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An adaptive finite element method to analyse contact problems

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Introduction

Contact analysis aims to calculate the contact area, divided into stick and slip regions, as well as the contact tractions. Using discretisation methods such as the finite element method discretisation errors are unavoidable.

In linear mechanics a couple of theories have been developed to estimate the discretisation error and to predict better meshes for improved results [1, 2, 3, 4, 5]. One of the most accepted methods has been proposed by ZIENKIEWICZ and ZHU [3]. But there are known problems to satisfy the boundary conditions. And applications to nonlinear problems are not reported until now.

Caused by these facts an improved error estimation method is proposed taking the boundary traction error into account. This proceeding seems to be natural because in engineering analysis the surface tractions are of major interest. The first applications on nonlinear contact problems show the advantage and the practicability of this method.

Theoretical Background

The finite element equations of static or a class of steady state problems in nonlinear elasticity are formulated incrementally,

$$\mathbf{K}\mathbf{u} = \mathbf{f}^{(e)} - \mathbf{f}^{(i)} \quad , \tag{1}$$

where $\mathbf{K}$ is the tangential stiffness matrix, $\mathbf{u}$ the incremental displacement and $\mathbf{f}^{(e)}$ and $\mathbf{f}^{(i)}$ are the external and internal forces, respectively. Additionally, sets of constraint equations are formulated to satisfy contact conditions, for example.

Our aim is now an a posteriori estimation of the discretisation error and the prediction of an optimal mesh leading to an acceptable accuracy with minimum numerical effort.

The Zienkiewicz/Zhu Error Estimation Method

The idea of error estimation proposed by Zienkiewicz and Zhu [3] is to approximate the strain energy error in each element by

$$||e||_i = \sqrt{\int_{\phi(\mathcal{B})_i} \Delta\boldsymbol{\sigma}^T \mathbf{C}^{-1} \Delta\boldsymbol{\sigma}\, \mathrm{d}v} \tag{2}$$

where

$$\Delta\boldsymbol{\sigma} = \overset{\star}{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}} \tag{3}$$

is the approximate error in stresses and $\mathbf{C}$ is the elasticity matrix. The stress distribution $\hat{\boldsymbol{\sigma}}$ is computed from $\hat{\boldsymbol{\sigma}} = \mathbf{C}\hat{\boldsymbol{\varepsilon}} = \mathbf{C}\mathbf{B}\hat{\mathbf{u}}$ and $\overset{\star}{\boldsymbol{\sigma}}$ is the projection of $\hat{\boldsymbol{\sigma}}$ onto the nodal points calculated by a variational recovery strategy [6],

$$\int_{\phi(\mathcal{B})} \mathbf{H}^T\mathbf{H}\,\mathrm{d}v\, \overset{\star}{\boldsymbol{\sigma}} = \int_{\phi(\mathcal{B})} \mathbf{H}^T \hat{\boldsymbol{\sigma}}\,\mathrm{d}v \quad . \tag{4}$$

$\overset{\star}{\boldsymbol{\sigma}}$ is C^0 continuous and gives a much better representation of the stress field than $\hat{\boldsymbol{\sigma}}$, it is interpreted as a least square fit of $\hat{\boldsymbol{\sigma}}$.

The *error estimator* of an assemblage of m elements is computed by

$$||e|| = \sqrt{\sum_{i=1}^{m} ||e||_i^2} \quad . \tag{5}$$

A problem independent *relative strain energy error* is calculated by

$$\eta_{SE} = \frac{||e||}{||u||} \quad , \quad \text{where} \quad ||u|| = \sqrt{\int_{\phi(\mathcal{B})} \overset{\star}{\boldsymbol{\sigma}}^T \mathbf{C}^{-1} \overset{\star}{\boldsymbol{\sigma}}\, \mathrm{d}v} \quad . \tag{6}$$

An *error indicator* giving a measure of which elements have to be rebuilt to reach a user defined precision η_{tol} is defined as

$$\xi_i = \frac{\eta_i}{\bar{\eta}_m} \qquad \text{with} \qquad \bar{\eta}_m = \frac{\eta_{tol}}{m} \quad . \tag{7}$$

Therefore, an element j should be refined when $\xi_j > 1$.

An optimal mesh with respect to accuracy and computational effort will be generated when for all elements $\xi \approx 1$. For this task the necessary element size can be calculated from

$$h = \frac{h_i}{\xi_i^p} \quad , \tag{8}$$

where p gives the polynomial order of the element description [6].

This method is physically simple and easily implemented into an existing finite element code. But there are some known disadvantages [4]. Especially, there is no good representation of the boundary conditions by $\overset{\star}{\boldsymbol{\sigma}}$. This may lead to poor results of the adaptive refinement at the surface. The method proposed in this paper will circumvent this lack. Based on an estimation of the surface traction error it is the aim to refine (redefine) a mesh so that the boundary conditions are fitted best.

Estimation of Surface Traction Error

Now an estimator of the *surface traction error* is constructed. It is stated that the error in surface stress distribution is approximated in a L_1–*(Tschebyscheff–) norm* by

$$|e_c|_i = \int\limits_{\partial_c\phi(\mathcal{B})_i} |\,\tilde{\boldsymbol{t}}_c - \overset{\star}{\boldsymbol{t}}_c\,|\,\mathrm{d}a \quad \text{and} \quad |e_c| = \int\limits_{\partial_c\phi(\mathcal{B})} |\,\tilde{\boldsymbol{t}}_c - \overset{\star}{\boldsymbol{t}}_c\,|\,\mathrm{d}a \;=\; \sum_{i=1}^{n} |e_c|_i \quad , \tag{9}$$

where the calculation is restricted to the contact region $\partial_c\phi(\mathcal{B})$ here. Then $|e_c|_i$ physically is interpreted as the error of local contact forces. The contact stress vector $\overset{\star}{\boldsymbol{t}}_c = \overset{\star}{\boldsymbol{\sigma}} \cdot \boldsymbol{n}_c$ is calculated from the projected stress field and $\tilde{\boldsymbol{t}}_c$ is an approximation of the contact stress distribution calculated from the equilibrium state [7],

$$\int\limits_{\partial_c\phi(\mathcal{B})} \mathbf{H}^T\mathbf{H}\,\mathrm{d}a\;\tilde{\mathbf{t}}_c = \mathbf{f}_c \quad . \tag{10}$$

The equivalent nodal forces $\mathbf{f}_c$ may be interpreted as the equilibrium forces in the contact region and are calculated from the right hand side of equation (1).

Again a relative error is defined as

$$\eta_{ci} = \frac{|e_c|_i}{|\mathbf{f}_c|} \quad \text{and} \quad \eta_c = \frac{|e_c|}{|\mathbf{f}_c|} = \sum_{i=1}^{n} \eta_{ci} \quad . \tag{11}$$

The error indication measure is given by

$$\xi_{ci} = \frac{\eta_{ci}}{\bar{\eta}_n} \quad , \text{ with } \quad \bar{\eta}_n = \frac{\eta_{ctol}}{n} \quad , \tag{12}$$

where n is the number of elements in contact.

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Refinement Strategies

Two different methods to do the refinement have been studied. The available preprocessing software has the abilities

1st to use the strain energy error indicators (eqn.(7)) as bases of remeshing, and

2nd to prescribe the global element size in the interior and local element sizes at all or parts of the boundary. Here the surface traction error indicator (eqn.(12)) is used to predict the local element size at the contact boundary.

In the case of nonlinear behavior the load is applied onto the initial (starting) mesh during a couple of incremental load steps. Based on the error estimation computed for this loading stage a new mesh is created. The results from the initial calculation, e.g. displacement and stress field, are mapped onto the adaptive refined meshes and thus only an additional equilibrium iteration is necessary to get improved results. This strategy is very efficient because the whole nonlinear loading history is computed by use of a rough model oeconomically, and the interesting results are corrected with an adaptive rebuilt mesh.

Numerical Examples

Hertzian Contact

Hertzian Contact is of nonlinear nature because neither the contact area nor the pressure distribution is known a priori.

Fig. 1 shows the model used to calculate the contact of an elastic circular disc with a rigid plane surface, or an identical counterpart. With respect to symmetry only a quarter of the disc has been dicretized.

The refinement based on the strain energy error indication takes some iterations to reach the wanted accuracy. Fig. 2 shows the convergence behavior of both the strain energy error and the contact traction error depending of the number of degrees of freedom ($NDOF$). In the first step the contact traction error is decreased to $\eta_c = 3\%$ but the strain energy error is not decreased as regarded. So it takes two more iterations to get an optimal mesh where both errors are nearly of the same order and below $\eta_{tol} = 5\%$.

A similar result can be reached by following the 2nd procedure in one single step. By prescribing a global element size averaged from eqn. (8) and local element length in the contact region predicted from eqn. (12)

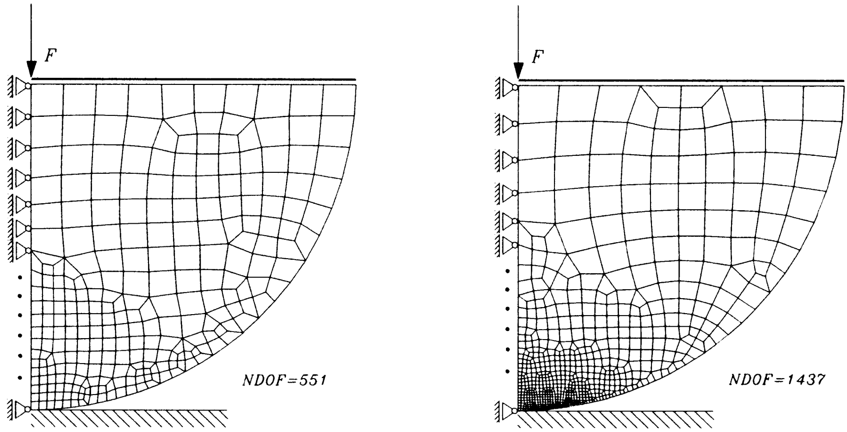

Figure 1: Initial and Improved Mesh for Hertzian Contact Analysis

the mesh plotted in fig. 1 (right) is generated. It leads to sufficient results shown as * and + in fig. 2.

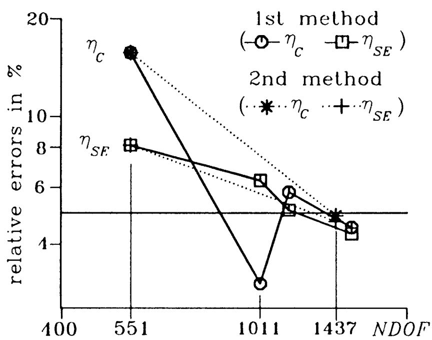

Figure 2: Convergence Behavior of Different Adaptive Methods

The normalized pressure distribution compared with the analytical solution is plotted in fig. 3. The slight difference between Hertzian solution and finite element results is caused by geometrical nonlinearity.

Fig. 4 shows the surface traction error indication in the contact region. It is shown that the contact edges, which are of great interest in contact analysis by use of discretisation methods, are detected very good.

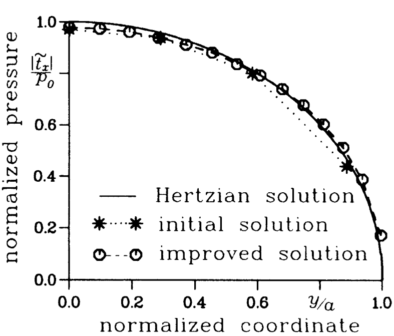

Figure 3: Pressure Distribution

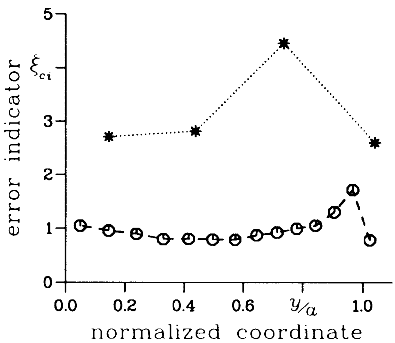

Figure 4: Surface Traction Error

Steadily Breaked Roller

To proof the capability of the surface traction error estimation to indicate the boundary between sliding and sticking part of the contact area a steadily breaked roller has been analysed. For the formulation of steady state rolling within finite element calculations it is referred to [8].

The roller consists of an elastic bond fixed on a rigid core. It is rolling with constant angular frequency Ω on a rigid plane surface and loaded with constant normal force P and breaking moment M so that partial slip occurs.

Left in Fig. 5 the initial finite element model is shown. At the final loading state the estimated errors are calculated as $\eta_{SE} = 10\%$ and $\eta_C = 7.1\%$. The predicted mesh by applying the 2nd method is plotted right in fig. 5. The errors have been decreased to $\eta_{SE} = 5\%$ and $\eta_C = 5\%$.

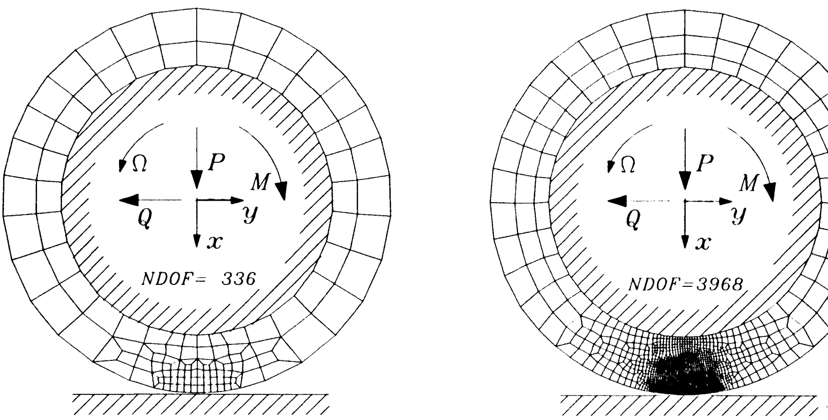

Figure 5: Initial and Improved Mesh for Rolling Contact Analysis

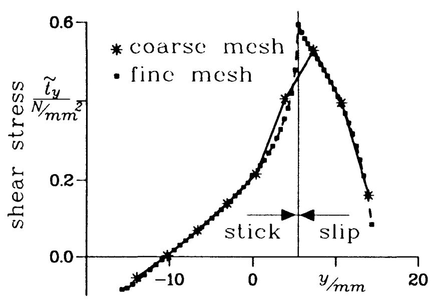

Figure 6: Shear Stress Distribution

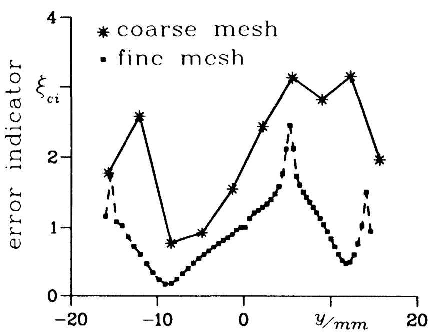

Figure 7: Surface Traction Error

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The corresponding shear stress distributions are plotted in fig. 6. The improvements near the contact edges and the sticking/sliding edge are obvious.

Fig. 7 shows the contact error indication. It is evident that the boundaries and the sticking/sliding edge are indicated clearly. At these points local refinement is necessary to get more accurate results.

Conclusions

Motivated by the known disadvantage of the Zienkiewicz/Zhu method in representing the surface stress state a new error estimation method has been proposed. Based on an approximation of the surface traction error the local element size at parts of the boundary is predicted to get results within a user defined precision and a minimum of computational effort. It has been shown that this method is capable to indicate contact boundaries and the edges between sticking and sliding zones as well as singularities in contact stress distributions.

By using this surface traction error estimation an efficient strategy for the analysis of nonlinear problems has been worked out. At first, the loading point is calculated by the use of a coarse mesh, then based on the error estimation the mesh is rebuilt and the results are mapped from the coarse mesh to the fine one. Thus, only an equilibrium iteration is necessary to get improved results with regarded accuracy.

One major disadvantage of the Zienkiewics/Zhu method has been removed, the poor representation of the surface tractions. Therefore, the proposed method may be applied to any other problems and for all boundaries with prescribed tractions and displacements or only parts of them, e.g. sharp corners, and will lead to more accurate results than the original one.

In engineering calculations we are mainly interested in stresses near the boundary, e.g. in contact analysis, because failure starts at the surface, usually. So it is essential to calculate results accurately at the boundary but not in the interior. The adaptive meshing based on the surface traction error is the best known method to succed this task.

No mathematical proof of asymptotic exactitude has been performed until now. This might be an interesting topic of future work. But even the physically distinction caused us to adapt this method to linear and nonlinear problems as well. The first practical experience shows the advance with respect to practicability and accuracy.

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