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AN ADJACENCY CRITERION FOR THE PRIME GRAPH OF A FINITE SIMPLE GROUP

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For every finite non-Abelian simple group, we give an exhaustive arithmetic criterion for adjacency of vertices in a prime graph of the group. For the prime graph of every finite simple group, this criterion is used to determine an independent set with a maximal number of vertices and an independent set with a maximal number of vertices containing 2, *and to define orders on these sets*; *the information obtained is collected in tables. We consider several applications of these results to various problems in finite group theory, in particular, to the recognition-by-spectra problem for finite groups.*

Let G be a finite group, $\pi(G)$ be the set of all prime divisors of its order, and $\omega(G)$ be the spectrum of G, that is, the set of all of its element orders. A graph $GK(G) = \langle V(GK(G)), E(GK(G))\rangle$, where $V(GK(G))$ is a vertex set and $E(GK(G))$ is an edge set, is called the *Gruenberg–Kegel graph* (or *prime graph*) of G if $V(GK(G)) = \pi(G)$ and the edge (r, s) is in $E(GK(G))$ iff $rs \in \omega(G)$. Primes $r, s \in \pi(G)$ are said to be *adjacent* if they are adjacent as vertices of $GK(G)$, that is, $(r, s) \in E(GK(G))$. Otherwise r and s are said to be *non-adjacent*.

Properties of the prime graph $GK(G)$ deliver rich information on the structure of G (see [1-4] and Secs. 5 and 7 below). The main objective of this paper is to specify an exhaustive arithmetic criterion of vertices being adjacent in the prime graph $GK(G)$ for every finite non-Abelian simple group G. Secs. 1-4 are devoted to this goal. In Sec. 5 we discuss some recent results concerning prime graphs of finite groups. Furthermore, we explain the importance of so-called prime graph independence numbers in research on a group structure. In Sec. 6 we calculate the invariants in question for all finite non-Abelian simple groups (the resulting tables are collected in Sec. 8). Applications of our results are discussed in Sec. 7.

1. PRELIMINARIES

If n is a natural number and π is a set of primes, then we denote the set of all prime divisors of n by $\pi(n)$, and the maximal divisor t of n such that $\pi(t) \subseteq \pi$ by n_π. Note that for a finite group G, $\pi(G) = \pi(|G|)$ by definition.

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TABLE 1. Non-Simple Groups of Lie Type

Group	Properties
$A_1(2)$	is soluble
$A_1(3)$	is soluble
$B_2(2)$	$B_2(2) \simeq \mathrm{Sym}_6$
$G_2(2)$	$[G_2(2), G_2(2)] \simeq {}^2A_2(3)$
${}^2A_2(2)$	is soluble
${}^2B_2(2)$	is soluble
${}^2G_2(3)$	$[{}^2G_2(3), {}^2G_2(3)] \simeq A_1(8)$
${}^2F_4(2)$	$[{}^2F_4(2), {}^2F_4(2)] = {}^2F_4(2)'$ is the Tits group

The adjacency criterion for two prime divisors in alternating groups is obvious and can be stated as follows.

Proposition 1.1. Let $G = \mathrm{Alt}_n$ be an alternating group of degree n.

(1) Let $r, s \in \pi(G)$ be odd primes. Then r and s are non-adjacent iff $r + s > n$.

(2) Let $r \in \pi(G)$ be an odd prime. Then 2 and r are non-adjacent iff $r + 4 > n$.

Information on the adjacency of vertices in a prime graph for every sporadic group and for the Tits group ${}^2F_4(2)'$ can be found in [5, 6]. Thus we need only consider simple groups of Lie type.

For Lie-type groups and for linear algebraic groups, we borrow the notation from [7, 8], respectively. Denote by G_{sc} a universal group of Lie type. Then every factor group G_{sc}/Z, where $Z \leq Z(G_{sc})$, is called a *group of Lie type*. In almost all cases $G_{sc}/Z(G_{sc})$ is simple, and we say that $G_{ad} = G_{sc}/Z(G_{sc})$ is of *adjoint type*. Some groups of Lie type over small fields are not simple. In Table 1 we present the data from [7, Thms. 11.1.2, 14.4.1] and specify all the exceptions. We write $A_n^{\varepsilon}(q)$, $D_n^{\varepsilon}(q)$, and $E_6^{\varepsilon}(q)$, where $\varepsilon \in \{+, -\}$, and $A_n^+(q) = A_n(q)$, $A_n^-(q) = {}^2A_n(q)$, $D_n^+(q) = D_n(q)$, $D_n^-(q) = {}^2D_n(q)$, $E_6^+(q) = E_6(q)$, and $E_6^-(q) = {}^2E_6(q)$.

If G is isomorphic to ${}^2A_n(q)$, ${}^2D_n(q)$, or ${}^2E_6(q)$, then we say that G is defined over a field $GF(q^2)$; if $G \simeq {}^3D_4(q)$ then we say that G is defined over $GF(q^3)$. And G will be defined over $GF(q)$ for all other finite Lie-type groups. The field $GF(q)$, in all cases, is called the *base field* of G. If G is a universal Lie-type group with the base field $GF(q)$, then there are a natural number N (equal to $|\Phi^+|$ in most cases) and a polynomial $f(t) \in \mathbb{Z}[t]$ such that $|G| = f(q) \cdot q^N$ and $(q, f(q)) = 1$ (see [7, Thms. 9.4.10, 14.3.1]). This polynomial is denoted by $f_G(t)$. If G is not universal then there is a universal group K with $G = K/Z$, where $Z \leq Z(K)$, and $f_G(t)$ is defined to be $f_K(t)$.

Assume that $\overline{G}$ is a connected simple linear algebraic group defined over an algebraically closed field of positive characteristic p. Let σ be an endomorphism of $\overline{G}$ such that $\overline{G}_\sigma = C_{\overline{G}}(\sigma)$ is a finite set. Then σ is called a *Frobenius map* and $G = O^{p'}(\overline{G}_\sigma)$ appears as a finite group of Lie type. Moreover, all finite groups of Lie type — split and twisted — can be obtained in just this way. Below, for every finite group G of Lie type, we fix (in some way) the linear algebraic group $\overline{G}$ and the Frobenius map σ, so that $G = O^{p'}(\overline{G}_\sigma)$. If $\overline{G}$ is simply connected, then $G = \overline{G}_\sigma = O^{p'}(\overline{G}_\sigma)$; if $\overline{G}$ is of adjoint type, then $\overline{G}_\sigma = \widehat{G}$ is the group of inner diagonal automorphisms of G (see [9, Sec. 12]).

If $\overline{R}$ is a σ-stable reductive subgroup of $\overline{G}$, then $\overline{R}_\sigma \cap G = \overline{R} \cap G$ is called a *reductive subgroup* of G. If $\overline{R}$ is of maximal rank, then $\overline{R}_\sigma \cap G$ is also said to be of *maximal rank*. Note that if $\overline{R}$ is σ-stable, reductive, and has maximal rank, then $\overline{R} = \overline{T} \cdot \overline{G}_1 * \ldots * \overline{G}_k$, where $\overline{T}$ is some σ-stable maximal torus in $\overline{R}$ and $\overline{G}_1, \ldots, \overline{G}_k$ are subsystem subgroups of $\overline{G}$. Furthermore, $\overline{G}_1 * \ldots * \overline{G}_k = \left[\overline{R}, \overline{R}\right]$. It is known that

$\overline{R}_\sigma = \overline{T}_\sigma G_1 * \ldots * G_m$, and $G_1, \ldots, G_m$ are referred to as *subsystem subgroups* of G. In general, $m \leqslant k$, and for all i, the base field of G_i is equal to $GF(q^{\alpha_i})$, where $\alpha_i \geqslant 1$. There is an effective algorithm in [10] determining all subsystem subgroups of $\overline{G}$ (see also [11]). We need to consider an extended Dynkin diagram of $\overline{G}$ and exclude any number of vertices. Connected components of the resulting graph are Dynkin diagrams of subsystem subgroups of $\overline{G}$, and the Dynkin diagrams of all the subsystem subgroups can be obtained in just this way.

If $\overline{T}$ is a σ-stable torus of $\overline{G}$ then $T = \overline{T} \cap G = \overline{T}_\sigma \cap G$ is called a *torus* of G. If $\overline{T}$ is maximal, then T is a *maximal torus* of G. If G is neither a Suzuki group nor a Ree group, then every maximal torus T satisfies $|\overline{T}_\sigma| = g(q)$, where $GF(q)$ is the base field of G, $g(t)$ is a polynomial of degree n dividing $f_G(t)$, and n is the rank of $\overline{G}$. For more details, see [12, Chap. 1].

In Lemmas 1.2 and 1.3 we collect information about maximal tori of finite simple groups of Lie type.

LEMMA 1.2 [13, Props. 7-10; 14]. Let $\overline{G}$ be a connected, simple, classical algebraic group of adjoint type and let $G = O^{p'}(\overline{G}_\sigma)$ be a finite simple classical group.

(1) Every maximal torus T of $G = A^\varepsilon_{n-1}(q)$ has the order

$$\frac{1}{(n, q-(\varepsilon 1))(q-(\varepsilon 1))}(q^{n_1} - (\varepsilon 1)^{n_1}) \cdot (q^{n_2} - (\varepsilon 1)^{n_2}) \cdot \ldots \cdot (q^{n_k} - (\varepsilon 1)^{n_k})$$

for an appropriate partition $n_1 + n_2 + \ldots + n_k = n$ of n. Moreover, for every partition, there exists a torus of corresponding order.

(2) Every maximal torus T of G, where $G = B_n(q)$ or $G = C_n(q)$, has the order

$$\frac{1}{(2, q-1)}(q^{n_1} - 1) \cdot (q^{n_2} - 1) \cdot \ldots \cdot (q^{n_k} - 1) \cdot (q^{l_1} + 1) \cdot (q^{l_2} + 1) \cdot \ldots \cdot (q^{l_m} + 1)$$

for an appropriate partition $n_1 + n_2 + \ldots + n_k + l_1 + l_2 + \ldots + l_m = n$ of n. Moreover, for every partition, there exists a torus of corresponding order.

(3) Every maximal torus T of $G = D^\varepsilon_n(q)$ has the order

$$\frac{1}{(4, q^n - \varepsilon 1)} \cdot (q^{n_1} - 1) \cdot (q^{n_2} - 1) \cdot \ldots \cdot (q^{n_k} - 1) \cdot (q^{l_1} + 1) \cdot (q^{l_2} + 1) \cdot \ldots \cdot (q^{l_m} + 1)$$

for an appropriate partition $n_1 + n_2 + \ldots + n_k + l_1 + l_2 + \ldots + l_m = n$ of n, where m is even, if $\varepsilon = +$, and m is odd if $\varepsilon = -$. Moreover, for every partition, there exists a torus of corresponding order.

LEMMA 1.3 [14, 15]. Let $\overline{G}$ be a connected, simple, exceptional algebraic group of adjoint type and let $G = O^{p'}(\overline{G}_\sigma)$ be a finite simple exceptional group of Lie type.

(1) Every maximal torus T of $G = G_2(q)$ has one of the following orders: $(q \pm 1)^2$, $q^2 - 1$, $q^2 \pm q + 1$. Moreover, for every number above, there exists a torus of corresponding order.

(2) Every maximal torus T of $G = F_4(q)$ has one of the following orders: $(q \pm 1)^4$, $(q \pm 1)^2 \cdot (q^2 \pm 1)$, $(q^2 \pm 1)^2$, $(q \pm 1)(q^3 \pm 1)$, $q^4 \pm 1$, $(q^2 \pm q + 1)^2$, $q^4 - q^2 + 1$. Hereinafter, $\pm$ signifies that we can choose either "+" or "−" independently for all multipliers, that is, $(q \pm 1)^2(q^2 \pm 1)$ is equal to $(q-1)^2(q^2-1)$, $(q+1)^2(q^2-1)$, $(q-1)^2(q^2+1)$, or $(q+1)^2(q^2+1)$. Moreover, for every number above, there exists a torus of corresponding order.

(3) For every maximal torus T of $G = E^\varepsilon_6(q)$, the number $(3, q - \varepsilon 1)|T|$ is equal to one of the following: $(q - \varepsilon 1)^k \cdot (q + \varepsilon 1)^{6-k}$, $2 \leqslant k \leqslant 6$; $(q^k - (\varepsilon 1)^k) \cdot (q^{6-k} - (\varepsilon 1)^{6-k})$, $1 \leqslant k \leqslant 5$; $(q^k - (\varepsilon 1)^k) \cdot (q - \varepsilon 1)^{6-k}$, $3 \leqslant k \leqslant 6$; $(q^3 - \varepsilon 1)(q^2 - 1)(q \pm 1)$; $(q^5 - \varepsilon 1)(q + \varepsilon 1)$; $(q^3 + \varepsilon 1)(q^2 \pm 1)(q - \varepsilon 1)$; $(q^4 + 1)(q^2 - 1)$; $(q^2 + 1)^2(q - \varepsilon 1)^2$; $(q^2 + \varepsilon q + 1)^3$; $(q^2 + \varepsilon q + 1)^2(q^2 - 1)$; $(q^4 - 1)(q + \varepsilon 1)^2$; $(q^3 + \varepsilon 1)(q^2 + \varepsilon q + 1)(q + \varepsilon 1)$; $(q^4 - q^2 + 1)(q^2 + \varepsilon q + 1)$;

$q^6+\varepsilon q^3+1$; $(q^2+\varepsilon q+1)(q^2-\varepsilon q+1)^2$. Moreover, for every number n above, there exists a torus T with $(3,q-\varepsilon 1)|T|=n$.

(4) For every maximal torus T of $G=E_7(q)$, the number $(2,q-1)|T|$ is equal to one of the following: $(q^{n_1}\pm 1)\cdot\ldots\cdot(q^{n_k}\pm 1)$, $n_1+\ldots+n_k=7$, and $(2,q-1)|T|\neq(q\pm 1)(q^6+1)$; $(q-\epsilon 1)\cdot(q^2+\epsilon q+1)^3$; $(q^5-\epsilon 1)\cdot(q^2+\epsilon q+1)$; $(q^3\pm 1)\cdot(q^4-q^2+1)$; $(q-\epsilon 1)\cdot(q^6+\epsilon q^3+1)$; $(q^3-\epsilon 1)\cdot(q^2-\epsilon q+1)^2$, where $\epsilon=\pm$. Moreover, for every number n above, there exists a torus T with $(2,q-1)|T|=n$.

(5) Every maximal torus T of $G=E_8(q)$ has one of the following orders: $(q^{n_1}\pm 1)\cdot\ldots\cdot(q^{n_k}\pm 1)$, $n_1+\ldots+n_k=8$, and $|T|\neq q^8+1$; $(q-\epsilon 1)\cdot(q^2+\epsilon q+1)^3\cdot(q\pm 1)$; $(q^5-\epsilon 1)\cdot(q^2+\epsilon q+1)\cdot(q\pm 1)$; $(q^3\pm 1)\cdot(q^4-q^2+1)\cdot(q\pm 1)$; $(q-\epsilon 1)\cdot(q^6+\epsilon q^3+1)\cdot(q\pm 1)$; $(q^3-\epsilon 1)\cdot(q^2-\epsilon q+1)^2\cdot(q\pm 1)$; q^8-q^4+1; $q^8+q^7-q^5-q^4-q^3+q+1$; $q^8-q^6+q^4-q^2+1$; $(q^4-q^2+1)^2$; $(q^6+\epsilon q^3+1)(q^2+\epsilon q+1)$; $q^8-q^7+q^5-q^4+q^3-q+1$; $(q^4+\epsilon q^3+q^2+\epsilon q+1)^2$; $(q^4-q^2+1)(q^2\pm q+1)^2$; $(q^2-q+1)^2\cdot(q^2+q+1)^2$; $(q^2\pm q+1)^4$, where $\epsilon=\pm$. Moreover, for every number above, there exists a torus of corresponding order.

(6) Every maximal torus T of $G={}^3D_4(q)$ has one of the following orders: $(q^3\pm 1)(q\pm 1)$, $(q^2\pm q+1)^2$, q^4-q^2+1. Moreover, for every number above, there exists a torus of corresponding order.

(7) Every maximal torus T of $G={}^2B_2(2^{2n+1})$ has one of the following orders: $q-1$, $q\pm\sqrt{2q}+1$, where $q=2^{2n+1}$. Moreover, for every number above, there exists a torus of corresponding order.

(8) Every maximal torus T of $G={}^2G_2(3^{2n+1})$ has one of the following orders: $q\pm 1$, $q\pm\sqrt{3q}+1$, where $q=3^{2n+1}$. Moreover, for every number above, there exists a torus of corresponding order.

(9) Every maximal torus T of $G={}^2F_4(2^{2n+1})$ with $n\geqslant 1$ has one of the following orders: $(q\pm 1)^2$, $q^2\pm 1$, q^2-q+1, $(q\pm\sqrt{2q}+1)^2$, $q^2-\epsilon q\sqrt{2q}+\epsilon\sqrt{2q}-1$, $q^2+\epsilon q\sqrt{2q}+q+\epsilon\sqrt{2q}+1$, where $q=2^{2n+1}$ and $\epsilon=\pm$. Moreover, for every number above, there exists a torus of corresponding order.

If q is a natural number, r is an odd prime, and $(r,q)=1$, then by $e(r,q)$ we denote a minimal natural number n with $q^n\equiv 1 (\mathrm{mod}\, r)$. If q is odd, we put $e(2,q)=1$, if $q\equiv 1(\mathrm{mod}\, 4)$, and $e(2,q)=2$ if $q\equiv -1(\mathrm{mod}\, 4)$.

The main technical tool in Sec. 6 is the following lemma, which is a consequence of Zsigmondi's theorem in [16].

LEMMA 1.4 Let q be a natural number greater than 1. Then for every $n\in\mathbb{N}$ there exists a prime r such that $e(r,q)=n$ but for the case where $n=6$ and $q=2$.

A prime r with $e(r,q)=n$ is called a *primitive prime divisor* of q^n-1. By Lemma 1.4, such a number exists except in the case mentioned above. If q is fixed, we denote by r_n some primitive prime divisor of q^n-1. (Obviously, q^n-1 can have more than one divisor of this kind.) By our definition, note, every prime divisor of $q-1$ is a primitive prime divisor of $q-1$ but for the sole exception — namely, 2 is not a primitive prime divisor of $q-1$ if $e(2,q)=2$. In the last-mentioned case 2 is a primitive prime divisor of q^2-1.

In view of [7, Thms. 9.4.10, 14.3.1], the order of any finite simple Lie-type group G of rank n over a field $GF(q)$ of characteristic p is equal to

$$|G|=\frac{1}{d}q^N(q^{m_1}\pm 1)\cdot\ldots\cdot(q^{m_n}\pm 1).$$

Therefore any prime divisor r of $|G|$ distinct from the characteristic p is a primitive divisor of q^m-1, for some natural m. Thus Lemma 1.4 allows us to 'find' prime divisors of $|G|$. Moreover, if G is neither a Suzuki group nor a Ree group, Lemmas 1.2 and 1.3 imply that for a fixed m, every two primitive prime divisors of q^m-1 are adjacent in $GK(G)$.

For Suzuki and Ree groups, we use following:

LEMMA 1.5. Let n be a natural number.

(1) Let $m_1(B,n) = 2^{2n+1}-1$, $m_2(B,n) = 2^{2n+1}-2^{n+1}+1$, and $m_3(B,n) = 2^{2n+1}+2^{n+1}+1$. Then $(m_i(B,n), m_j(B,n)) = 1$ if $i \neq j$.

(2) Let $m_1(G,n) = 3^{2n+1}-1$, $m_2(G,n) = 3^{2n+1}+1$, $m_3(G,n) = 3^{2n+1}-3^{n+1}+1$, and $m_4(G,n) = 3^{2n+1}+3^{n+1}+1$. Then $(m_1(G,n), m_2(G,n)) = 2$ and $(m_i(G,n), m_j(G,n)) = 1$ otherwise.

(3) Let $m_1(F,n) = 2^{4n+2}-1$, $m_2(F,n) = 2^{4n+2}+1$, $m_3(F,n) = 2^{4n+2}-2^{2n+1}+1$, $m_5(F,n) = 2^{4n+2}-2^{3n+2}+2^{2n+1}-2^{n+1}+1$, and $m_6(F,n) = 2^{4n+2}+2^{3n+2}+2^{2n+1}+2^{n+1}+1$. Then $(m_1(F,n), m_3(F,n)) = 3$ and $(m_i(F,n), m_j(F,n)) = 1$ otherwise.

The **proof** is by direct computations. $\square$

By Lemma 1.3, every prime divisor s (distinct from the characteristic) of the order of the Suzuki group ${}^2B_2(2^{2n+1})$ divides one of the numbers $m_i(B,n)$ defined in Lemma 1.5. The same is true for the Ree groups ${}^2G_2(3^{2n+1})$ and ${}^2F_2(2^{2n+1})$ and for all prime divisors of the numbers $m_i(G,n)$ and $m_i(F,n)$, respectively. Thus Lemma 1.5 allows us to find prime divisors of orders in Suzuki and Ree groups. Moreover, Lemma 1.3 implies that for a fixed k, every two prime divisors of $m_k(B,n)$ are adjacent in $GK({}^2B_2(2^{2n+1}))$. The same is also true for Ree groups and for all prime divisors of $m_k(G,n)$ and $m_k(F,n)$.

2. ADJACENT ODD PRIMES

In this section we argue to verify whether two odd primes distinct from the characteristic are adjacent in the Gruenberg–Kegel graph of a finite group of Lie type.

Proposition 2.1. Let $G = A_{n-1}(q)$ be a finite simple group of Lie type over a field of characteristic p. Let r and s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r,q)$ and $l = e(s,q)$ and assume that $2 \leqslant k \leqslant l$. Then r and s are non-adjacent if and only if $k + l > n$ and k does not divide l.

Proof. Note first that, for any odd prime $c \neq p$, the following condition is met:

$$c \text{ divides } q^x - 1 \text{ iff } e(c,q) \text{ divides } x. \qquad (1)$$

Indeed, by definition, c divides $q^{e(c,q)} - 1$ and does not divide $q^y - 1$ for all $y < e(c,q)$, that is, $e(c,q)$ is the order of q in the multiplicative group $GF(c)^*$ of a finite field $GF(c)$. Therefore if c divides $q^z - 1$, then $q^z = 1$ in $GF(c)^*$; hence, $e(c,q)$ divides z. Now assume that $e(c,q)$ divides z. Then $q^z - 1 = (q^{e(c,q)} - 1)f(q)$ for some $f(t) \in \mathbb{Z}[t]$; so, c divides $q^z - 1$.

Suppose $k + l \leqslant n$. Consider a maximal torus T of G of order

$$\frac{1}{(n, q-1)(q-1)}(q^k - 1) \cdot (q^l - 1) \cdot (q-1)^{n-k-l}.$$

The torus T is an Abelian subgroup of G, and $r, s \in \pi(T)$. It follows that T contains an element of order rs, and hence r and s are adjacent. If k divides l then both r and s divide $q^l - 1$, and so a maximal torus of order $\frac{1}{(n,q-1)}(q^l - 1)(q-1)^{n-l-1}$ contains an element of order rs.

Assume now that $k + l > n$ and k does not divide l, letting g be an element of G of order rs. Then $(|g|, p) = 1$, and hence g is semisimple. Therefore there exists a maximal torus T such that $g \in T$. By Lemma 1.2, the order of T is equal to

$$\frac{1}{(n, q-1)(q-1)}(q^{n_1} - 1) \cdot (q^{n_2} - 1) \cdot \ldots \cdot (q^{n_x} - 1)$$

for an appropriate partition $n_1 + n_2 + \ldots + n_x = n$ of n. Since r and s are primes, there exist n_i and n_j such that r divides $q^{n_i} - 1$ and s divides $q^{n_j} - 1$. In view of (1), $n_i = a \cdot k$ and $n_j = b \cdot l$ for some

$a, b \geqslant 1$. Moreover, since $k+l>n$ and $k \leqslant l$, we have $b=1$. Indeed, otherwise $n_j \geqslant l+l \geqslant k+l>n$, a contradiction with $n_1+\ldots+n_x=n$. Since k does not divide l, it follows that $n_i \neq n_j$. Hence $n_1+n_2+\ldots+n_x \geqslant n_i+n_j=a \cdot k+l>n$, which is a contradiction. □

Proposition 2.2. Let $G={}^2A_{n-1}(q)$ be a finite simple group of Lie type over a field of characteristic p. Define

$$\nu(m)=\begin{cases} m & \text{if } m \equiv 0 (\operatorname{mod} 4); \\ \frac{m}{2} & \text{if } m \equiv 2 (\operatorname{mod} 4); \\ 2m & \text{if } m \equiv 1 (\operatorname{mod} 2). \end{cases}$$

Let r and s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k=e(r, q)$ and $l=e(s, q)$ and suppose that $2 \leqslant \nu(k) \leqslant \nu(l)$. Then r and s are non-adjacent iff $\nu(k)+\nu(l)>n$ and $\nu(k)$ does not divide $\nu(l)$.

Proof. Note first that, for any odd prime $c \neq p$, the following is met:

$$c \text{ divides } q^x-(-1)^x \text{ iff } \nu(e(c, q)) \text{ divides } x. \tag{2}$$

Assume first that $e(c, q)$ is odd. If c divides $q^z-(-1)^z$ then c divides $q^{2z}-1$, and hence $e(c, q)$ divides $2z$ by (1). Since $e(c, q)$ is odd, it divides z, and so c divides q^z-1 by (1). For c is odd, q^z+1 is not divisible by c, and hence $q^z-(-1)^z=q^z-1$, that is, z is even. But $e(c, q)$ is odd, and so $2e(c, q)=\nu(e(c, q))$ divides z. Now assume that $\nu(e(c, q))$ divides z. Then z is even, hence $q^z-(-1)^z=q^z-1$ and c divides $q^z-(-1)^z$ by (1). Therefore (2) is true in this instance.

Suppose that $e(c, q) \equiv 2 (\operatorname{mod} 4)$. If c divides $q^z-(-1)^z$ then c divides $q^{2z}-1$, and hence $e(c, q)$ divides $2z$. But $\nu(e(c, q))=\frac{e(c, q)}{2}$, and so $\nu(e(c, q))$ divides z. If $\nu(e(c, q))$ divides z, and z is odd, then $q^z-(-1)^z=q^z+1$. Since $e(c, q)$ divides $2z$, c divides $q^{2z}-1$. We have $q^{2z}-1=(q^z-1) \cdot (q^z+1)$. For z is odd, $e(c, q)$ does not divide z, so c does not divide q^z-1 by (1), and hence c divides $q^z+1=q^z-(-1)^z$. If $\nu(e(c, q))$ divides z, and z is even, then $2\nu(e(c, q))=e(c, q)$ divides z, and hence c divides $q^z-(-1)^z=q^z-1$ by (1). Thus (2) is true in this case too.

Lastly, assume that $e(c, q) \equiv 0 (\operatorname{mod} 4)$. If c divides $q^z-(-1)^z$ then, as above, c divides $q^{2z}-1$, and so $e(c, q)$ divides $2z$ and $\frac{e(c, q)}{2}$ divides z. But $\frac{e(c, q)}{2}$ is even, and hence z is even and $q^z-(-1)^z=q^z-1$. It follows that $e(c, q)=\nu(e(c, q))$ divides z. If $\nu(e(c, q))=e(c, q)$ divides z then z is even, and by (1), c divides $q^z-(-1)^z=q^z-1$.

Suppose that $\nu(k)+\nu(l) \leqslant n$. Consider a maximal torus T of G of order

$$\frac{1}{(n, q+1)(q+1)}(q^{\nu(k)}-(-1)^{\nu(k)}) \cdot (q^{\nu(l)}-(-1)^{\nu(l)}) \cdot (q+1)^{n-\nu(k)-\nu(l)}.$$

The torus T is an Abelian subgroup of G, and $r, s \in \pi(T)$. It follows that T contains an element of order rs, and so r and s are adjacent. If $\nu(k)$ divides $\nu(l)$ then both r and s divide $q^{\nu(l)}-(-1)^{\nu(l)}$; so, a maximal torus of order $\frac{1}{(n, q+1)}(q^{\nu(l)}-(-1)^{\nu(l)})(q+1)^{n-\nu(l)-1}$ contains an element of order rs.

Assume now that $\nu(k)+\nu(l)>n$, $\nu(k)$ does not divide $\nu(l)$, and $g \in {}^2A_{n-1}(q)$ is an element of order rs. Then $(|g|, p)=1$, and hence g is semisimple. Therefore there exists a maximal torus T such that $g \in T$. Using (2) and Lemma 1.2, we arrive a contradiction as in the proof of Prop. 2.1. □

Proposition 2.3. Let G be one of finite simple groups of Lie type, $B_n(q)$ or $C_n(q)$, over a field of characteristic p. Define

$$\eta(m)=\begin{cases} m & \text{if } m \text{ is odd}; \\ \frac{m}{2} & \text{otherwise}. \end{cases}$$

Let r and s be odd primes with $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r, q)$ and $l = e(s, q)$ and suppose that $1 \leqslant \eta(k) \leqslant \eta(l)$. Then r and s are non-adjacent if and only if $\eta(k) + \eta(l) > n$ and $\eta(k)$ and $\eta(l)$ satisfy the following condition:

$$\text{either } \frac{\eta(l)}{\eta(k)} \text{ is not an odd integer or } \eta(k) = \eta(l) \text{ and } k \neq l. \tag{3}$$

Note that (3) holds in the following cases:

(1) both k and l are even, and either $\eta(k)$ does not divide $\eta(l)$ or $\frac{\eta(l)}{\eta(k)}$ is even; then $q^{\eta(k)}+(-1)^k = q^{\eta(k)}+1$ does not divide $q^{\eta(l)} + (-1)^l = q^{\eta(l)} + 1$;

(2) both k and l are odd, and $\eta(k)$ does not divide $\eta(l)$; then $q^{\eta(k)} + (-1)^k = q^k - 1$ does not divide $q^{\eta(l)} + (-1)^l = q^l - 1$;

(3) k is odd and l is even; then $q^{\eta(k)} + (-1)^k = q^k - 1$ does not divide $q^{\eta(l)} + (-1)^l = q^{\eta(l)} + 1$;

(4) k is even, l is odd, and either $\eta(k)$ does not divide $\eta(l)$ or $\frac{\eta(l)}{\eta(k)}$ is even or equal to 1; then $q^{\eta(k)}+(-1)^k = q^{\eta(k)} + 1$ does not divide $q^{\eta(l)} + (-1)^l = q^l - 1$.

The argument above shows that $\eta(k)$ and $\eta(l)$ satisfy (3) iff $q^{\eta(k)} + (-1)^k$ does not divide $q^{\eta(l)} + (-1)^l$.

Proof. First we prove that for any odd prime $c \neq p$, the following is met:

$$\text{if } c \text{ divides } q^x \pm 1 \text{ then } \eta(e(c, q)) \text{ divides } x. \tag{4}$$

Assume first that $e(c, q)$ is odd and c divides $q^z \pm 1$. In view of (1), $e(c, q)$ divides $2z$. But $e(c, q)$ is odd, and hence $e(c, q) = \eta(e(c, q))$ divides z. Assume now that $e(c, q)$ is even and c divides $q^z \pm 1$. Then c divides $q^{2z} - 1$, and so $e(c, q)$ divides $2z$ and $\frac{e(c,q)}{2} = \eta(e(c, q))$ divides z.

If $\eta(k) + \eta(l) \leqslant n$ then we may consider a maximal torus T of order $\frac{1}{(2,q-1)}(q^{\eta(k)} + (-1)^k) \cdot (q^{\eta(l)} + (-1)^l) \cdot (q - 1)^{n-\eta(k)-\eta(l)}$. The torus T is an Abelian group and $r, s \in \pi(T)$. Hence T contains an element of order rs. If $\eta(k)$ and $\eta(l)$ do not satisfy (3), then $q^{\eta(k)} + (-1)^k$ divides $q^{\eta(l)} + (-1)^l$, and so both r and s divide $q^{\eta(l)} + (-1)^l$. Thus a maximal torus of order $\frac{1}{(2,q-1)}(q^{\eta(l)} + (-1)^l)(q - 1)^{n-\eta(l)}$ contains an element of order rs.

Suppose $\eta(k) + \eta(l) > n$, $\eta(k)$ and $\eta(l)$ satisfy (3), and there exists an element $g \in G$ of order rs. Since $(|g|, p) = 1$, it follows that g is semisimple. Therefore there exists a maximal torus T containing g. In view of Lemma 1.2, the order $|T|$ is equal to

$$\frac{1}{(2, q-1)}(q^{n_1} \pm 1) \cdot (q^{n_2} \pm 1) \cdot \ldots \cdot (q^{n_x} \pm 1)$$

for an appropriate partition $n_1 + n_2 + \ldots + n_x = n$ of n. Using (4), we arrive at a contradiction as in Prop. 2.1. □

Proposition 2.4. Let $G = D_n^{\varepsilon}(q)$ be a finite simple group of Lie type over a field of characteristic p, and let the function $\eta(m)$ be defined as in Prop. 2.3. Suppose r and s are odd primes and $r, s \in \pi(D_n^{\varepsilon}(q)) \setminus \{p\}$. Put $k = e(r, q)$, $l = e(s, q)$, and $1 \leqslant \eta(k) \leqslant \eta(l)$. Then r and s are non-adjacent if and only if $2 \cdot \eta(k) + 2 \cdot \eta(l) > 2n - (1 - \varepsilon(-1)^{k+l})$ and $\eta(k)$ and $\eta(l)$ satisfy (3).

The **proof** is as above.

Proposition 2.5. Let G be a finite simple exceptional group of Lie type over a field of characteristic p, suppose that r and s are odd primes, and assume that $r, s \in \pi(G) \setminus \{p\}$, $k = e(r, q)$, $l = e(s, q)$, and $1 \leqslant k \leqslant l$. Then r and s are non-adjacent if and only if $k \neq l$ and one of the following holds:

(1) $G = G_2(q)$, and either $r \neq 3$ and $l \in \{3, 6\}$ or $r = 3$ and $l = 9 - 3k$;

(2) $G = F_4(q)$, and either $l \in \{8, 12\}$, or $k \in \{3, 4\}$ and $l = 6$, or $k = 3$ and $l = 4$;

(3) $G = E_6(q)$, and either $l = 5$ and $k \geqslant 3$, or $l = 6$ and $k = 5$, or $l = 8$ and $k \geqslant 3$, or $l = 8$ and $r = 3$ for $(q-1)_3 = 3$, or $l = 9$, or $l = 12$, $k \neq 3$, and $r \neq 3$;

(4) $G = {}^2E_6(q)$, and either $l = 8$ and $k \geqslant 3$, or $l = 8$ and $r = 3$ for $(q+1)_3 = 3$, or $l = 10$ and $k \geqslant 3$, or $l = 12$, $k \neq 6$, and $r \neq 3$, or $l = 18$;

(5) $G = E_7(q)$, and either $l \in \{14, 18\}$ and $k \neq 2$, or $l \in \{7, 9\}$ and $k \geqslant 2$, or $l = 8$ and $k = 7$, or $l = 10$ and $k \geqslant 3$, $k \neq 4, 6$, or $l = 12$ and $k \geqslant 4$, $k \neq 6$;

(6) $G = E_8(q)$, and either $l \in \{7, 14\}$ and $k \geqslant 3$, or $l = 9$ and $k \geqslant 4$, or $l \in \{8, 10, 12\}$ and $k \geqslant 5$, $k \neq 6$, or $l = 18$ and $k \neq 1, 2, 6$, or $l = 20$ and $r \cdot k \neq 20$, or $l \in \{15, 24, 30\}$;

(7) $G = {}^3D_4(q)$, and either $l = 6$ and $k = 3$ or $l = 12$.

Proof. As for the classical groups of Lie type, prime divisors $r, s \in \pi(G)$ satisfying the conditions of the proposition are adjacent iff rs divides the order of some maximal torus in G. Therefore we can prove this proposition as the previous ones, but with Lemma 1.3 used in place of Lemma 1.2. □

Proposition 2.6. Let G be a finite simple Suzuki or Ree group over a field of characteristic p, r and s be odd primes, and $r, s \in \pi(G) \setminus \{p\}$. Then r and s are non-adjacent if and only if one of the following holds:

(1) $G = {}^2B_2(2^{2n+1})$, r divides $m_k(B, n)$, s divides $m_l(B, n)$, and $k \neq l$;

(2) $G = {}^2G_2(3^{2n+1})$, r divides $m_k(G, n)$, s divides $m_l(G, n)$, and $k \neq l$;

(3) $G = {}^2F_4(2^{2n+1})$, r divides $m_k(F, n)$, s divides $m_l(F, n)$, $k \neq l$, and if $\{k, l\} = \{1, 3\}$, then $r \neq 3 \neq s$.

Numbers $m_i(B, n)$, $m_i(G, n)$, and $m_i(F, n)$ are defined as in Lemma 1.5.

The **proof** is as above.

3. ADJACENCY TO THE CHARACTERISTIC

In this section we verify whether a prime r and the characteristic p of the base field of a finite Lie-type group are adjacent.

Proposition 3.1. Let $G = O^{p'}(\overline{G}_\sigma)$ be a finite simple classical group of Lie type defined over a field of characteristic p, and let $r \in \pi(G)$ and $r \neq p$. Then r and p are non-adjacent if and only if one of the following holds:

(1) $G = A_{n-1}(q)$, r is odd, and $e(r, q) > n - 2$;

(2) $G = {}^2A_{n-1}(q)$, r is odd, and $\nu(e(r, q)) > n - 2$ (the function $\nu(m)$ is defined as in Prop. 2.2);

(3) $G = C_n(q)$, $\eta(e(r, q)) > n - 1$ (the function $\eta(m)$ is defined as in Prop. 2.3);

(4) $G = B_n(q)$, $\eta(e(r, q)) > n - 1$;

(5) $G = D_n^\varepsilon(q)$, $\eta(e(r, q)) > n - 2$;

(6) $G = A_1(q)$, $r = 2$;

(7) $G = A_2^\varepsilon(q)$, $r = 3$, and $(q - \varepsilon 1)_3 = 3$.

Proof. It is evident that 2 and p are adjacent in all classical simple groups except $A_1(q)$, and we may assume that r is odd.

First we outline the general idea behind the proof. We use the notation of [13]. In order to prove that r and p are adjacent, we find a connected reductive subgroup R of maximal rank in G such that $R = T(G_1 * G_2)$, where T is a maximal torus and both G_1 and G_2 are non-trivial groups of Lie type for which $r \in \pi(G_1)$. Then G_1 contains an element of order r, which centralizes G_2. Since G_2 is non-trivial, it contains an element of order p, and so $G \geq R$ contains an element of order rp.

In order to prove that r and p are non-adjacent, we consider an arbitrary element g of order r and its connected centralizer $G \cap C_{\overline{G}}(g)^0$. Recall that $C_{\overline{G}}(g)^0 = \overline{S} * \overline{L}$, where $\overline{S} = Z(C_{\overline{G}}(g)^0)$ is a central torus and $\overline{L}$ is a semisimple part. Clearly, $g \in \overline{S} \cap G \leq \overline{S}_\sigma$, and hence r divides $|\overline{S}_\sigma|$. This implies that $\overline{L}_\sigma$ is trivial, and so $C_{\overline{G}}(g)^0$ does not contain unipotent elements. But every unipotent element of $C_{\overline{G}}(g)$ is contained in $C_{\overline{G}}(g)^0 \leq N_{\overline{G}}(\overline{T})^0 = \overline{T}$. Therefore none of the unipotent elements of G centralize g. Now we consider all classical groups one by one.

$A_1(q)$. It is known that if g is an element of order $r \neq p$, then $(|C_{A_1(q)}(g)|, p) = 1$ (see, [13, Prop. 7]). Therefore r and p are non-adjacent for every $r \in \pi(A_1(q)) \setminus \{p\}$.

$A_2(q)$. Using [13, Prop. 7], we see that only prime divisors of $\frac{q-1}{(3,q-1)}$ are adjacent to p, and the proposition is true in this instance.

$A_{n-1}(q)$ and $n \geqslant 4$. In this case T is a Cartan subgroup, $G_1 = A_{n-3}(q)$, and $G_2 = A_1(q)$. The existence of such a subgroup is proved using [13, Prop. 7]. Thus every prime r with $e(r,q) \leqslant n-2$ divides $|G_1|$ and, hence, is adjacent to p.

Now let $e(r,q) = n-1$ and let g be an element of order r. In view of (1), $q^{n-1}-1$ must divide $|\overline{S}_\sigma| \cdot (q-1)$. It follows from [13, Prop. 7] that $G \cap C_{\overline{G}}(g)^0$ is a maximal torus of order $\frac{1}{(n,q-1)}(q^{n-1}-1)$. Hence $|\overline{L}_\sigma| = 1$ and r and p are non-adjacent. If $e(r,q) = n$ and g is an element of order r, then $q^n - 1$ should divide $|\overline{S}_\sigma| \cdot (q-1)$ by (1). Again using [13, Prop. 7], we see that $G \cap C_{\overline{G}}(g)^0$ is a maximal torus of order $\frac{1}{(n,q-1)}\frac{q^n-1}{q-1}$. Therefore $|\overline{L}_\sigma| = 1$ and every r with $e(r,q) > n-2$ is non-adjacent to p.

${}^2A_2(q)$. Appealing to [13, Prop. 8], we state that only prime divisors of $\frac{q+1}{(3,q+1)}$ are adjacent to p, and the proposition is true in this instance.

${}^2A_{n-1}(q)$ and $n \geqslant 4$. Again T is a Cartan subgroup, $G_1 = {}^2A_{n-3}(q)$, and $G_2 = A_1(q)$. The existence of such a subgroup is proved using [13, Prop. 8]. Thus every prime r with $\nu(e(r,q)) \leqslant n-2$ divides $|G_1|$ and, hence, is adjacent to p. Now let $\nu(e(r,q)) = n-1$ and g be an element of order r. Then $q^{n-1}-(-1)^{n-1}$ divides $|\overline{S}| \cdot (q+1)$ in view of (2). It follows from [13, Prop. 8] that $G \cap C_{\overline{G}}(g)^0$ is a maximal torus of order $\frac{1}{(n,q+1)}(q^{n-1}-(-1)^{n-1})$. Hence $|\overline{L}_\sigma| = 1$ and r and p are non-adjacent. If $\nu(e(r,q)) = n$ and g is an element of order r, then $q^n - (-1)^n$ divides $|\overline{S}_\sigma| \cdot (q+1)$ by (2). Using [13, Prop. 8], we conclude that $G \cap C_{\overline{G}}(g)^0$ is a maximal torus of order $\frac{1}{(n,q+1)}\frac{q^n-(-1)^n}{q+1}$. Therefore $|\overline{L}_\sigma| = 1$ and every prime r with $\nu(e(r,q)) > n-2$ is non-adjacent to p.

$C_n(q)$. Choose R so that T is a Cartan subgroup, $G_1 = C_{n-1}(q)$, and $G_2 = A_1(q)$. Such a subgroup R exists in view of [13, Props. 9, 12]. Again every prime r with $\eta(e(r,q)) \leqslant n-1$ divides $|G_1|$ and, hence, is adjacent to p. If $\eta(e(r,q)) = n$ and g is an element of order r, then (4) implies that either q^n-1 or q^n+1 divides $|\overline{S}_\sigma|$. From [13, Props. 9, 12], we see that $G \cap C_{\overline{G}}(g)^0$ is a maximal torus of order $\frac{1}{(2,q-1)}(q^n \pm 1)$, so $|\overline{L}_\sigma| = 1$ and r and p are non-adjacent.

$B_n(q)$. Since $B_2(q) \simeq C_2(q)$ and $B_n(2^t) \simeq C_n(2^t)$, we may assume that p is odd and $n \geqslant 3$. We can take T to be a Cartan subgroup, letting $G_1 = B_{n-2}(q)$ and $G_2 = D_2(q)$. Such a subgroup R exists in view of [13, Prop. 11]. Since every prime r with $\eta(e(r,q)) \leqslant n-2$ divides the order of G_1, we state that r and p are non-adjacent if $\eta(e(r,q)) \leqslant n-2$. If $\eta(e(r,q)) = n-1$, then, again by [13, Prop. 11], there exists a reductive subgroup R such that $|\overline{S}_\sigma| = q^{\eta(e(r,q))} + (-1)^{e(r,q)}$ and $\overline{L}_\sigma = B_1(q) \simeq A_1(q)$. Hence r and p are adjacent in G. Assume now that $\eta(e(r,q)) = n$ and that g is an element of order r in G. Then the order $|\overline{S}_\sigma|$, defined in [13, Prop. 11], is equal to $\prod\limits_i (q^{n_i} \pm 1)$, where $\sum\limits_i n_i \leqslant n$. Using (4), we conclude that $\eta(e(r,q)) = n$ divides n_i, for some i. Hence either q^n-1 or q^n+1 divides $|S_\sigma|$. In view of [13, Prop. 11], we have $|\overline{L}_\sigma| = 1$, and hence r and p are non-adjacent.

$D_n^\varepsilon(q)$. Let T be a Cartan subgroup, $G_1 = D_{n-2}^\varepsilon(q)$, and $G_2 = A_1(q)$. The existence of such a subgroup

R is proved using [13, Prop. 10]. Therefore r and p are adjacent for all r with $\eta(e(r,q)) \leqslant n-2$ except when $\eta(e(r,q)) = n-2$ and r divides $q^{n-2}+\varepsilon 1$. In the last-mentioned case there exists a reductive subgroup R such that $|\overline{S}_\sigma| = q^{n-2}+\varepsilon 1$ and $\overline{L}_\sigma \simeq D_2^\varepsilon(q)$. Hence in this exceptional case r and p are adjacent also. Now if $\eta(e(r,q)) \geqslant n-1$ and g is an element of G of order r, then again $q^{\eta(e(r,q))} + (-1)^{e(r,q)}$ divides $|\overline{S}_\sigma|$, and [13, Prop. 10] implies that $\overline{L}_\sigma$ is trivial. Therefore r and p are non-adjacent in this instance. □

Proposition 3.2. Let G be a finite simple exceptional group of Lie type over a field of characteristic p. Let $r \in \pi(G)$, $k = e(r,q)$, and $r \neq p$. Then r and p are non-adjacent if and only if one of the following holds:

(1) $G = G_2(q),\ k \in \{3,6\}$; (2) $G = F_4(q),\ k \in \{8,12\}$;
(3) $G = E_6(q),\ k \in \{8,9,12\}$; (4) $G = {}^2E_6(q),\ k \in \{8,12,18\}$;
(5) $G = E_7(q),\ k \in \{7,9,14,18\}$; (6) $G = E_8(q),\ k \in \{15,20,24,30\}$;
(7) $G = {}^3D_4(q),\ k = 12$.

Proof. All statements above are obtained using information about centralizers of semisimple elements given in [15, 17].

Proposition 3.3. Let G be a finite simple Suzuki or Ree group over a field of characteristic p, and let $r \in \pi(G) \setminus \{p\}$. Then r and p are non-adjacent if and only if one of the following holds:

(1) $G = {}^2B_2(2^{2n+1})$ and r divides $m_k(B,n)$;
(2) $G = {}^2G_2(3^{2n+1})$, r divides $m_k(G,n)$, and $r \neq 2$;
(3) $G = {}^2F_4(2^{2n+1})$, r divides $m_k(F,n)$, $r \neq 3$, and $k > 2$.

Numbers $m_i(B,n)$, $m_i(G,n)$, and $m_i(F,n)$ are defined as in Lemma 1.5.

Proof. All statements above are obtained using information about centralizers of semisimple elements given in [15].

4. ADJACENCY OF 2 AND AN ODD PRIME r

In this section we verify whether 2 and an odd prime r are adjacent if both of the numbers are not equal to the characteristic p of the base field. We start with groups $A_n^\varepsilon(q)$. Recall that for these groups, an adjacency criterion for prime divisors of $q-\varepsilon 1$ was not taken up in Sec. 2. The reason is that it seems more natural to consider it in tandem with the criterion for 2, which we will do in the following two propositions.

Proposition 4.1. Let $G = A_{n-1}(q)$ be a finite simple group of Lie type, r be a prime divisor of $q-1$, and s be an odd prime distinct from the characteristic. Put $k = e(s,q)$. Then s and r are non-adjacent if and only if one of the following holds:

(1) $k = n$, $n_r \leqslant (q-1)_r$, and if $n_r = (q-1)_r$, then $2 < (q-1)_r$;
(2) $k = n-1$ and $(q-1)_r \leqslant n_r$.

Proof. First we argue for the following statement:

$$\begin{array}{lll} \left(\frac{q^n-1}{q-1}\right)_r = n_r & \text{if } (q-1)_r \geqslant n_r \text{ and } (q-1)_r > 2; \\ \left(\frac{q^n-1}{q-1}\right)_2 > 2 & \text{if } (q-1)_2 = n_2 = 2. \end{array} \tag{5}$$

Assume that r is odd. Then $n = r^k \cdot l$, where $(l,r) = 1$ and $q = r^k \cdot m + 1$. Now $q^n - 1 = (q^{r^k} - 1)\cdot(q^{n-r^k} + q^{n-2r^k} + \ldots + q^{r^k} + 1)$. The second multiplier is the sum of l numbers of the form q^i, and

$q^i \equiv 1 (\bmod r)$ for all i. Since $(l,r)=1$, the second multiplier is coprime to r. Thus $\left(\frac{q^n-1}{q-1}\right)_r = \left(\frac{q^{r^k}-1}{q-1}\right)_r$. Since $q = r^k \cdot m + 1$, we arrive at

$$\frac{q^{r^k}-1}{q-1} = (r^k \cdot m)^{r^k-1} + r^k \cdot (r^k \cdot m)^{r^k-2} + \ldots + \frac{1}{2} r^k (r^k-1)(r^k \cdot m) + r^k.$$

For r^{k+1} divides all but the last summand, and r^{k+1} does not divide r^k, we conclude that r^k divides $\left(\frac{q^{r^k}-1}{q-1}\right)_r$, but r^{k+1} does not divide $\left(\frac{q^{r^k}-1}{q-1}\right)_r$. Therefore $\left(\frac{q^{r^k}-1}{q-1}\right)_r = n_r$ in this instance.

Assume now that $r=2$, $n = 2^k \cdot l$, where l is odd, and $q = 2^k \cdot m + 1$. Then

$$\left(\frac{q^n-1}{q-1}\right)_2 = \left(\frac{q^l-1}{q-1}\right)_2 \cdot (q^l+1) \cdot (q^{2l}+1) \cdot \ldots \cdot (q^{2^{k-1}l}+1).$$

Since l is odd, $\left(\frac{q^l-1}{q-1}\right)_2 = 1$. If $(q-1)_2 > 2$ then $(q^i+1)_2 = 2$ for all i. Therefore

$$\left(\frac{q^n-1}{q-1}\right)_2 = (q^l+1)_2 \cdot (q^{2l}+1)_2 \cdot \ldots \cdot (q^{2^{k-1}l}+1)_2 = 2^k = n_2.$$

The case $(q-1)_2 = n_2 = 2$ is obvious.

Suppose that $k \leqslant n-2$. Lemma 1.2 implies that there exists a maximal torus T of G of order $\frac{1}{(n,q-1)}(q^k-1)(q-1)^{n-k-1}$. The torus T is an Abelian subgroup of G, and both r and s divide $|T|$. Therefore T contains an element of order rs.

Assume that $k=n$. By (1) and Lemma 1.2, every element of order s is contained in a maximal torus T of order $\frac{1}{(n,q-1)}\frac{q^n-1}{q-1}$. In view of (5), r does not divide $|T|$ iff condition (1) of the present proposition holds.

Suppose that $k=n-1$. By (1) and Lemma 1.2, every element of order s is contained in a maximal torus T of order $\frac{1}{(n,q-1)}(q^{n-1}-1)$. We have $(q^{n-1}-1) = (q-1)(q^{n-2}+q^{n-3}+\ldots+q+1)$ and $(q^{n-2}+q^{n-3}+\ldots+q+1)_r = 1$. Therefore r does not divide $|T|$ iff $(q-1)_r \leqslant n_r$, which agrees with condition (2) of the proposition. □

Proposition 4.2. Let $G = {}^2A_{n-1}(q)$ be a finite simple group of Lie type, r be a prime divisor of $q+1$, and s be an odd prime distinct from the characteristic. Put $k = e(s,q)$. Then s and r are non-adjacent if and only if one of the following holds:

(1) $\nu(k) = n$, $n_r \leqslant (q+1)_r$, and if $n_r = (q+1)_r$, then $2 < (q+1)_r$;

(2) $\nu(k) = n-1$ and $(q+1)_r \leqslant n_r$.

The function $\nu(m)$ is defined as in Prop. 2.2.

Proof. As in the previous proposition, we show that

$$\begin{array}{ll} \left(\frac{q^n-(-1)^n}{q+1}\right)_r = n_r & \text{if } (q+1)_r \geqslant n_r \text{ and } (q+1)_r > 2; \\ \left(\frac{q^n-(-1)^n}{q+1}\right)_2 > 2 & \text{if } (q+1)_2 = n_2 = 2. \end{array} \quad (6)$$

Assume that r is odd. Then $n = r^k \cdot l$, where $(l,r)=1$ and $q = r^k \cdot m - 1$. Now $q^n - (-1)^n = (q^{r^k}+1) \cdot (q^{n-r^k} - q^{n-2r^k} + \ldots + (-1)^{l-1} q^{r^k} + (-1)^l)$. The second multiplier is the sum of l numbers of the form $(-1)^t q^{n-(t+1)r^k}$, and $q^{n-(t+1)r^k} \equiv (-1)^{n+t-1} (\bmod r)$ for all t. Hence the second multiplier is comparable with $(-1)^{n-1}l$ modulo r. Since $(l,r)=1$, the second multiplier is coprime to r. Thus $\left(\frac{q^n-(-1)^n}{q+1}\right)_r = \left(\frac{q^{r^k}+1}{q+1}\right)_r$. At the same time $q = r^k \cdot m - 1$, and hence

$$\frac{q^{r^k}+1}{q+1} = (r^k \cdot m)^{r^k-1} - r^k \cdot (r^k \cdot m)^{r^k-2} + \ldots + (-1)^{k-2}\frac{1}{2} r^k (r^k-1)(r^k \cdot m) + (-1)^{k-1} r^k.$$

Since r^{k+1} divides all but the last summand, and r^{k+1} does not divide r^k, we see that r^k divides $\left(\frac{q^{r^k}+1}{q+1}\right)_r$, but r^{k+1} does not divide $\left(\frac{q^{r^k}+1}{q+1}\right)_r$. Therefore $\left(\frac{q^{r^k}+1}{q+1}\right)_r = n_r$ in this instance.

Suppose now that $r=2$, $n=2^k\cdot l$, where l is odd, and $q=2^k\cdot m-1$. Then

$$\left(\frac{q^n-(-1)^n}{q+1}\right)_2 = \left(\frac{q^l+1}{q+1}\right)_2 \cdot (q^l-1)\cdot(q^{2l}+1)\cdot\ldots\cdot(q^{2^{k-1}l}+1).$$

Since l is odd, $\left(\frac{q^l+1}{q+1}\right)_2 = 1$. If $(q+1)_2>2$ then $(q^{2i}+1)_2=2$, for all $i\geqslant 1$, and $(q^l-1)_2=2$. Therefore

$$\left(\frac{q^n-(-1)^n}{q+1}\right)_2 = (q^l-1)_2\cdot(q^{2l}+1)_2\cdot\ldots\cdot(q^{2^{k-1}l}+1)_2 = 2^k = n_2.$$

The case $(q+1)_2=n_2=2$ is obvious.

Assume that $\nu(k)\leqslant n-2$. Lemma 1.2 implies that there exists a maximal torus T of G of order $\frac{1}{(n,q+1)}(q^{\eta(k)}-(-1)^{\eta(k)})(q+1)^{n-k-1}$. The torus T is an Abelian subgroup of G, and both r and s divide $|T|$. Therefore T contains an element of order rs.

Suppose that $\nu(k)=n$. By (2) and Lemma 1.2, every element of order s is contained in a maximal torus T of order $\frac{1}{(n,q+1)}\frac{q^n-(-1)^n}{q+1}$. In view of (6), r does not divide $|T|$ iff condition (1) of the present proposition holds.

Assume now that $\nu(k)=n-1$. By (2) and Lemma 1.2, every element of order s is contained in a maximal torus T of order $\frac{1}{(n,q+1)}(q^{n-1}-(-1)^{n-1})$. We have $(q^{n-1}-(-1)^{n-1})=(q+1)(q^{n-2}-q^{n-3}+\ldots+(-1)^{n-3}q+(-1)^{n-2})$ and $(q^{n-2}-q^{n-3}+\ldots+(-1)^{n-3}q+(-1)^{n-2})_r=1$. Therefore r does not divide $|T|$ iff $(q+1)_r\leqslant n_r$, which agrees with condition (2) of the proposition. □

Proposition 4.3. Let G, equal to $B_n(q)$ or to $C_n(q)$, be a finite simple group of Lie type over a field of odd characteristic p. Let r be an odd prime divisor of $|G|$, $r\neq p$, and $k=e(r,q)$. Then 2 and r are non-adjacent if and only if $\eta(k)=n$ and one of the following holds:

(1) n is odd and $k=(3-e(2,q))n$;

(2) n is even and $k=2n$.

The function $\eta(m)$ is defined as in Prop. 2.3.

Proof. If $\eta(k)\leqslant n-1$, then Lemma 1.2 implies that there exists a maximal torus T of order $\frac{1}{2}(q^{\eta(k)}+(-1)^k)(q-1)^{n-\eta(k)}$. The torus T is an Abelian subgroup of G, and both of the numbers 2 and r divide $|T|$. Hence T contains an element of order $2r$, and 2 and r are adjacent.

Thus we may assume that $\eta(k)=n$. In view of (4) and Lemma 1.2, every element g of order r is contained in a maximal torus T of order $\frac{1}{2}(q^n+(-1)^k)$. Therefore 2 and r are non-adjacent iff 2 does not divide $|T|$. □

Proposition 4.4. Let $G=D_n^{\varepsilon}(q)$ be a finite simple group of Lie type over a field of odd characteristic p. Let r be an odd prime divisor of $|G|$, $r\neq p$, and $k=e(r,q)$. Then 2 and r are non-adjacent if and only if one of the following holds:

(1) $\eta(k)=n$ and $(4,q^n-\varepsilon 1)=(q^n-\varepsilon 1)_2$;

(2) $\eta(k)=k=n-1$, n is even, $\varepsilon=+$, and $e(2,q)=2$;

(3) $\eta(k)=\frac{k}{2}=n-1$, $\varepsilon=+$, and $e(2,q)=1$;

(4) $\eta(k)=\frac{k}{2}=n-1$, n is odd, $\varepsilon=-$, and $e(2,q)=2$.

The function $\eta(m)$ is defined as in Prop. 2.3.

Proof. First note that if $\eta(k) \leqslant n-2$, then Lemma 1.2 implies that there exists a maximal torus T of order $\frac{1}{(4,q^n-\varepsilon 1)}(q^{\eta(k)}+(-1)^k)(q+(-\varepsilon 1)^k)q-1)^{n-\eta(k)-1}$. Both of the numbers 2 and r divide $|T|$ and, hence, are adjacent.

Now consider the case where $G = {}^2D_n(q)$ (i.e., $\varepsilon = -$). If $\eta(k) = n-1$ then, by (4) and Lemma 1.2, every element of order r is contained in a maximal torus T of order $\frac{1}{(4,q^n+1)}(q^{n-1}+(-1)^k)(q-(-1)^k)$. Hence 2 does not divide $|T|$ iff condition (4) of the proposition holds. If $\eta(k) = n$ then, by (4) and Lemma 1.2, every element of order r is contained in a maximal torus T of order $\frac{1}{(4,q^n+1)}(q^n+1)$; in particular, $k = 2\eta(k)$. Hence 2 does not divide $|T|$ iff $(4, q^n+1) = (q^n+1)_2$, which proves the proposition for twisted groups.

Assume now that $G = D_n(q)$. If $\eta(k) = n-1$ then (4) and Lemma 1.2 imply that every element of order r is contained in a maximal torus T of order $\frac{1}{(4,q^n-1)}(q^{n-1}+(-1)^k)(q+(-1)^k)$, and $k = n-1$ if k is odd. Then 2 does not divide $|T|$ iff (2) or (3) holds. If $\eta(k) = n$, then n is odd, and by (4) and Lemma 1.2, every element of order r is contained in a maximal torus T of order $\frac{1}{(4,q^n-1)}(q^n-1)$. Clearly, 2 does not divide $|T|$ iff $(4, q^n-1) = (q^n-1)_2$. $\square$

Proposition 4.5. Let G be a finite simple exceptional group of Lie type over a field of odd characteristic p. Let r be an odd prime divisor of $|G|$, $r \neq p$, and $k = e(r,q)$. Then 2 and r are non-adjacent if and only if one of the following holds:

(1) $G = G_2(q)$, $k \in \{3, 6\}$;

(2) $G = F_4(q)$, $k = 12$;

(3) $G = E_6(q)$, $k \in \{9, 12\}$;

(4) $G = {}^2E_6(q)$, $k \in \{12, 18\}$;

(5) $G = E_7(q)$, and either $k \in \{7, 9\}$ and $e(2,q) = 2$ or $k \in \{14, 18\}$ and $e(2,q) = 1$;

(6) $G = E_8(q)$, $k \in \{15, 20, 24, 30\}$;

(7) $G = {}^3D_4(q)$, $k = 12$;

(8) $G = {}^2G_2(3^{2n+1})$, and r divides $m_3(G,n)$ or $m_4(G,n)$, where $m_i(G,n)$ are defined as in Lemma 1.5.

The **proof** follows immediately from Lemmas 1.3 and 1.5.

5. THE INTERPLAY BETWEEN PROPERTIES OF THE PRIME GRAPH $GK(G)$ AND THE STRUCTURE OF G

Let $GK(G)$ be the prime graph of a finite group G. Obviously, the spectrum $\omega(G)$ determines the structure of $GK(G)$ uniquely. Denote by $s(G)$ the number of connected components in $GK(G)$, and by $\pi_i(G)$, $i = 1, \ldots, s(G)$, the ith connected component of $GK(G)$. If G has even order then we put $2 \in \pi_1(G)$. Denote by $\omega_i(G)$ the set of numbers $n \in \omega(G)$ such that each prime divisor of n belongs to $\pi_i(G)$.

THEOREM 5.1 (Gruenberg–Kegel; see [1]). If G is a finite group with a disconnected graph $GK(G)$, then one of the following conditions holds:

(a) $s(G) = 2$ and G is a Frobenius group;

(b) $s(G) = 2$ and G is a 2-Frobenius group (i.e., $G = ABC$, where A and AB are normal subgroups of G; AB and BC are Frobenius groups with cores A and B and complements B and C, respectively);

(c) there exists a non-Abelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$, where K is a maximal normal soluble subgroup of G. Furthermore, K and $\overline{G}/S$ are $\pi_1(G)$-subgroups, the graph $GK(S)$ is disconnected, $s(S) \geqslant s(G)$, and for every i, $2 \leqslant i \leqslant s(G)$, there is j, $2 \leqslant j \leqslant s(S)$, such that $\omega_i(G) = \omega_j(S)$.

Along with the classification of finite simple groups with disconnected prime graph obtained in [1, 2], the Gruenberg–Kegel theorem gave rise to a number of important consequences (see, e.g., [1, Thms. 3-6;

2, Thms. 2, 3]). In recent years this theorem has been used in proving recognizability of finite groups by spectra (for details, see [3, 4]).

The proof of the Gruenberg–Kegel theorem makes essential use of the fact that in a group G (if its order is even), there exists an element of odd order, disconnected in $GK(G)$ with a prime 2. It turns out that the requirement for the prime graph $GK(G)$ to be disconnected can, in most cases, be replaced by the weaker requirement that 2 is non-adjacent to at least one odd prime.

Denote by $t(G)$ a maximal number of prime divisors of the order of G that are pairwise non-adjacent in $GK(G)$. In other words, $t(G)$ is a maximal number of vertices in independent sets of $GK(G)$. (A vertex set is said to be *independent* if its elements are pairwise non-adjacent.) In graph theory, this number is usually called an *independence number* of the graph. By analogy, we denote by $t(r, G)$ a maximal number of vertices in independent sets of $GK(G)$ containing a prime r. And call this number an *r-independence number*.

THEOREM 5.2 [4]. Let G be a finite group satisfying the following two conditions:

(a) there exist three primes in $\pi(G)$ which are pairwise non-adjacent in $GK(G)$, that is, $t(G) \geqslant 3$;

(b) there exists an odd prime in $\pi(G)$ which is non-adjacent to 2 in $GK(G)$, that is, $t(2, G) \geqslant 2$.

Then there exists a finite non-Abelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$ for a maximal normal soluble subgroup K of G. Furthermore, $t(S) \geqslant t(G) - 1$ and one of the following conditions holds:

(1) $S \simeq \mathrm{Alt}_7$ or $A_1(q)$ for some odd q, and $t(S) = t(2, S) = 3$.

(2) for every prime $r \in \pi(G)$ non-adjacent to 2 in $GK(G)$, the Sylow r-subgroup of G is isomorphic to the Sylow r-subgroup of S; in particular, $t(2, S) \geqslant t(2, G)$.

Note that Theorem 5.2(a) implies the insolubility of G, and by the Feit–Thompson Odd Theorem, G is of even order. Moreover, (a) can be replaced by a weaker condition — of G being insoluble (see [4, Props. 2, 3]). Combined with [18, Thm. 2], Theorem 5.2 yields

THEOREM 5.3 (see [4, Prop. 4]). Let L be a finite simple group with $t(2, L) \geqslant 2$, which is non-isomorphic to $A_2(3)$, ${}^2A_3(3)$, $C_2(3)$, and Alt_{10}. Let G be a finite group with $\omega(G) = \omega(L)$. Then G satisfies the conclusion of Theorem 5.2. In particular, G has a unique non-Abelian composition factor.

6. INDEPENDENCE NUMBERS

In this section, based on results obtained in Secs. 1-4, we compute values of independence numbers and 2-independence numbers for all finite non-Abelian simple groups. Furthermore, for every finite non-Abelian simple group of Lie type over a field of characteristic p, we determine a p-independence number $t(p, G)$.

For a finite group G, we denote by $\rho(G)$ (by $\rho(r, G)$) some independent set in $GK(G)$ (containing r) with a maximal number of vertices. Thus $|\rho(G)| = t(G)$ and $|\rho(r, G)| = t(r, G)$. For a given finite non-Abelian simple group, we point out some sets $\rho(G)$ and $\rho(2, G)$ (which may be not uniquely defined) and, hence, determine $t(G)$ and $t(2, G)$.

Proposition 6.1. Let G be a sporadic group. Then $\rho(G)$, $t(G)$, $\rho(2, G)$, and $t(2, G)$ are as in Table 2. Furthermore, $\rho(2, G)$ is uniquely determined.

The **proof** follows readily using [5] or [6].

Now we deal with simple alternating groups.

Proposition 6.2. Let $G = \mathrm{Alt}_n$ be an alternating group of degree $n \geqslant 5$. Let s'_n be the largest prime not exceeding $n/2$ and s''_n be the smallest prime greater than $n/2$. Set $\tau(n) = \{s \mid s \text{ is a prime}, n/2 < s \leqslant n\}$, and set $\tau(2,n) = \{s \mid s \text{ is a prime}, n-3 \leqslant s \leqslant n\}$. Then $\rho(G)$, $t(G)$, $\rho(2,G)$, and $t(2,G)$ are as in Table 3. Furthermore, $\rho(2,G)$ is uniquely determined.

The **proof** follows immediately from Prop. 1.1.

Now we consider groups of Lie type. Since the problem here is much more complicated, we solve it step by step. Our main tools are Lemmas 1.4 and 1.5 and results scattered over Secs. 2-4. Recall that for a given natural number q, r_n denotes some primitive prime divisor of q^n-1 (if any). Note that such a divisor r_{2m} of $q^{2m}-1$ divides q^m+1 and does not divide q^k+1, for every natural $k<m$. It also seems useful to remind that for primes r and s, the equality $r^n = s^k+1$ can hold only in the following cases: $r^n=9$ and $s^k=8$, or $r=2^k+1$ is a Fermat prime and $n=1$, or $s=2^n-1$ is a Mersenne prime and $k=1$. (Like Lemma 1.4, this fact is a consequence of Zsigmondi's theorem; see [16].)

Since the argument in the situation where 2 is the characteristic of the field of definition fits with every characteristic p, we start with determining $t(p,G)$.

Proposition 6.3. Let G be a finite simple group of Lie type over a field of characteristic p and order q. Then $\rho(p,G)$ and $t(p,G)$ are listed in Table 4 for classical groups and in Table 5 for exceptional groups.

Proof. (1) $G = A_{n-1}^{\varepsilon}(q)$. Let $G = A_1(q)$. Since $A_1(2)$ and $A_1(3)$ are not simple, we may assume that $q>3$. By Lemma 1.4, there exist primitive prime divisors r_1 and r_2 of $q-1$ and $q+1$, which are non-adjacent in $GK(G)$. By Proposition 3.1, these are non-adjacent to p. Hence $\rho(p,G)=\{p,r_1,r_2\}$.

Suppose that $n>2$. In order to treat groups $A_{n-1}(q)$ and ${}^2A_{n-1}(q)$ together, we define a new function such as

$$\nu_\varepsilon(m) = \begin{cases} m & \text{if either } \varepsilon=+ \text{ or } \varepsilon=- \text{ and } m\equiv 0 (\operatorname{mod} 4);\\ \frac{m}{2} & \text{if } \varepsilon = - \text{ and } m \equiv 2 (\operatorname{mod} 4);\\ 2m & \text{otherwise.}\end{cases}$$

Obviously, $\nu_\varepsilon(m)$ is the identity function, if $\varepsilon=+$, and $\nu_\varepsilon(m)=\nu(m)$ if $\varepsilon=-$. It is easy to check that ν_ε is a bijection onto $\mathbb{N}$, that is, ν_ε^{-1} is well defined.

Let $n=3$ and $G=A_2^\varepsilon(q)$. Proposition 3.1 implies that $r_{\nu_\varepsilon^{-1}(2)}$ and $r_{\nu_\varepsilon^{-1}(3)}$ are non-adjacent to p. Moreover, if $(q-\varepsilon 1)_3 = 3$, then 3 is non-adjacent to p by Proposition 3.1, and to $r_{\nu_\varepsilon^{-1}(2)}$ and $r_{\nu_\varepsilon^{-1}(3)}$ by Props. 4.1 and 4.2. By Lemma 1.4, $r_{\nu_\varepsilon^{-1}(3)}$ exists for all q except the case where $\varepsilon=-$ and $q=2$. But ${}^2A_2(2)$ is not simple and so omitted. In this case, therefore, $r_{\nu_\varepsilon^{-1}(3)}$ exists for all simple groups. If there is an odd primitive divisor $r_{\nu_{\varepsilon^{-1}(2)}}$ then it is not connected with the characteristic, and we are faced with one of the first two possibilities among four below. If the sole primitive divisor $r_{\nu_{\varepsilon^{-1}(2)}}$ is equal to 2 then $q+\varepsilon 1$ is a power of 2, that is, $q+\varepsilon 1$ is a Mersenne prime, if $\varepsilon=+$, and is a Fermat prime or 9 if $\varepsilon=-$. By Proposition 3.1, 2 is adjacent to the characteristic, and so one of the last two possibilities among the four is realized. Thus we have

$$\rho(p,G) = \begin{cases} \{p,3,r_{\nu_\varepsilon^{-1}(2)},r_{\nu_\varepsilon^{-1}(3)}\} & \text{if } (q-\varepsilon 1)_3 = 3 \text{ and } q+\varepsilon \neq 2^k;\\ \{p,r_{\nu_\varepsilon^{-1}(2)},r_{\nu_\varepsilon^{-1}(3)}\} & \text{if } (q-\varepsilon 1)_3 \neq 3 \text{ and } q+\varepsilon \neq 2^k;\\ \{p,3,r_{\nu_\varepsilon^{-1}(3)}\} & \text{if } (q-\varepsilon 1)_3 = 3 \text{ and } q+\varepsilon = 2^k;\\ \{p,r_{\nu_\varepsilon^{-1}(3)}\} & \text{if } (q-\varepsilon 1)_3 \neq 3 \text{ and } q+\varepsilon = 2^k.\end{cases}$$

Note that we avoid using $e(r,q)$ and ν_ε in tables in Sec. 8.

Let $G=A_4(2)$, $A_5(2)$, or ${}^2A_3(2)$. Since 2^6-1 has no primitive prime divisors, by Proposition 3.1 we have $\rho(2,A_5(2))=\{2,31\}$, $\rho(2,A_6(2))=\{2,127\}$, and $\rho(2,{}^2A_3(2))=\{2,5\}$.

In all other cases, by Lemma 1.4, there exist primitive prime divisors $r_{\nu_\varepsilon^{-1}(n-1)}$ and $r_{\nu_\varepsilon^{-1}(n)}$. In view of Lemma 1.2, we have $r_{\nu_\varepsilon^{-1}(n-1)}, r_{\nu_\varepsilon^{-1}(n)} \in \pi(G)$. By Propositions 2.1 and 2.2, these divisors are non-adjacent in $GK(G)$. And Proposition 3.1 yields $\rho(p,G)=\{p,r_{\nu_\varepsilon^{-1}(n-1)},r_{\nu_\varepsilon^{-1}(n)}\}$ for $G=A_{n-1}^\varepsilon(q)$.

(2) $G = C_n(q)$ or $B_n(q)$. In view of Propositions 2.3, 3.1, and 4.3, the prime graphs of $C_n(q)$ and $B_n(q)$ coincide. We consider these groups together, and for brevity, use the symbol $C_n(q)$ for both.

Let $G = C_3(2)$. Since there are no primes r with $e(r, 2) = 6$, only 7, being a primitive prime divisor of $2^3 - 1$, is not adjacent to 2.

Let $G = C_n(q)$, $n \geqslant 2$, and $(n, q) \neq (3, 2)$. If n is even, then only primitive prime divisors of $q^n + 1$ are non-adjacent to p, as follows by Prop. 3.1. Thus $\rho(G) = \{p, r_{2n}\}$ in this instance. If n is odd, then Propositions 2.3 and 3.1 yield $\rho(G) = \{p, r_n, r_{2n}\}$.

(3) $G = D_n^\varepsilon(q)$. With due regard for well-known isomorphisms between groups with small Lie ranks, we may assume that $n \geqslant 4$. Let $n = 4$ and $q = 2$. Since there are no primitive prime divisors of $2^6 - 1$, only 7, being a primitive prime divisor of $2^3 - 1$, is non-adjacent to 2 in case $G = D_4(2)$, and only 7 and 17, being primitive divisors of $2^3 - 1$ and $2^8 - 1$, are non-adjacent to 2 in case $G = {}^2D_4(2)$. All other cases could be easily described directly in terms of Props. 2.4 and 3.1. Results of such a consideration are collected in Table 4.

(4) For exceptional groups distinct from Suzuki's and Ree's, the result can be obtained using Props. 2.5 and 3.2 and Lemma 1.4.

(5) G is a finite simple Suzuki or Ree group. Let $G = {}^2B_2(2^{2m+1})$ and let s_i be a prime divisor of $m_i(B, n)$, where $i = 1, 2, 3$ (see Lemma 1.5). By Proposition 2.6, primes s_i and s_j are adjacent iff $i = j$. On the other hand, every s_i, where $i = 1, 2, 3$, is non-adjacent to $p = 2$. Thus $\rho(p, G) = \{p, s_1, s_2, s_3\}$ and $t(p, G) = 4$ in this instance. The same argument can be applied to Ree groups and to prime divisors of $m_i(G, n)$ and $m_i(F, n)$. $\square$

In general, a primitive prime divisor r_m of $q^m - 1$ can be chosen in a number of ways. Therefore the set $\rho(p, G)$ cannot be uniquely determined for a finite simple group G of Lie type. However, it turns out that values of $e(r, q)$, for all primitive prime divisors r in $\rho(p, G)$, are invariants for the given group G of Lie type.

Proposition 6.4. Let G be a finite simple group of Lie type over a field of characteristic p and order q. Assume that G is non-isomorphic to ${}^2B_2(q)$, ${}^2G_2(q)$, ${}^2F_4(2)'$, and ${}^2F_4(q)$. Let $\rho(p, G) = \{p, s_1, s_2, \ldots, s_m\}$ be an independent set in $GK(G)$ containing p and having a maximal number of vertices, and let $k_i = e(s_i, q)$. Then the set $\{k_1, \ldots, k_m\}$ is determined uniquely.

The **proof** follows from results given in Secs. 2-4.

If we replace primitive prime divisors by the divisors of numbers defined in Lemma 1.5 we obtain a similar statement for Suzuki and Ree groups.

Proposition 6.5. Let G be a finite simple Suzuki or Ree group over a field of characteristic p, and let $\rho(p, G) = \{p, s_1, \ldots, s_k\}$.

(1) If $G = {}^2B_2(2^{2n+1})$ then, up to reordering, s_i divide $m_i(B, n)$. In particular, $k = 3$.

(2) If $G = {}^2G_2(3^{2n+1})$ then all s_i are odd, and up to reordering, s_i divide $m_i(G, n)$. In particular, $k = 4$.

(3) If $G = {}^2F_4(2^{2n+1})$ then, up to reordering, s_i divide $m_{i+2}(F, n)$, and $s_1 \neq 3$. In particular, $k = 3$.

Numbers $m_i(B, n)$, $m_i(G, n)$, and $m_i(F, n)$ are defined as in Lemma 1.5.

We determine $t(2, G)$. Obviously, for a group of Lie type over a field of even characteristic, p-independence coincides with 2-independence. Thus we may assume that G is defined over a field of odd characteristic.

Proposition 6.6. Let G be a finite simple group of Lie type over a field of odd characteristic p and order q. Then $\rho(2, G)$ and $t(2, G)$ are listed in Table 6 for classical groups and in Table 7 for exceptional groups.

Proof. (1) $G = A^{\varepsilon}_{n-1}(q)$. Let $G = A_1(q)$ and $q > 3$. Since q is odd, every prime divisor $r \neq p$ of $|G|$ divides $(q-1)/2$ or $(q+1)/2$. If $e(2,q) = 1$, then 2 is non-adjacent to some prime divisor r_2 of $(q+1)/2$, and $\tau(G) = \{2, p, r_2\}$. If $e(2,q) = 2$, then 2 is non-adjacent to some divisor r_1 of $(q-1)/2$, and $\tau(G) = \{2, p, r_1\}$.

Let $G = A^{\varepsilon}_{n-1}(q)$ and $n \geqslant 3$. If $(q-\varepsilon 1)_2 < n_2$, then only the primitive prime divisor $r_{\nu_\varepsilon^{-1}(n-1)}$ is non-adjacent to 2, as follows by Props. 4.1 and 4.2. In this case $n_2 \geqslant 4$, since the inequality $(q-\varepsilon 1)_2 < n_2$ is impossible for $n_2 \leqslant 2$. By Lemma 1.4, the primitive prime divisor $r_{\nu_\varepsilon^{-1}(n-1)}$ always exists. Thus $\rho(2,G) = \{2, r_{\nu_\varepsilon^{-1}(n-1)}\}$.

If $(q-\varepsilon 1)_2 > n_2$ or $(q-\varepsilon 1)_2 = n_2 = 2$, then every primitive prime divisor $r_{\nu_\varepsilon^{-1}(n)}$ is non-adjacent to 2, which follows by Props. 4.1 and 4.2. Therefore $\rho(2,G) = \{2, r_{\nu_\varepsilon^{-1}(n)}\}$.

Lastly, let $(q-\varepsilon 1)_2 = n_2 > 2$. By Propositions 4.1 and 4.2, only primitive prime divisors $r_{\nu_\varepsilon^{-1}(n-1)}$ and $r_{\nu_\varepsilon^{-1}(n)}$ are non-adjacent to 2. On the other hand, primes $r_{\nu_\varepsilon^{-1}(n-1)}$ and $r_{\nu_\varepsilon^{-1}(n)}$ are non-adjacent by Props. 2.1 and 2.2. Thus $\rho(2,G) = \{2, r_{\nu_\varepsilon^{-1}(n-1)}, r_{\nu_\varepsilon^{-1}(n)}\}$.

(2) $G = C_n(q)$ or $B_n(q)$. Results collected in Table 7 follow directly from Prop. 4.3.

(3) $G = D^{\varepsilon}_n(q)$. These results are a direct consequence of Prop. 4.4. Note that the equality $t(2,G) = 2$ holds for the majority of groups of type D_n over fields of odd order. Exceptions are the following: n is odd and $q \equiv 5 (\bmod 8)$, for $G = D_n(q)$, and $q \equiv 3 (\bmod 8)$ for $G = {}^2D_n(q)$.

(4) In order to complete the proof of the proposition, we can use Proposition 4.5 and Lemma 1.5 for Suzuki and Ree groups, and Lemma 1.4 for other exceptional groups. □

Now we look at the uniqueness of $\rho(2,G)$. The situation here is very similar to one with $\rho(p,G)$.

Proposition 6.7. Let G be a finite simple group of Lie type over a field of odd characteristic p and order q. Assume that G is non-isomorphic to groups $A_1(q)$ and ${}^2G_2(q)$. Let $\rho(2,G) = \{2, s_1, s_2, \ldots, s_m\}$ be an independent set in $GK(G)$ containing 2 and having a maximal number of vertices, and let $k_i = e(s_i, q)$. Then the set $\{k_1, \ldots, k_m\}$ is uniquely determined.

The **proof** follows from results given in Secs. 2-4.

Proposition 6.8. Let G be either $A_1(q)$, with q odd, or ${}^2G_2(3^{2n+1})$, and let $\rho(2,G) = \{2, s_1, s_2\}$.

(1) If $G = A_1(q)$, q is odd, then, up to renumbering, $s_1 = p$ and $e(s_2, q) = 3 - e(2,q)$.

(2) If $G = {}^2G_2(3^{2n+1})$, then, up to renumbering, $s_i = m_{i+2}(G,n)$.

Numbers $m_i(G,n)$ are defined as in Lemma 1.5.

The **proof** follows from results given in Secs. 2-4.

Lastly, for every finite simple group G of Lie type, we determine some independent set $\rho(G)$ in $GK(G)$ with a maximal number of vertices.

Proposition 6.9. Let G be a finite simple group of Lie type over a field of characteristic p and order q. Then $\rho(G)$ and $t(G)$ are listed in Table 8 for classical groups and in Table 9 for exceptional groups.

Proof. (1) $G = A^{\varepsilon}_{n-1}(q)$. If $G = A_1(q)$ or $G = A^{\varepsilon}_2(q)$, then arguing as in the proof of Proposition 6.3 yields $\rho(G) = \rho(p,G)$.

Suppose $n = 4$. In view of Proposition 6.3 and Table 4, we have $t(p,G) = 3$ in all cases except for the group ${}^2A_3(2)$. Using Proposition 6.6 and Table 6, we see that $t(2,G) \leqslant 3$. By Propositions 4.1 and 4.2, it follows that $t(r_{\nu_\varepsilon^{-1}(1)}, G) \leqslant 3$. Furthermore, there exist at most three other primitive divisors $r_{\nu_\varepsilon^{-1}(2)}$, $r_{\nu_\varepsilon^{-1}(3)}$, and $r_{\nu_\varepsilon^{-1}(4)} = r_4$. Therefore $t(G) = t(p,G) = 3$ except the case where $t({}^2A_3(2)) = t(2, {}^2A_3(2)) = 2$.

Let $G = A^{\varepsilon}_{n-1}(q)$, $n \geqslant 5$, and $q \neq 2$. By Lemma 1.4, there exist primitive prime divisors $r_{\nu_\varepsilon^{-1}(k)}$ for every $k > 2$. Denote by m a number $[n/2]$, that is, the integral part of $n/2$. By Propositions 2.1 and 2.2, the set

$$\rho = \{r_{\nu_\varepsilon^{-1}(m+1)}, r_{\nu_\varepsilon^{-1}(m+2)}, \ldots, r_{\nu_\varepsilon^{-1}(n)}\}$$

is independent in $GK(G)$. On the other hand, by Propositions 2.1, 2.2, 3.1, 4.1, and 4.2, every two prime divisors of $\pi\left(q\prod_{i=1}^{m}(q^i-(\varepsilon 1)^i)\right)$ are adjacent in $GK(G)$. Furthermore, each of these is adjacent to at least one number in ρ. Since every prime $s\in\pi(G)\setminus\left(\rho\cup\pi\left(q\prod_{i=1}^{m}(q^i-(\varepsilon 1)^i)\right)\right)$ is of the form r_i for some $i>m$, it follows that every independent set in $GK(G)$ with a prime divisor of $\pi\left(q\prod_{i=1}^{m}(q^i-1)\right)$ contains at most $|\rho|$ vertices. Thus $\rho(G)=\rho$ and $t(G)=|\rho|=[(n+1)/2]$.

Let $q=2$. Since $\nu_+^{-1}(6)=\nu_-^{-1}(3)=6$, results of the previous paragraph hold true for $A_{n-1}(2)$ with $n\geqslant 12$, and for ${}^2A_{n-1}(2)$ with $n\geqslant 6$. If $n=5$ and $\varepsilon=-$, then $\rho(G)=\rho(2,G)=\{2,5,11\}$ and $t(G)=3$ for ${}^2A_4(2)$. Therefore we can suppose that $G=A_{n-1}(2)$. If $n=5,6$ then $\rho(G)=\{r_3,r_4,r_5\}=\{5,7,31\}$ and $t(G)=3$. If $7\leqslant n\leqslant 11$ then we must eliminate divisors of 2^6-1 from $\rho(G)$, and then argue as in the previous paragraph. In this case $\rho(G)=\{r_i\mid i\neq 6,\ [n/2]<i\leqslant n\}$ and $t(G)=[(n-1)/2]$.

(2) $G=C_n(q)$ or $B_n(q)$. Since $GK(C_n(q))=GK(B_n(q))$, we consider these groups together.

$G=C_2(q)$, $q>2$. Since every two prime divisors of $q(q^2-1)$ are adjacent, we have $\rho(G)=\rho(p,G)=\{p,r_4\}$.

Let $n\geqslant 3$, if $q>2$, and $n\geqslant 7$ if $q=2$. Define the set

$$\rho=\{r_{2i}\mid [n+1/2]<i\leqslant n\}\cup\{r_i\mid [n/2]<i\leqslant n,\ i\equiv 1(\operatorname{mod}2)\}.$$

Using results of Secs. 1-4 and following the line of argument used in the proof for $A_{n-1}^{\varepsilon}(q)$, we obtain $\rho(G)=\rho$, and it is now easy to verify that $t(G)=[(3n+5)/4]$.

If $q=2$ and $3\leqslant n\leqslant 5$, the bulk of the argument remains the same; we need only eliminate divisors of type r_6 from ρ. Thus $t(G)=[(3n+1)/4]$ in this instance. Lastly, if $(n,q)=(6,2)$, we must eliminate a divisor of type r_6 from ρ, possibly adding a primitive prime divisor 7 of 2^3-1 instead. Thus $t(G)=[(3n+5)/4]$ as in the general case.

(3) $G=D_n^{\varepsilon}(q)$. The argument is the same as in the previous parts of the proof. Using Proposition 2.4, we obtain a set $\rho(G)$ in the common situation, and then consider some exceptions arising from the exceptions in Lemma 1.4.

(4) Results for exceptional groups are obtained using Props. 2.5 and 2.6, Tables 5 and 6, and Lemmas 1.4 and 1.5. □

7. APPLICATIONS

Here we apply our results in the spirit of Sec. 5. First, our study shows that the condition $t(2,G)>1$ is realized for an extremely wide class of finite simple groups.

THEOREM 7.1. Let G be a finite non-Abelian simple group with $t(2,G)=1$; then G is an alternating group Alt_n, where n is such that $\tau(2,n)=\{s\mid s \text{ is a prime},\ n-3\leqslant s\leqslant n\}=\varnothing$

Proof. See Props. 6.1-6.9 and corresponding tables in Sec. 8.

Thus we may apply results of Theorems 5.2 and 5.3 as follows.

COROLLARY 7.2. Let L be a finite non-Abelian simple group distinct from groups $A_2(3)$, ${}^2A_3(3)$, $C_2(3)$, Alt_{10}, and from groups Alt_n with $\tau(2,n)=\varnothing$. Let G be a finite group satisfying $\omega(G)=\omega(L)$. Then there exists a finite non-Abelian simple group S such that $S\leq\overline{G}=G/K\leq\mathrm{Aut}(S)$ for a maximal normal soluble subgroup K of G. Furthermore, $t(S)\geqslant t(G)-1$ and one of the following conditions holds:

(1) $S\simeq\mathrm{Alt}_7$ or $A_1(q)$ for some odd q, and $t(S)=t(2,S)=3$;

(2) for every prime $r \in \pi(G)$ non-adjacent in $GK(G)$ to 2, the Sylow r-subgroup of G is isomorphic to the Sylow r-subgroup of S; in particular, $t(2, S) \geqslant t(2, G)$.

The **proof** follows directly from Theorems 5.3 and 7.1.

If G is one of the groups $A_2(3)$, ${}^2A_3(3)$, $C_2(3)$, or Alt_{10}, then $t(2, G) \geqslant 2$, and we also have the following:

COROLLARY 7.3. Let L be a finite simple group distinct from Alt_n with $\tau(2, n) = \varnothing$. Let G be a finite group satisfying $\omega(G) = \omega(L)$. Then G has at most one non-Abelian composition factor.

For an arbitrary subset ω of the set $\mathbb{N}$ of natural numbers, we denote by $h(\omega)$ the number of pairwise non-isomorphic finite groups G such that $\omega(G) = \omega$. We say that for a finite group G, *the recognition problem is solved* if we know the value of $h(\omega(G))$ (for brevity, we write $h(G)$). In particular, the group G is said to be *recognizable by spectrum* (briefly, *recognizable*) if $h(G) = 1$.

In a span of the last twenty years, the recognition problem has been resolved for many finite non-Abelian simple and almost simple groups (see a review in [3]). However, most of these groups have a disconnected prime graph, since essential use in proofs was made of the Gruenberg–Kegel theorem and the classification of finite simple groups with disconnected prime graph obtained by Williams and Kondratiev. The main theorem in [4] (Theorem 5.2 here) and results of the present paper grant us a possibility for dealing with groups whose prime graph is connected.

A finite non-Abelian simple group L is said to be *quasirecognizable by spectrum* if every finite group with the same spectrum has exactly one non-Abelian composition factor S, isomorphic to L. Thus studying into quasirecognizability is an important step in determining whether a given group is recognizable by spectrum. In [4], a sketchproof is given for the following:

THEOREM 7.4 [4, Prop. 5]. Let $L = {}^2D_n(q)$, $q = 2^k$, k and n be natural numbers, n be even, and $n \geqslant 16$. Then L is quasirecognizable by spectrum.

Actually, this result has been proven modulo results given in Sec. 6. So now we are done away with it.

We draw the reader's attention to one statement mentioned above.

Proposition 7.5. Let $G = B_n(q)$ and $H = C_n(q)$. Then the prime graphs $GK(G)$ and $GK(H)$ coincide.

The **proof** follows from results given in Secs. 2-4.

Finally, it is worth mentioning a recent result on prime graphs of finite groups. Recall that a vertex set of a graph is called a *clique* if all vertices in that set are pairwise adjacent. In [18, Thm. 1], a description is furnished for all finite non-Abelian simple groups whose prime graph connected components are cliques. In checking the list of such groups versus results of the present paper, however, we found out that it contains some mistakes. As a matter of fact, for groups $G = A_2^{\varepsilon}(q)$, where $q = 2^k - \varepsilon 1$ and $(q - \varepsilon 1)_3 = 3$, we have $3, p \in \pi_1(G)$ and $3p \notin \omega(G)$. Therefore the component $\pi_1(G)$ is not a clique for these groups, which clashes with the statement of Theorem 1 in [18]. Below, in Corollary 7.6, we give a revised list of the groups in question.

COROLLARY 7.6. Let G be a finite non-Abelian simple group, and let all connected components of its prime graph $GK(G)$ be cliques. Then G is one of the groups in the following list:

(1) sporadic groups M_{11}, M_{22}, J_1, J_2, J_3, and HiS;

(2) alternating groups Alt_n, where $n = 5, 6, 7, 9, 12, 13$;

(3) groups of Lie type $A_1(q)$, where $q > 3$; $A_2(4)$; $A_2(q)$, where $(q-1)_3 \neq 3$, $q+1 = 2^k$; ${}^2A_3(3)$; ${}^2A_5(2)$; ${}^2A_2(q)$, where $(q+1)_3 \neq 3$, $q-1 = 2^k$; $C_3(2)$, $C_2(q)$, where $q > 2$; $D_4(2)$; ${}^3D_4(2)$; ${}^2B_2(q)$, where $q = 2^{2k+1}$; $G_2(q)$, where $q = 3^k$.

Proof. All connected component of the prime graph $GK(G)$ of a finite group G are cliques iff the number $s(G)$ of components is equal to an independence number $t(G)$ of $GK(G)$. For every finite non-Abelian simple group, the values of $s(G)$ and $t(G)$ are now known. Thus, using Tables 2a-2c in [3] for values of $s(G)$ and appealing to Tables 2-9 in Sec. 8 for $t(G)$, we obtain the results required. □

8. TABLES

In tables below, n and k are assumed to be natural. By $[x]$ we denote the integral part of x. For a finite group G, we write $\rho(G)$ ($\rho(r, G)$) for some independent set in $GK(G)$ (containing r) with a maximal number of vertices, and put $t(G) = |\rho(G)|$ and $t(r, G) = |\rho(r, G)|$. In Table 3, $\tau(n)$ denotes the set $\{s \mid s \text{ is a prime, } n/2 < s \leqslant n\}$, and $\tau(2, n)$ denotes the set $\{s \mid s \text{ is a prime, } n-3 \leqslant s \leqslant n\}$. Denote by s'_n the largest prime not exceeding $n/2$, and by s''_n the smallest prime greater than $n/2$. In Tables 4-9, we assume that G is a finite non-Abelian simple group of Lie type over a field of characteristic p and order q. By r_m we denote the primitive prime divisor of $q^m - 1$. If p is odd then we say that 2 is a primitive prime divisor of $q - 1$ if $q \equiv 1 (\operatorname{mod} 4)$, and that 2 is a primitive prime divisor of $q^2 - 1$ if $q \equiv -1 (\operatorname{mod} 4)$.

TABLE 2. Sporadic Groups

G	$t(G)$	$\rho(G)$	$t(2,G)$	$\rho(2,G)$
M_{11}	3	$\{3,5,11\}$	3	$\{2,5,11\}$
M_{12}	3	$\{3,5,11\}$	2	$\{2,11\}$
M_{22}	4	$\{3,5,7,11\}$	4	$\{2,5,7,11\}$
M_{23}	4	$\{3,7,11,23\}$	4	$\{2,5,11,23\}$
M_{24}	4	$\{5,7,11,23\}$	3	$\{2,11,23\}$
J_1	4	$\{5,7,11,19\}$	4	$\{2,7,11,19\}$
J_2	2	$\{5,7\}$	2	$\{2,7\}$
J_3	3	$\{5,17,19\}$	3	$\{2,17,19\}$
J_4	7	$\{7,11,23,29,31,37,43\}$	6	$\{2,23,29,31,37,43\}$
Ru	4	$\{5,7,13,29\}$	2	$\{2,29\}$
He	3	$\{5,7,17\}$	2	$\{2,17\}$
McL	3	$\{5,7,11\}$	2	$\{2,11\}$
HN	3	$\{7,11,19\}$	2	$\{2,19\}$
HiS	3	$\{5,7,11\}$	3	$\{2,7,11\}$
Suz	4	$\{5,7,11,13\}$	3	$\{2,11,13\}$
Co_1	4	$\{7,11,13,23\}$	2	$\{2,23\}$
Co_2	4	$\{5,7,11,23\}$	3	$\{2,11,23\}$
Co_3	4	$\{5,7,11,23\}$	2	$\{2,23\}$
Fi_{22}	4	$\{5,7,11,13\}$	2	$\{2,13\}$
Fi_{23}	5	$\{7,11,13,17,23\}$	3	$\{2,17,23\}$
Fi'_{24}	6	$\{7,11,13,17,23,29\}$	4	$\{2,17,23,29\}$
O'N	5	$\{5,7,11,19,31\}$	4	$\{2,11,19,31\}$
LyS	6	$\{5,7,11,31,37,67\}$	4	$\{2,31,37,67\}$
F_1	11	$\{11,13,17,19,23,29,31,41,47,59,71\}$	5	$\{2,29,41,59,71\}$
F_2	8	$\{7,11,13,17,19,23,31,47\}$	3	$\{2,31,47\}$
F_3	5	$\{5,7,13,19,31\}$	4	$\{2,13,19,31\}$

TABLE 3. Simple Alternating Groups

G	Conditions	$t(G)$	$\rho(G)$	$t(2,G)$	$\rho(2,G)$
Alt_n	$n = 5, 6$	3	$\{2, 3, 5\}$	3	$\{2, 3, 5\}$
	$n = 8$	3	$\{2, 5, 7\}$	3	$\{2, 5, 7\}$
	$n \geqslant 7,\ s'_n + s''_n > n$	$\lvert\tau(n)\rvert + 1$	$\tau(n) \cup \{s'_n\}$	$\lvert\tau(2, n)\rvert + 1$	$\tau(2, n) \cup \{2\}$
	$n \geqslant 9,\ s'_n + s''_n \leqslant n$	$\lvert\tau(n)\rvert$	$\tau(n)$	$\lvert\tau(2, n)\rvert + 1$	$\tau(2, n) \cup \{2\}$

TABLE 4. p-Independence Numbers for Finite Simple Classical Groups

G	Conditions	$t(p,G)$	$\rho(p,G)$
$A_{n-1}(q)$	$n = 2,\ q > 3$	3	$\{p, r_1, r_2\}$
	$n = 3,\ (q-1)_3 = 3,\ \text{ and } q+1 \neq 2^k$	4	$\{p, 3, r_2, r_3\}$
	$n = 3,\ (q-1)_3 \neq 3,\ \text{ and } q+1 \neq 2^k$	3	$\{p, r_2, r_3\}$
	$n = 3,\ (q-1)_3 = 3,\ \text{ and } q+1 = 2^k$	3	$\{p, 3, r_3\}$
	$n = 3,\ (q-1)_3 \neq 3,\ \text{ and } q+1 = 2^k$	2	$\{p, r_3\}$
	$n = 6,\ q = 2$	2	$\{2, 31\}$
	$n = 7,\ q = 2$	2	$\{2, 127\}$
	$n > 3\ (n, q) \neq (6, 2), (7, 2)$	3	$\{p, r_{n-1}, r_n\}$
${}^2A_{n-1}(q)$	$n = 3,\ q \neq 2,\ (q+1)_3 = 3,\ \text{ and } q-1 \neq 2^k$	4	$\{p, 3, r_1, r_6\}$
	$n = 3,\ (q+1)_3 \neq 3,\ \text{ and } q-1 \neq 2^k$	3	$\{p, r_1, r_6\}$
	$n = 3,\ (q+1)_3 = 3,\ \text{ and } q-1 = 2^k$	3	$\{p, 3, r_6\}$
	$n = 3,\ (q+1)_3 \neq 3,\ \text{ and } q-1 = 2^k$	2	$\{p, r_6\}$
	$n = 4,\ q = 2$	2	$\{2, 5\}$
	$n \equiv 0 (\mathrm{mod}\, 4),\ (n, q) \neq (4, 2)$	3	$\{p, r_{2n-2}, r_n\}$
	$n \equiv 1 (\mathrm{mod}\, 4)$	3	$\{p, r_{n-1}, r_{2n}\}$
	$n \equiv 2 (\mathrm{mod}\, 4),\ n \neq 2$	3	$\{p, r_{2n-2}, r_{n/2}\}$
	$n \equiv 3 (\mathrm{mod}\, 4),\ n \neq 3$	3	$\{p, r_{(n-1)/2}, r_{2n}\}$
$B_n(q)$ or $C_n(q)$	$n = 3,\ q = 2$	2	$\{2, 7\}$
	n is even	2	$\{p, r_{2n}\}$
	$n > 1$ is odd, $(n, q) \neq (3, 2)$	3	$\{p, r_n, r_{2n}\}$
$D_n(q)$	$n = 4,\ q = 2$	2	$\{2, 7\}$
	$n \equiv 0 (\mathrm{mod}\, 2),\ n \geqslant 4,\ (n, q) \neq (4, 2)$	3	$\{p, r_{n-1}, r_{2n-2}\}$
	$n \equiv 1 (\mathrm{mod}\, 1),\ n > 4$	3	$\{p, r_n, r_{2n-2}\}$
${}^2D_n(q)$	$n = 4,\ q = 2$	3	$\{2, 7, 17\}$
	$n \equiv 0 (\mathrm{mod}\, 2),\ n \geqslant 4,\ (n, q) \neq (4, 2)$	4	$\{p, r_{n-1}, r_{2n-2}, r_{2n}\}$
	$n \equiv 1 (\mathrm{mod}\, 2),\ n > 4$	3	$\{p, r_{2n-2}, r_{2n}\}$

TABLE 5. p-Independence Numbers for Finite Simple Exceptional Groups of Lie Type

G	Conditions	$t(p,G)$	$\rho(p,G)$
$G_2(q)$	$q > 2$	3	$\{p, r_3, r_6\}$
$F_4(q)$	none	3	$\{p, r_8, r_{12}\}$
$E_6(q)$	none	4	$\{p, r_8, r_9, r_{12}\}$
${}^2E_6(q)$	none	4	$\{p, r_8, r_{12}, r_{18}\}$
$E_7(q)$	none	5	$\{p, r_7, r_9, r_{14}, r_{18}\}$
$E_8(q)$	none	5	$\{p, r_{15}, r_{20}, r_{24}, r_{30}\}$
${}^3D_4(q)$	none	2	$\{p, r_{12}\}$
${}^2B_2(2^{2n+1})$	$n \geqslant 1$	4	$\{2, s_1, s_2, s_3\}$, where s_1 divides $2^{2n+1} - 1$, s_2 divides $2^{2n+1} - 2^{n+1} + 1$, and s_3 divides $2^{2n+1} + 2^{n+1} + 1$
${}^2G_2(3^{2n+1})$	$n \geqslant 1$	5	$\{3, s_1, s_2, s_3, s_4\}$, where $s_1 \neq 2$ divides $3^{2n+1} - 1$, $s_2 \neq 2$ divides $3^{2n+1} + 1$, s_3 divides $3^{2n+1} - 3^{n+1} + 1$, and s_4 divides $3^{2n+1} + 3^{n+1} + 1$
${}^2F_4(2^{2n+1})$	$n \geqslant 1$	4	$\{2, s_1, s_2, s_3\}$, where $s_1 \neq 3$ and divides $2^{4n+2} - 2^{2n+1} + 1$, s_2 divides $2^{4n+2} + 2^{3n+2} + 2^{2n+1} + 2^{n+1} + 1$, and s_3 divides $2^{4n+2} - 2^{3n+2} + 2^{2n+1} - 2^{n+1} + 1$
${}^2F_4(2)'$	none	2	$\{2, 13\}$

TABLE 6. 2-Independence Numbers for Finite Simple Classical Groups of Characteristic $p \neq 2$

G	Conditions	$t(2,G)$	$\rho(2,G)$
$A_{n-1}(q)$	$n = 2$, $q \equiv 1 (\mathrm{mod}\, 4)$	3	$\{2, r_2, p\}$
	$n = 2$, $q \equiv 3 (\mathrm{mod}\, 4)$, $q \neq 3$	3	$\{2, r_1, p\}$
	$n \geqslant 3$ and $n_2 < (q-1)_2$	2	$\{2, r_n\}$
	$n \geqslant 3$, and either $n_2 > (q-1)_2$ or $n_2 = (q-1)_2 = 2$	2	$\{2, r_{n-1}\}$
	$2 < n_2 = (q-1)_2$	3	$\{2, r_{n-1}, r_n\}$
${}^2A_{n-1}(q)$	$n_2 > (q+1)_2$	2	$\{2, r_{2n-2}\}$
	$n_2 = 1$	2	$\{2, r_{2n}\}$
	$2 < n_2 < (q+1)_2$	2	$\{2, r_n\}$
	$n \geqslant 3$, $2 = n_2 \leqslant (q+1)_2$	2	$\{2, r_{n/2}\}$
	$2 < n_2 = (q+1)_2$	3	$\{2, r_{2n-2}, r_n\}$
$B_n(q)$ or	$n > 1$ is odd and $(q-1)_2 = 2$	2	$\{2, r_n\}$
$C_n(q)$	n is even or $(q-1)_2 > 2$	2	$\{2, r_{2n}\}$

TABLE 6 (Continued)

G	Conditions	$t(2,G)$	$\rho(2,G)$
$D_n(q)$	$n \equiv 0 (\mathrm{mod}\, 2)$, $n \geqslant 4$, $q \equiv 3 (\mathrm{mod}\, 4)$	2	$\{2, r_{n-1}\}$
	$n \equiv 0 (\mathrm{mod}\, 2)$, $n \geqslant 4$, $q \equiv 1 (\mathrm{mod}\, 4)$	2	$\{2, r_{2n-2}\}$
	$n \equiv 1 (\mathrm{mod}\, 2)$, $n > 4$, $q \equiv 3 (\mathrm{mod}\, 4)$	2	$\{2, r_n\}$
	$n \equiv 1 (\mathrm{mod}\, 2)$, $n > 4$, $q \equiv 1 (\mathrm{mod}\, 8)$	2	$\{2, r_{2n-2}\}$
	$n \equiv 1 (\mathrm{mod}\, 2)$, $n > 4$, $q \equiv 5 (\mathrm{mod}\, 8)$	3	$\{2, r_n, r_{2n-2}\}$
${}^2D_n(q)$	$n \equiv 0 (\mathrm{mod}\, 2)$, $n \geqslant 4$	2	$\{2, r_{2n}\}$
	$n \equiv 1 (\mathrm{mod}\, 2)$, $n > 4$, $q \equiv 1 (\mathrm{mod}\, 4)$	2	$\{2, r_{2n}\}$
	$n \equiv 1 (\mathrm{mod}\, 2)$, $n > 4$, $q \equiv 7 (\mathrm{mod}\, 8)$	2	$\{2, r_{2n-2}\}$
	$n \equiv 1 (\mathrm{mod}\, 2)$, $n > 4$, $q \equiv 3 (\mathrm{mod}\, 8)$	3	$\{2, r_{2n-2}, r_{2n}\}$

TABLE 7. 2-Independence Numbers for Finite Simple Exceptional Groups of Lie Type of Characteristic $p \neq 2$

G	Conditions	$t(2,G)$	$\rho(2,G)$
$G_2(q)$	none	3	$\{2, r_3, r_6\}$
$F_4(q)$	none	2	$\{2, r_{12}\}$
$E_6(q)$	none	3	$\{2, r_9, r_{12}\}$
${}^2E_6(q)$	none	3	$\{2, r_{12}, r_{18}\}$
$E_7(q)$	$q \equiv 1 (\mathrm{mod}\, 4)$	3	$\{2, r_{14}, r_{18}\}$
	$q \equiv 3 (\mathrm{mod}\, 4)$	3	$\{2, r_7, r_9\}$
$E_8(q)$	none	5	$\{2, r_{15}, r_{20}, r_{24}, r_{30}\}$
${}^3D_4(q)$	none	2	$\{2, r_{12}\}$
${}^2G_2(3^{2n+1})$	none	3	$\{2, s_1, s_2\}$, where s_1 divides $3^{2n+1} - 3^{n+1} + 1$, s_2 divides $3^{2n+1} + 3^{n+1} + 1$

TABLE 8. Independence Numbers for Finite Simple Classical Groups

G	Conditions	$t(G)$	$\rho(G)$
$A_{n-1}(q)$	$n = 2$, $q > 3$	3	$\{p, r_1, r_2\}$
	$n = 3$, $(q-1)_3 = 3$, and $q + 1 \neq 2^k$	4	$\{p, 3, r_2, r_3\}$
	$n = 3$, $(q-1)_3 \neq 3$, and $q + 1 \neq 2^k$	3	$\{p, r_2, r_3\}$
	$n = 3$, $(q-1)_3 = 3$, and $q + 1 = 2^k$	3	$\{p, 3, r_3\}$
	$n = 3$, $(q-1)_3 \neq 3$, and $q + 1 = 2^k$	2	$\{p, r_3\}$
	$n = 4$	3	$\{p, r_{n-1}, r_n\}$
	$n = 5, 6$, $q = 2$	3	$\{5, 7, 31\}$
	$7 \leqslant n \leqslant 11$, $q = 2$	$\left[\frac{n-1}{2}\right]$	$\{r_i \mid i \neq 6, \left[\frac{n}{2}\right] < i \leqslant n\}$
	$n \geqslant 5$ and $q > 2$ or $n \geqslant 12$ and $q = 2$	$\left[\frac{n+1}{2}\right]$	$\{r_i \mid \left[\frac{n}{2}\right] < i \leqslant n\}$

TABLE 8 (Continued)

G	Conditions	$t(G)$	$\rho(G)$
${}^2A_{n-1}(q)$	$n=3, q\neq 2, (q+1)_3=3,$ and $q-1\neq 2^k$	4	$\{p,3,r_1,r_6\}$
	$n=3,$ $(q+1)_3\neq 3,$ and $q-1\neq 2^k$	3	$\{p,r_1,r_6\}$
	$n=3,$ $(q+1)_3=3,$ and $q-1=2^k$	3	$\{p,3,r_6\}$
	$n=3,$ $(q+1)_3\neq 3,$ and $q-1=2^k$	2	$\{p,r_6\}$
	$n=4,$ $q=2$	2	$\{2,5\}$
	$n=4,$ $q>2$	3	$\{p,r_4,r_6\}$
	$n=5,$ $q=2$	3	$\{2,5,11\}$
	$n\geqslant 5$ and $(n,q)\neq(5,2)$	$\left[\frac{n+1}{2}\right]$	$\{r_{i/2}\mid\left[\frac{n}{2}\right]<i\leqslant n,$ $i\equiv 2(\mathrm{mod}\,4)\}\cup$ $\{r_{2i}\mid \left[\frac{n}{2}\right]<i\leqslant n,$ $i\equiv 1(\mathrm{mod}\,2)\}\cup$ $\{r_i\mid\left[\frac{n}{2}\right]<i\leqslant n,$ $i\equiv 0(\mathrm{mod}\,4)\}$
$B_n(q)$ or $C_n(q)$	$n=2$ and $q>2$	2	$\{p,r_4\}$
	$n=3$ and $q=2$	2	$\{5,7\}$
	$n=4$ and $q=2$	3	$\{5,7,17\}$
	$n=5$ and $q=2$	4	$\{7,11,17,31\}$
	$n=6$ and $q=2$	5	$\{7,11,13,17,31\}$
	$n>2,$ $(n,q)\neq(3,2),(4,2),(5,2),(6,2)$	$\left[\frac{3n+5}{4}\right]$	$\{r_{2i}\mid\left[\frac{n+1}{2}\right]\leqslant i\leqslant n\}\cup$ $\{r_i\mid\left[\frac{n}{2}\right]<i\leqslant n,$ $i\equiv 1(\mathrm{mod}\,2)\}$
$D_n(q)$	$n=4$ and $q=2$	2	$\{5,7\}$
	$n=5$ and $q=2$	4	$\{5,7,17,31\}$
	$n=6$ and $q=2$	4	$\{7,11,17,31\}$
	$n\geqslant 4,$ $(n,q)\neq(4,2),(5,2),(6,2)$	$\left[\frac{3n+1}{4}\right]$	$\{r_{2i}\mid\left[\frac{n+1}{2}\right]\leqslant i<n\}\cup$ $\{r_i\mid\left[\frac{n}{2}\right]<i\leqslant n,$ $i\equiv 1(\mathrm{mod}\,2)\}$
${}^2D_n(q)$	$n=4$ and $q=2$	3	$\{5,7,17\}$
	$n=5$ and $q=2$	3	$\{7,11,17\}$
	$n=6$ and $q=2$	5	$\{7,11,13,17,31\}$
	$n=7$ and $q=2$	5	$\{11,13,17,31,43\}$
	$n\geqslant 4,$ $n\not\equiv 1(\mathrm{mod}\,4),$ $(n,q)\neq(4,2),(6,2),(7,2)$	$\left[\frac{3n+4}{4}\right]$	$\{r_{2i}\mid\left[\frac{n}{2}\right]\leqslant i\leqslant n\}\cup$ $\{r_i\mid\left[\frac{n}{2}\right]<i\leqslant n,$ $i\equiv 1(\mathrm{mod}\,2)\}$
	$n>4,$ $n\equiv 1(\mathrm{mod}\,4),$ $(n,q)\neq(5,2)$	$\left[\frac{3n+4}{4}\right]$	$\{r_{2i}\mid\left[\frac{n}{2}\right]<i\leqslant n\}\cup$ $\{r_i\mid\left[\frac{n}{2}\right]<i<n,$ $i\equiv 1(\mathrm{mod}\,2)\}$

TABLE 9. Independence Numbers for Finite Simple Exceptional Groups of Lie Type

G	Conditions	$t(G)$	$\rho(G)$
$G_2(q)$	$q > 2$	3	$\{p, r_3, r_6\}$
$F_4(q)$	$q = 2$	4	$\{5, 7, 13, 17\}$
	$q > 2$	5	$\{r_3, r_4, r_6, r_8, r_{12}\}$
$E_6(q)$	$q = 2$	5	$\{5, 13, 17, 19, 31\}$
	$q > 2$	6	$\{r_4, r_5, r_6, r_8, r_9, r_{12}\}$
${}^2E_6(q)$		5	$\{r_4, r_8, r_{10}, r_{12}, r_{18}\}$
$E_7(q)$		7	$\{r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{18}\}$
$E_8(q)$		11	$\{r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{15}, r_{18}, r_{20}, r_{24}, r_{30}\}$
${}^3D_4(q)$	$q = 2$	2	$\{2, 13\}$
	$q > 2$	3	$\{r_3, r_6, r_{12}\}$
${}^2B_2(2^{2n+1})$	$n \geqslant 1$	4	$\{2, s_1, s_2, s_3\}$, where s_1 divides $2^{2n+1} - 1$, s_2 divides $2^{2n+1} - 2^{n+1} + 1$, and s_3 divides $2^{2n+1} + 2^{n+1} + 1$
${}^2G_2(3^{2n+1})$	$n \geqslant 1$	5	$\{3, s_1, s_2, s_3, s_4\}$, where $s_1 \neq 2$ divides $3^{2n+1} - 1$, $s_2 \neq 2$ divides $3^{2n+1} + 1$, s_3 divides $3^{2n+1} - 3^{n+1} + 1$, and s_4 divides $3^{2n+1} + 3^{n+1} + 1$
${}^2F_4(2^{2n+1})$	$n \geqslant 2$	5	$\{s_1, s_2, s_3, s_4, s_5\}$, where $s_1 \neq 3$ and divides $2^{2n+1} + 1$, s_2 divides $2^{4n+2} + 1$, $s_3 \neq 3$ and divides $2^{4n+2} - 2^{2n+1} + 1$, s_4 divides $2^{4n+2} - 2^{3n+2} + 2^{2n+1} - 2^{n+1} + 1$, and s_5 divides $2^{4n+2} + 2^{3n+2} + 2^{2n+1} + 2^{n+1} + 1$
${}^2F_4(2)'$	none	3	$\{3, 5, 13\}$
${}^2F_4(8)$	none	4	$\{7, 19, 37, 109\}$

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