbol{P}$ of equation (62), $\boldsymbol{P}=-\boldsymbol{P}$ we get
\[

\bigcup_{wake}^{U(x, y, z)=P(x, y, z)-2 P (intersection point)} $$
\begin{aligned}
& \text { lower force }
\end{aligned}
$$
\]

The term (intersection point) indicates the point at which the horizontal path of integration up to the field point ( $x, y, z$ ) crosses the disc.

The induced velocity components $\boldsymbol{V}$ and $\boldsymbol{W}$ are evaluated numerically.
The wake geometry resulting from this steady induced velocity field is also calculated numerically, as described in Chapter 3.

### 3.1 General Description

This section briefly describes the blade-loads program, BLP2 of CAL, as modified at UR. This program used to calculate blade loads based on the two different wake geometries.

The BLP2 is in two parts. Part 1 first generates the blade segment coordinates and the wake element coordinates for the given flight condition. Using these coordinates, Part l then calculates (using the Biot-Savart field law) the induced velocity coefficients $S_{m_{k}}$; and $T_{N_{i}}$ appearing in equations (105) and (106),
$\left(m=0,3 ; K=1, N_{r a} ; j=1, N_{r a} ; i=1, N_{a}\right)$. These $S$ and $T$ coefficients are then written in binary form on TAPE 2. Figure 11 shows the logical flow diagram for Part 1 of the unmodified CAL progran..

Part 2 uses the TAPE 2 data set generated in Part 1 as input. In addition, all the dynamic properties of the blades and control inputs to the craft are supplied in Part 2. It then solves the equations of rotor dynamics, blade aerodynamics, and wake-induced effects by iteration. (The essential elements of the representation cf these three phenomena in the numerical program are sumnarized in Sections 3.21, 3.22, and 3.23 respectively.) After achieving convergence, the blade loads and other quantities of interest (as well as heir harmonic analyses) are written on TAPE 3 . Figure 12 shows the major steps in the iterative procedures for solving the equations. Figure 13 shows the flow diagram for Part 2 of BLP2. It will be noticed from Figure 12 or 13 that the Part 2 procedure involves two iterations, one "inside" the other. The iterative solution of the simultaneous equations (109) in the bound circulations Fe takes place "inside" the iterative solution of the blade dymamical equations (96).

Figure 14 schematically shows the BLP2 as modified at UR. By comparison with Figure 1 , it may be noted that the control settings, vehicle attitude, and motion have ceased to be unknowns. (This means that BLP2 ignores the "Pilot" and "Vehicle Dynamics" blocks in Figure 1 ) Similarly, the uniform induced velocity is needed if the CAL rigid wake geometry is used; and some information on the blade loads is needed if the UR wake geometry is to be computed. Since the modification to the CAL procedure is essentially the use of an alternative wake geometry, the necessary changes and/or additions to the CAL program occur in Part 1 of BLP2, as can be


Figure 11. Flow diagram for Part 1 of BLP'2. (From Reference 2)


Figure 12. Major Steps in the Iterative Procedure for Solving the Equations in Part 2 of ELP2. (From Reference 2)


Figure 13. Flow Diagram for Part 2 of BLP2. (From Reference 2)

(9) -Wake-induced velocity effects on bound circulation 6-Uniform induced velocity for positioning the CAL wake
(5) - Information on blade loads for choosing pressure field for calculating UR wake
(See Figure 1 for other quantities of interest.)
Figure 14. Schematic of the Modified BLP2.
noticed from the lower right-hand portion of Figure 14. Appendix II shows the FORTRAN IV source listings for subroutine SUBl4 of Part 1 , as modified at UR, and for subroutines RWAKE, COEFS, STMLN, VLCTS, CHAMP, FRWRD, ADVNC, and CRSNG, which comprise the UR written wakegeometry program.

Essentially no changes were made in Part 2 of BLP2.

### 3.2 Representation of Various Physical Phenomena in the CAL Program

This section briefly describes the representation of the phenomena of rotor dynamics, blade aerodynamics, wake-induced effects, and wake transport in the original BLP2. An alternative to the CAL wake transport was added at $U R$, while the other three representations were left unchanged.

### 3.21 Rotor Dynamics

For a numerical procedure in which all pertinent quantities are computed at a finite number $\mathrm{Ne}_{\mathrm{a}}$ of discrete instants around the azimuth for a finite number $N_{r}$ of blade radial stations, it is convenient to harmonically analyze the generalized forces $G_{k}$
mentioned in Section l.2l:

$$
\begin{equation*}
G_{k}(t)=\bar{A}_{o_{k}}+\sum_{n=2}^{N_{n}}\left(\bar{A}_{n_{k}} \cos n \Omega t+\bar{B}_{n} \sin n \Omega t\right) \tag{92}
\end{equation*}
$$

$$
\cdots k=1, N_{m}
$$

wh.are $N_{h}$ is the number of harmonics retained. (The maximum number would be $\underline{N}_{g}-1$. . . The components $\bar{A}_{\mu_{k}},\left(\bar{A}_{n_{k}}, \mathcal{N}_{n_{h}}, \dot{n}=\mathbf{2}, N_{h}\right)$ are defined by equatiofs (4) and (92), except that hereafter, we shall neglect the torsional displacements $(\boldsymbol{\otimes}(r, t)$, so that the aerodynamic pitching moments will not do any "work" on the blade modes, which now involve only flapping displacements. Thus,

$$
\begin{align*}
& \bar{A}_{n_{k}}=\sum_{j=1}^{N_{r}} P_{n j} y_{j}^{(k)} \Delta r_{j}  \tag{93}\\
& \bar{B}_{n_{k}}=\sum_{j=1}^{N_{r}} Q_{n j} z_{j}^{(k)} \Delta r_{j}
\end{align*}
$$

 position $\mathcal{J} ; \boldsymbol{y}_{j} \boldsymbol{k}$ is the flapping displacement at $\boldsymbol{r} j$ in the $K$ th normal mode; and $P_{n} j$ and $Q_{n j}$ are the $n$th azimuthal cosine and sine components of the lift density $\boldsymbol{P}$ at the $\boldsymbol{j}$ th spanwise station, such that

$$
\begin{equation*}
P\left(r_{j}, t\right)=P_{0 j}+\sum_{n=1}^{N_{n}}\left(P_{n j} \cos n \Omega t+Q_{n} ; \sin n \Omega t\right) \tag{94}
\end{equation*}
$$

The equations of motion (3) for the $N_{m}$ modes an now be solved separately for each of the harmonic components of the steady-state response, resulting from the corresponding component of the forcing function, yielding:

$$
\begin{aligned}
& a_{O_{k}}=\frac{\bar{A}_{0 k}}{M_{k} \omega_{k}^{2}} \\
& a_{n_{k}}=\frac{2}{M_{k}} \cdot \frac{\bar{A}_{n k}\left(\omega_{k}^{2}-n^{2} \Omega^{2}\right)+\bar{B}_{n k} g_{k} \omega_{k}^{2}}{\left(\omega_{k}^{2}-n^{2} \Omega^{2}\right)^{2}+g_{k}^{2} \omega_{k}^{4}} \\
& \ldots n=1, N_{m} \\
& b_{n_{k}}=\frac{2}{M_{k}} \cdot \frac{\bar{B}_{n A}\left(\omega_{k}^{2}-n^{2} \Omega^{2}\right)-\bar{A}_{n k} g_{k} \omega_{k}^{2}}{\left(\omega_{k}^{2}-n^{2} \Omega^{2}\right)^{2}+g_{k}^{2} \omega_{k}^{2}}
\end{aligned}
$$

These define the generalized flapping displacements $q_{t}$ :

$$
\begin{equation*}
q_{k}(t)=a_{k}+\sum_{n=1}^{N_{n}}\left(a_{n_{k}} \cos n \Omega t+b_{n_{k}} \sin n \Omega t\right) \tag{96}
\end{equation*}
$$

The flapping displacement and corresponding "plunging" velocity of a blade :dial station can now be computed according to equation (1):

$$
\begin{array}{r}
y_{j}(t)=\sum_{k=1}^{N_{m}} y^{(t)}\left[a_{0_{k}}+\sum_{n=1}^{N_{n}}\left(a_{n_{k}} \cos n \Omega t+b_{n_{k}} \sin n \Omega t\right)\right]  \tag{97}\\
\ldots j=1, N_{r}
\end{array}
$$

$\left.\left.y_{j}(t)=\sum_{k=1}^{N_{n}} y_{i}^{(4)}\right) \sum_{n=2}^{\mu_{n} n} n\left(b_{k} \cos n t-a_{k} \sin n \Omega t\right)\right]$
The plunging velocity affects the quasi-steady component $\boldsymbol{I}_{\boldsymbol{K}}$ of the bound circulation, as can be seen from Figure 14.

### 3.22 Blade Aerodynamics

The blade representation of Reference 2 used trigonometric series expressions for the distributions $\overrightarrow{\mathscr{W}}\left(x_{1}\right)$ and $\boldsymbol{\gamma}(x$,$) mentioned in Section$ 1.22:

$$
\begin{align*}
& \gamma_{K}(\theta)=2\left[A_{o_{K}} \cot \frac{\theta}{2}+\sum_{n=1}^{\infty} A_{n_{K}} \sin n \theta\right]  \tag{99}\\
& \bar{\omega}_{K}(\theta)=\bar{v}_{o_{K}}+\sum_{n=1}^{\infty} \bar{v}_{n_{K}} \cos n \theta=1, N_{r a}
\end{align*}
$$

where $\theta$ is the blade-chord based Glauert coordinate (see Figure 2(c)) defined by

$$
\begin{equation*}
x,=-b \cos \theta \tag{101}
\end{equation*}
$$

The subscript $\mathcal{K}$ refers to the particular discrete disc point to which the $\gamma(\theta), A_{0}, A_{n}, \bar{w}(\theta), \overline{v_{0}}, \bar{v}_{n}$ above belong. $K$ ranges from 1 to $N_{r a}$, where $N_{r a}=N_{0} \cdot N_{r}$. See Figure 15(a). (Note that the vorticity distribution in equation (99) identically satisfies the Kutta-condition of equation (6).) In the numerical computations, the infinite series in equations (99) and (100) were truncated after the third term, ie., $n$ ranged from 1 to 3 . (It turns cut that the higher harmonic components in $7 \kappa(0)$ do not affect the theoretical $\boldsymbol{F}, \vec{F}$, and $\boldsymbol{P} c / 2$ as evaluated from equations (14), (15) and (16).)

Substituting equations (99) and (100) into equation (5), the coefficients $A_{n_{K}}$ can be related to the components $\mathscr{V}_{O_{K}}$ of the ampressed normal velocity:


Figure 15. K-Subscript Notation, Wake Configuration and Wake vortex Strengths. (From Reference 2)

$$
\begin{align*}
& A_{O_{K}}=\bar{v}_{O_{K}} \\
& A_{z_{K}}=-\bar{v}_{I_{K}}  \tag{102}\\
& A_{n_{K}}=-\bar{v}_{n_{K}}
\end{align*}
$$

The total bound circulation is now obtained by evaluating the integral in equation (14):

$$
\begin{equation*}
T_{K}=\left(\frac{\partial c_{l}}{\partial \alpha}\right)_{K} \cdot b \cdot\left(A_{o_{K}}+\frac{3}{2} A_{I_{K}}\right) \tag{103}
\end{equation*}
$$

where $\left(\frac{\partial G}{\partial \alpha}\right)_{K}$ is the measured lift curve slope for the airfoil
section.
The numerical program also limits the total bound circulation $\boldsymbol{T}_{\boldsymbol{L}}$ at disc point $\mathcal{K}$ to a maximum value $\Gamma_{m_{K}}$, which is determined from the previously known stall angle of attack $\boldsymbol{\alpha}$ max for the particular blade section:

$$
\begin{equation*}
T_{m_{K}}=\left(\frac{\partial c_{l}}{\partial \alpha}\right)_{K} b_{K}\left(V_{1_{K}} \sin \alpha_{\max }\right) \tag{104}
\end{equation*}
$$

When the $T_{K}$ as computed from equation (103) is found to exceed this $\boldsymbol{P}_{\boldsymbol{m}}$, it is recognized that the blade is stalled at that position and the computed $\boldsymbol{T}_{\mathcal{K}}$ is replaced by $\boldsymbol{T}_{\boldsymbol{N}} \boldsymbol{K}$. The circulatory lift, as computed using equation (15), is augmented at the stalled segment by a cross-flow drag force.

The other "output" from the blade aerodynamics block in Figure 1 is the wake vorticity strengths. The wake configuration used in the CAL program is shown in Figure 15(b). The wake consists of a mesh wake behind each blade for $N_{s} V$ azimuthal segments, as well as concentrated tip (and root) vortices behind these meshes. The actual locations in space of the various points in the wake model
are determined in the computations of the "wake vorticity positions"; see Figure l. The strengths of the mesh-vortex segments are related to the bound vorticities ac the $\boldsymbol{N}_{\mathrm{ra}}$ disc points as shown in Figure 15(c). The strength of a concentrated tip (or root) vortex segment is taken to be simply the maximum value of the bound vorticity on the corresponding blade when at the appropriate azimuthal position.

### 3.23 Wake-Induced Effects

For the purposes of the numerical program, the vorticity strengths on all the filaments in the mesh wake can be expressed in terms of the Nra bound vorticity strengths as shown above. Further, the wake-induced normal velocities $W_{k}$ were needed only at the $N_{r o}$ points on the disc in the form of components similar to the $\bar{V}_{n \mu}$ in equation (100). For a given wake geometry, therefore, it was possible to evaluate coefficients $\mathcal{S}_{n_{k}}$. such that the induced nomal velocity at the $\mathcal{K}$ th disc point due to the mesh wake is given by

$$
\begin{equation*}
w_{k_{s}}(\theta)=\sum_{j=2}^{N_{n}}\left(s_{o_{k_{j}}}+\sum_{n=1}^{3} s_{n_{k_{j}}} \cos n \theta\right) r_{j} \tag{105}
\end{equation*}
$$

$\ldots K=1, N_{r a}$
Similarly, for the concentrated trailed vortices part of the wake, coefficients $T_{n_{N_{i}}}$ are computed such that the corresponding induced normal velocity ${ }^{\text {d }}$ due to the trailed wake is given by

$$
\begin{equation*}
\omega_{k_{T}}(\theta)=\sum_{i=1}^{N_{0}}\left(T_{K_{K_{i}}}+\sum_{n=1}^{3} T_{n_{i}} \cos n \theta\right) \bar{T}_{i} \tag{106}
\end{equation*}
$$

$$
\ldots K=1, N_{r o}
$$

$\overline{F_{i}}$ is the maximum bound vorticity on a blade when at azimuthal position $i$. The induced velocity coefficients $5^{5} n_{N_{j}}$ and $T_{N_{i}}$ are evaluated in Subroutine SuBl4 of Part 1 of the $C A L$ program. $\mathcal{C l}_{6}$

The "impressed" normal flow component $\bar{W}_{\infty}(\theta)$ is broken up into two parts: one, $V_{\mu}(\theta)$, resulting from all the contributing quantities on page 6 except the wake vorticity; and the other, $\omega_{K}(\theta)$, resulting exclusively from the wake vorticity. Thus,

$$
\begin{equation*}
\bar{w}_{K}(\theta)=v_{K}(\theta)+w_{K}(\theta) \tag{107}
\end{equation*}
$$

$$
\ldots K=1, N_{r o}
$$

and

$$
\begin{equation*}
w_{K}(\theta)=w_{K_{S}}(\theta)+w_{K_{T}}(\theta) \tag{108}
\end{equation*}
$$

Combining equations (102), (103), (107), and (108), it is possible to write

$$
\begin{equation*}
T_{K}=I_{K}+\sum_{j=1}^{N_{r O}} \sigma_{\kappa j} \Gamma_{j} \quad \ldots K=1, N_{r a} \tag{129}
\end{equation*}
$$

which is a set of $N_{r a}$ equations for the $N_{r a}$ unknowns $\Gamma_{\mathcal{K}}$. $I_{K}$ is the bound circulation resulting from the normal flow $v_{\mathcal{K}} \mathcal{K}_{\text {, }}$, and ${ }^{\mathcal{K}} \sigma_{\kappa_{j}} r_{j}$ the induce velocity due to the wake vorticity segments whose strengths depend on the bound circulation at disc point $j$. It may be noted that the $\boldsymbol{I}_{\mathcal{K}}$ depend on the blade response, since the $v_{k}$ include the blade-flapping velocities, torsional displacements, and torsional velocities. Equations (109) are solved by iteration in subroutine GAMMA of Part 1 of the CAL program.

### 3.24 Wake Transport

Representation of the wake transport in the CAL program consists of determining the positions of the points defining the mesh and trailed vortices in the wake model shown in Figure 15. The rigidwake calculations reported in Reference 2 used the following formulas for the $\boldsymbol{x}^{\prime}, \boldsymbol{y}_{2}$ and $\boldsymbol{z}^{\prime}$ coordinates of the point $\boldsymbol{R}_{i} ; \boldsymbol{k}_{\boldsymbol{e}}$ in the wake model, ( $i \geq 2$ ):

$$
\begin{align*}
& x_{i j k_{Q}}^{\prime}=-V_{F s} \cdot \frac{\Delta \psi}{\Omega} \cdot(i-y+\delta) \cdot \cos \alpha-r_{j} \cdot \cos \left[\left(t_{Q}-i+\delta\right) \Delta \psi\right]-\frac{3}{2} b_{j} \sin \left[\left(k_{a}-i+\delta\right) \Delta \psi\right] \\
& y_{i\left(j_{a}\right.}=r_{j} \cdot \sin \left[\left[t_{a}-i+\sigma\right) \Delta \psi\right]-\frac{2}{2} b_{j} \cos \left[\left(k_{a}-i+\delta\right) \Delta \psi\right]  \tag{lIr}\\
& z_{i j k_{a}}^{\prime}=\left(V_{r s} \sin \alpha-\underline{-u}\right) \cdot \frac{\Delta \psi}{\Omega} \cdot(i-7-\delta)-\psi_{o c}(j) \quad \ldots i \geqslant 2
\end{align*}
$$

where $\Delta \boldsymbol{\psi}$ is the azimuthal increment, $\Delta \psi_{x} \frac{2 \pi}{N_{a}} . \mathbb{E}_{a}$ is the azimuthal position of the blade shedding the wake. $\boldsymbol{b}_{;}$is the semichord at the blade segment end paint $j . \mathcal{H}_{o c}\left(r_{j}\right)$ is the builtin coning plus steady deflection of the blade segment end points, positive upwards. CI is the assumed uniform induced velocity (quantity $4^{\prime \prime}$ in Figure 14) over the disc. $\delta$ is the fraction of a time interval through which the wake is advanced with respect to the true blade position $k_{a} \cdot \delta$, which is quasi-empirical, was determined in such a way that for a two-dimensional oscillating airfoil, this discrete representation of the shed wake gave results that were in "reasonabile agreement" in the "reduced frequency range of interest" with the classical solution of the problem. (See Reference 2 for details of this determination of $\boldsymbol{\sigma}^{\prime}$.) For $i=1$, the coordinates $\left(x^{\prime}, y, z^{\prime}\right) i j k_{a}$ are simply the points on the trailing edge of the blade at azimuthal position $\boldsymbol{k}_{\boldsymbol{a}}$.

It will be noticed that this wake transport ignores the displacements in the sideways direction. Also neglected is the variation, over the disc, in the induced velocity in the $\boldsymbol{z}^{\prime}$ direction.

### 3.3 Algorithm for Computing the UR Wake Geometry

It can be seen from equations (110) that the point on the disc* $\boldsymbol{R}_{i j k_{e}}\left(x^{\prime}, y, z^{\prime}\right)$ where the fluid particle now at the point $\boldsymbol{R}_{i} \boldsymbol{j}_{\boldsymbol{a}}$ in the wake used co be $(\boldsymbol{c}-1+\boldsymbol{\delta})$ time intervals before the blades assumed the azimuthal position $k_{a}$, has the coordinates

$$
\begin{align*}
& x_{i j k_{a}}^{0^{\prime}}=-r_{j} \cos \left[\left(k_{a}-i+\delta\right) \Delta \psi\right]-\frac{3}{2} b_{j} \sin \left[\left(k_{a}-i+\delta\right) \Delta \psi\right] \\
& y_{i j k_{a}}^{0}=r_{j} \sin \left[\left(k_{a}-i+\delta\right) \Delta \psi\right]-\frac{3}{2} b_{j} \cos \left[\left(k_{a}-i+\delta\right) \Delta \psi\right]  \tag{111}\\
& z_{i j k_{a}}^{0}=-y_{o c}\left(r_{j}\right) \quad \ldots i \geqslant 2
\end{align*}
$$

[^9]The difference between the expressions (110) and (j11) defines the movement of this fluid particle from $R_{i j k_{a}}\left(x^{0}, y^{\circ}, z^{0 \prime}\right)$ to $\boldsymbol{R}\left(x^{\prime}, y, z^{\prime}\right)$, under the influence of $V_{F S}$ and $\underline{w}$, in $(i-1-\delta)$ time intervals.

In the UR wake-geometry computations, the points of "injection", $R_{i} ; k_{a}$, can be retained* as in (111), but the movement under the influence of both $V_{\text {Fs }}$ and the induced velocity was computed differently, through an actual calculation of the streamlines through the injection points. The steady pressure field used for calculating the streamlines always had the pressure discontinuity on the flat disc.

### 3.31 Subroutine RWAKE

Subroutine RWAKE, which is the entry point for calculating the uR wake geometry, first chooses the steady pressure field through formulas such as those developed in Sections 2.33 and 2.34. Figure 16 shows the flow diagram for subroutine RWAKE. Table III shows the various options for the choice of the steady pressure field based on various physical quantities.
It may be noticed from Appendix I that the values of $\boldsymbol{Q}_{n}^{\boldsymbol{\prime}}(i \boldsymbol{\eta})$ are purely imaginary quantities when $n$ is even. The corresponding $C$ and $\boldsymbol{D}^{\boldsymbol{m}}$ also turn out to be purely imaginary quantities. ( $\boldsymbol{p}_{n}(\nu){ }^{n}$ are always real.) In such cases, the quantities actually handled under QMN (J) were $i Q_{n}^{m}$. To offset this factor $i$, the correspondingquantities $C M N(J)$ and $\operatorname{DMN}(J)$ in core were set equal to $-i C_{n}^{m}$ and- $i D_{n}^{m}$ respectively. Thus, all the computations were done in "reals", "and the products $\operatorname{CMN}(J) * \operatorname{QMN}(J), \operatorname{DMN}(J) * Q M N(J)$ always agreed with their theoretical counterparts $C_{n}^{m} Q_{n}^{m}(i \psi), D_{n}^{m} Q_{n}^{m}(i \geqslant)$ for all J; i.e., for all combinations of $m$ and $n$.

After choosing the steady pressure field, RWAKE sequentially calculates, using equations (lll), the disc-injection points $\boldsymbol{R}_{\mathrm{i}} \mathrm{k}_{a}$ for the mesh and trailed wake grid positions (see Figure 15(b) ); it then calls sibrcutine STMLN, which calculates the streamline through a given disc puint. At the end of RWAKE, all the wake coordinates, disc injortion point coordinates, total normal velocity values at disc points, and induced velocity values at the disc points are written on TAPE8.

[^10]

Figure 16. Flow Diagram for Subroutine RWAKE.


Subroutine STMLN(KEND) follows the fluid injected from a given disc injection point for a number (KEND-1) of time intervals, saving the coordinates of the position after each time interval. The field is assumed steady, and the particle paths and streamlines are identical. A time interval is equivalent to the rotation of a blade from one azimuthal position to the next. Thus,

$$
\begin{equation*}
\Delta t=\frac{2 \pi}{\Omega \cdot N_{a}} \text { seconds } \tag{112}
\end{equation*}
$$

To nondimensionalize time, we must use a normalization that is compatible with the normalizations (30) and (81) for distance and velocity respectively. This calls for normalizing $t$ on the time taken by $\boldsymbol{V}_{\text {fs }}$ to travel a distance $\boldsymbol{R}$; i.e.,

$$
\begin{equation*}
\Delta t=\frac{2 \pi}{\Omega N_{Q}} \cdot \frac{V_{F s}}{R}=\frac{\Delta \psi}{\mu} \tag{113}
\end{equation*}
$$

where $\Delta \psi_{\text {is }}$ the angle (radians) between two azimuthal positions and $\mu$ is, again, the tip-speed ratio. For accurate streamline calculations, the program has the provision for further reducing the time increment through a factor NINC. Thus, each time increment TINC is (I/NINC) times the time interval, and

$$
\begin{equation*}
\operatorname{TINC}=\frac{\Delta \psi}{\mu \cdot N / N C} \tag{114}
\end{equation*}
$$

Equation (114) represents the "usual" time increment value. An exception is the very first interval of a streamline. (See Figure 17) For a frartional wake advance $\delta$, the TINC for the first NINC increments of each streamline will be ( $1-\delta^{\prime}$ ) times that in equation (114). NINC is one of the accuracy parameters read in at the beginning of RWA KE.

Subroutine VLCTS numerically calculates the induced-velocity components $V$ and $W$ at a given point $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ by the procedure indicated in Section 2.35. XUP (the location of "infinity" upstream of the disc) and XINC (the value of the incremental length used for the numerical integration of (83) ) are the accuracy parameters used in VLCTS. Both XUP and XINC are also read in at the beginning of RWAKE.


Figure 17. Flow Diagram for Subroutine STMLN(KEND).

The quantities KFWD and KRVRS appearing in Figure 17 are "counters" for the number of flat-disc crossings made by the particle being followed, in the forward (i.e., normal, or downward) and reverse (ie., upward) directions respectively.
Subroutine CHAMP computes $P, \frac{\partial \rho}{\partial Y}$, and $\frac{\partial \rho}{\partial Z}$ at a given point $(\boldsymbol{\nu}, \boldsymbol{\psi}, \psi)$, using the formulas from Section 2.35. It may he noticed that the multiplier appearing in equation (87) for $\frac{\partial \beta}{\partial \rho}$ will make the quantity real since, as explained in Section 3.31 , 票ther $C_{n}^{m}, D_{n}^{m}$ or $Q_{n-1}^{m}(i \neq)$, but not both, will be imaginary for each value of $n$. The quantity handled in QMNMI (J) will always be real, being equal to $i Q_{n \rightarrow 7}^{m}(i \geqslant)$ if $n$ is odd, and equal to $Q_{n-1}^{\infty}(i \geqslant)$ if $n$ is even. his leads to agreemont between the products $\operatorname{CMN}(\tau) * Q M N M I(J), \operatorname{DMN}(J) * Q M N M I(J)$ and their theoretical counterparts

$$
C_{n}^{m} Q_{n-1}^{m}(i \eta), D_{n}^{m} Q_{n-1}^{m}(i \eta) \text { for all J. }
$$

### 3.33 Subroutine FRWRD

Subroutine FRWRD, which is called in subroutine STMLN, computes the next position on the streamline by invoking the (call ADVNC. - call CHAMP) cycle NINC times. Figure 18 shows the flow chart for subroutine FRWRD. Subroutine ADVNC saves the current values of coordinates, velocities, pressure, etc., and calculates the coordinates of the particle after a time increment TINC, using the current values of the velocity. If a flat-disc crossing is recognized on ADVNC-ing, FRWRD compensates for it by calling CRSNG. Subroutine CRSNG assures that the values of the velocity are continuous across the flat disc, although the pressure has a discontinuity at the flat-disc surface. This is achieved in CRSNG by first finding out the exact point of crossing, and then modifying the $U$ component of the velocity through an operation similar to equation (91).

### 3.4 Integration With the CAL Program

It is seen from Figure 11 that the CAL wake coordinates are generated in subroutine SUBl4 of Part 1 of the original BLP2. (SUB14 calls another subroutine COORD for actually computing these coordinates.) Since our modifications to the BLP2 essentially consist of generating and using an alternate wake geometry, the interfacing between the UR-written wake-computing part of the program and the BLP2 occurs in subroutine SUP14. The FORTRAN IV source listing for SUBI4 appears in Appendix II. Figure 19 shows the flow diagram

*ADVNC saves the values of various quantities and calculates coordinates of new position (after a time increment of TINC) using current velocity values.
**CRSNG compensates for crossing of the disc, so that velocity values on either side of the disc are continuous, although pressure is not.

Figure 18. Flow Diagram for Subroutine FRWRD.

*RWAKE computes the coordinates of the ur wake and writes them on TAPE8

Figure 19. Flow Diagram for Subroutine SUBl4 as Modified at UR.
for subroutine SUB14 as modified at UR. The changes and additions to the original program are shown in dotted lines. The changes were made so as to make it possible to compute and use either of the two wake geometries. The UR wake geometry is generated (and written on TAPE8) in subroutine RWAKE. Provision was also made to make it possible to generate just the UR wake and then terminate the job. Conversely, a UR wake generated in a previous run can be directly read off TAPEB and used for computing the induced velocity coefficients in Fart l. All these options are accomplished through an input integer variable MMSN. (See the source listing for subroutine RWAKE in Appendix II for a description of these options.)

### 3.5 Peripheral Program

Two peripheral programs had to be written in addition to the $U R$ wake computation described in Section 3.3. These were named RWAKEPIC and RSLTSPLT. Both make extensive use of the plotter subroutines package a ailable on the University of Rochester disc library.

RWAKEPIC reads the UR-computed wake geometry from the TAPE8 data set generated in subroutine RWAKE of Part 1 and, using the plotting routines, pictorially displays the wake geometry. Three options are available as to the type of picture produced: pictures of streamlines, instantaneous pictures of the wake generated by a finite number of blades, and progressive distortion of stream-tubes originating (from points equispaced around the azimuth) on the disc. RWAKEPIC, which is a separate main program, was found to be useful in interpreting the UR wake geometry.

RSLTSPLT, which is also a separate main program, reads the converged airloads, blade-bending moments, bound vorticity strengths, etc., from the TAPE3 data set generated toward the end of Part 2 (see page 48). Using the plotter routines, RSLTSPLT then creates curves showing the time histories as well as harmonic analyses of these converged quantities which are the results of the BLP2. RSLTSPLT was found to be helpful in comparing the blade-loads predictions resulting from a UR wake geometry with those from the CAL wake geometry and with experimentally measured blade loads.

## CHAP'RER 4

## COMPUTED RESULTS AND INTERPRETATION

The UR-written wake-geometry program described in Chapter 3 was first used for calculating the induced velocity at a small number of points on a flat, lifting disc at an angle of attack of $2.5^{\circ}$. Figure 20 shows the computed $\boldsymbol{Z}$-component of this velocity. The induced velocity is upward on a crescent-shaped area near the leading edge of the cisc and on an area immediately behind the center. These results agree weli with the results of Mangler's computations shown in Figures 3 and 4.

The wake-geometry program was then used to predict three wake geometries for two flight conditions of the UH-l rotor:
(1) $\mu=0.26$. (Wake geometry computed from streamlines emanating from the flat disc, thus ignoring the steady deflections $Z_{0 c}\left(r_{j}\right)$. See footnote, page 61.)
(2) $\mu=0.26$. (Streamlines emanating from the steady deflected positions of the blades.)
(3) $\mu=0.08$. (Streamlines emanating from the steady deflected positions of the blades.)

The CAL-computed values for the airloads and blade flapping moments are available (Reference 2) for these flight conditions. (These were computed using a wake geometry input calculated from a uniform downwash.) Also available are experimental measurements for the airloads and flapping moments.

The modified BLP2 was then used to obtain the blade-loads predictions resulting from the UR-computed wake geometries. Except for the wake-geometry inputs, all other inputs for these computations were identical to those in the corresponding CAL runs. Even for the wake-geometry inputs, the wake models used were the same.
$4.1 \quad$ UH-1 at $\not \mu=0.26$ (Condition 67 of Reference 15 ), Injection
$\boldsymbol{V}_{\text {F }}=188$ feet per second (forward speed)
$\boldsymbol{\alpha}=5.8$ degrees (disc angle of attack)


Contours of zero downwash are shown by dotted lines. Numbers shown in the figure are $\mathcal{E}$-direction induced velocity values, positive downward, normalized on the forward velocity $V_{\text {fs }}$.

Comparison with Mangler's induced velocity values from Figure 3

| location number | $\begin{gathered} \text { Mangler's } \\ \frac{\mu}{V C_{t}} \end{gathered}$ | $\begin{aligned} & \mathrm{UR} \\ & \text { e } \mathrm{VCt} \end{aligned}$ |
| :---: | :---: | :---: |
| (2) | 0.2 | 0.194 |
| (3) | 0.24 | 0.182 |
| 0 | 0.06 0.34 | 0.04 0.24 |

Figure 20. $Z$-Component of the Induced Velocity on a Lifting Disc, Using the UR Program.

```
\(N W=2\) (number of revolutions of wake)
    \(\boldsymbol{\sigma}=0.7\) (wake advance)
```

The induced velocity on the disc, as computed by RWAKE, showed an upwash area near the forward edge and just behind the center of the disc. Figure 21 shows a typical instantaneous "picture" of wake trails. It is clear that the trails deviate appreciably from a skewed circular helix. Figure 22 shows streamlines starting from $90 \%$ radius on the flat disc. All the streamlines leave the disc in the downward direction, but those in the region $/ Y />0.8$ soon come under the influence of a tip vortex effect, the ones farther out doing so sooner. Streamines in the $/ Y /<0.8$ region continue downward. The result is a separation of the streamines into two wing-tip type systems emanating from the tip areas, and a body of streamlines emanating from the inboard area of the circular wing. The "wing-tip" grouping of streamlines exhibited a tendency to go higher (and move laterally) relative to the latter group, which moves downward comparatively undisturbed by the presence of the disc. Similar separation was observed in the water-tunnel experiments reported in Reference 16. The rolling up of the tip vortices is clearly visible in Figures 23 and 24 , which show the progressive distortion of an initially circular fluid contour, released from $90 \%$ radius on the flat disc, one and two rotor revolutions after release, respectively.

Figure 25 shows a comparison of the unsteady parts of the airloads (time histories), computed from this UR wake geometry, with the CAL-computed airloads and with experimentally observed values. It can be seen that the airloads predictions from the two different wake geometries are only slightly different. The UR wake airloads do not seem to suggest a decidedly better (or worse) agreement with experiment.

In Figure 2l, a tip-trailed vortex is seen passing close to about the $75 \%$ radial station when the blade is at $\%=75^{\circ}$. The induced velocity at that station due to the nearest tip-vortex trail segment was -10.13 fps(CAL)/-3.93 fps(UR). (The negative sign indicates an upward velocity, since our convention has positive $\boldsymbol{z}$ downward. See Figure 6.) The induced velocity at this station due to the entire wake was $+9.48 \mathrm{fps}(\mathrm{CAL}) /+5.745 \mathrm{fps}(\mathrm{UR})$. As compared to these, the normal component $\boldsymbol{V}_{\boldsymbol{Z}} \cdot \boldsymbol{\alpha}_{g}$ due to the control input was 920 fps at that station. The total normal component* was $+43.74 \mathrm{fps}(\mathrm{CAL}) /+48.60$ fps(UR).

[^11]
Figure 21. Typical Instantaneous Geometry of Wake Trails (UH-1, $\mu=0.26, ~$
$N_{b}=2$, One of the Blades at $\psi=75^{\circ}$ )

Figure 22. Geometry of Streamlines in the Steady Flow Field

Figure 23. Three Views of a Fluid Contour, Released From $30 \%$ Radius on the
Disc, One Rotor Revolution After Release (UH-1, $\mu$ Disc, One Rotor Revolution After Release (UH-1, $\mu=0.26$ ).


Figure 24. Three Views of a Fluid Contour, Released From $90 \%$ Radius on the
Disc, Two Rotor Revolutions After Release (UH-1, $\mu=0.26$ ).


Figure 25. Time Histories of the Unsteady Parts of Measured and Computed Airloads (UH-1, $\mu=0.26$ ).

Conversely, at $21.2 \%$ radius and $255^{\circ}$ azimuth, where significant differences between the two computed airloads are apparent in Figure 25 , the induced velocity due to the three nearest segments of the tip-trail from the preceding blade is found to be -4.85 fps(CAL)/-3.41 fps(UR). By comparison, the induced velocity due to the entire wake is -25.1 fps(CAL)/-9.918 fps(UR). This difference of about 15 fps is probably due to the closeness of the mesh wake, which curls around close to tie retreating blade at small radii, combined with the fact that this UR wake computation used streamlines emanating from the flat disc, thus placing the wake lower than the CAL computation which accounted for $\mathcal{Y}_{0}$.
The slight differences between the airload predictions from the two wake geometries are more observable at smaller radii. This is probably attributable to the smaller flapping (plunging) velocities and smaller tangential velocities (and hence smaller $V_{1} \cdot \alpha_{g}$ contribution) at small radii, which would make differences in the wakeinduced velocity more noticeable as seen at the blade aerodynamic section. Contributions to induced velocity by specific vortex filament segments are given in Appendix III.

Figure 26 shows a comparison of harmonic breakdowns of the airloads. Figures 27 and 28 show the flapwise bending moments (time histories of unsteady parts and harmonic components, respectively) for the two computations and the experimental results.

## $4.2 \quad \mathrm{H}-1$ at $\mu=0.26$ (Condition 67 of Reference 15), Injection at Steady Deflected Positions of Blades <br> $V_{\text {FS }}=188$ feet per second (forward speed) <br> $\alpha=5.8$ degrees (disc angle of attack) <br> $N W=2$ (number of revolutions of wake) <br> $\delta=0.7$ (wake advance)

The computed wake geometry for this case showed all the qualitative features of Figures 21 through 24. The tip-vortex effect, however, was delayed (i.e., became discernible farther downstream) since the streamlines, which emanated from the "coned" blade surface this time, had to travel some distance before they entered the regime influenced by the pressure discontinuity on the flat dise which is the source of the tip vortices.

When the computed geometry was used as input to BLP2, the blade motions did not converge. The iterative procedure in Part 2 exhibited a sustained-oscillations type of behavior. These itera-tion-to-iteration oscillations in the generalized blade displacements


Figure 26. Harmonic Analyses of Measured and Computed Airloads (UH-1, $\mu=0.26$ ).


Figure 27. Time Histories of the Unsteady Parts of Measured and Computed Blade-Flapwise Bending Moments (UH-1, $\mu=0.26$ ).


Figure 28. Harmonic Analyses of Measured and Computed BladeFlapwise Bending Moments (UH-1, $\mu=0.26$ ).
were less than $1.5 \%$ for all structural modes except the highest (2nd antisymmetric), for which they were about $8.6 \%$. The nonconverged loads, however, were found to be quite close to the CALcomputed loads. At the $255^{\circ}$ azimuth of the $21.2 \%$ radial station, the airload prediction agreed better with the CAL-computed value than did the prediction of Section 4.1 (see Figure 25).

## $4.3 \quad \mathrm{UH}-1$ at $\mu=0.08$ (Condition 65 of Reference 15), Injection at Steady Deflected Positions of Blades

$$
\begin{aligned}
V_{\mathcal{F}} & =55.1 \text { feet per second (forward speed) } \\
\boldsymbol{\alpha} & =2.5 \text { degrees (disc angle of attack) } \\
\mathrm{NW} & =6 \text { (number of revolutions of wake) } \\
\boldsymbol{\delta} & =0.7 \text { (wake advance) }
\end{aligned}
$$

Realizing that the differences in the conputed airloads for the $\mu=0.26$ flight condition from the two wake-geometr: inputs were only slight, it was decided to carry out a similar comparison for a slower $(\mu=0.08)$ flight condition. For this case, the induced velocity-to-forward speed ratio shiculd be higher than for the faster flight case (with comparable disc loading). Pirioad predictions for this slower flight condition, therefore, would be expected to show more inarked differences for one wake geometry as compared to the other.

During the wake-geometry computation, the domwash-to-forward speed ratio was found to be as high as 1.11 in certain regions toward the rear of the disc. 1"nis makes the entire calculation of the induced velocity field (and the resultant streamlines and wake geometry) suspect, since the theory was baseci on a linearization of the momentum equations. By way of comparison, the highest value of the downwash-to-forward speed ratio was about 0.12 for the $\mu=0.26$ flight condition. Thus, the degree of confidence in the computed wake geometry for the $\mu=0.08$ filght condition must be lower than for $\mu=0.26$.

As expected, the upwash area of the disc was abcut the same as before, viz., a thin crescent near the leading edge and a small area behind the disc center. Many of the streamlines emanatirg from these regions started in the upward direction. (This did not happen in the $\mu=0.26$ case because the induced upwash, bcing trily a small perturbation to the fluid velocity, was more than compensated for by $V_{\text {Fs }}$ sina, which is the normal (downward) component of the forward speed. For the $\mu=0.26$ case, not only was $\gamma_{\text {fo }}$ hi.gher,
but $\propto$ was also somewhat larger, $5.8^{\circ}$, than the $2.5^{\circ}$ for the $\mu=$ 0.08 flight.)

Figures 29 and 30 show typical wake trails for the $\lambda=0.08$ flight condition. The deviations from a helical wake are more prominent in this case because of the higher ratios of all the induced velocity components to the forward speed. Figure 31 shows the streamlines starting from $90 \%$ radius. The wing-tip phenomenon is again evident. (The upward-starting streamlines mentioned above are not noticeable in this picture because the leading edge crescent with large induced upwash was located outboard of about $92 \%$ radius, and the upwash region behind the center of the disc extended only up to about $50 \%$ radius.) This rolling-up-of-the-tip-vortex effect is more clearly seen in Figures 32 through 35, which show the progressive distortion of an originally circular fluid contour released from $90 \%$ radius, one through four rotor revolutions after release, respectively.

Figures 36 and 37 show plots of the unsteady parts of the airloads and bending moments, respectively, for $\mu=0.08$. It can be seen that the airload predictions from the two different wake geometries differ sigrificantly. The airload predictions from the UR wake seem to be in better agreement with experiment in the region around $90^{\circ}$ azimuth. While the rest of the azimuthal positions show the airload histories to be qualitatively similar, the increase in higher harmonic content with the UR wake seems clear. This is emphasized in the blade response in Figure 37.

The wake-induced velocity at the $75 \%$ radius, when the blade is at $\psi=75^{\circ}$, is found to be 14.17 fps , and the total normal velocity at that point is +55.81 fps. Thus, for this slower flight condition, the wake-induced velocity is, as expected, a more significant fraction of the normal velocity experienced at the blade section,


$84$

Figure 30. Typical Instantaneous Geometry of a Wake Trail (UH-1, $\mu=0.08$,

Figure 31. Geometry of Streamines in the Steady Flow Field

Figure 32. Three Views of a Fluid Contour, Released From $90 \%$ Radius on the
Disc, One Rotor Revolution After Release (UH-1, $\mu=0.08$ ).

Figure 33. Tnree Views of a Fluid Contour, Released From $90 \%$ Radius on the

Figure 34. Three Views of a Fluid Contour, Released From $90 \%$ Radius on the Disc, Three Rotor Revolutions After Release (UH-1, $\mu=0.08$ ).

Figure 35. Three Views of a Fluid Contour, Released From $90 \%$ Radius on the Disc, Four Rotor Revolutions After Release (UH-1 $\mu=0.08$ ).


Figure 36. Time Histories of the Unsteady Parts of Measured and Computed Airloads ( $\mathrm{UH}-1, \mu=0.08$ ).


Figure 37. Time Histories of the Unsteady Parts of Measured and Computed Blade-Flapwise Bending Moments (UH-1, $\mu=0.08$ ).

## CONCLUSIONS AND POSSIBILITIES FOR FURTHER WORK

The wake-geometry computations reported in Chapter 4 show that the flow field behind a lifting actuator disc features tip vortices similar to those from a lifting wing.

The computed airloads for the $\mu=0.26$ flight condition of the UH-1 teetering rotor seem to be less sensitive to wake geometry than are those for the $\mu=0.08$ flight condition. This is indicated by the slight differences between the airload predictions resulting from two different wake geometries (CAL rigid helical wake and UR distorted wake) for the $\mu=0.26$ condition and by the significant differences between similar predictions for the $\mu=0.08$ condition. The accuracy of the UR-computed wake geometry for the slower ( $\mu=0.08$ ) flight condition is questionable because of applying ínearized equations to an inapplicable amplitude regime. Still, the airload predictions of the blade-loads program, even if this wake geometry input is considered "arbitrary", are useful in providing some insight as to the differences which will result from wake-geometry differences.

This stronger sensitivity to wake geometry in the case of a slower flight condition calls for some quantitative discussion. As suggested near the end of Chapter 4 , the wake-induced velocity in the case of the $\mu=0.08$ condition forms a more significant fraction of the normal velocity experienced by the blades. This suggests an examination of the variation of contributions due to wake effects and those due to all other effects.

The faster flight condition certainly has larger contributions from the geometric pitch input and of the blade flapping to the total nomal velocity at the blades. Thus, for example, at the $85 \%$ radial station, the collective pitch was 0.2011 rad at $\mu=0.26$, as compared to 0.1 .209 rad at $\mu=0.08$. The amplitude of the cyclic pitch input was 0.1195 rad at $\mu=0.26$, as compared to 0.04 rad at $\mu=0.03$. Similarly, the amplitude of the first azimuthal harmonic component of the displacement in the rigid flapping of the teetering blade (normalized on tip deflection) was 0.6096 at $\mu=0.26$, as compared to 0.1830 at $\mu=0.08$. At the rotor speed of $33 \mathrm{rad} / \mathrm{sec}$, this flapping component results in a maximum flapping velocity of 20.1 fps at $\mu=0.26$, as compared to 6.04 fps at $\mu=$ 0.08 .

The corresponding quasi-steady bound vorticity strength $I_{k}$ comparisons (see Figure 14 and equation (109;) for the $85 \%$ radial station are (sq ft/Sec):
steady component 331.3 ( $\mu=0.26$ ) as compared to $249.9(\mu=0.08)$ first harmonic 158.3 ( $\mu=0.26$ ) as compared to 75.34 ( $\mu=0.08$ )
second harmonic 33.2 ( $\mu=0.26$ ) as compared to 8.9 ( $\mu=0.08$ ) third harmonic $28.91(\mu=0.026$ ) as compared to $13.75(\mu=0.08)$

These values show that in the case of the faster flight condition, the variations in the quasi-steady bound vorticity are larger. This same point is illustrated in another way in tible IV, which compares the contributions of the two terms on the right-hand side of equation (109). The ratio of the influences of the quasi-steady and wake-induced quantities on the bound vorticity is quite high for the first two harmonic variations at $\mu=0.26$, but not so in the casc of $\mu=0.08$. For the 3rd, 4th, and 5th harmonics, the ratic is amcller at either value of $\mu$, because there are no cyclic variations ili $\alpha g$ or $V_{g}$ above the ist harmonic, and the blade flapping displacements are smaller in harmonic components above the 2nd.

| TABLE IV. | COMPARISON OF CON'TRIBUTIONS OF THE QUASI-STEADY |
| :--- | :--- | :--- |
|  | AND WAKE-INDUCED QUANTITIES TO THE BOUND |
|  | VORTICITY STRENGTH COMPONENTS AT THE 85\% RADIAL |
|  | STATION, UH-1, $\mu=0.26$ AND $\mu=0.08$ |

As regards the amount of wake vorticity remaining close to the blades, Figures 21,29 , and 30 show how much more rapidly the distance between a given blade and the wake increases for the higher speed case. The strength of the wake vorticity, of course, also enters these considerations, but this is a more complex and less influential variable as regards the effects of formard speed.

It can therefore be concluded that the relative insensitivity of the blade loads to wake geometry in the case of the faster flight condition ( $\mu=0.26$ ) is due mainly to (1) the wake being blown away relatively quickly and (2) a predominance of the cyclic pitch input, tangential velocity variation, and blade flapping displacements in the tetal time-varying aerodynamic environment at the blade sections.

It may be note that all the computations reported here pertained to the UH-1 two-bladed teetering rotor. It would probably be worthwhile to carry out similar comparative airload computations for a fully articulated rotor such as the four-bladed Sikorsky H-34. This might facilitate comparisons of the importance of blade displacements, cyclic pitch and tangential velocity variations and wake-induced effects. Since the H-34 has four blades, there will be more wake vorticity near the disc, and hence wake geometry could be more influential than it seemed in the $\mu=0.25$ case for the UH-l. If this does seem worthwhile, the wake computation can be repeated using the converged blade loads to establish the time-average pressure field which is the basic input to the UR wake geometry determination.

If the UR wake-computing program is to be extensively used with the CAL BLP2, considerable saving in running time could be achieved by writing the UR wake geometry onto a temporary directaccess data set at the beginning of subroutine SuBl4 (see Figure 19) and reading it from this data set instead of repeatedly reading it sequentially from TAPE8.

As alluded to earlier, a fundamental difficulty exists in tle work reported here. Linearizing the equations of motion on the basis of disturbance velocities being small compared to forward speed means that the wake computations will be more reliable, for a given tip speed, at the higher advance ratios. On the other hand, since we are using an assumed steady pressure field for calculating the paths of particles which are really unsteady, sufficient time must elapse to allow an averaging of the unsteady effects to take place. Thus, the streamline positions--and the wake geometry obtained therefrom-will be most realistic (in the case of a finite number of blades) only when each vorticity element leaving the blades is subjected to the effect of a large number of blade passages. For a given number of blades and a given tip speed, this calls for a small advance ratiof. Put another way, the method explored here will be most accurate for a lightly loaded rotor (i.e., low thrust per disc area) with many blades at high forward speed and high tip speed. Like all other rotor vortex aerodynamic theories, the results will reflect changes in vortex strength and position only when the number of blades is sufficiently high to leave vortices close to succeeding
blades and yet low enough to keep the strength of the vortices from becoming too low. The scope of this research has not allowed definition of these limits, if in fact they exist. It has suggested, however, that wake details are important for tip speed ratios of 0.1 and below, and are unimportant at around $\mu=0.25$ and above.

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## APPENDIX

I-1 Legendre Functions of the First Kind, $P_{n}(z)$

$$
\begin{aligned}
& P_{0}(z)=1 \\
& P_{1}(z)=z \\
& P_{2}(z)=\frac{1}{2}\left(3 z^{2}-1\right) \\
& P_{3}(z)=\frac{z}{2}\left(5 z^{2}-3\right)
\end{aligned}
$$

Particular values:

$$
P_{n}(0)=\left\{\begin{array}{cc}
(-1)^{\frac{n}{2}} \cdot \frac{(n-1) \|^{* *}}{2^{*} \cdot\left(\frac{n}{2}\right)!} & \ldots n^{2} \text { even } \\
0 & \ldots n \text { odd }
\end{array}\right.
$$

$$
P_{n}(1)=1
$$

Asymptotic values for $|z| \rightarrow \infty$ :

$$
P_{n}(z) \sim \frac{(2 n-1)!!}{n!} z^{n}
$$

[^12]For $\mathcal{Z}$ outside the segment $(-1,+1)$ :

$$
\begin{aligned}
& Q_{0}(z)=\frac{1}{2} \log \frac{z+1}{z-1} \\
& Q_{2}(z)=\frac{z}{2} \log \frac{z+1}{z-1}-1 \\
& Q_{0}(z)=\frac{1}{4}\left(3 z^{2}-1\right) \log \frac{T+1}{z-1}-\frac{3}{2} z \\
& Q_{3}(z)=\frac{7}{4}\left(5 z^{2}-3\right) \log \frac{z+1}{z-1}-\frac{1}{6}\left(15 z^{2}-4\right)
\end{aligned}
$$

For $-1<z<+1$, replace $\log \frac{\pi+1}{2-7}$ in the above formulas by $\log \frac{1+2}{1-7}$. Particular values for $\epsilon \rightarrow 0(\epsilon>0)$ :

$$
Q_{n}(7+6) \sim \frac{1}{2} \log \frac{2}{6}
$$

Asymptotic values for $/ \mathbb{Z} / \rightarrow \infty$ :

$$
Q_{n}(z) \sim \frac{n!}{(2 n-1)!!} \cdot \frac{1}{2^{n+1}}
$$

(Note: from the above formulas, $Q_{n}(i \geqslant)$ take the forms:

$$
\begin{aligned}
& a_{0}(i \eta)=-i \tan ^{-1} \frac{1}{\eta} \\
& Q_{1}(i \eta)=\eta \tan ^{-1} \frac{1}{\eta}-1 \\
& a_{2}(i \eta)=\frac{i}{2}\left(3 \eta^{2}+1\right) \tan ^{-1} \frac{1}{\eta}-\frac{3}{2} i \eta \\
& a_{3}(i \eta)=-\frac{\eta}{2}\left(5 \eta^{2}+3\right) \tan ^{-1} \frac{1}{\eta}+\frac{5}{2} \eta^{2}+\frac{2}{3}
\end{aligned}
$$

and the values on the disc $(\boldsymbol{\eta}=0)$ are:

$$
\begin{aligned}
& Q_{1}(i 0)=-1 \\
& Q_{3}(i 0)=\frac{2}{3}
\end{aligned}
$$

I-3 Associated Functions of Legendre of the First Kind, $P_{n}^{m}(z)$
For $-7<\pi<7$ :

$$
\begin{aligned}
& P_{n}^{m}(z)=(-1)^{m}\left(1-z^{2}\right)^{m / 8} \frac{d^{m} P_{n}(z)}{d z^{m}} \quad \text { (Hobson's notation) } \\
& P_{1}^{\prime}(z)=-\sqrt{1-z^{2}} \\
& P_{2}^{\prime}(z)=-3 z \sqrt{1-z^{2}} \\
& P_{3}^{1}(z)=-\frac{3}{2} z\left(5 z^{2}-1\right) \sqrt{1-z^{2}} \\
& P_{4}^{1}(z)=-\frac{5}{2} z\left(7 z^{2}-3\right) \sqrt{1-z^{2}} \\
& P_{2}^{2}(z)=3\left(1-z^{2}\right) \\
& P_{3}^{2}(z)=15 z\left(1-z^{2}\right)
\end{aligned}
$$

For $Z$ outside the segment $(-1,+1)$, replace $(-1)^{m} \cdot\left(1-Z^{2}\right)^{m / 2}$ in the above formulas by ( $\boldsymbol{I}^{2}-1$ ) $\quad$ ma

Particular values:

$$
P_{n}^{m}(0)=\left\{\begin{array}{cl}
(-1)^{\frac{n+m}{2}} \cdot \frac{(n+m-1)!!}{2^{\frac{n-m}{2}} \cdot\left(\frac{n-m}{2}\right)!} & \cdots(n-m) \text { even } \\
0 & \cdots(n-m) \text { odd }
\end{array}\right.
$$

For $\epsilon \rightarrow O(\epsilon>0)$ :

$$
\begin{aligned}
& P_{n}^{m}(1-\epsilon)=(-1)^{m} \cdot \frac{(n+m)!}{m!(n-m)!} \cdot\left(\frac{\epsilon}{2}\right)^{m / 2} \\
& P_{n}^{m}(1+\epsilon)=\frac{(n+m)!}{m!(n-m)!} \cdot\left(\frac{\epsilon}{2}\right)^{m / 2}
\end{aligned}
$$

Asymptotic values for $/ z / \rightarrow \infty$ :

$$
p_{n}^{m}(z) \sim \frac{(2 n-1)!!}{(n-m)!} \cdot z^{n}
$$

I-4 Associated Functions of Legendre of the Second Kind, $Q_{n}^{m}(Z)$
For $\boldsymbol{Z}$ outside the segment $(-1,+1)$ :

$$
\begin{aligned}
& Q_{n}^{m}(z) \triangleq\left(z^{2}-1\right)^{m / s} \cdot \frac{d^{m} Q_{n}(z)}{d z^{m}} \\
& Q_{y}^{\prime}(z)=\frac{1}{2} \sqrt{z^{2}-1} \log \frac{z+1}{z-1}-\frac{z}{\sqrt{z^{2}-1}} \\
& Q_{2}^{1}(z)=\frac{3}{2} z \sqrt{z^{2}-1} \log \frac{z+1}{z-1}-3 \sqrt{z^{2}-1}-\frac{1}{\sqrt{x^{2}-1}} \\
& Q_{3}^{1}(z)=\psi^{3}\left(5 z^{2}-1\right) \sqrt{z^{2}-1} \log \frac{z+1}{z-1}-\frac{15}{2} z \sqrt{z^{2}-1}-\frac{z}{\sqrt{E^{2-1}}} \\
& Q_{4}^{\prime}(z)=\frac{5}{4} z\left(7 z^{2}-3\right) \sqrt{z^{2}-1} \log \frac{z+1}{z-1}-\frac{5}{6}\left(21 z^{2}-2\right) \sqrt{z^{2}-1}-\frac{1}{\sqrt{z^{2}-1}} \\
& Q_{2}^{2}(z)=\frac{3}{2}\left(x^{2}-1\right) \log \frac{x+1}{z-1}-3 z+\frac{2 z}{z^{2}-1} \\
& Q_{3}^{2}(z)=\frac{15}{2} z\left(z^{2}-1\right) \log \frac{z+1}{\frac{z-1}{102}-15 z^{2}+10+\frac{2}{z^{2}-1}}
\end{aligned}
$$

For $-1<\boldsymbol{Z}<+1$, replace the multiplicative factor $\left(\boldsymbol{Z}^{2}-1\right) \boldsymbol{m} / 2$
in the above formulas by ( -1 ) $\left(1-Z^{2}\right) m / 2$; and replace $\log \frac{Z+1}{Z-1}$ by $\log \frac{1+1}{1-2}$.
Particular values for $\epsilon \rightarrow 0, \quad(\epsilon>0, m>0)$ :

$$
Q_{n}^{m}(1+\epsilon) \sim(-1)^{m} \cdot 2^{m / 2-1} \cdot \frac{(m-1)^{!}}{\epsilon^{m / p}}
$$

Asymptotic values for $/ \mathbb{F} / \rightarrow \infty$ :

$$
Q_{n}^{m}(z) \sim(-1)^{m} \cdot \frac{(n+m)!}{(2 n+1)!!} \cdot \frac{1}{z^{n+1}}
$$

Note: From the above formulas, $Q_{n}^{m}(i \eta)$ take the forms:

$$
\begin{aligned}
& Q_{1}^{\prime}(i \eta)=\sqrt{1+\eta^{2}} \tan ^{-1} \frac{1}{7}-\frac{\eta}{\sqrt{7+\eta^{2}}} \\
& Q_{2}^{\prime}(i \eta)=3 i \eta \sqrt{7+\eta^{2}} \tan ^{-1} \frac{1}{7}-3 i \sqrt{1+7^{2}}+\frac{i}{\sqrt{7+\eta^{2}}} \\
& Q_{3}^{1}(i \eta)=-\frac{3}{2}\left(5 \eta^{2}+1\right) \sqrt{7+\eta^{2}} \tan ^{-1} \frac{1}{7}+\frac{15}{2} \eta \sqrt{1+\eta^{2}}-\frac{\eta}{\sqrt{7+\eta^{2}}} \\
& Q_{4}^{1}(i \eta)=-\frac{5}{2} i \eta\left(7 \eta^{2}+3\right) \sqrt{7+\eta^{2}} \tan ^{-1} \frac{1}{7}+\frac{5}{6} i\left(2 \eta^{2}+2\right) \sqrt{1+\eta^{2}}+\frac{i}{\sqrt{1+\eta^{2}}} \\
& Q_{2}^{2}(i \eta)=3 i\left(1+\eta^{2}\right) \tan ^{-1} \frac{7}{7}-3 i \eta-\frac{2 i \eta}{1+\eta^{2}} \\
& Q_{3}^{2}(i \eta)=-15 \eta\left(7+\eta^{2}\right) \tan ^{-1} \frac{1}{7}+15 \eta^{2}+10-\frac{2}{1+\eta^{2}}
\end{aligned}
$$

and the values on the disc ( $\boldsymbol{\eta}=0$ ) are:

$$
\begin{aligned}
& Q_{2}^{1}(i 0)=-2 i \\
& Q_{4}^{1}(i 0)=\frac{8}{3} i \\
& Q_{3}^{2}(i 0)=8
\end{aligned}
$$

I-5 Some Properties of the Associated Functions of Legendre,

$$
\begin{aligned}
& \frac{P_{n}^{m}(z)^{a n d} Q_{n}^{m}(z)}{P_{n}^{m}(-z)=(-1)^{m+n} P_{n}^{m}(z)} \\
& Q_{n}^{m}(-z)=(-1)^{m+n+1} Q_{n}^{m}(z)
\end{aligned}
$$

Orthogonality

$$
\begin{aligned}
& \int_{0}^{1} p_{n}^{m}(z) p_{n}^{m}(z) d z=0 \quad \text { for } n \neq n^{\prime} \\
& \int_{0}^{1} p_{n}^{m}(z) p_{n}^{m}(z) \frac{d z}{1-z^{2}}=0 \quad \text { for } m \neq m^{\prime}
\end{aligned}
$$

Normalization

$$
\begin{aligned}
& \int_{0}^{1}\left[p_{n}^{n}(z)\right]^{2} d z=\frac{1}{2} \int_{-1}^{1}\left[P_{n}^{m}(z)\right]^{2} d z=\frac{1}{2 n+1} \cdot \frac{(n+m)!}{(n-m)!} \\
& \int_{0}^{1}\left[P_{n}^{m}(z)\right]^{2} \frac{d z}{1-z^{2}}=\frac{1}{2} \int_{-1}^{1}\left[P_{n}^{m}(z)\right]^{2} \frac{d z}{1-z^{2}}=\frac{1}{2 m} \cdot \frac{(n+m)!}{(n-m)!}
\end{aligned}
$$

Differentiation

$$
\frac{d p_{n}^{m}(z)}{d z}=\frac{n z}{z^{2}-1} p_{n}^{m}(z)-\frac{n+m}{z^{2}-1} p_{n-1}^{m}(z)
$$

For $\mathcal{Z}$ outside the segment $(-1,+1)$,

$$
\begin{aligned}
& \frac{d Q_{n}^{m}(z)}{d z}
\end{aligned}=\frac{n z}{z^{2}-1} Q_{n}^{m}(i)-\frac{n+m}{z^{2}-1} Q_{n-1}^{m}(Z)
$$

Subroutine SUB14 (as modified at UR) ..... 107
Subroutine RWAKE ..... 118
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Subroutine STMLN ..... 137
Subroutine VICTS ..... 145
Subroutine CHAMP ..... 150
Subroutine FRWRD. ..... 155
Subroutine ADVNC ..... 159
Subroutine CRSNG ..... 162
Subroutine VARFMT ..... 166
Mainprog RWAKEPIC ..... 167
Mainprog RSLTSPLT ..... 180



 * $O X, 3 H V B=14$, ox, \# $8 x, 3 H Y B=1$
$13 H N H=14,7$
 3 3 r.vi $=F 8$.




$$
\rightarrow R A=N R * \nabla A
$$

****LLSW AUTNBYUR ACTNBYUR
*****

$$
\begin{aligned}
& \text { GU TU (700,60U), LLSH } \\
& \text { GOO CALL OVOCHK(11) } \\
& \text { TOO CUNTINUF }
\end{aligned}
$$




KTRAC=REII
DELX $=V F * \operatorname{COS}(A T R) * D T$
RH(J) = (KE(J) +RE(JPI))/2.0
SCM(J)=(SCE(J)+SCE(JP1)//2.0
SCDK(J) = SCM(J)/(REIJPI) - RE(JI)
SCMD2(J) $=$ SCM(J)/2.U
STDFM(J) $=$ STOFE
STDFM(J) $=(S T D F E(J)+S T D F E(J ? l)) / 2.0$
CONTINUE
CONTINUE
DO $171 \quad J=1$, NRP1
SCEIS $(J)=1 . j * S C E(J)$
continue
VO $194 K A=1, N A$
$K D T=(K A-1): V R$
LO $190 \mathrm{~J}=1, \cup R$
$D=K D T+J$
ABKKDINAK
169
171
$B L D R A B=R E(N R P 1)$
DU $243 \mathrm{KA}=1$, NA
$X C=(F L O A T(K A)+W K A M 1) / F N A$
[O] $241 \mathrm{~N}=1,4$
ARG $=1$
AMUUI
222 CNPS I(N,KA) $=\operatorname{COS}(A R G)$
(FIABS(CNPSI(N,KA)).LE.U.OLO)CNPSI(N,KA) $=0.0$
IF $\{A B S I S N P S I(', K, K A)$.LE.U.UIUISNPSI(N,KA) $=0.0$
241 CONTINUE


$$
\begin{aligned}
& \text { 340 } 1 \text { WG } 436 \text { 1BN = 1,NB } \\
& K A=I R T R+(18 V-1 \\
& 344 \text { LC } 350 \mathrm{KI}=1 \text {, NR }
\end{aligned}
$$



Aitingut

YOARNITV
GOARNITV
MSWWFAF*
actnayur

| Alt VBrur |
| :---: |
| ALITNEYUR |
| AUT NBYUR |
| ALT UBYUR |
| AUTNBYUR |
| AUINBYUR |
| ALT NBYUR |
| ALTNBYUR |
| ALITNBYUR |
| ACUNBYUR |
| AOTNBYUR |
| *****NTV |
| ADTNBYUR |
| ALITABYUR |
| ADINBYUR |
| ADI NBYUR |
| ALTNBYUR |
| ALSNBYUR |
| adtnayur |
| AUT MEYUR |
| ALITNBYUR |
| ALIT NGYUR |
| ALTIVBYUR |
| AUTNBYUR |
| AOTNBYUR |
| AOTNBYUR |
| ALINBYUR |


$|M|=I-1$
|MI $=1$
$J A L E K=I A$
$J A L E K=I A L-I N I$
IFIJALENSSIt, It, ग¿U


+IAII.11 = LiK\#YUTFRIJALER,II
HASI, I)
CETAII, 5 5y LUNTIVUE
$1=G R$ \% (-CUSAL* LU(IFR(JALER,I) +SINAL*XUUFR(JAZER,I)

$$
\text { GU } 101522.5231 \text {, NIV }
$$

 LU ,24 l=NSV, IwAPI

$$
\begin{aligned}
& \text { JALER=1AL-IN1 } \\
& \text { IFIJA } E R: 520,25,526
\end{aligned}
$$

$$
\begin{aligned}
& 527 \text { IFIJALERIS23OS25,526 } \\
& 5: 5 \text { JARYR=JARFR +NA }
\end{aligned}
$$

 r【A(I,2) = U《*YUOFR(JALER, () $\quad$ (JALER,I) +SINAL*XUOFR(JAZER, I)) LETAII,

S?4 CUWI 1:vUE
522 REWIMJ :
312 GUNI INut


***SUR 14

R[AT)(2) ((XIII, J),EIA(I,J),(EIA(I,J),I=1,NSV), J=1,NRPI)
RFAL(c) $((X I(I, J), E T A(I, J), L E T A(I, J), I=N S V, N W A P I!, J=I, N J V)$


CONTINII.
430 CONTINII:
434 REWIND
436 CUNTINUT
450 CUNTINUF
REWIND 4
REWIND
RETURN
RETURN
ENU
402 WRITE(6, 4(14)IAV, VIV, ((XI(I, J), EIAII, J), LETA(I, J), J=I, ATV), दU4 FIRMATIIHU


41 j ClivTINUE
$I W R I=2$
$\because$

***RWAKE

> SUBRGUTINE RWAKE STARTS HERE.


[^13]IHIS SUBKOUTINF WILL if CALLED ONLY IF MMSN $=3$ CR 4.

COMMENT
FPMS FPMC 2 FPMS2. LAVERAGE MUMENIUM CUNDITICN SAII PFIEU.) COL,CU3,C12, I $12, C 14, D 14, C 23,023$ CUMPUIEU FROM DR,RH:, $\forall F S, ~ B C, ~$ PIMJM, ROLMO, IPMZO,FPMCI,FPMSI,FPMCZ,FPMS2.
( AVEIAGE MUMENTUM CONUITION NOT SATISFIEU
CUI, CU3,C12,U12,C23, U23 COMPUTEU FRCM UR,RHG.
SIEAUY,FIRSI HARMUNIC ANO SECUND HARMINIC CCMPONENTS IF IHE
 SPANWISE LIFI DENSITY (IN LUWT/IN AT $40.75,85,90$ ANL IS
PER CENT KADIUS.
( AVERAGE MOMENTUM CONOITIOM SATISFIEG.I
CO1,C(13,C12,L12,C14,D14,C23,D23 COMPUTEU FRUM DR,RHU, VFS, HC, PIMIJM, ROLMO AND IHE LIFT UENSITY COMPCNENIS IN $T$ ABUVE. APPLICABLE FUR HU-I LIFI UENSITY COMPLNENIS IN 7 ABUVE-
( AVERAGE MGMCNIUM CONDITION NCI SATISIIED.)
CURNELL WAKE GEUME TRY USEU IN THIS RUN.
U UF K WAKE UEOMEIRY, WHICH WAS GFNERAIEO IN A PREVIUUS RUIG,
WAS USED IN THIS RUN. HENCE SURKUUIIN. RWAKE WAS VUT CALLED. SUBRUUTINF GUIJRU TAS NEVER CALLED IN IHIS RUN.
U DF K WAKE GEDMETRY WAS GENERATEU. JL W WAS IERMINATEL LN REIURN FRUM SULRDUTINE RHAKE. THUS, ÜENERATICN GF THE LTAPA
UATASET WAS IHE SOLE PIRPOSE OF THIS KUN. UATASET WAS IHE SOLE PIJRPOSE OF IHIS KUN.
U OF $N$ WAKE GEDAETRY WAS GENFKAIEU ANL US U OF $\sim$ WAKE GEGAETRY WAS GENFRATEU ANL USED IN THIS RUN.
HENCE, SUBRUUTINE RWAKE WAS CALLED. SUGRCLIINE COORD WAS NEVER CALLED IN IHIS RUN.
VILUCITIES AI POINIS IN THE MAKE WERE COMPUTED USING IHE
INCORKECI FUNMULA,WHICH IGNORFD THE UX AND UY DISPLACEMENIS. STREAMLINES WERE INITIATED FRUM THE FLAI UISK, THUS IUVURING THE STEAUY IJEFLECTIONS STDFEIJI.
AN INIEGKAIION FRGM INFINITY' WAS PERFOKMED EACH TIME IHI
STAEAMLINTS aERE INITIATEUFRUM THE FLAT ISISK, THUS IGVURING STILL THE PRESSURE-DISCONTINUITY SURFACE. THUS, IHE SILAUY DEFLELTIUNS STUFE (j) UF THE BLAUES WFRE ACCOUNTEC FOR.
11
0

$\frac{7}{2}$
VNSW
\[

$$
\begin{aligned}
& \qquad 1 \text { vVSw MIITER, JNLY IF THE U OF R WAKE UECMETRY IS COMPUTEL; } \\
& \text { I.EEO ONLY IF MMSW }=3 \text { OR } 4 . \text { I }
\end{aligned}
$$
\]

LIMENSION DUMCYY $250: 1$


## 

-tTA1 151.11) -LETA1151.11)
-YM(IU) EM(LO)

- SCDR(10)
- S2(10.250)
, 11(10.25)
, T1(10,25)
, FMT(12)
, NRPI ,NTV


,VF
XUCRR(25,151), YU)FR(25,151i, 2 UOFR(25,151), VNRML112,251

URN: ( 0 ), (D, M: ( $(6)$, PNNMi(6), (UMNMI (6), RUNNO(4), TIILE (14)
FMI $3(12)$, XUISK(12,25), YaISK(12,25), 201SK(12, <5),FMT4(12) ULISK(12, 25),VDISK(12,25), WUISK(12,25), IISW, JJSW
KKSH,LLSin, MMSH, iUNSW, XINJN, YINJN, ZINJN, JRAL, JALER, JPTSL
AK.LIHOLJUT,LIAP?,LIAP4,LIAPE,IHROL, IMNOL, ISCGL
XUP, XMAX, VINC,BL,LG,FNINC, PSIND, TINC, JMAX, AMUO, PS IO
PSIUC, AMU, ATA, PSI, EPS, $X, Y, L, X P, Z P, R P, U V E L, V Y E L, W Y E L, P, T$
YIME,JPI,JINC,KIAG,KFWO,KRYRS,DPOY,OPOL
M. P jOL, PGLE, VELL, VOLO, HOLL, TULU, TIMOL, KOLE
$x_{1} L 1, Y O L 0, Z O L U, X P i a L O, Z P O L D, A M U O L, A T A C L, P S I G L, E P S C L$
$X S A J E, Y S A V E, Z S A V E, U S A V E, V S A V E, W S A V E, A M U S V, A T A S V, P S I S V$
EPSSV,PSAVE, MEPSV,NH,MEPS,RNOM,ALUEG,ALPHA, COSAL, SINAL
rGUIVALi-NCE (SZ, DUM $Y$ Y)

EM( 1$)=0$.
$F M(2)=0$.
$E M(3)=1$.
$E M(4)=1$.
$F M(5)=2$.
$E M(0)=0$.
FN(I)=1.
$E N(2)=3$.
$f N(3)=2$.
$F N(4)=4$.
$f N(5)=3$.
$E N(0)=0$.
WN2O(1) $=-1$.
GMVU(2) $=2.13$.
$G M+U U(3)=2$.
$W M N / 13(4)=-8.83$.
$W M N \angle 0(5)=8$.
GMNLU( 6$)=0$.
nidl is the number if increments per alimuthal position.
REACCS, 1201 IXUP, XMAK, NINC, ERLDLG, XUNIM
$F G R \mu A T I J X, F 4, I, 6 X, F 4,1,6 X, 13,7 X, F 5,3,7 X, F 7,41$
$A K=V F S /(D R *$ (JMEGR)
$H N=F L O A I(N B)$
F JINC=FLDAT(NIVC)
$P S I V[1=36,0, /\|F: 1 A+F V I V C\|$
I/NC=AK*TPI/! (Fiyt*FNINC)
JMAX $=(()$ XUP + I.) $/$ II IN(.) +FLUAI(NWAPI) *FNINC) $\# 1.25$

 IH XMAX $=, F 4.1,10 X$, GHNINC $=, 13,10 X, 8 H B L D L G=, F 5.3$.
$10 X, \operatorname{Hi} X(1 / T M=, F 7.4 / / 5 H 4 N=$

I4//סH ALLEG $=, F 10.5 .1$ NX. 7 HALPHA $=, E 15.81$



## CALL COEFS

IEWINO 0
WRIIE(8) RUNNG, TITLE,NA,NA,NR,NRPI,NW,NSV,NTV,AR,ALGEG,OR,VFS, BLDLG, XUP,NINC,ULTV,OLRV, (RE(I),SCEIS(I),I=I,NRPI)

101204 JJRAU = I, NRPI
JRAD=JJRAD
RNDM = (SORT(NFiJRAU)*RF(JRALI) SCELS(JRAD)*SCELS(JRAE)))/OR IF (KNDM.GT.1.99) RNDM $=0.99$

AMUU $=-$ SORT (I.-RVOM*RNGM)
L,ELTLG=-ATAHI, CELI(JRAD)/RE(JRADI)
LU 1207 JJACE $=1$, VA
JALCR=JJALER
PSIU=(TPI* (IFLJAT(JALER-1)+BLDLG)/FNA) +DELTLG
IF(PI-ABS(PS[1j) 1120:, 1210,1210
PSIO=PSIO\#(1.-(IPI/ABS(PSIJ))
GO IO 120 H
CuNIINUE
$X P=-$ RNDM $=\cos (r S I(0)$
$Y=$ RNDM*SINIPSIO)

| $n$ |
| :--- |
| $\underset{~}{2}$ |
|  |

1211
$\angle P=-0.1 E-3$
U! TO 1223
$\angle P=-S T D+E(J K A D) / U_{R}$
CONTINUE
KKENO=NSV
CUNTINUE

1211
1212
1213
1213
211
arilf(g) (lKUUFRIKA/ER,KPIJL), YUOFRIKALER,KPTSL), (L.
12, CuVIIVUE



LELIL.=-AIAISSELOIVRDII/(XE(VRPI)-DLIV)) 11: $1213 \mathrm{JJALER}=1,10 \mathrm{~A}$


1214
1215
1216,

$$
\begin{aligned}
& \text { CU:11 } 1214 \\
& \text { CENTINUE }
\end{aligned}
$$

 $Y=$ RNUM $F \operatorname{SIN}(P S I G)$
$12<42 P=-0.1 \mathrm{c}-3$
$1227 \angle P=-$ STUFE(DiRPI)/ON
1230 CLNTIVU-
KKENU $=\operatorname{NOAPI} 1$

のRITE(B) (IXIMFR(KACER,KPTSL), YLDFR(KALER,KPISL),
*****vTV

12CO fOREATIIHI/////' DISTRIBUTIUN OF VVEL ON THE DISK. (IYOISKIKKAD,KAL

जRIIE(6,FMT4) ((VOISK(KRAU,KAZER), KRAD=1,NHBRDS), KALZR=1, NA)



12:U FIKGATIHI/////CDISTR




VUISK(KRAD,KAZEK), WDI SK(KRAD,KAZER),
VINRML (KRAD,KALE?),KRAD=1, NMBRDS), KALER=1, NA) tNe fllt 8
KEM (N1) B
METURV
tvo

***CUEFS
SUBROUTIVE GJLFS

 I, SCMD2(10), SCE15(11), STUFM(10), CNPSI(4, 25), SNPS (44, 25), w(275) UIMENSIUN QMN2O(6)
COMMON RM(10)





- IUSTAT.IAL
$. D I \quad . D S T R T$
- LNIS - LNES
$\begin{array}{ll}-B M & \text { - BN } \\ -A A S & A A B\end{array}$
-IWN ISTZ
DPRIDS NB
$-P I \quad$ ITPI
. BLURIU.ALER

$3<$
. 25
$1(10,250)$
$3(10,250)$
(2140,25)
LOMMOIV
YKI
-BL
KB
IBV
SVL
$K A$
FNA
RIRAI
X:NTM
I.UMMUN XUOFR(25,151), YUUFR(25,151), ZUOFR(25,151), VNRML(12.25)

, LUIVALLVLE (SL, UUMaY)


KSI:N $=1$
IFIKKS


$$
\begin{aligned}
& 148 \\
& 740
\end{aligned}
$$



$$
\begin{aligned}
& \begin{array}{l}
0 \\
0 \\
H \\
\underset{3}{3} \\
0 \\
7 \\
3
\end{array}
\end{aligned}
$$

$19 \cos 2=0$.
$99520=0$.
p95c. $1=0$.

$P+5 S 2=0$.
$10112 \quad 1=1.0$
CMN(I) $=0$.
$M(I)=E M(I)+0.05$
NiN(I) =EM(I) +0.03
CMNII)=0.
CMN(I) $=0$. CONIINUF.
112
Lu 16 ( $722,704,105,706,707,708,709,710,733,734,735,736,731,738,739$
TU4 REAU(S, 1O2) CMN(I), (IMN(I)
TUZ HURMATII 6, EIO.3,16X:EIU.
'II HUKMATCIX, SSHCOEFFICIEMIS GMNIII AND DMNIII WERE SUPHLIED AS INFUI
, (H) ro 701
7.3) KHAU(S,721) Cl,CMK,CMP

$\operatorname{CMN(3)}=-0.625 * \operatorname{CMP*CT/(AR*AR)}$
$\operatorname{BMN}(3)=-0.625 * \operatorname{CMR*CI/(AR*AR)}$
GU TO 701


117 FURMAIILX, 4 HHTHE FIJLLUWING QUANIITIES WERE SUPPLIED AS INPUT / / $C T=T T * A R * A R /(3.141593 * R H O *(j R * U R * V F S * V F S)$
$C M R=R O L M D /(T T * D R)$
$C M P=P I M O M /(T T * D R)$
CU TU 715


$$
\begin{aligned}
& \text { LAPPING MOMENT }= \\
& \text { FLAPPING MOMENT }=
\end{aligned}
$$

SLUSS/CFI.
ROLMO $=$ EIU.3. VFS $=$. ROLMO= $X, 14 H$ IHESE LEAU TO
 1.4 FCRMATIIX, 4 JHIHE FULLOWING UUANJITIES WERE JUPPLIED AS INPUT
FURMAIIIX, IIH STESDY FLAPPING MOMENT = , ELS. B. SH LGFT/
11

$$
\text { WKIIFIG, } 7 \text { SUIFPMLO, FYMCL,FPMSL, FPMC 2 IFPMS2 }
$$

FLAPPIVG MOMENJ=
FLAPPING MUMENT $=$

$$
1 / 11
$$

$$
\begin{aligned}
& \text {-E15.8.5H LOFT/ } \\
& \text { F15.8.5H LHFT/ }
\end{aligned}
$$

$$
\begin{aligned}
& \text { EI5.8.5H LnFT/ } \\
& , E\{5.8,5 \mathrm{H} \text { LHFT/// }
\end{aligned}
$$






| 2 | 1x，6HP4062 $=$ | ．E10．3， 1 UHLIS／IN， |  | P40S2＝ | ，E10．3．7HLBS／IN | 1 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | LX，ohp $75 \angle U=$ | ，E10．3，7hLis．fing | 1 |  |  |  |
| $\checkmark$ | 1x，6HP75cl＝ |  |  | P7551＝ | ，E10．3．7HLBS／IN | 1 |
|  | $1 \times, 6$ ¢P 7 ¢ $2=$ | －1 LO．3．1rHLSS／IN． |  | P75S2＝ | ，E10．3，7HLES／IN | 1 |
|  | 1x．6HP ¢5 $20=$ | ，Elu．3．7hLiss／IN， | ／ |  |  |  |
| $\gamma$ | 1x，6HPB5心 $=$ | ，EIC．3，1鱽；S／IN． |  | P8yS：$=$ | －E10．3，7HLBS／IN | 1 |
| 1 | （x，大HP S SC）$=$ | ，E10．3．1chios／IN． |  | P6） $22=$ | ，ELO．3，7HLES／IN | 1 |
| 4 | $1 x, 6 H P y 010=$ | －F10．3，7HLBS／IN． | 1 |  |  |  |
| 1 | $1 \mathrm{x}, 6 \mathrm{HP90C} 1=$ | ，El1）．3，16HL，S／IN． |  | P90S1＝ | ，E10．3，7HLBS／IN | 1 |
| l | 1x，6tip $7012=$ | －LIU．3，1 rehilis／IN． |  | P90 $22=$ | －EIU．3．7HLBS／IN | 1 |
|  | Lx，6HPY； $20=$ | －EIU．3，7HLBS／IN． | $/$ |  |  |  |
| 1 | $1 x .6$＋P45C1 $=$ |  |  | P9551＝ | －E10．3，7HLBS／IN | 1 |
| † | $1 \times, 6 \mathrm{HP95C2}=$ | －E10．3．18HL； $5 / I N$ ． |  | P95S2＝ | ，E10．3．7HLBS／IN | ／1／ |
|  | IX，14HTHESE | LEAO TJ |  |  |  |  |
|  |  |  |  |  |  |  |
|  | ＋0．045＊P 002 | ＋0．U475¢P957．j） |  |  |  |  |
|  |  |  |  |  |  |  |
| I $\quad$ C．045＊P90C1＋0．0475＊P＊5C1） |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| $1 \quad 0.045 * 99051+0.0475 * P 95511$ |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| FPMS2＝12．＊DK＊DR＊（0．126672＊P40S2＊0．162078＊P15S2＊0．0628125＊P85S2 |  |  |  |  |  |  |
| $1+0.045 * 2+052+0.0475 * p 95321$ |  |  |  |  |  |  |
| （i）1n 731 |  |  |  |  |  |  |

[^14]

$x 0=50$
$\times 1=11.5$
$\times 2=10$.
$x \angle=10$.
$Y 1=3.5$
$Y 2=4.5$
UL $741 \quad 1=1.20$
RNDMC(I) =YI-14.*AIM1/25.)
RNDMS(I) $=$ Y2-14.*AIM1/25.)
$\operatorname{PMN}(4)=-2.5 * A M U Z O *(7 . * A M U Z U * A M U Z O-3) * S Q R T.(1 .-A M U Z C * A M U Z O)$
PMNIS $=15 . *$ AMUZO* 1 1。-AMUZ (1*AMUZO)
P2O(1) =(CMN(1)*PMN(1)*OMNZ.3(1) +CMN(2)*PMN(2)*GMNZU(2)) *2.



CALL SCALEIPZO,26,5.,PZUMN,SCLZO,11
CALL SCALE(PSI.26,5., PSIMN, SCLSI, 1)
CALL SCALF(PC2,26,3.,PC $2 M N, S C L C 2,11$
CALL SCALEIPS2,26,3.,PSZMN,SCLS2,11 CALL AXISIXI,YIP, 2OHALIMUTHALLY CONSTANT,-20.5..0., PRZOMN,SCLZC) CALL LIME\{PLU,KNUMC, 26,1,5,1)
$\pm$
 CiU $742 \quad 1=1,20$
PLO(I)=P $20(1)+X U$ PCI(I)=PCI(I)+XI 4HRNOM ,-4.4.,270.,0.,0.251
 PSIII)=PSI(I)+XI $P C 2(I)=H C 2(I)+X 2$
$P S 2(I)=P S 2(I)+X 2$

(4) PS2(I) $\begin{aligned} & \text { CAS } 2(1)+ \\ & \text { CALL AXIS(XU,Y }\end{aligned}$
not reproducible

$$
\begin{aligned}
& \text { CALL AXIS(XI, YI, 4RRIUM },-4,40,270.00 .00 .25)
\end{aligned}
$$

$$
\begin{aligned}
& \text { CALL } \triangle \times I S(x 1, Y 2 P, 1 C H F I T S T \text { SIVE ,-10,5., O., PSIMN,SCLSII }
\end{aligned}
$$

$$
\begin{aligned}
& \text { LALL LINE PLL } 2, H \text { NOMC, } 26,1,2,3 \text { ) }
\end{aligned}
$$

> CALL LIVC(PS?,RAGM, $2 t, 1,5,5$ )
> $\mathrm{r}_{1}=\mathrm{rlP}_{1} \mathrm{P}-3$.
> $Y 1=Y L-0$
> GALL SYMBOLXO,YI,U. 14,4 PHCOMPONENTS OF PRESSURE-UIFFERENTIAL CN I
CALL SYMACL $(\times 0, Y 2 P, 0.14,30$, FREE-STREAM OYNAMIC PRESSURE.,0.,301
CALL PLOTBu..O...-3)

144 cuntitive
 4HUMVI, $|1,2 H|=,[10.3 \mid$ 122 EVTURN
END
 1，SCMLZ（10），SCE1，ग111，STIFM（10），CNPSI（4，25），SNPSI（4，25），W（275） 1，IMEVSITA UMNO（6）

| i，JMMJd | R -103 |
| :---: | :---: |
| 1 | x（1451，11） |
| $\angle$ | x4（16） |
| 3 | Ux（10） |
| 4 | $S<(10,2,0)$ |
| ， | $33(10,250)$ |
| \％ | 17110，23） |
| 1 | LPSI（25） |


| I．IMMADN | HMPAN | －wkADV | －IUSTR | －1A2 | －NRP 1 | －NTV | －DLJV | －DLRV |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | UELK | －VFSAI | － 01 | －USTRT | ，NA | ，1 | ，J | －18 |
|  | J H | －KI | －Liv 15 | －LN2S | －2N3S | ，FPI | －AL | －AM |
|  | AV | －乚 L | － 3 H | －BN | －CC1 | ，CC4 | ${ }_{-C \mathrm{Cl}}$ | －CC1O |
|  | RA | －K ${ }^{\text {d }}$ | －AAS | －AA 6 | －XVZ | ，XV1 | ，XV2 | ，XV3 |
|  | IRT | －IBN | －IWN | －IST2 | －DMPRAO | －IVZ | －TVI | －TV2 |
|  | 『『3 | －SVI | －SVI | －SV2 | －SV3 | －KA2 | ，KA3 | －ISW4 |
|  | ISW ${ }^{\text {S }}$ | －KA | －DPRU | －N8 | －NR | －NSV | ，NADB | －NRA |
|  | NiwAP1 | －FVA | －PI | －IPI | ，LNI | －LN2 | －2A3 | ，VF |
|  | OMET | ，KT：くAl： | －HEDRA | －ALER | －1 SW 3 | －1 SWI |  |  |
| CUMMON | $x \cdot \sim N T$ |  |  |  |  |  |  |  |
| COMMEN | XuOt | R（25，1 | 51），YU | 2125,1 | 1），LU0 | R（25 | ），$V$ | L112 |
| COMMON | EMIt | 1， $5 \cup 61$ | ，JMiNZ | $61, P M N$ | 6）．QMN | 61.6 | ）．OP | ） |
| LUMMON | Sipmes |  | INTolg | NMMI（6） | UMNM1 6 | ）©RUN | （4）． 1 | LE（14 |

（IPMP（ $s$ ），DiJMN（ ol，PMNMI（6），GMNMI（6），RUNNO（4），TITLE（14）
$\begin{array}{ll}\text { E4 } 31 \\ \text { ES } & 55\end{array}$ FMI 3(12), XDISK(12,25), YDISK(12,25),20ISK(12,25), FMT4(12) UUISX(12,25), VUISK (12,25), WDISK(12,25), 11SW, JJSW
 t UUIVALENCE (SL, DUMMY)
tquivalence ( DUMMY(1),RE), (DUMMY(12),SCE), (DUMAY(23),STDFE),


(DR BLURAD) (VF VFSI (AI
FUUVALFNCF (AMUD, AYU0), (UMNZO(1), QMNZU
 $X=X P * C O S A L-\angle P * S I V A L$
$L=X P * S I: N A L+\angle P * C I S A L$
PSIUD=P;II*LEU./PI
I RAIOJFGRMAFIUI RRGM OISK CJIRRUINATES TO ELLIPSGEDAL CCCRDINATES
P) $I=A T A N 2(Y,-A P)$
$A R A$
$A B C U=1,-X P * K P-Y * Y-L p * L P$
$K(A B C D=S Q R T(A B C D * A B C D+4 * *\langle P * \angle P)$
IF(IABCOHRTABCO).GT.0.1 GO 50777
$A M U=0$ -

ISMLN=60.*TIMER(1,1)



IH YIVJV=, FIU. 3, TH ZINJN=, EIU.3,11H XUPSTREAN = ,E10.3,3X,1HI
 IIX, IHI, IIX, IHP, 10X, 4HIVVL, $8 X, 4 H V V E L, 8 X, 4 H W V E L / / / 1$
WH: I' PUIVI JUSI UPSTREAM UF INJECTIUN PGINT.
$x=x \operatorname{INJN}+U_{0} 1-2$
LALL VLGT
$V V F L=V V I L-(1 \cdot 1 t-2) * 1 P D Y$
wVEL = WVEL-(U.1E-2)*1.P1/Z
wbiAIV SIARIIdG POINT FUR THE STREAMLINE.
VPEFF=(I.-UVELI*SIVAL*WVEL*COSAL
$V$ VRML (JRAC, JALFR ) = VPEFF
IF (VPEFFI $141 E, 1419$, 1420
WKIIEI $O$, 14EIIVPEFF
FORMATIF JHE VELUCITY

1419
1421
MSS = SIGVRI. $2 P$
$\cdots$
H
3
ATA=SURTI-AHCUI



G1) IO 4444
333 AMU $=0.7071067812 \neq E P S * S G K T(A B C O+R T A B C O)$
ATA $=-2 P / A M U$
4444 MEP $=1.0$ O\&EPS
Call chama

$+リ S=-1$.
MEP ${ }_{3}=-1$
KFWD=KF~M
1404 UVEL=P
PPS = 1 .
1473 PLLL: $=$ P
JIVC=NIVC
JIMS:
KIAtu $=1$
WRIIF 16,1412$)$
1412 FURMAI $/ / / 1 \times, 0$

LuII FiRGiAIIIHt,40X.'.'///I

XUOFR(JALER, 1$)=x$


142
XLISK(JRAD, JALER)=X
I INC=(I--GLDLG)*AR*TPI/(FNA*FNINC)
JP T $=2$
LUISKIJRAD ARER =2


rall frwnd
XUOFR(JALER, 2$)=X$
YUOFR(JALER,2)=Y
TINC=AR*TPI/(INA*FNINC)
IFIKEND-311413,1414,1414
1414 CO $1415 \mathrm{JJPI}=3$, KENO

ZUCNFRISALER, JP TI $=2$
1415 CUNTINUE
1413 WRITE(6,1416)JKAD,JAZFR, ISMLN, KFWD,KRVKS


$$
5177 A * * *
$$

 UOISK(12,25), VDISKI 12,25$)$, WDISK(12,256, IISW, JJSH KKSW, LLSH, HMSW, NISW, XINJN, YINJN, LINJN; JRAL, JAZER,
 $P S I U D, A M U, A T A, P S I, E P S, X, Y, Z, X P, Z P, R P, U V E L, V V E L, W V E L, P, I$ TIME, JPT, JINC, KIAG, KF HO, KR VRS, DPDY, DPUZ
MEPSOL, POLD, UULO, VOLO, WOLU, TOLD, TIMOL, KOLU
XCLU, YOLD, ZOLE, XPULU, ZPULD, AMUOL, ATADL, PSIOL, EPSCL
XSAVE, YSAVE, ZSAVE, USAVE, VSAVE, WSAVE, AMUSV, ATASV, PSISV EPSSV,P SAVE, MEPSV,NW, MEPS,RNDM, ALDEG, ALPHA, COSAL, SINAL <br> > tUUIVALENCE (SZ, DUMMY) <br> \section*{EUUIVALENCE (SZ, DUMMY)} <br> \section*{EUUIVALENCE (SZ, DUMMY)}
LOMMON
LOMMON
CUMMON
LOMMUN
CGMMON
COMMON
LOMMON
COMmDN
Climmon
COMMON
tQUIVALENCF ( LUMMY(1),RE), (UUMMY(12), SCE), (DUMMY(23), STDFE), UUMMY(34), W2), (UUMMY(45), hC), OUMMY(404), STOFM), (OUMMY(414), CNPSI),
FGUIVALEVCE (LR, BLDRADI, (VF,VFSI, (AI,ALDEG), (AIR,ALPHA) EUUIVAL ENCE (AMUS, AMUOI, (OMNZG(I),QMNZOII), (PSIO,PSIO)
Xenfig must be a vegative number
ARUPT = AMU
AI API $=A T A$
PSIPT $=P S I$
$V V=u$ 。
$h h=0$.
XMOPT $=$ XUP $-0.5 *$ XGYTM
$X$ MQMD2 $=x-0.5 * \operatorname{XGNTM}$
$\angle S I N A L=2 * S I N A L$
$Z \operatorname{COSAL}=Z * \operatorname{COS} A L$

UVEL $=U V+L+P-P I L i$,
VVEL $=V V+D L L T A X * V P D Y$
$W V E L=W W+C E L T A X * I P G Z$
IFIMEPMPT.EU.NEPSI 6O TU 103
1,14 FORMAT: 1 \# MEPMPT ANO MEP: DU NOT AGREE JUST BEFGRE RETURNING FRU
 212,",MEPS=',1/, **/'//1 1נ3 REIURA
END

SUBROUTINE GHAMP


## 5


-A8(250)
2ETA(151.11)

- LETA(15LgIE
- ZM(IG)
-SCDR(10)
$.52(10.2501$
$.51(10.25)$
-FMIII2)
- 


$.13(10,25)$
. SPSI(25)
IUSTRT,IAL
$.0 T$ USTRT , ZHIS . ZNZS
AAS -AAC

- IWN :SVZ
2
2
2
0
0
0
0
0
-PI PTPI

$\mathrm{X}, \mathrm{V}$ TM
XLINFR
XLINFR(25,151), YULER(25,151), ZUOFR(25,151), UNKML(12,25) EN(6), EV(6), QMNZO(6), PMN(6), QMN(6), CMN(6), UPN(6)
CIMMMUN
2
2
3
4
$i$
0
1
CUMMON
$c^{2}$
2
4
5
1
$i$
$y$
$y$
COMMOV
common

COMMON
ELUIVALENCE (SL, DUMMY)

EGUIVAL EVCE (AMUD,AMUO), (QMMZO(I), QMNZO(I)),(PSIO,PSIO)
AMUAMU=AMU*AMU
AMSU=1.-AMUAMU
RTAMSQ=SORT(AMSG)
ATAATA=ATA*ATA
ATSU= I OATAATA
RTATSU=SURT(ATSU)
IF(ATA.LT.O.U) GO TW 1002

***CHAMP
1 ,
, E1s.7,'
ATA=
?
 SUBKOUIINE FRGIND
UIMENSIDM OUMMY(250U)
 TSCMUZ(10), SCE15(11), STLFM(10), CNPSI(4,25), SNPS(14, 25), W(275)
UIMENSIUN QANZU(6)
-SCM(10) , AB(250) -
- ZETA(151.11)
LM(10)
5
0
0
0
0
0
.52110 .2501
.$T 1110.251$
,FMT(12)

XGNTM
XURFR
COMMON XUNFR(25,151), YULFR(25,151), ZUOFR(25,151), UARML112,25)


$\because$ UJIVAL.NEF IST, UURMYI

IF(ifs)lbus, loje, loul

514
CIS
150
SUB FURMAT(IX,2I3,3x,11,2X,12,3x,9E12.4)
lsus CONTINUK
RETUKN
ENU
$\sum_{\infty}^{\infty} \sum_{\infty}^{\infty} \sum_{-\infty}^{\infty}$

3NAOY***




$$
\angle P=-x * \operatorname{SiNAL}+\angle * \operatorname{CCS} A L
$$


***CRSNG

EQUIVALENCE (SZ, DUMMY)

PSAVE $=P$
AMUS $V=A M U$
ATASV $=$ ATA
USAVE = UVEL
XSAVE =X
$\angle S A: E=Z$
$615 x P=x P U L U+(x P-x P U L U) * 2 P G L D /(\angle P O L D-\angle P)$
SPECIAL
SPECIAL

PSI =AIA $A, 2(Y,-x, P)$
$A B C L=1 .-\angle P * A P-Y * Y$
IFABCD.GI....) oit




G6Gt CALL CHIMP
JJSNC=JIVL-L

611 WRITEIG,B11)
GII FRRMAIIIX.5ITM
oll frrmatilix. 5 itmeinheile, back at the new position of the particle -
$A M U=A M U S V$
2
4
4
4
4
4
4
4
$P=P S A V E$
$V$
$X=X: A \cup F$
$Y=Y ; A V F$

164
***CRsNG
$z=$ SiAVE
KEIUNV
tive
$\checkmark$


**VARtMT
JATA DMLM $1 \mathrm{HL}, 1 \mathrm{H} 2,1 \mathrm{H} 3,1 \mathrm{H} 4,2 \mathrm{H}, 1 \mathrm{H}, 1 \mathrm{H}, 1 \mathrm{H} 8,1 \mathrm{H} 9,2 \mathrm{HIO}, 2 \mathrm{HI}, 2 \mathrm{HI} 2$,
H13, $2 \mathrm{H} 14,2 \mathrm{H} 15,2 \mathrm{H} 16,2 \mathrm{H} 17,2 \mathrm{H} 18,2 \mathrm{H} 19,2 \mathrm{H} 2 \mathrm{O}$
FinCLMA.GT. 20)GU is 101
(i) $100 \quad 1=1$, 3
UNFMTII)= DMCHIII

100

> iMAFPT(7)=LM(1)
> $\begin{aligned} & \text { RETURIN } \\ & \text { NRITE } 6,1 U 2) \text { NLLMN }\end{aligned}$

$$
\begin{aligned}
& \text { XIV IN SU } \\
& \text { KEIURA } \\
& \text { xII IN SUBRIUTINE VIRFM. } \\
& \text { END }
\end{aligned}
$$


**AKPIC PICNO (4), RUANO (4), IITLE(14), RE(11), SCE15(11) XUISK(12,25),YDISK(12,25), ZOISK(12,25), JJKAD(11)


XTRAI (2,25, $15(1), Y \operatorname{TRAL}(2,25,151), 2 T K A L(2,25,151)$
BILD(2000), XPOS 400$), Y \operatorname{POS}(400), 2 \operatorname{POS}(400), X 6 \operatorname{POS}(400)$ FMT4(12)
IR:VISNI JHI צOS OGNIVIGO TIVYI GYVM JO G\&חIJId SnngNVINVISNI CORRESPONDING TO JAZB1.
PRIIUUCES PICTURES OF THE (NNKAU*NNB) STKEAMLINES THROUGH THE
( (JJRAUII),K), I=L,NNRAD). THE LUHEST OF THESE NNB AZIMUTHACLY
PERIOUIC ALIMUTHAL POSITIONS IS JALBI. IHE LENGTH CF EACH STAEAMLINE IS NNSV OR NNWAPI AZIMUTHAL INTERVALS.
1 DID YUU REALLY UNDERSTAND THIS DESCRIPTION
 NIVSV AZIMUTHAL INGREMENTS.
DIMENSIOY
 GIMEN'S IUN DIMENSIUN
DIMENSION DIMENSIUN Im?


いつ
mRIte RIN-lliejtiflcatiens.


$$
\begin{aligned}
& \operatorname{MiRP} P=1 \\
& \operatorname{siKP}=1=?
\end{aligned}
$$

$\triangle W A P I=(: W W * N A)+1$

341 HURMAT(12x, 121
342 FURMATI//////' TUTAL VUMBER OF PICTURE-PASSES IN THIS RWAKEPIC RUN $1=$ NNPASS $=.13,1 / 1 \%$ NMBROS $=1,121$

WRIIF 6,34 ? $)$ NIVPASS, PMISRUS
342 FURMAI $/ / / / / / /$ TUTAL VUMBE


[^15] 1
TKAILIVG JURTEX GJ-IIRUINATES.

 (5x,2(3E17.4,10X))
. 20,NMBKUS,20HE1I.311
317 FURMAIIIHI////1 DISTRIBUTIJN UF UVEL ON THE DISK. IIULISKIKRAD,KAZE

WRITE ( 6 ,FMT4) (IUCISK(KRAD, KAZE $\Psi$ ), KRAD =1, NMARDSI,KALER=1,NA) b.RITE(6, 336)NMBRDS, NA
316 FIRMAIIIHI////C DISTRIBUTION DF VVEL DN THE DISK.IIVLISKIKRAD,KAZE $(R), K K A U=1,{ }^{\prime},\left(\left\langle 0^{\prime}\right), K A Z E R=1,,^{\prime}, 12, \prime^{\prime}\right)^{\prime}, / / / 11$


FURMAI (IHI////' DISTRIBUTIJN OF VNRML ON THE DISK. (IVNRML(KRAD,KA
LLERI,KKAL=1,

- RIIE © , FMT4) ( $(V, \forall R M L(K R A D, K A L E R), K R A D=1, N M B R O S), K A Z E R=1, N A)$
FVA= FLOAT(NA)
$P I=3.1415926516$ ALPHA=ALDEG*P//180.
CALL PLOTS(BILD(200U), 2000)
$x 1=3$.
$Y 1=3$.
$\times 2=3$.
$Y 2=2$.
$x 3=\times 2-2 * \operatorname{CaS}(A L P H A)$
$Y 3=r 2-2 * S I(A L P H A)$
$x_{4}=x 2+2$. $* \operatorname{CoS}(A L P H A)$
$x_{6}=17$
$\mathrm{Y} 4=\mathrm{Y} 2+2 . * S$ IN(ALPHA)
365 FURMAII/IO SISCE IISN= $, 11, \circ$, IHIS PIGTURE-PASS PRCLUCEC A PICTURE 1 !HL STREAMLINES.'//I
GUTU 36C
363 WRITEC6.36
366 FOKMATI// SINCE IISW=',11,", THIS PICTURE-PASS PRCCUCED A PICTURE ( A HLUIU COINTUUR. $/ / /$
360 :NNS $V=$ NNSV +1

> OQ 340 IIPASS=1,NNPASS
> KEAD (5, 304IF, IISW, NINB, JALE I, NNH, NNSU, NNTV, NNRAU, $\begin{aligned} & 1 \text { (AJRAD(I) } 1=1 \text {, NNRAD) } \\ & 304 \text { FGRMAI(F5.3, } 5 x, 12,5 x, 12, ~\end{aligned}$
> 304 FGRMAIIF5, 3, $5 \mathrm{X}, 12,5 \mathrm{x}, 12,7 \mathrm{X}, 12,5 \mathrm{x}, 12,6 \mathrm{X}, 13,6 \mathrm{X}, 12,7 \mathrm{X}, 12,7 \mathrm{X}, 6121$

$$
\begin{aligned}
& 1 \text { (JJRAO(I), I=1,NNRAD) } \\
& \text { WKIIE(G, SOTIIIPASS,F,IISN, NNB, JALBI, NNW, NNSV, NNTV,NARAD. } \\
& \text { OD } 340 \text { IIPASS = I,NNPASS }
\end{aligned}
$$

TEST FIAR CUAFLICI IN THE PARAMETERS FGR IHE CRMLIMCU ANO THIS 'PIC• JUG.
VUCHK=G

IF (JALBL.GT.INA/:VNB ) I YOCHK=?
IF (innoct.vm)

if $312 \mathrm{kKAi}=1, \mathrm{~N}$, RAD
IF( (JJRADIKRAOI).GT. (VRPIt JTVI)NUCHK=6

LALL EXIT
313 CONIINUT.

$$
N: W A P I=(N \cup H * N A)+1
$$

PSIGI=(JALBI-1)*2.*PI/FNA
315 PSIBI=PSINL*(1.-(2.*PI/ABSIPSIB1))
316 PbIIG=180.*RSİ1/PI
CALL FACTUR(f)
ERAW IOENTIFIC, VIIUV,PICTURE PARAMETERS.

 CALL SYMBOL (0.,8.7,0.14,RUVVO,0.,16)


DRAn AXES, ELLIPSE,ETC. LO $334 \mathrm{~J}=1,31$
AJJ=FLOAT $(\mathrm{J}-1)$ r=2.*AJJ/50.
$Y$ PuS(J) $=Y 1+Y$ $Y P() \leq(102-J)=Y 1+Y$ $Y$ PCSS $(100+1)=Y 1-Y$ YPOS $X=S U R T(4 \bullet-Y * Y) * C O S I A L P H A 1$
KPOSIJI $=X 1+X$
xPOSIJI = XL $+x$
$X P C S(202-J)=X 1+X$
$\times P O S(100+51=x 1-X$
XPOS $(100+5)=x 1-x$
$\times P D S(102-J i=x 1-x$
XPOS $1102-J i=\times 1-X$
CONTINUE
CONTL LINE
CALL LINE (XPOS, YPUS, 201, 1,0,0)
CALL PLOT $3.03,0,3)$
CALL PLOT $3 .+3 .+3$ )
$N$
+
$\vdots$
$\vdots$
$\vdots$
$\vdots$
2
2
2
2
3
3
$\stackrel{5}{m}$
CALL AXIS $10.5,0.5,4 \mathrm{H} \quad Y, 4,5 ., 90 .,-1.25,0.51$ CALL SYMBCL $16.54,9.5,0.21,14$ HPORT SIUE VIEh.0...14)
CALL SYMBOL $6.54,9.3,0.21,14 H P O R T$ SIUE VIEW,0..144
CALL SYMBRL $17.44,0.3,0.21,4 H P L A N, 0.44)$
CALL SYMBOL $\left(X_{3}, Y 3,0.07,1 H, U .+1\right)$
LALL SYMBOL (X4, Y4,0.07, 1H,0., 1)
CALL PLUT(3.08., +3 )
P111 $3-0 .+31$
$\stackrel{1}{2}$
$\vdots$
$\vdots$
$\vdots$
$\vdots$
$\vdots$
CALL AXIS(0.5,6.5,4H $\quad$, 4, 3.,90., 0.75,-0.5)

60 10 1344, 345, 346), IISW
344 CONTIVIL
WAKE-TRAIL. WAKE-TRAIL.






$$
\text { IF(JRAO,EH. (NXP } 1+1) \text { )GO } 10332
$$

$$
352 \text { U0,j3 JPT = 1, VVaAP }
$$

11

CALL LITEIXPIIS, YPUS, NNWAPI,I,NA,KISNI
340 CUNTIVUE
CALL PLUT(22..U...-3)
GO TU 340
CUNTUJPS.
GIJNTOURS.

> CUNT GURS.
346 CUNTINUE

## 43 CUNI IVUE 47 CONI INUF

## CALL SYMBUL (16.,9.5,0.21,LOHFROHT VIEW,0., 101

 CALL SYMBUL $1 \times 6 M 2, Y 2,0.07,1 H, 0 ., 11$ CALL SYMBUL (X6,Y4,0.07,1H,U.OI)CALL SYMBOLIX6,YL,0.C7,1H 10.911
CALL SYMBOL $1 \times 6 \mathrm{P} 2, Y 2,0.07,1 H, 0 ., 1$
JRAD=JJRAU(KRAD)
IFIJRAD.GT -VRPII GI) TI
 YPOSTJPII $=$ Y $1+2$ **YMESHIJRAD,JPT,NNSVI LPOS(JPT)=Yく-L.* LMESH(JRAU,JPT,NNSY)


## $355 \mathrm{JIOR}=1$ <br> 153

HIJRAD.ED.(VRP I+1):OU 10359
II OK=2
TU 357
3う) in 3b

xPGs(JPT)=X1-2.*XTKAL(JJOK, JPT, NNSV)
YPOS $\operatorname{LPOSPI}=Y 1+2 \bullet *$ YTRAL JTOR,JPT, NNSV)
 (PGS (NAPI) = XPUS ( 11
YOOS (NAPI) $=$ YPISS 11
$2 P O$ INAP $11=\angle P O S(1)$
$\therefore \quad$ a $\angle P G S I N A P I I=X 6 P C S I I I$
357 CALL LINE (XPOS, IPUS,NAP $1,1,0,1)$
LALL LINFIXCMUS, ZPOS,NAPI,1,0,1

## 354 CLNJ Jidut

LALL PLUT(2L., U., - 3 )
WNITF16,367
hormati/:
Cuntinut
aKIft ( 0.335 )
367 FIRMATI/' FINISHEU PLJTTIN: FUR THIS PICTURE-PASS. $1 / 1 H I)$
$34 U$ CUNTINUE
340 CUNTINUE FINISHEU PLJTTING FUR THIS PICTURE-PASS. '/IHI)
335 Furmatilhi.'SUCLESSFUL COAPLEIION UF THIS RGAKEPIC RUM.'
***AK.IIC
$\underset{7}{7}$

＊あれから，

imi3（12）


twIVALIVC．
ivUIVALINE．
there are a tulal if lifigriafo plissible quantitits rhat can be read uff A．RUDYNAMIC LIFT，LBWI／FI




REAL(3) RUVLUL,RUNNO, TITLE, NR, NA, NB, BLURAD, NRM, NRA, NRMA, NAMIDZ, JI, J=1, NRM)

CALL SYMBGL(0.5,1.,010.14,RUNLOL,90.,801



$13: 16$ FCRMAT(12x, 12.7x,1212)


IF (ABS (XLHTY).LI.O.IE-2) IUVKMW=1 IF (ABS (XUSTU).LT.U. IE-2)IUNKNW=1
$\begin{aligned} \text { SCLHTY } & =1 . \\ \text { SCLJTI } & =1 .\end{aligned}$
'MIIN $=0.1234 .3671$ $S C L L=0.76543211$
NOT REPRODUCIBLE






NUMPS=NURLOC + ,


( $\mathrm{TY}(1, J), J=1, N R 1, I=1, N A)^{\prime}, j x,{ }^{\prime} \lambda R=1,12,5 x$,
-vA=•, (21)
WRITE(totmII)
WRIIE(tormil) (IGTY(I,J),J:I,VR),I=1,NA)
ls/l flikichtillllivenamloc WRIIEIOFFMIIIGAVV.J. NKITE(G, 137\%)•JR,NAMIU?
 A. $\langle$ ! TE(G,FMTI)(IU(VV', JJ), JW=1, NR), NNQ = I, NAMIOZ)

 NRIIE(G,FMTII (ALFR (J),J=1,NR)

$$
1,0 \text { If: } 1376
$$

1309 NRIIE16,1374: (HOVOIII,I =NUMLOC,NOMP5),NRMPNA



 WRIfE(G,FNTB) (IA(NJQ,JU),JU=1,NRM), NNU=1, NAMLO2)



WRITES6.13TIINHM

WRIIE(G,FMI I) (ALERI,(J), J= (,NKM)
1318 LU $1361 \mathrm{~J}=1$, ViAUIII
NOT REPROOUCIBLE
1.6 $13671=10$ iv
(38, WTY(I.J)=UTY(I,J)-ALERO(J)


INOY=+1 MEANS IHAT IHE UEXI PLOT IS TU UE MADE GN TH = UPPER LEVIL.
IF(NNKAU.ET:OU) WU TI 1332
CALL دYMEUL(0.,-1.5,0.21,PLGINO,90.,16)

CALL PLIII ( $6.5,4.8,0.14,1$ OHI TIME-HISTORIES), 0.,16)
$A \times L G T H=2.3398112316 * X A Z$
$F=A \times L G T H / 4$.
LO 1332 KRAL=1.NNRAU

C 1341 PCT=PCTRM(JRAD)
IF IJQNI.EQ. 2 IPCT = PC TROM (JRAO)
CALL SYMBOL $11.65,3.5,0.14,4 H \$ R A L, 0.7+4)$

## UO $1333 \mathrm{JPO}=1 . \forall \mathrm{VA}$ <br> YPOS (JPOS) $=U T Y(J P U S, J R A U) / S C L H T Y$ $Y P O S(N A P I)=Y P U S(1)$

## IFIIUNくVB.EL.I)GO IC. 1331

CALL FLOT10..2..-3)
CALL LINE (XALM H, YPIIS,NAPI, $1,0,1$ )
CALL PLOTIO...-2.,-31
ORAN ALIMUTH AXIS. 31
ALL PLUTIO..2...2;
(ALL PLOTIO..U.,-3)
CALL FACTOR(F)
CALL AXIS (O.,C...IH,-1.4.,0...0.,90.1
CALL SYMBOL (3.2,-0.31,0.14, (H.,0.,31)
T.ALL SYMHOL 13.0 : $,-0.45,0.14,7 \mathrm{HALIMUTH}, 0 ., 7)$
CALL PLOI(O.:U.:-3)
NOT REPRODUCIble

- SIPV AJP ? HI "Vril
WMIV = - $⿻$ \& SLIHTY


$\begin{array}{ll}\text { (13)2 } & \text { IVLY }=-1 \text { IUY } \\ C & I V D Y=-1 \quad M: A\end{array}$


1334 LalL Plilrlu., -4.5.5-31
1334 LALL PLIITU, $-4,5,-31$
1336 CALL PLOITS 1332
$C$
IFIIVUY $11340,1335,1347$
1332 GUNTIIVUE
GOTU $134 H$
1347 CALL PLUT(5.0-3.0-3)
1543 FORMAII/10 +I'ISHEU TIML-IISTIRIES PLOTS FUR KGNTY=1,I 1
1348 WRIIE(6y 1343 ) $\because$ GNIY

1346
1347
CALL PLUT(3.0.3..-3)
STEADY CUMPuNE:T.
 If (lunk it. LC.W) 63 I: 1373
CALL LIJEIXKKM, ALERT,NR, 1,...1)
Bh IL 1321
CALL LINE


 (1) 1354
1353 SCLSTU=SSTOLL (ANTI/XOSTV
SLLHRM = SHRMLL (JUNT)/XOHKM

CALL LINTIXXRM, ALSNT,NK, 1, U, 1:
$1355 L C 1358 J=1$.NKM
ALTRGIJ) = ALEHII(J)/SLLSTL
CALL LINF (XARCM, ALERO,NRM
CRA* IHE RAGJUS AXIS.
$A X L G T H=1 . / S L L X K$
$F=A X L G T H / 2$.
CALL PLCHR(F)
(ALL AXISIO..U.,IH, -1,2., U.,0.,0. 0.50 )
CALL SYMBOLII.2,-0.45,U.14, GHKAUIUS,0..61
CALL PLITIO.,O...-3)
URAN THE UTY AXIS.

1354 IFINMHAKM.FU.UIFU TU 1322
$\mathrm{F}=\mathrm{AXLGIH} / 2$.
(0) $1 \mathrm{SL2}$ KHARM $=1$, NNHARM
JHARM $=$ JJHAKM (KHARM)
IF (JHARM.LE.NAMIOZ) GO 10 1339
WRIIE(6, 1340 ) KHAKM, JHAKM, VAMIUZ
 ( $\cdot$, NAM 102 $=$-.(3)


If (IUNKNmeta.ullo To 1359
If(Junt.en.2) 60 Til 1324
CALL LINE XXRM, YPUS, NR, $1,0,11$
$n$
2
2
2
2
CALL LINE (XXRDM, YPOS, NRM,I,U, I)
1325 CALL AXIS $10.00 .1 \mathrm{IH}, 1,4 \ldots, 90$.,AMIN, SCLA $)$
CALL AXIS $10.00 ., 4 H_{N A D M,-4,3 ., 0 ., 0 ., S C L X X)}$
uO IU 1302
1359 CALL PLUT10..2.,.31
IFIJUNT.EU.2)GO TO 1360
CALL LINEIXXRM, YPOS,NR, $1,0,11$
GO IO 1361
CALL LINEIXXRDM,YPUS,NRM,I,C,I)
DRAW THE RAUII AXIS.
CALL PLOT (AXLGTH,O.,*3)
0
$\underset{\sim}{0}$
$\sim$

CALL PLOT (0.,-2.,-3)
CALL FACTOR(F)
CALL FACIOR(F)
CALL AXISIO.:
CALL SYMBOLI $1.2,-0.45,0.14,6$ HRALIUS,0.,61 CALL PLOT $0 .=0.1$-3i
CALL FACTORII.i
TMIN $=-1$ SCL HRM
OTMIN=-L.*SCLHRH
CALL AXISIO.:O.O

- 

CALL AXISIO.:0., IH . I.4..99..9UTMIN, SCLHRM)
C 1362 CALL PLOT $10 .,-4.5,-31$
sint compuneni plut.
1326 YPOS(JPGS) $=$ G(JHARM, JPOS)/SLLHRM
CO 1326 JPOS=1,NRADII
NOML OC=4 JHARM-
CALL SYMBOL (U.2S, 3.5,0.14, COMPNTINOMLOCI, U., 8) IFIIUNKNG -EU.O):BO Ti' 1363
CALL LINE XXRA, YPOS,NR,I,U,I)
io 101317
1316 CALL LINE $1 \times X R O M, Y P O S, N R M, 1, U, 1)$
1317 CALL AXIS $10 ., 0 ., 1 H, 1,4 \ldots 90 ., A M I$
CALL AXIS (0., 0., IH, 1,4..,90.,AMIN,SCLA)
LALL AXIS(0., U., 4HRNDM, -4, 1., O., O., SCLXX)
LO IO 1366


## APPENDIX III

## INDUCED VELOCITY AT SELECTED POSITIONS ON THE DISC DUE TO

SELECTED SEGMENTS OF THE NEAREST TIP TRAIL

Table $V$ shows the induced velocity due to nearby filaments of the trailed vorticity. Five instantaneous positions were chosen for points on the blades (see Figure 38), and the induced velocities resulting at these points due to selected short segments of the tip vortices (as positioned by the two geometries, CAL and UR) were computed for the $\mu=0.26$ flight condition of $u H-1$.


Coordinates ( $x^{\prime}, y, z^{\prime}$ ) of blade stations $1,2,3$ :

1. $(+1.514,-4.434,-0.44)$
2. $(-3.649,+12.40,-0.679)$
3. $(-4.574,+15.86,-0.854)$

See Table $V$ for coordinates of wake segments as represented in the two wake geometries.

Figure 38. Positions of Selected Blade Stations and Selected Wake Trail Segments (UH-1, $\mu=0.26$ ).

| TABLE V. INDUCED VELOCITY AT SELECTED BLADE STATIONS DUE TOSELECTED TIP TRAIL SEGMENTS (UH-1, $\mu=0.26$ ) |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | J | k | Wake | Coordinates of Point A | Coordinates of Point B | ГAB | $w^{\prime}$ at M |
| 1 | 4 | 5 | CAL | -5.694, -12.14, +1.771 | -6.180, +2.494, +2.370 | 325.0 | -4.322 |
| 1 | 4 | 5 | UR | -6.767,-12.07, +3.079 | -7.773, +2.571, +3.454 | 327.6 | -3.136 |
| 1 | 7 | 8 | CAL | +12.23, -12.14, -0.625 | +11.75, +2.494, -0.026 | 325.0 | +2.828 |
| 1 | 7 | 8 | UR | +12.11, -12.03, +0.353 | +11.42, +2.470, +1.007 | 327.6 | +2.909 |
| 2 | 9 | 11 | CAL | +4.878, +11.99, +0.374 | -6.221, +18.28, +0.773 | 231.1 | +7.161 |
| 2 | 9 | 11 | UR | +4.475, +11.76, +1.344 | $-6.574,+17.81,+1.835$ | 229.4 | +6. 150 |
| 2 | 5 | 6 | CAL | -6.180, +2.494, +2.370 | -13.05, +11.99, +2.770 | 279.1 | -2.725 |
| 2 | 5 | 6 | UR | -7.773, +2.571, +3.454 | -14.25, +11.92, +4.027 | 284.2 | -2.016 |
| 3 | 10 | 11 | CAL | -0.258, +15.67, +0.573 | -6.221, +18.28, +0.773 | 220.0 | +9.085 |
| 3 | 10 | 11 | UR | -0.644, +15.31, +1.506 | -6.574, +17.81, +1.835 | 215.6 | +3.488 |
|  |  |  |  | $\theta_{M}^{i}$ |  |  |  |
| See Figure 38. |  |  |  |  |  |  |  |


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| OHIGINATINGACTIVITY (Cotporate Author)University of Rochester $\quad$Ze. REPOAT SECUAITr CLASSIFICATION <br> Unclassified <br> UROUP |  |
| AN ACTUATOR-DISC ANALYSIS OF HELICOPTER WAKE GEOMETRY AND THE CORRESPONDING BLADE RESPONSE |  |
| 4. DESCRIP TIVE NOTES (Type of report end incluaive defee) Final Report |  |
| 8. AUTMONAB) (First name, difdie fniffef, leof nome) <br> M. Joglekar <br> R. Loewy |  |
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| CH. CONTRACT OR GAANT NO. <br> DAAJ02-68-C-0047 NEW <br> b. PROJECT NO. <br> c. Task 1Fl6C204A13904 <br> $d$ |  |
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| In this analytical research, it is assumed that a helicopter rotor in forward flight can be represented by a flat plate. Expressions are developed relating the pressure field to the steady aerodynamic thrust and moment and the time-dependent flapping moment This pressure field satisfies the zero pressure condition at the center and edge of the disc, the first and second harmonic variation in lift, and Laplace's equation. <br> A computer program was written to calculate the velocity field and streamlines. Two sample cases corresponding to high- and low-speed conditions for the UH-1 rotor were chosen for computation. Both cases showed the tip-vortex phenomenon. <br> In order to examine the sensitivity of blade load calculations to wake geometry, an existing airloads program was modified to accept the computed wake geometry. A comparison of results of this modified program with experimental data indicated that the effect of distorted wake is greater at low advance ratios than at high advance ratios. The reduced sensitivity at high advance ratios may be due to the wake being blown farther behind the rotor and/or the predominance of other aeroelastic parameters. |  |

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[^0]:    *By "information", we mean here "knowledge about the values (steady or otherwise) of specific physical quantities of interest".

[^1]:    *We are here assuming the blade to be infinitely rigid in the chordwise direction. The computations reported herein used the additional structural simplifications of Reference 2, ignoring the dependence of torsional displacements on airloads, thus assuming the torsional response of the blades to be completely known beforehand.

[^2]:    *This is tantamount to requiring a finite velocity at the trailing edge. It has been corroborated by experiment, in that the pressure distributions on the surfaces of the airfoil as predicted from such a choice of the bound vorticity distribution agree with experimental measurements (see, for example, Reference 4).

[^3]:    *The "disc" here means the surface described by the blades as they rotate. Due to the preconing built into the blades, the steady deflection under load, and the flapping displacements, this surface will not, in general, be a plane circular surface.

[^4]:    *Material for this section has been drawn from Reference 2i.

[^5]:    *Cornell Aeronautical Laboratory, Inc., Buffalo, New York, hereafter abbreviated as CAL.
    **Meaning Piziali's work, Reference 2, according to our list of References.

[^6]:    *Referred to as BLP2 of CAL (Blade-Loads Program 2 of Cornell Aeronautical Laboratory). Some details of this program appear in Sections 3.1 and 3.2.

[^7]:    *Hereafter referred to as UR.

[^8]:    *or, "no finite hanges in velocity can occur in the infinitesimal
    time it takes a particle to cross the disc."

[^9]:    *The "disc" is not a flat surface, since blade steady deflection ans builtin coning are accounted for, through $\psi / x$.

[^10]:    *An option in the uR wake computation program also enables the particles to be injected on the flat disc, thus ignoring $Y_{o c}$.

[^11]:    * $\bar{w}$, normal to the blade chord. See Figure 2(a).

[^12]:    * Reference 24
    ** (2n-1)!! 1e3e5e....(2n-1), with (-1)!!

[^13]:    XUUFRIJALER.JPISLI IS THE X-COURDINATE OF POSITICN NO. JPISL CN THE STREAMLIVE, I.E THIS PARIICLE WAS SHLD FRCM THE PARTICULAR RADIAL POSITIUN WIEN THE SHEUGING BLAUE WAS AT AZIMUTHAL POSIIION NU. JAZER.
    IHAI WAS IJPISL-I-BLULGI IIME INTERVALS BEFORE THIS'PARIICLE REACHEC THIS PUSIIIGN.

[^14]:    11：）KEAリ（S，128）KHU，BC，PIMOM，RGLMO，
    
    1 P4UZO，P4UC1，P40S1，P40C2，P40S2，F7520，P75C1，P7ちSI，P75C2．P75S2．

[^15]:    ( ( ATRAL (NRP2, KALER,KPTSL),YTRAL(NRPZ,KAZEK,KPTSL),

