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AN ALGEBRAIC APPROACH TO ISOPARAMETRIC HYPERSURFACES IN SPHERES I

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Introduction. An isoparametric hypersurface in a sphere is an orientable submanifold of the sphere which has codimension 1 and constant principal curvatures. Cartan was the first to study such hypersurfaces e.g. [1]. The subject seems to have been forgotten till it was revived by Nomizu, who published a survey on E. Cartan's theory of isoparametric hypersurfaces [9]. Takagi and Takahashi applied results of Hsiang and Lawson on orbits of codimension 1 to classify all homogeneous hypersurfaces in spheres [12]. This classification includes the description of all hypersurfaces with at most 3 distinct principal curvatures since Cartan had shown that all such hypersurfaces are homogeneous. In [7] and [8], Münzner proved that the number g of distinct principal curvatures of an isoparametric hypersurface in a sphere is 1, 2, 3, 4 or 6. Moreover, refining ideas of Cartan, he showed that each such hypersurface is an open submanifold of a level surface of a homogeneous polynomial of degree g and characterized these polynomials by two differential equations. Obviously, it remains to consider the cases $g = 4$ and $g = 6$ and to classify the corresponding polynomials. Of course, this would be superfluous if all isoparametric hypersurfaces in a sphere were homogeneous. As mentioned above, for $g = 1, 2, 3$ all hypersurfaces are homogeneous. However, there exist non-homogeneous examples. The first non-homogeneous examples were found by Ozeki and Takeuchi [10], [11]. They constructed two infinite series of non-homogeneous isoparametric hypersurfaces. Recently, Ferus, Karcher, and Münzner found—for $g = 4$—a new type of examples (constructed from representations of a Clifford algebra) which includes all known non-homogeneous examples and—with the exception of two manifolds—all homogeneous examples [6]. They even constructed infinitely many infinite series of non-homogeneous hypersurfaces.

In this paper we develop a new algebraic approach to isoparametric hypersurfaces in spheres. We concentrate on the case $g = 4$, but the

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case $g = 6$ could be treated similarly.

We start with the observation that on a finite-dimensional Euclidean vector space $(V, \langle\cdot, \cdot\rangle)$ every homogeneous polynomial $F: V \to \boldsymbol{R}$ of degree 4 can be written in the form

$$F(x) = 3\langle x, x\rangle^2 - (2/3)\langle\{xxx\}, x\rangle$$

where $\{\cdots\}: V\times V\times V \to V$ is a trilinear map (i.e., a triple system) satisfying

$$\{x_1x_2x_3\} = \{x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}\} \qquad \text{for every permutation } \sigma \text{ and}$$
$$\langle\{xyz\}, w\rangle = \langle z, \{xyw\}\rangle \qquad \text{for all } x, y, z, w \in V\,.$$

Such triple systems are called symmetric. If F is the polynomial associated with an isoparametric hypersurface in the unit sphere of $(V, \langle\cdot, \cdot\rangle)$, then the Cartan-Münzner differential equations translate into identities for the triple system $(V, \{\cdots\})$. Triple systems satisfying these identities are called isoparametric triple systems. These definitions, simple consequences and examples are contained in §1. We point out that the homogeneous examples are in close relation to simple compact Jordan triple systems of rank 2. A typical example here is $V = \mathrm{Mat}(2, r; \boldsymbol{C})$, $r \geqq 2$ with $\langle A, B\rangle = (1/2)\,\mathrm{trace}\,(A\bar{B}^t + B\bar{A}^t)$ and

$$F(A) = 3\langle A, A\rangle^2 - \langle A\bar{A}^tA, A\rangle\,,$$
$$\{ABC\} = A\bar{B}^tC + C\bar{B}^tA + B\bar{A}^tC + C\bar{A}^tB + A\bar{C}^tB + B\bar{C}^tA\,.$$

In §2 we consider Peirce decompositions of symmetric triple systems relative to minimal and maximal tripotents. In the example $V = \mathrm{Mat}(2, r; \boldsymbol{C})$ a minimal tripotent is E_{11}, the usual matrix unit, and the corresponding Peirce decomposition is the eigenspace decomposition of the endomorphism $A \to \{E_{11}E_{11}A\}$. In the next section (§3) we compare our method with the work of Ozeki and Takeuchi [10]. Using our setting we derive a slightly improved version of one of their main results. In §§4, 5 we introduce the main tool for our approach, Peirce decompositions relative to orthogonal tripotents (e_1, e_2). We prove that an isoparametric triple system $(V, \{\cdots\})$ always contains two orthogonal minimal tripotents (e_1, e_2) and that (e_1, e_2) induce a Peirce decomposition of V:

$$V = V_{11} \oplus V_{10} \oplus V_{12}^+ \oplus V_{12}^- \oplus V_{22} \oplus V_{20}\,.$$

In the example $V = \mathrm{Mat}(2, r; \boldsymbol{C})$ minimal orthogonal tripotents are (E_{11}, E_{22}). Putting $V_{12} = V_{12}^+ \oplus V_{12}^-$ the corresponding Peirce decomposition can symbolically be written in the form

$$V = \left(\begin{array}{c|c|c} V_{11} & V_{12} & V_{10} \\ \hline \underbrace{V_{12}}_{1} & \underbrace{V_{22}}_{1} & \underbrace{V_{20}}_{r-2} \end{array}\right)\begin{matrix}\}1 \\ \}1 \\ \end{matrix}\,.$$

The main result of this paper, Theorem 5.22, shows that a symmetric triple system is isoparametric if and only if it has a vector space decomposition $V = V_{11} \oplus V_{10} \oplus V_{12}^{+} \oplus V_{12}^{-} \oplus V_{22} \oplus V_{20}$ such that

(1) each element of V_{11}, V_{10}, V_{22} and V_{20} is a scalar multiple of a minimal tripotent

(2) each element of V_{12}^{+}, V_{12}^{-} is a scalar multiple of a maximal tripotent

(3) $\langle \{x_{11}x_{22}x_{12}\}, \{x_{10}x_{20}x_{12}\}\rangle + \langle \{x_{11}x_{20}x_{12}\}, \{x_{10}x_{22}x_{12}\}\rangle = 0$

(4) there exist positive integers m_1, m_2 such that $\dim (V_{11} \oplus V_{10}) = m_2 + 1 = \dim (V_{22} \oplus V_{20})$ and $\dim V_{12} = 2m_1$.

This result replaces the quite complicated triple system identities by the more useful notions of tripotents and Peirce decompositions, a well-known tool in nonassociative algebra.

We will use the results of this paper to get detailed insight into the algebraic structure of isoparametric hypersurfaces in spheres, i.e., into isoparametric triple systems. In [2] we explicitly work out the Peirce decompositions relative to tripotents from V_{ij}. In [4] we thoroughly investigate FKM-triples, the isoparametric triple systems associated with the isoparametric hypersurfaces first defined in [6]. Finally, in [3] we classify isoparametric triple systems which have a Peirce decomposition with $V_{10} = 0 = V_{20}$. Such triples are equivalent to triples satisfying condition (A) of Ozeki and Takeuchi but not necessarily their quite technical condition (B). We thus generalize the results of [10]. As it turns out, every isoparametric triple system satisfying (A) is homogeneous or equivalent to an FKM-triple.

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1. The equivalence of isoparametric hypersurfaces in spheres with isoparametric triple systems. In this section we show how to describe isoparametric hypersurfaces in spheres with 4 distinct principal curvatures by isoparametric triple systems and vice versa.

1.1. Throughout the paper let V denote a finite-dimensional real vector space provided with a scalar product $\langle \cdot, \cdot \rangle$.

By definition, an *isoparametric hypersurface in the* (*unit*) *sphere* S^* of V is an oriented submanifold M of S^* which has codimension 1 and constant principal curvatures. Such hypersurfaces are studied in [1]–[12]. We state some of the results which will be used in the sequel. Let M always denote an isoparametric hypersurface in S^*.

(a) ([5, Proposition 6]). Let M have the distinct principal curvatures $\lambda_1 < \cdots < \lambda_g$ with corresponding multiplicities $m_1, \cdots, m_g$. Then, taking subscripts mod g, we have $m_i = m_{i+2}$. If g is odd, then all multiplicities are equal. Note that we have

$$\dim V = 2 + (g/2)(m_1 + m_2)\ . \tag{1.1}$$

(b) ([7], see also [5, Theorem 10] and [10]). There exists a unique maximal family $\mathscr{T} = \{M_t, t \in (-1, 1)\}$ of isoparametric hypersurfaces in S^* such that each M_t is closed in S^* and M is an open submanifold of $M_{\bar{t}}$ for some $\bar{t} \in (-1, 1)$. Further, there exists a homogeneous polynomial function $F\colon V \to \boldsymbol{R}$ of degree g such that the following equations hold

$$\langle \operatorname{grad} F(x), \operatorname{grad} F(x)\rangle = g^2\langle x, x\rangle^{g-1} \tag{1.2}$$

$$\Delta F(x) = (1/2)(m_2 - m_1)g^2\langle x, x\rangle^{g/2-1}\ . \tag{1.3}$$

Moreover, the maximal family $\mathscr{T}$ is given by

$$\mathscr{T} = \{M_t = F^{-1}(t) \cap S^*; t \in (-1, 1)\}\ . \tag{1.4}$$

Conversely, for each homogeneous polynomial $F\colon V \to \boldsymbol{R}$ satisfying (1.1) to (1.3) with positive integers m_1 and m_2, the family $\mathscr{T}$ given by (1.4) defines a maximal family of isoparametric hypersurfaces in S^* with g distinct principal curvatures and multiplicities m_1, m_2.

(c) ([8]). The only possible values for g are 1, 2, 3, 4 and 6.

(d) ([1]). If $g \leqq 3$, then M is homogeneous.

(e) The homogeneous isoparametric hypersurfaces are classified in [12].

According to (c)-(e) only the cases $g = 4$ and $g = 6$ remain to be investigated. In this paper we begin the study of the case $g = 4$.

(f) Two maximal families $\mathscr{S}$ and $\mathscr{T}$ of isoparametric hypersurfaces given by the polynomials $F_{\mathscr{S}}$ and $F_{\mathscr{T}}$ (according to (b)) are said to be *equivalent* if there exists an orthogonal transformation $\phi\colon V_{\mathscr{S}} \to V_{\mathscr{T}}$ such that $F_{\mathscr{T}}(\phi x) = \pm F_{\mathscr{S}}(x)$ for all $x \in V_{\mathscr{S}}$.

1.2. We give a general procedure for attaching to every homogeneous polynomial of degree 4 a triple product on V and vice versa.

By definition, a triple product on V is a trilinear map $\{\cdots\}\colon V \times V \times V \to V$. Generalizing the "left multiplications" of an algebra, we define endomorphisms $T(u, v) \in \operatorname{End} V$ by $T(u, v)w := \{uvw\}$, $u, v, w \in V$. We sometimes write $T(u) := T(u, u)$ for short.

Assume $F\colon V \to \boldsymbol{R}$ is a homogeneous polynomial of degree 4. There exists a unique totally symmetric 4-linear form $\widetilde{F}\colon V \times V \times V \times V \to \boldsymbol{R}$ which satisfies $\widetilde{F}(x, x, x, x) = 3\langle x, x\rangle^2 - F(x)$, $x \in V$. With $\widetilde{F}$ we define a triple product $\{\cdots\}_F$ on V by the relation

$$\tilde{F}(u, v, w, x) = (2/3)\langle\{uvw\}_F, x\rangle\ , \qquad u, v, w, x \in V\ .$$

The left multiplications of this triple product are sometimes denoted by $T_F(u, v)$.

REMARK. It might seem more natural to define $\tilde{\tilde{F}}$ such that $\tilde{\tilde{F}}(x, x, x, x) = F(x)$ holds and to define $\{\cdots\}_{\tilde{\tilde{F}}}$ on V by the relation $\tilde{\tilde{F}}(u, v, w, x) = \langle\{uvw\}_{\tilde{\tilde{F}}}, x\rangle$. However, for the homogeneous isoparametric hypersurfaces with $g = 4$ we will see in 1.5 that the natural triple product (induced by a compact Jordan triple system) has $\tilde{F}(u, v, w, x) = (2/3)\langle\{uvw\}, x\rangle$. To preserve the fruitful analogy with the homogeneous case, we define the triple product in the general situation using $\tilde{F}$ instead of $\tilde{\tilde{F}}$.

The main objects of this paper are *isoparametric triple systems*, i.e., a triple $(V, \langle\cdot, \cdot\rangle, \{\cdots\})$ where $(V, \langle\cdot, \cdot\rangle)$ is a finite-dimensional Euclidean space and $\{\cdots\}$ is a triple product on V (with "left multiplications" $T(u, v)$) which has the following properties:

(ISO 1) $\{\cdots\}$ is totally symmetric,

(ISO 2) $\langle\{xyz\}, w\rangle = \langle z, \{xyw\}\rangle$,

(ISO 3) $\langle\{xxx\}, \{xxx\}\rangle - 9\langle x, x\rangle\langle\{xxx\}, x\rangle + 18\langle x, x\rangle^3 = 0$,

(ISO 4) there exist positive integers m_1 and m_2 such that

(a) trace $T(x, y) = 2(3 + 2m_1 + m_2)\langle x, y\rangle$

(b) $\dim V = 2(1 + m_1 + m_2)$.

If no confusion is possible, we write V or $(V, \{\cdots\})$ instead of $(V, \langle\cdot, \cdot\rangle, \{\cdots\})$. More generally we call a triple $\{\cdots\}$ on V satisfying only (ISO 1) and (ISO 2) a *symmetric triple system*. An easy computation shows

LEMMA 1.1. *Let F: $V \to \boldsymbol{R}$ be a homogeneous polynomial of degree* 4 *and $\{\cdots\}_F$ defined as above. Assume further that m_1 and m_2 are positive integers such that* $\dim V = 2(m_1 + m_2 + 1)$. *Then F satisfies* (1.2) *and* (1.3) *with $g = 4$ if and only if*

(a) $\langle\{xxx\}_F, \{xxx\}_F\rangle - 9\langle x, x\rangle\langle\{xxx\}_F, x\rangle + 18\langle x, x\rangle^3 = 0$ *and*

(b) trace $T_F(x, y) = 2(3 + 2m_1 + m_2)\langle x, y\rangle$ *for all $x, y \in V$.*

Hence the polynomials describing maximal families of isoparametric hypersurfaces with 4 distinct principal curvatures are in 1-1 correspondence with isoparametric triple systems.

We note that F is determined by $\{\cdots\}_F$, $F(x) = 3\langle x, x\rangle^2 - (2/3)\langle\{xxx\}, x\rangle$.

1.3. A very important feature of isoparametric triple systems is that they occur in pairs.

LEMMA 1.2. *Let $(V, \{\cdots\})$ be a symmetric triple system and define $\{\cdots\}'$ on V by*

$$\{xyz\}' = 3(\langle x, y\rangle z + \langle y, z\rangle x + \langle z, x\rangle y) - \{xyz\} . \tag{1.5}$$

Then $(V, \langle\cdot, \cdot\rangle, \{\cdots\}')$ is again a symmetric triple system. Further $\{\cdots\}'' = \{\cdots\}$. If F is the polynomial associated with $(V, \{\cdots\})$, then the polynomial associated with $(V, \{\cdots\}')$ is $-F$. If $(V, \{\cdots\})$ is an isoparametric triple, then $(V, \{\cdots\}')$ is also isoparametric with $m_1' = m_2$ and $m_2' = m_1$.

We call $(V, \langle\cdot, \cdot\rangle, \{\cdots\}')$, where $\{\cdots\}'$ is defined in Lemma 1.3, the *dual triple system* of $(V, \langle\cdot, \cdot\rangle, \{\cdots\})$ and abbreviate it by V'. The left multiplications in V' are denoted by $T'(u, v)$. The dual triple system naturally occurs when one translates the notion of "equivalence of isoparametric hypersurfaces" into the language of triple systems (see Lemma 1.3).

Let V and W be symmetric triple systems. We call $(V, \{\cdots\}_V)$ and $(W, \{\cdots\}_W)$ isomorphic (as triple systems), if there exists an orthogonal map $\phi: V \to W$ satisfying $\phi(\{xyz\}_V) = \{\phi x, \phi y, \phi z\}_W$ for all $x, y, z \in V$. We say V and W are *equivalent* if V is isomorphic to W or to W', i.e., if there exists an orthogonal map $\phi: V \to W$ such that $\phi\{xxx\}_V = \{\phi x, \phi x, \phi x\}_W$ or $\phi\{xxx\}_V = 9\langle x, x\rangle \phi x - \{\phi x, \phi x, \phi x\}_W$. This is, obviously, an equivalence relation.

LEMMA 1.3. *Let $\mathscr{S}$ and $\mathscr{T}$ be two maximal families of isoparametric hypersurfaces in the unit sphere of V with $g = 4$, let $V_{\mathscr{S}}$ and $V_{\mathscr{T}}$ be the corresponding isoparametric triple systems. Then $\mathscr{S}$ and $\mathscr{T}$ are equivalent if and only if $V_{\mathscr{S}}$ and $V_{\mathscr{T}}$ are equivalent.*

PROOF. Let $F_{\mathscr{S}}$ and $F_{\mathscr{T}}$ be the polynomials describing $\mathscr{S}$ and $\mathscr{T}$ according to 1.1.(b). By definition $\mathscr{S}$ and $\mathscr{T}$ are equivalent if and only if there exists an orthogonal $\phi: V_{\mathscr{S}} \to V_{\mathscr{T}}$ such that $F_{\mathscr{T}}(\phi x) = \pm F_{\mathscr{S}}(x)$ for all $x \in V_{\mathscr{S}}$. But this is equivalent to $3\langle x, x\rangle^2 - (2/3)\langle\{\phi x, \phi x, \phi x\}_{\mathscr{T}}, \phi x\rangle = \pm(3\langle x, x\rangle^2 - (2/3)\langle\{xxx\}_{\mathscr{S}}, x\rangle)$ and therefore (by differentiation) to $9\langle x, x\rangle x - 2\phi^{-1}(\{\phi x, \phi x, \phi x\}_{\mathscr{T}}) = \pm(9\langle x, x\rangle x - 2\{xxx\}_{\mathscr{S}})$. From this the assertion easily follows.

1.4. We will use the defining identity (ISO 3) of an isoparametric triple system in its linearized form. Linearization means that we replace x by $x + \lambda u$ for $x, u \in V$, $\lambda \in \boldsymbol{R}$ and equate the coefficients of the different powers of λ in the resulting expression. In our setting linearization is the same as differentiation. We use the abbreviation uv^* for the linear map $w \to \langle w, v\rangle u$. We get

(1.6) $\{xx\{xxx\}\} - 6\langle x, x\rangle\{xxx\} - 3\langle\{xxx\}, x\rangle x + 18\langle x, x\rangle^2 x = 0\ .$

(1.7) $3T(x)^2 + 2T(x, \{xxx\}) - 18\langle x, x\rangle T(x) - 3\langle x, \{xxx\}\rangle Id$
$\quad - 12(x\{xxx\}^* + \{xxx\}x^*) + 18\langle x, x\rangle^2 Id + 72\langle x, x\rangle xx^* = 0\ .$

(1.8) $T(x, u)T(x) + T(x)T(x, u) + T(x, \{xxu\}) + (1/3)T(u, \{xxx\})$
$\quad - 6\langle x, u\rangle T(x) - 6\langle x, x\rangle T(x, u) - 2\langle\{xxx\}, u\rangle Id$
$\quad - 2(u\{xxx\}^* + \{xxx\}u^*) - 6(x\{xxu\}^* + \{xxu\}x^*)$
$\quad + 12\langle x, x\rangle\langle x, u\rangle Id + 24\langle x, u\rangle xx^* + 12\langle x, x\rangle(xu^* + ux^*) = 0\ .$

(1.9) $2T(x, u)T(x, v) + 2T(x, v)T(x, u) + T(u, v)T(x) + T(x)T(u, v)$
$\quad + T(v, \{xxu\}) + 2T(x, \{xuv\}) + T(u, \{xxv\}) - 12\langle x, u\rangle T(x, v)$
$\quad - 12\langle x, v\rangle T(x, u) - 6\langle u, v\rangle T(x) - 6\langle x, x\rangle T(u, v)$
$\quad - 6\langle\{xxv\}, u\rangle Id - 6(u\{xxv\}^* + \{xxv\}u^*)$
$\quad - 6(v\{xxu\}^* + \{xxu\}v^*) - 12(x\{xuv\}^* + \{xuv\}x^*)$
$\quad + 12\langle x, x\rangle\langle u, v\rangle Id + 24\langle x, u\rangle\langle x, v\rangle Id + 24\langle u, v\rangle xx^*$
$\quad + 24\langle x, u\rangle(xv^* + vx^*) + 24\langle x, v\rangle(xu^* + ux^*)$
$\quad + 12\langle x, x\rangle(vu^* + uv^*) = 0\ .$

(1.10) $T(w, u)T(x, v) + T(x, u)T(w, v) + T(w, v)T(x, u) + T(x, v)T(w, u)$
$\quad + T(u, v)T(x, w) + T(x, w)T(u, v) + T(v, \{uwx\}) + T(w, \{xuv\})$
$\quad + T(x, \{wuv\}) + T(u, \{xwv\}) - 6\langle x, u\rangle T(w, v) - 6\langle w, u\rangle T(x, v)$
$\quad - 6\langle w, v\rangle T(x, u) - 6\langle x, v\rangle T(w, u) - 6\langle u, v\rangle T(x, w)$
$\quad - 6\langle x, w\rangle T(u, v) - 6\langle\{xwv\}, u\rangle Id - 6(u\{xwv\}^* + \{xwv\}u^*)$
$\quad - 6(v\{xwu\}^* + \{xwu\}v^*) - 6(w\{xuv\}^* + \{xuv\}w^*)$
$\quad - 6(x\{wuv\}^* + \{wuv\}x^*) + 12(\langle x, w\rangle\langle u, v\rangle + \langle w, u\rangle\langle x, v\rangle$
$\quad + \langle x, u\rangle\langle w, v\rangle)Id + 12\langle u, v\rangle(xw^* + wx^*)$
$\quad + 12\langle w, u\rangle(xv^* + vx^*) + 12\langle x, u\rangle(wv^* + vw^*)$
$\quad + 12\langle w, v\rangle(xu^* + uw^*) + 12\langle w, x\rangle(vu^* + uv^*)$
$\quad + 12\langle x, v\rangle(wu^* + uw^*) = 0\ .$

REMARK. Clearly, equations (1.7)–(1.10) are all equivalent to (1.6). We list them here because we will use them frequently.

We will often consider symmetric triple systems. The failure of an arbitrary symmetric triple system V to satisfy the identity (ISO 3) is measured by the polynomial $\mathscr{M}\colon V \to V$, where

(1.11) $\mathscr{M}(x) = \{xx\{xxx\}\} - 6\langle x, x\rangle\{xxx\} - 3\langle\{xxx\}, x\rangle x + 18\langle x, x\rangle^2 x\ .$

Since $\mathscr{M}$ is homogeneous of degree 5 there exists a uniquely determined totally-symmetric five-linear map $\mathscr{M}: V \times V \times V \times V \times V \to V$ satisfying $\mathscr{M}(x, x, x, x, x) = \mathscr{M}(x)$, $x \in V$. We will use the abbreviations $\mathscr{M}(x; u)$ or $\mathscr{M}(x; u, v) \cdots$ to denote $\mathscr{M}(x, x, x, x, u)$ or $\mathscr{M}(x, x, x, u, v) \cdots$. We now express the five-linear map by the triple product. Indeed, since (1.7) to (1.10) are the successive linearizations of (1.6) we see that

(1.12) $5\mathscr{M}(x; u) =$ left hand side of (1.7) applied to u

(1.13) $(10/3)\mathscr{M}(x; u, v) =$ left hand side of (1.8) applied to v

(1.14) $10\mathscr{M}(x; u, v, w) =$ left hand side of (1.9) applied to w

(1.15) $10\mathscr{M}(x, u, v, w, y) =$ left hand side of (1.10) applied to y.

LEMMA 1.4. *Let V be a symmetric triple system.*
(a) *For $x_i \in V$, $i = 1, \cdots, 6$ we have*

$$\langle \mathscr{M}(x_1, x_2, x_3, x_4, x_5), x_6 \rangle = \langle \mathscr{M}(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}), x_{\sigma(6)} \rangle$$

for every permutation $\sigma \in \mathfrak{S}_6$.
(b) *Let $\mathscr{M}'$ be defined for the dual system V' in the same way as $\mathscr{M}$ is defined for V. Then $\mathscr{M}(x) = \mathscr{M}'(x)$ for all $x \in V$.*

PROOF. (a) We define the polynomial function $h: V \to \boldsymbol{R}$, $h(x) = \langle \mathscr{M}(x), x \rangle$. Using (ISO 1) and (ISO 2) it is easy to see that $d_x h(u) = 6\langle \mathscr{M}(x), u \rangle$ and $d_x^2 h(u, v) = 6 \cdot 5 \langle \mathscr{M}(x; v), u \rangle$. Since $d_x^2 h(u, v)$ is symmetric in u and v we get $\langle \mathscr{M}(x; v), u \rangle = \langle \mathscr{M}(x; u), v \rangle$ which easily implies (a).
(b) A straightforward computation shows $\langle \mathscr{M}(x), x \rangle = \langle \mathscr{M}'(x), x \rangle$. Since $6\mathscr{M}(x) = \operatorname{grad} h(x)$ we get (b).

1.5. We close this section by presenting all known examples of isoparametric triple systems.

(a) *Homogeneous isoparametric triple systems.* An isoparametric triple is called *homogeneous* if the corresponding isoparametric hypersurfaces are homogeneous. We recall from [7] that an isoparametric hypersurface M of the sphere in the Euclidean space $(V, \langle\ ,\ \rangle)$ is called *homogeneous* if there exists a group of orthogonal transformations of $(V, \langle\ ,\ \rangle)$ which leaves M invariant and acts transitively on M. The results of [12] show that every homogeneous isoparametric triple system is equivalent to one of the following three types:

(a.1) Let $\boldsymbol{F}$ be $\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$ (the quaternions). We consider the real vector space $V = \operatorname{Mat}(p, r; \boldsymbol{F})$ of $p \times r$ matrices with coefficients in $\boldsymbol{F}$. For $x = (x_{ij}) \in V$ define $\bar{x} = (\bar{x}_{ij})$ where "–" is the canonical involution in $\boldsymbol{F}$. Then $\langle x, y \rangle = (1/2)\operatorname{trace}(x\bar{y}^t + \bar{x}^t y)$ is a scalar product on V and

(1.16) $$\{xyz\} = x\bar{y}^t z + z\bar{y}^t x + y\bar{x}^t z + z\bar{x}^t y + x\bar{z}^t y + y\bar{z}^t x$$

is a totally symmetric triple system. It is isoparametric if $p = 2$ and $r \geqq 3$ in case $\boldsymbol{F} = \boldsymbol{R}$, or $r \geqq 2$ in case $\boldsymbol{F} = \boldsymbol{C}, \boldsymbol{H}$.

(a.2) For $\boldsymbol{K} = \boldsymbol{R}, \boldsymbol{C}$ define $O(5; \boldsymbol{K}) = \{x \in \mathrm{Mat}\,(5, 5; \boldsymbol{K}); x^t = -x\}$. It is easy to check that $O(5; \boldsymbol{K})$ is closed under the triple product (1.16); hence, the triple product (1.16) induces a triple system on $O(5; \boldsymbol{K})$ which is isoparametric.

(a.3) Let $\boldsymbol{O}$ be the division Cayley algebra over $\boldsymbol{R}$ and by $\boldsymbol{O}^C$ its complexification. Since $\boldsymbol{O}^C$ is again a Cayley algebra it has a canonical involution $x \to x'$. Every $x \in \boldsymbol{O}^C$ has unique decomposition $x = a + ib$ with $a, b \in \boldsymbol{O}$. We define $\bar{x} = a - ib$ and consider $V := \boldsymbol{O}^C \oplus \boldsymbol{O}^C$ as a real vector space. The elements of V are written in the form $x = (x_1, x_2)$ with $x_i \in \boldsymbol{O}^C$. Then $\langle x, y\rangle = \mathrm{Re}\ (x_1\bar{y}_1' + y_1\bar{x}_1' + x_2\bar{y}_2' + y_2\bar{x}_2')$ is a scalar product on V and $\{xxx\} = 6(x_1\bar{x}_1'x_1 + x_1\bar{x}_2 \cdot x_2', x_2\bar{x}_2'x_2 + x_1 \cdot \bar{x}_1x_2)$ defines, by linearization, a totally symmetric triple on V. It can be shown that it is a homogeneous isoparametric triple system. Summing up what we have extracted from [9], we have the following list of all homogeneous isoparametric triple systems (up to equivalence):

V	$\dim V$	m_1	m_2
$\mathrm{Mat}\,(2, r; \boldsymbol{R})$, $r \geqq 3$	$2r$	1	$r-2$
$\mathrm{Mat}\,(2, r; \boldsymbol{C})$, $r \geqq 2$	$4r$	2	$2r-3$
$\mathrm{Mat}\,(2, r; \boldsymbol{H})$, $r \geqq 2$	$8r$	4	$4r-5$
$O(5; \boldsymbol{R})$	10	2	2
$O(5; \boldsymbol{C})$	20	4	5
$(\boldsymbol{O}^C \oplus \boldsymbol{O}^C)_{\boldsymbol{R}}$	32	6	9

(b) *Isoparametric triple systems of FKM-type.* Let $(V, \langle\cdot, \cdot\rangle)$ be an Euclidean space. Assume $P_0, \cdots, P_m$, $m \geqq 1$, is a Clifford system ([6, 3.2]). If $\dim V = 2(m_1 + m_2 + 1)$ with $m = m_1$ and a positive integer m_2, then

$$\{xxx\} = 3\left[\langle x, x\rangle x + \sum_{r=0}^{m} \langle P_r x, x\rangle P_r x\right]$$

defines an isoparametric triple system on V. The corresponding isoparametric hypersurfaces were first considered in [6].

2. Peirce decomposition relative to a single tripotent. Throughout this section let $V = (V, \{\cdots\})$ be a symmetric triple system. We introduce the notion of a tripotent of V and study its Peirce decomposition.

2.1. As in §1.2 we associate to $(V, \{\cdots\})$ the polynomial $F(x) = 3\langle x, x\rangle^2 - (2/3)\langle\{xxx\}, x\rangle$.

LEMMA 2.1. (a) *The extremal points* $c \in S^*$ *of* $F|S^*$ *satisfy* $\{ccc\} \in \boldsymbol{R}c$.

(b) *Let* $c \in S^*$ *with* $\{ccc\} = \kappa c$ *for some* $\kappa \in \boldsymbol{R}$. *Then the following are equivalent*:

(1) $\mathscr{M}(c) = 0$,

(2) $\kappa \in \{3, 6\}$,

(3) $F(c) = 1$ (*for* $\kappa = 3$) *or* $F(c) = -1$ (*for* $\kappa = 6$).

(c) *If* $F|S^*$ *is not constant and if* $\mathscr{M}(c) = 0$ *for each extremal point of* $F|S^*$, *then* $F(S^*) = [-1, 1]$ *and* $F^{-1}(-1) \cap S^*$ *are the points of* S^* *where* $F|S^*$ *is minimal and* $F^{-1}(1) \cap S^*$ *are the points of* S^* *where* $F|S^*$ *is maximal.*

PROOF. (a) Obviously, $d_x[F(x) - \zeta(\langle x, x\rangle - 1)] = 0$ for all $u \in V$ if and only if $0 = d_xF(u) - 2\zeta\langle x, u\rangle = 12\langle x, x\rangle\langle x, u\rangle - (8/3)\langle\{xxx\}, u\rangle - 2\zeta\langle x, u\rangle$ for all $u \in V$. This is equivalent with $\{xxx\} = \kappa x$ for some $\kappa \in \boldsymbol{R}$.

(b) is straightforward.

(c) By assumption $F(S^*) = [\alpha, \beta]$, $\alpha, \beta \in \boldsymbol{R}$, $\alpha < \beta$. Each point c in $F^{-1}(\alpha) \cap S^*$ or $F^{-1}(\beta) \cap S^*$ is an extremal point. Then $\mathscr{M}(c) = 0$, whence $F(c) = \pm 1$ by (b). This implies $\alpha = -1$, $\beta = 1$.

For an element $c \in V$ with $\{ccc\} = \kappa c$ we define

$$V_\mu(c) = \{x \in V;\ T(c)x = \mu x,\ \langle x, c\rangle = 0\}\,, \qquad \mu \in \boldsymbol{R}\,.$$

Obviously, $V = \boldsymbol{R}c \oplus (\bigoplus_{\mu \in \boldsymbol{R}} V_\mu(c))$. From (1.5) we derive for $\langle c, x\rangle = 0$ that $\{ccx\}' = 3x - \{ccx\}$. Thus $\{ccx\} = \mu x$ iff $\{ccx\}' = (3 - \mu)x$, i.e.,

$$V'_{3-\mu}(c) = V_\mu(c) \quad \text{for all} \quad \mu \in \boldsymbol{R}\,. \tag{2.1}$$

THEOREM 2.2. (a) *Let* $c \in V$ *with* $\langle c, c\rangle = 1$ *and* $\{ccc\} = 6c$. *Then the following are equivalent*:

(1) $\mathscr{M}(c; u) = 0$ *for all* $u \in V$

(2) $V = \boldsymbol{R}c \oplus V_2(c) \oplus V_0(c)$

(3) $V' = \boldsymbol{R}c \oplus (V')_1(c) \oplus (V')_3(c)$.

In this case, we have $(V')_1(c) = V_2(c)$, $(V')_3(c) = V_0(c)$.

(b) *Let* $e \in V$ *with* $\langle e, e\rangle = 1$ *and* $\{eee\} = 3e$. *Then the following are equivalent*:

(1) $\mathscr{M}(e; u) = 0$ *for all* $u \in V$

(2) $V = \boldsymbol{R}e \oplus V_3(e) \oplus V_1(e)$

(3) $V' = \boldsymbol{R}e \oplus (V')_0(e) \oplus (V')_2(e)$.

In this case, we have $(V')_0(e) = V_3(e)$, $(V')_2(e) = V_1(e)$.

(c) *Assume there are positive integers* m_1, m_2 *such that* $\dim V = 2(1 + m_1 + m_2)$.

(1) *Let* c *satisfy one of the equivalent conditions of* (a). *Then* trace $T(c) = 2(3 + 2m_1 + m_2)$ *iff* $\dim V_2(c) = 2m_1 + m_2$ *iff* $\dim V_0(c) = m_2 + 1$.

(2) *Let* e *satisfy one of the equivalent conditions of* (b). *Then* trace $T(e) = 2(3 + 2m_1 + m_2)$ *iff* $\dim V_3(e) = m_1 + 1$ *iff* $\dim V_1(e) = 2m_2 + m_1$.

PROOF. (a) By (1.12) we know that (1) is equivalent to (1.7), hence to $T(c)^2 - 2T(c) - 24cc^* = 0$. Since $T(c)|\boldsymbol{R}c = 6\mathrm{Id}$ we get $T(c)(\boldsymbol{R}c)^\perp \subset (\boldsymbol{R}c)^\perp$. Therefore the last equation is satisfied iff the restricted endomorphism satisfies the equation $\tau^2 - 2\tau = 0$, which is obviously equivalent to (2). The statement (2) $\Leftrightarrow$ (3) follows from (2.1).

(b) By Lemma 1.7.b we have $\mathscr{M}(e; u) = 0$ iff $\mathscr{M}'(e; u) = 0$. Hence (b) follows from (a) applied to V' and e instead of V and c.

(c) is a consequence of (a) resp. (b) and (ISO 4).

2.2. In this subsection we consider an element $c \in V$ with $\langle c, c\rangle = 1$, $\{ccc\} = 6c$ and $\mathscr{M}(c; u) = 0$ for all $u \in V$. Then every $x \in V$ has a decomposition

$$x = \alpha c \oplus x_2(c) \oplus x_0(c) \tag{2.2}$$

with $\alpha \in \boldsymbol{R}$ and $x_\mu \in V_\mu(c)$. When it is clear which element c is referred to we simply write V_μ instead of $V_\mu(c)$ and x_μ instead of $x_\mu(c)$. It is also convenient to introduce the abbreviation $x \circ y := \{xcy\}$, $x, y \in V$. We remark that $\circ$ defines a commutative algebra on V. From the context it will always be clear which element c is used to define $x \circ y$.

THEOREM 2.3. *Let* $c \in V$, $\langle c, c\rangle = 1$, $\{ccc\} = 6c$ *and* $\mathscr{M}(c; u) = 0$ *for all* $u \in V$.

(a) $\mathscr{M}(c; u, v) = 0$ *for all* $u, v \in V$ *iff the following multiplication rules hold for* $u_\mu, v_\mu \in V_\mu$, $\mu = 0, 2$:

$$u_0 \circ v_0 = 0 \tag{2.3}$$

$$u_0 \circ v_2 \in V_2 \tag{2.4}$$

$$u_2 \circ v_2 = 2\langle u_2, v_2\rangle c + (u_2 \circ v_2)_0 . \tag{2.5}$$

(b) *Assume* $\mathscr{M}(c; u, v) = 0$ *for all* $u, v \in V$. *Then* $\mathscr{M}(c; u, v, w) = 0$ *for all* $u, v, w \in V$ *iff the following identities are satisfied for all* $u_\mu, v_\mu, w_\mu \in V_\mu$.

$$\{u_0 v_0 w_0\} = 2(\langle u_0, v_0\rangle w_0 + \langle v_0, w_0\rangle u_0 + \langle w_0, u_0\rangle v_0) \in V_0(c) \tag{2.6}$$

$$\{u_0 v_0 w_2\} = u_0 \circ (v_0 \circ w_2) + v_0 \circ (u_0 \circ w_2) \in V_2(c) \tag{2.7}$$

$$\{u_0 v_2 w_2\} = \langle u_0, v_2 \circ w_2\rangle c + [v_2 \circ (w_2 \circ u_0) + w_2 \circ (v_2 \circ u_0)]_0 + \{u_0 v_2 w_2\}_2 \tag{2.8}$$

$$\begin{aligned}\{u_2 v_2 w_2\} = {} & 2(\langle u_2, v_2\rangle w_2 + \langle v_2, w_2\rangle u_2 + \langle w_2, u_2\rangle v_2) - u_2 \circ (v_2 \circ w_2)_0 \\ & - v_2 \circ (w_2 \circ u_2)_0 - w_2 \circ (u_2 \circ v_2)_0 + \{u_2 v_2 w_2\}_0 \in V_2(c) \oplus V_0(c) .\end{aligned} \tag{2.9}$$

PROOF. We choose u, v and w in the various eigenspaces of $T(c)$

and evaluate (1.8) and (1.9).

For a symmetric triple system V an element $c \in V$ with the properties $\langle c, c \rangle = 1$, $\{ccc\} = 6c$ and $\mathscr{M}(c; u, v, w) = 0$ for all $u, v, w \in V$ is called a *minimal tripotent.* We recall from Theorem 2.2.a that the condition $\mathscr{M}(c; u, v, w) = 0$ for all $u, v, w \in V$ implies the existence of the decomposition $V = \boldsymbol{R}c \oplus V_2(c) \oplus V(c)$ which we call the *Peirce decomposition of V relative to c* and the validity of the formulas (2.3) to (2.9) which we refer to as *Peirce multiplication rules.* The decomposition (2.2) is called the *Peirce decomposition of x* (*relative to c*) and the spaces $V_\mu(c)$ are said to be the *Peirce spaces of c.*

REMARK 2.4. (a) The notion of a tripotent in a triple system is analogous to the notion of an idempotent in an algebra. In many important classes of algebras an idempotent induces a "Peirce decomposition" of the algebra. Because of this we choose the name Peirce decomposition also in the case of triple systems.

(b) If V is not only symmetric, but even isoparametric, then a minimal tripotent of V is just an element c of V with $\langle c, c \rangle = 1$ and $\{ccc\} = 6c$. By (ISO 4) and Theorem 2.2.c we know in this case $\dim V_2(c) = 2m_2 + m_1 > 0$ and $\dim V_0(c) = m_2 + 1 > 0$.

Let F be the polynomial associated to the isoparametric triple V according to Lemma 1.1. Then (ISO 4) implies that F restricted to the unit sphere S^* of V is not constant. Therefore $F^{-1}(-1) \cap S^*$ are the minima of $F|S^*$ by Lemma 2.1.c. Moreover, Lemma 2.1.c shows that $F^{-1}(-1) \cap S^*$ is the set of the minimal tripotents of V. This justifies the adjective "minimal".

It is known (see e.g. [5]) that the family of isoparametric hypersurfaces described by F has exactly two focal manifolds, $M_- = F^{-1}(-1) \cap S^*$ and $M_+ = F^{-1}(1) \cap S^*$. Therefore the set of minimal tripotents of V coincides with the focal manifold M_-.

(c) To give examples of minimal tripotents we consider the isoparametric triple system Mat $(2, r; \boldsymbol{F})$ as defined in §1.5.(a.1). We denote by E_{ij} the usual matrix units. Then each E_{ij} is a minimal tripotent. The Peirce spaces $V_\mu(E_{1j})$ are $V_2(E_{1j}) = \{aE_{1j}; a \in \boldsymbol{F}^-\} \oplus \boldsymbol{F}E_{2j} \oplus \bigoplus_{k \neq j} \boldsymbol{F}E_{1k}$ where $\boldsymbol{F}^-$ denotes the orthogonal complement of 1 in $\boldsymbol{F}$, e.g. $\boldsymbol{R}^- = 0$, and $V_0(E_{1j}) = \bigoplus_{k \neq j} \boldsymbol{F}E_{2k}$.

(d) If c is a minimal tripotent of the symmetric triple system V, every $f \in V_0(c)$ with $\langle f, f \rangle = 1$ satisfies $\{fff\} = 6f$ by (2.6). In particular, if V is isoparametric, every $f \in V_0(c)$ with $\langle f, f \rangle = 1$ is a minimal tripotent.

(e) The multiplication rules (2.4) and (2.8) are consequences of the remaining ones.

(f) Let c be a minimal tripotent of the symmetric triple system V. From Theorem 2.3 it is clear that all triple products are known once $u_0 \circ v_2$, $(u_2 \circ v_2)_0$, $\{u_0 v_2 w_2\}_2$ and $\{u_2 v_2 w_2\}_0$ are known for all u_μ, v_μ. But $(u_2 \circ v_2)_0$ is determined by $u_0 \circ v_2$ via $\langle u_2 \circ v_2, u_0\rangle = \langle u_2, u_0 \circ v_2\rangle$ and similarly $\{u_0 v_2 w_2\}_2$ is determined by $\{u_2 v_2 w_2\}_0$. Hence all triple products are known as soon as $u_0 \circ v_2$ and $\{u_2 v_2 w_2\}_0$ are known for all $u_\mu, v_\mu, w_\mu \in V_\mu$.

2.3. This subsection is the dual version of § 2.2, i.e., we consider an element e of the symmetric triple system V with $\langle e, e\rangle = 1$, $\{eee\} = 3e$ and $\mathscr{M}(e; u) = 0$ for all $u \in V$. Hence, by Theorem 2.2.b, we have a decomposition $V = \boldsymbol{R}e \oplus V_3(e) \oplus V_1(e)$ where $V_\mu(e) = \{x \in V;\ T(e)x = \mu x,\ \langle e, x\rangle = 0\}$. Correspondingly, each element x of V has a decomposition $x = \alpha e \oplus x_3(e) \oplus x_1(e)$. When no confusion is possible we often write V_μ instead of $V_\mu(e)$ and x_μ instead of $x_\mu(e)$. As in § 2.2, we introduce an algebra "$\square$" on V, depending on e, via the definition

$$x \square y = \{xey\}\,, \qquad x, y \in V\,.$$

THEOREM 2.5. *Assume $e \in V$, $\langle e, e\rangle = 1$, $\{eee\} = 3e$ and $\mathscr{M}(e; u) = 0$ for all $u, v \in V$.*

(a) *Then $\mathscr{M}(e; u, v) = 0$ for all $u, v \in V$ iff the following multiplication rules hold for all $u_\mu, v_\mu \in V_\mu$, $\mu = 1, 3$:*

$$u_3 \square v_3 = 3\langle u_3, v_3\rangle e \tag{2.10}$$

$$u_3 \square v_1 \in V_1 \tag{2.11}$$

$$u_1 \square v_1 = \langle u_1, v_1\rangle e + (u_1 \square v_1)_3\,. \tag{2.12}$$

(b) *Suppose $\mathscr{M}(e; u, v) = 0$ for all $u, v \in V$. Then $\mathscr{M}(c; u, v, w) = 0$ for all $u, v, w \in V$ iff the following identities are satisfied for all $u_\mu, v_\mu, w_\mu \in V_\mu$*

$$\{u_3 v_3 w_3\} = \langle u_3, v_3\rangle w_3 + \langle v_3, w_3\rangle u_3 + \langle w_3, u_3\rangle v_3 \in V_3(e) \tag{2.13}$$

$$\{u_3 v_3 w_1\} = 3\langle u_3, v_3\rangle w_1 - u_3 \square (v_3 \square w_1) - v_3 \square (u_3 \square w_1) \in V_1(e) \tag{2.14}$$

$$\begin{aligned}\{u_3 v_1 w_1\} = {} & \langle v_1 \square w_1, u_3\rangle e + 3\langle v_1, w_1\rangle u_3 \\ & - [v_1 \square (w_1 \square u_3) + w_1 \square (v_1 \square u_3)]_3 + \{u_3 v_1 w_1\}_1\end{aligned} \tag{2.15}$$

$$\begin{aligned}\{u_1 v_1 w_1\} = {} & u_1 \square (v_1 \square w_1) + v_1 \square (w_1 \square u_1) + w_1 \square (u_1 \square v_1) \\ & + \{u_1 v_1 w_1\}_3 \in V_3(e) \oplus V_1(e)\,.\end{aligned} \tag{2.16}$$

PROOF. This can be proved in the same way as Theorem 2.3 or one considers the dual system V' and uses Theorem 2.3 for V' and e.

An element e of a symmetric triple V satisfying $\langle e, e\rangle = 1$, $\{eee\} = 3e$

and $\mathscr{M}(e; u, v, w) = 0$ for all $u, v, w \in V$ is called a *maximal tripotent*. If V is isoparametric, then, by Lemma 2.1.c, every maximal tripotent is a maximum of $F|S^*$ where F is the polynomial associated to V. As for minimal tripotents, the decomposition $V = \boldsymbol{R}e \oplus V_3(e) \oplus V_1(e)$ is said to be the *Peirce decomposition of V relative to e*, the corresponding decomposition of $x \in V$ is the *Peirce decomposition of x relative to e*, the spaces $V_\mu(e)$ are called *Peirce spaces*, and the identities (2.10) to (2.16) are referred to as *Peirce multiplication rules*. Analogous remarks as Remark 2.4.(b), (d), (e) and (f) apply in the case of maximal tripotents. Besides these we have

REMARK 2.6. (a) It is easy to see that for a symmetric triple system V the following are equivalent:

(i) e is a maximal (resp. minimal) tripotent of V

(ii) e is a minimal (resp. maximal) tripotent of V'.

(b) Examples for maximal tripotents of the isoparametric triple system $M(2, r; \boldsymbol{F})$ are $(\sqrt{2})^{-1}(E_{ij} + E_{kl})$ for $\{i, j\} \cap \{k, l\} = \varnothing$. The Peirce spaces for $e = (\sqrt{2})^{-1}(E_{11} + E_{22})$ are

$$V_3(e) = \boldsymbol{R}(E_{11} - E_{22}) \oplus \boldsymbol{F}(E_{12} + E_{21})$$
$$V_1(e) = \boldsymbol{F}^-E_{11} \oplus \boldsymbol{F}^-E_{22} \oplus \boldsymbol{F}(E_{12} - E_{21}) \oplus (\bigoplus_{k,l \geqq 3} \boldsymbol{F}E_{1k} \oplus \boldsymbol{F}E_{2l})$$

where $\boldsymbol{F}^-$ is the orthogonal complement of 1 in $\boldsymbol{F}$.

(c) Similarly to Remark 2.4.(f) one gets: Let e be a maximal tripotent of the symmetric triple system V. Then all triple products are known once $v_3 \square v_1$ and $\{v_1 v_1 v_1\}_3$ are known for every $v_i \in V_i$.

2.4. In this subsection we define the subspace $V_2^\circ(c)$ of $V_2(c)$ where c is a minimal tripotent of the symmetric triple system V. If V is isoparametric, every $f \in V_2^\circ(c)$ with $\langle f, f \rangle = 1$ has essentially the same Peirce decomposition as c. We point out that the results of this subsection are not used in §3.

The formulas (2.3), (2.4) and (2.5) imply that the algebra "$\circ$" defined by $x \circ y = \{xcy\}$ is determined by the operation of $V_0(c)$ on $V_2(c)$. In case V is isoparametric one can show that this operation is faithful.

In the general situation where c is a minimal tripotent of the symmetric triple system V we introduce that part of $V_2(c)$ where the operation of $V_0(c)$ is trivial:

$$V_2^\circ(c) = \{x \in V_2(c);\ V_0(c) \circ x = 0\}\,. \tag{2.17}$$

It will be convenient to have also the following abbreviation.

$$V_2^K(c) := \boldsymbol{R}c \oplus V_2^\circ(c)\,. \tag{2.18}$$

We give an example for $V_2^\circ(c)$: Let $V = \mathrm{Mat}(2, r; \boldsymbol{F})$ as defined in §1.5.(a.1). Using Remark 2.4.(c) it is easy to check that $V_2^\circ(E_{ij}) = \boldsymbol{F}^- E_{ij}$ where $\boldsymbol{F}^-$ is the orthogonal complement of 1 in $\boldsymbol{F}$.

In the following lemma we use the notation $U \ominus W$ to denote the orthogonal complement of W in the Euclidean space U.

LEMMA 2.7. *Let c be a minimal tripotent of the symmetric triple system V.*

(a) *If $x_2^\circ \in V_2^\circ(c)$, then $x_2^\circ \circ y = 0$ for all $y \in V \ominus (\boldsymbol{R}c \oplus V_2^\circ(c))$.*

(b) *For every $y \in V_0(c)$, $\langle y, y \rangle = 1$, we have*

$$V_2^K(c) \subset V_0(y) = \{v \in V,\ T(y)v = 0\}\ .$$

(c) *If $V_0(c)$ contains a minimal tripotent, then*

$$\{uvw\} = 2(\langle u, v\rangle w + \langle v, w\rangle u + \langle w, u\rangle v)$$

for all $u, v, w \in V_2^\circ(c)$.

(d) *We assume that $V_0(c)$ has a basis consisting of minimal tripotents. If $u \in V_2^K(c)$ is a minimal tripotent with* $\dim V_0(c) = \dim V_0(u)$, *then*

$$V_0(u) = V_0(c) \quad \textit{and} \quad V_2(u) = (V_2(c) \oplus \boldsymbol{R}c) \ominus \boldsymbol{R}u\ .$$

Moreover, $\{V_0(c), u, V_0(c)\} = 0$.

PROOF. (a) By the definition of $V_2^\circ(c)$ we have $x_2^\circ \circ V_0(c) = 0$, and for $y_2 \in V_2(c)$ we conclude $\langle x_2^\circ \circ y_2, z_0\rangle = 0$ hence $x_2^\circ \circ y_2 = 2\langle x_2^\circ, y_2\rangle c$ by (2.5). This shows (a).

(b) By (2.3) and (2.7) we get $T(y)c = y \circ y = 0$ and $T(y)u_2^\circ = 2y \circ (y \circ u_2^\circ) = 0$.

(c) follows from (b) and (2.6) applied to the minimal tripotent $y \in V_0(c)$.

(d) We write u in the form $u = \alpha c + u_2^\circ$ and let y be a minimal tripotent of $V_0(c)$. Then $T(u)y = \alpha^2 T(c)y + 2\alpha u_2^\circ \circ y + T(u_2^\circ)y = 0$ where the last summand is zero because of (b) and (2.3) applied to y. This shows $V_0(c) \subset V_0(u)$ and therefore $V_0(u) = V_0(c)$ by our assumption. Since $V_2(u) = V \ominus (\boldsymbol{R}u \oplus V_0(u))$ the second assertion follows. Finally, $\{V_0(c), u, V_0(c)\} = \{V_0(u), u, V_0(u)\} = 0$ by (2.3) applied to u.

The assumptions of parts (c) and (d) of Lemma 2.7 hold when V is isoparametric:

COROLLARY 2.8. *Let c be a minimal tripotent of the isoparametric triple system V.*

(a) *For all $u, v, w \in V_2^K(c)$ we have*

$$\{uvw\} = 2(\langle u, v\rangle w + \langle v, w\rangle u + \langle w, u\rangle v) .$$

(b) *Let* $u \in V_2^K(c)$ *and* $\langle u, u\rangle = 1$. *Then*

$$V_0(u) = V_0(c) \quad and \quad V_2(u) = (V_2(c) \oplus \boldsymbol{R}c) \ominus \boldsymbol{R}u .$$

Moreover,

$$\{V_0(c)u V_0(c)\} = 0 .$$

In analogy to (2.17) and (2.18) we define

$$V_1^\circ(e) = \{x \in V_1(e), x \,\Box\, V_3(e) = 0\} \tag{2.19}$$

$$V_1^K(e) = \boldsymbol{R}e \oplus V_1^\circ(e) . \tag{2.20}$$

It is straightforward how to translate Lemma 2.7 and Corollary 2.8 to the case of maximal tripotents.

3. Comparison with the work of H. Ozeki and M. Takeuchi.

3.1 A decomposition of a symmetric triple system relative to a maximal tripotent as developed in § 2.3 also appears in [10], however in a different setting. In this subsection we describe the procedure used in [10] and identify the fundamental notions of [10]. This will help the reader to translate the results of [10] into the language of triple systems. *Throughout* § 3.1 *V denotes a symmetric triple system.*

In [10] a point e of the unit sphere S^* of V is picked where the restriction to S^* of the polynomial F associated to V is maximal. In case V is isoparametric, e is a maximal tripotent. Therefore *we will assume in the sequel that e is a maximal tripotent of the symmetric triple system V.* Then one considers the map $t \to F(te + x)$ where $x \in X = V \ominus \boldsymbol{R}e$. Obviously,

$$F(te + x) = f_0(x) + tf_1(x) + t^2 f_2(x) + t^3 f_3(x) + t^4 f_4(x)$$

where f_j is a homogeneous polynomial $f_j: X \to \boldsymbol{R}$ of degree $4 - j$. It is easy to see $f_4(x) = F(e)$ and $f_3 = 0$.

Using Theorem 2.2.(c) the following lemma becomes obvious (it is essentially identical with [10, Lemma 5]).

LEMMA 3.1. *The quadratic form* $f_2 = A = (1/2)d_x^2 F(e, e)$ *can be written as*

$$f_2(x) = 2(\langle x_1, x_1\rangle - 3\langle x_3, x_3\rangle)$$

where $x_1 \in V_1(e) = Y$ *and* $x_3 \in V_3(e) = W$. *Moreover,* $\dim Y = 2m_2 + m_1$ *and* $\dim W = m_1 + 1$ *iff* trace $T(c) = 2(3 + 2m_1 + m_2)$.

In the sequel we assume that $\dim Y = 2m_2 + m_1$ *and* $\dim W = m_1 + 1$.

According to the procedure in [10] we now consider $f_1 = B$. Making use of our abbreviation $a \square b = \{aeb\}$ we get

$$f_1(x) = d_x F(e) = -(8/3)\langle x_1 \square x_1 + 2x_1 \square x_3 + x_3 \square x_3,\ x_1 + x_3\rangle .$$

Using (2.10), (2.11) and (2.12) it follows

(3.1) $\qquad f_1(x) = -8\langle x_3 \square x_1, x_1\rangle$ for $x = x_3 + x_1$, $x_j \in V_j(e)$.

Let (w_3^α) be an orthogonal basis of $W = V_3(e)$. Then

(3.2) = [10, (3.7)] $\qquad B(x) = f_1(x) = 8 \sum_\alpha p_\alpha \langle w_3^\alpha, x_3\rangle$

where

(3.3) $$p_\alpha = -\langle w_3^\alpha \square x_1, x_1\rangle .$$

Finally, we consider $f_0 = C$. Since C is of degree 4 on $X = V_1(e) \oplus V_3(e)$ we may write

(3.4) = [10, (3.8)] $\qquad C = \sum_{h=0}^{4} C_h$

where C_h is the homogeneous part of degree h on $W = V_3(e)$ (and hence of degree $4 - h$ on $Y = V_1(e)$). Moreover, we define $m_1 + 1$ cubic forms $q_0, \cdots, q_{m_1}$ on Y by

(3.5) = [10, (3.9)] $\qquad C_1(x) = 8 \sum_{\alpha=0}^{m_1} q_\alpha \langle x_3, w_3^\alpha\rangle$.

In the following lemma we identify the forms q and compute the explicit expressions for C_h which are also contained in [10, Lemma 7].

LEMMA 3.2. *The following formulas hold*

(3.6) $$C_0 = F(x_1) = \langle x_1, x_1\rangle^2 - 2\langle (x_1 \square x_1)_3 \square x_1, x_1\rangle = \langle x_1, x_1\rangle^2 - 2 \sum_\alpha p_\alpha^2$$

(3.7) $$q_\alpha = (1/3)\langle \{x_1 x_1 x_1\}, w_3^\alpha\rangle$$

(3.8) $$\begin{aligned} C_2 &= 8\langle x_3 \square x_1, x_3 \square x_1\rangle - 6\langle x_1, x_1\rangle\langle x_3, x_3\rangle \\ &= 2 \sum_\alpha \langle \operatorname{grad} p_\alpha, \operatorname{grad} p_\beta\rangle\langle w_3^\alpha, x_3\rangle\langle w_3^\beta, x_3\rangle - 6\langle x_1, x_1\rangle\langle x_3, x_3\rangle \end{aligned}$$

(3.9) $$C_3 = 0$$

(3.10) $$C_4 = \langle x_3, x_3\rangle^2 .$$

PROOF. We expand $C(x_1 + x_3) = F(x_1 + x_3)$ and collect the terms of degree h in x_3:

(*) $$\begin{aligned} F(x_1 + x_3) = F(x_1) &- (8/3)\langle \{x_1 x_1 x_1\}, x_3\rangle + 6\langle x_1, x_1\rangle\langle x_3, x_3\rangle \\ &- 4\langle \{x_1 x_1 x_3\}, x_3\rangle - (8/3)\langle \{x_3 x_3 x_3\}, x_1\rangle + F(x_3) . \end{aligned}$$

Using (2.16) and (2.12) we obtain $C_0 = \langle x_1, x_1\rangle^2 - 2\langle (x_1 \square x_1)_3 \square x_1, x_1\rangle$. Since $(x_1 \square x_1)_3 = \sum_\alpha \langle x_1 \square x_1, w_3^\alpha\rangle w_3^\alpha$ we get $\langle (x_1 \square x_1)_3 \square x_1, x_1\rangle = \sum_\alpha p_\alpha^2$.

From (*) we derive $C_1 = -(8/3)\langle\{x_1x_1x_1\}, x_3\rangle = -(8/3)\sum_\alpha \langle\{x_1x_1x_1\}, w_3^\alpha\rangle \langle x_3, w_3^\alpha\rangle$, which implies (3.7).

Next, we conclude from (*) and (2.14) that $C_2 = 8\langle x_3 \square x_1, x_3 \square x_1\rangle - 6\langle x_1, x_1\rangle\langle x_3, x_3\rangle$.

Moreover, $\langle x_3 \square x_1, x_3 \square x_1\rangle = \sum_{\alpha,\beta} \langle x_3, w_3^\alpha\rangle \langle x_3, w_3^\beta\rangle \langle w_3^\alpha \square x_1, w_3^\beta \square x_1\rangle$. Because grad $p_\alpha = -2T(e, w_3^\alpha)$ we have (3.8).

Finally, (2.13) implies $C_3 = -(8/3)\langle\{x_3x_3x_3\}, x_1\rangle = 0$ and $C_4 = F(x_3) = 3\langle x_3, x_3\rangle^2 - (2/3)\langle\{x_3x_3x_3\}, x_3\rangle = \langle x_3, x_3\rangle^2$. Thus (3.9) and (3.10) follow.

REMARK 3.3. As already mentioned in Remark 2.6.(c) the entire triple system is determined by the functions $V_3 \times V_1 \to V_1$: $(x_3, x_1) \to x_3 \square x_1$ and $V_1 \to V_3$: $x_1 \to \{x_1x_1x_1\}_3$ i.e., in view of (3.3) and (3.7) by the quadratic forms (p_α) and the cubic forms (q_α). In [10, Theorem 1] Ozeki and Takeuchi give a list of conditions on (p_α) and (q_α) which are necessary and sufficient for the corresponding triple system to be isoparametric. Rather than just translating their result in the language of triple systems we prefer to present a more direct proof of this result in the next subsections. Thereby we also can derive a slightly improved version of [10, Theorem 1].

In view of Section 5.5 we prefer to prove the theorem in §3.2 in terms of minimal rather than maximal tripotents. In §3.3 we formulate our results in terms of maximal tripotents and carry out the comparison with [10].

3.2. In this subsection we consider a symmetric triple system V and a minimal tripotent c of V with corresponding Peirce decomposition $V = \boldsymbol{R}c \oplus V_2 \oplus V_0$. The formula (2.6) is equivalent to saying that every element $f \in V_0$ with $\langle f, f\rangle = 1$ satisfies $\{fff\} = 6f$. In particular, if V is isoparametric then f is a minimal tripotent. Since we want to characterize isoparametric triple systems, a first step is to investigate in general when every $f \in V_0$ with $\langle f, f\rangle = 1$ is a minimal tripotent.

LEMMA 3.4. *Let c be a minimal tripotent of the symmetric triple system V. Then every $f \in V_0(c)$ with $\langle f, f\rangle = 1$ is a minimal tripotent iff for all $v_0 \in V_0$, $v_2 \in V_2$ the following identities hold*:

(a) $T(v_0, c)^3 v_2 = \langle v_0, v_0\rangle T(v_0, c)v_2$

(b) $\langle v_0 \circ v_2, T(v_2)v_0\rangle = 0$

(c) $3\langle T(v_2)v_0, T(v_0, c)^2 v_2\rangle = 2\langle v_0, v_0\rangle\langle T(v_2)v_2, v_0\rangle$

(d) $3\langle T(v_2)^2 v_0, v_0\rangle + 2\langle T(v_0)v_2, T(v_2)v_2\rangle - 3\langle v_0, v_0\rangle\langle v_2, T(v_2)v_2\rangle$
$-18\langle v_2, v_2\rangle\langle v_2, T(v_0)v_2\rangle + 18\langle v_0, v_0\rangle\langle v_2, v_2\rangle^2 = 0.$

PROOF. By definition, f is a minimal tripotent if $\mathscr{M}(f, f, u, v, w) = 0$ for all $u, v, w \in V$. We put $h(x, y, z, u, v, w) = \langle \mathscr{M}(x, y, z, u, v), w \rangle$. Then h is linear in each variable and totally symmetric. We say that h is of type (ijk) if it is of degree i resp. j resp. k in c resp. v_2 resp. v_0. Clearly, $i + j + k = 6$. It is also obvious that every $f \in V_0(c)$ with $\langle f, f \rangle = 1$ is a minimal tripotent iff all the expressions of type (ijk) with $k \geqq 2$ vanish. Since c is a minimal tripotent we already know $h(c, c, x, y, z, w) = 0$. Moreover, $\langle \mathscr{M}(v_0), x \rangle = 0$ by (2.6) and Lemma 2.1.b. Therefore we only need to consider the types (ijk) where $0 \leqq i \leqq 1$ and $2 \leqq k \leqq 4$. By (1.12) and (1.15) this can be done by using the identities (1.7) and (1.8) and taking scalar products: For (1.14) we apply (1.7) for $x = v_0$ to v_2 and take the scalar product with c: $0 = 3\langle T(v_0)v_2, v_0 \circ v_0 \rangle + 2\langle T(v_0)v_0, v_0 \circ v_2 \rangle - 18\langle v_0, v_0 \rangle \langle v_0 \circ v_0, v_2 \rangle$. Since $v_0 \circ v_0 = 0$ by (2.3), $T(v_0)v_0 = 6\langle v_0, v_0 \rangle v_0$ by (2.6) and $v_0 \circ v_2 \in V_2$ by (2.4), each of the three summands vanishes. The remaining cases (123), (132), (042), (033) and (024) follow similarly.

We will characterize when V satisfies (ISO 3), i.e., when $\mathscr{M} \equiv 0$ on V.

LEMMA 3.5. *Let c be a minimal tripotent of the symmetric triple system V. Then V satisfies* (ISO 3) *iff every $f \in V_0(c)$ with $\langle f, f \rangle = 1$ is a minimal tripotent and the following identities* (a), (b) *and* (c) *hold for all $v_j \in V_j$:*

(a) $\langle \{v_2 v_2 v_2\}, v_2 \circ v_2 \rangle = 0$

(b) $\langle \{v_2 v_2 v_2\}, \{v_2 v_2 v_0\} \rangle = 6\langle v_2, v_2 \rangle \langle \{v_2 v_2 v_2\}, v_0 \rangle$

(c) $\langle \mathscr{M}(v_2), v_2 \rangle = 0$.

PROOF. We use the notation of the proof of the previous lemma. Obviously, (ISO 3) is fulfilled iff h vanishes for all possible types (ijk). But since c is a minimal tripotent we know already that h is zero if $i \geqq 2$. Moreover, h vanishes for (ijk) with $k \geqq 2$ iff every $f \in V_0(c)$ with $\langle f, f \rangle = 1$ is a minimal tripotent. Hence the lemma follows if we can show that h vanishes for all expressions of type (ijk) with $0 \leqq i \leqq 1$, $0 \leqq k \leqq 1$, $i + j + k = 6$, iff (a), (b) and (c) hold. This is shown by considering each type individually.

REMARK 3.6. Part (a) of Lemma 3.4 and (2.4) show that the linear family of endomorphisms $H(x_3) := T(x_3, e) | V_1(e)$ has the property $H(x_3)^3 = \langle x_3, x_3 \rangle H(x_3)$, i.e., $\{H(x_3); x_3 \in V_3\}$ is a cubic space in the following sense: Let $(U, \langle \cdot, \cdot \rangle)$ and $(V, \langle \cdot, \cdot \rangle)$ be finite-dimensional Euclidean vector spaces and $H: U \to \operatorname{End} V$ a linear map satisfying $\langle H(u)v, w \rangle = \langle v, H(u)w \rangle$ and $H(u)^3 = \langle u, u \rangle H(u)$. Then $H(U)$ is called a *cubic space*.

LEMMA 3.7. *Let* $(V, \langle\cdot, \cdot\rangle)$ *and* $(U, \langle\cdot, \cdot\rangle)$ *be finite-dimensional Euclidean spaces and let* $H: U \to \operatorname{End} V$ *be a linear map such that* $H(U)$ *is a cubic space. Then for all* $u, v \in U$:

(a) $H(u)H(v)^2 + H(v)^2H(u) + H(v)H(u)H(v) = 2\langle u, v\rangle H(v) + \langle v, v\rangle H(u)$.

(b) *If* $\dim U \geqq 2$, *then* $\operatorname{trace} H(u) = 0$.

(c) $\operatorname{trace} H(u)^2 = m\langle u, u\rangle$ *where* m *is an integer.*

PROOF. (a) follows by linearization of $H(u)^3 = \langle u, u\rangle H(u)$.

(b) Since $\dim U \geqq 2$ there exists a $v \in U$ with $\langle v, v\rangle = 1$ and $\langle u, v\rangle = 0$. Putting $H_u = H(u)$, $H_v = H(v)$, we get $H_u = H_uH_v^2 + H_v^2H_u + H_vH_uH_v$, hence $\operatorname{trace} H_u = 3 \operatorname{trace} H_uH_v^2 = 3 \operatorname{trace}[(H_uH_v^2 + H_v^2H_u + H_vH_uH_v)H_v^2] = 9 \operatorname{trace} H_uH_v^2$, which implies (b).

(c) We can assume $\dim U \geqq 2$. For every $u \in U$ with $\langle u, u\rangle = 1$ we know that $H(u)^2$ is an orthogonal projection. Therefore $\operatorname{trace} H(u)^2$ is an integer. On the other hand $u \to \operatorname{trace} H(u)^2$ is a continuous function. This implies (c).

The following lemma implies that in Lemma 3.4 we may substitute $\langle \mathscr{M}(v_2), v_2\rangle = 0$ for $3\langle T(v_2)v_0, T(v_0, c)^2v_2\rangle = 2\langle v_0, v_0\rangle\langle T(v_2)v_2, v_0\rangle$.

LEMMA 3.8. *Let* c *be a minimal tripotent of the symmetric triple system* V. *We assume*

(1) $T(v_0, c)^3v_2 = \langle v_0, v_0\rangle T(v_0, c)v_2$ *and*

(2) $\langle \mathscr{M}(v_2), v_2\rangle = 0$

for all $v_j \in V_j$. *Let* $f \in V_0$ *satisfy*

(3) $\langle f \circ v_2, T(v_2)f\rangle = 0$

for all $v_2 \in V$. *Then*

(a) $3\langle T(v_2)f, T(f, c)^2v_2\rangle = 2\langle f, f\rangle\langle f, T(v_2)v_2\rangle$ *and*

(b) $\operatorname{trace} T(v_2, f) = 0$

for all $v \in V_2$.

PROOF. (a) We may assume $\langle f, f\rangle = 1$. Then the self-adjoint endomorphism $T(f, c)|V_2$ has the eigenvalues ± 1 and 0: $V_2 = A \oplus B \oplus Z$ where $A = \{a \in V_2; f \circ a = a\}$, $B = \{b \in V_2;\ f \circ b = -b\}$ and $Z = \{z \in V_2; f \circ z = 0\}$, correspondingly $v_2 = a + b + z$ for $v_2 \in V_2$. From (3) we derive $0 = \langle a - b, \{a + b + z, a + b + z, f\}\rangle$. We expand this expression, collecting terms of the same homogeneity in a, b and z:

$$0 = \langle f, \{aaa\} + (2-1)\{aab\} + 2\{aaz\} + (1-2)\{abb\} + \{azz\} + (2-2)\{abz\} - \{bbb\} - 2\{bbz\} - \{bzz\}\rangle .$$

Since all the summands have different degrees, their scalar products vanish individually, i.e.,

$$(*) \qquad \begin{aligned} 0 &= \langle f, \{aaa\}\rangle = \langle f, \{aab\}\rangle = \langle f, \{aaz\}\rangle = \langle f, \{abb\}\rangle \\ &= \langle f, \{azz\}\rangle = \langle f, \{bbb\}\rangle = \langle f, \{bbz\}\rangle = \langle f, \{bzz\}\rangle . \end{aligned}$$

We use $(*)$ to simply $3\langle T(v_2)f, T(f, c)^2v_2\rangle = 3\langle T(v_2)f, a + b\rangle = 3\langle f, \{v_2, v_2, a + b\}\rangle = 3\langle f, 2\{bza\} + 2\{azb\}\rangle = 12\langle f, \{abz\}\rangle$. Moreover $2\langle f, T(v_2)v_2\rangle = 2\langle f, 6\{abz\} + \{zzz\}\rangle$. Therefore the two expressions are equal iff

$$(**) \qquad 0 = \langle f, \{zzz\}\rangle$$

holds, which we prove now. We choose an orthonormal basis (w_0^α) of V_0 in such a way that $f = w_0^0$. Putting $H_\alpha = T(w_0^\alpha, c)|V_2$ we derive from Lemma 3.6.a for $\alpha > 0$ that $\langle H_\alpha z, z\rangle = \langle (H_0^2H_\alpha + H_\alpha H_0^2 + H_0H_\alpha H_0)z, z\rangle = 0$. Obviously, $\langle H_0z, z\rangle = 0$, hence $0 = \langle x_0 \circ z, z\rangle = \langle x_0, z \circ z\rangle$ for every $x_0 \in V_0$, i.e., $(z \circ z)_0 = 0$. By (2.9) this implies $T(z)z = 6\langle z, z\rangle z + \{zzz\}_0$. We now use (2) for $x_2 = z$ and get $0 = 36\langle z, z\rangle^3 + \langle \{zzz\}_0, \{zzz\}_0\rangle - 54\langle z, z\rangle^3 + 18\langle z, z\rangle^3 = \langle \{zzz\}_0, \{zzz\}_0\rangle$. Thus $\{zzz\}_0 = 0$, which clearly implies $(**)$.

(b) From the Peirce multiplication rules for c we get trace $T(v_2, f) = \operatorname{trace}(T(v_2, f)|V_2)$. To compute trace $(T(v_2, f)|V_2)$ we choose an orthonomal basis (y_2^β) of V_2 which is a union of orthonormal bases of A, B and Z. Then $\operatorname{trace}(T(v_2, f)|V_2) = \sum_\beta \langle y_2^\beta, \{v_2 f y_2^\beta\}\rangle = \sum_\beta \langle f, \{y_2^\beta y_2^\beta v_2\}\rangle$. Since every summand in the last expression has at least degree two in a, b or z it follows from $(*)$ and $(**)$ that every summand is zero. This proves (b).

REMARK. The decomposition $V_2 = A \oplus B \oplus Z$ used in the proof of Lemma 3.8 will be refined in the next sections.

We will now describe when a symmetric triple system satisfies (ISO 4). The following preliminary result will be used later:

LEMMA 3.9. *Let c be a minimal tripotent of the symmetric triple V with* $\dim V_0(c) \geqq 2$ *and assume* $T(v_0, c)^3v_2 = \langle v_0, v_0\rangle T(v_0, c)v_2$ *for all* $v_i \in V_i$. *Then*

$$\operatorname{trace} T(c, v) = 0 \textit{ for every } v \in V \textit{ with } \langle v, c\rangle = 0 .$$

PROOF. Since $v = v_2 + v_0$ with $v_j \in V_j$ it is enough to prove trace $T(c, v_j) = 0$. The Peirce multiplication rules show trace $T(c, v_2) = 0$ and trace $T(c, v_0) =$ trace $T(c, v_0)\pi_2$, where π_2 denotes the orthogonal projection of V onto V_2. Because $\dim V_0 \geqq 2$ we can apply Lemma 3.7.b to $H(x_0) = T(x_0, c)|V_2$ and get trace $T(c, v_0) = 0$.

LEMMA 3.10. *Let c be a minimal tripotent of the symmetric triple system V and assume, in addition, for every $v_i \in V_i$*

$$T(v_0, c)^3v_2 = \langle v_0, v_0\rangle T(v_0, c)v_2 .$$

Then (ISO 4) *is fulfilled iff there exist positive integers* m_1, m_2 *such that*

(a) $\dim V_0 = m_2 + 1$, $\dim V_2 = 2m_1 + m_2$,

(b) $\operatorname{trace}(T(f_0, c)^2 | V_2) = 2m_1$ *for some* $f_0 \in V_0$ *with* $\langle f_0, f_0\rangle = 1$,

(c) $\operatorname{trace} T(v_0, v_2) = 0$ *for all* $v_i \in V_i$.

PROOF. By Theorem 2.2.c we know that (ISO 4b) and (ISO 4a) for $x = y = c$ is equivalent to (a). Therefore it remains to show that, if (a) and the assumptions of the lemma hold, the conitions (b) and (c) are equivalent to

(1) $\operatorname{trace} T(v_i) = 2(3 + 2m_1 + m_2)\langle v_i, v_i\rangle$ for $i = 0, 2$,

(2) $\operatorname{trace} T(c, v_i) = 0 = \operatorname{trace} T(v_2, v_0)$, $i = 0, 2$.

Here the first part of (2) follows from Lemma 3.9 and the second part is just (c). To compute the trace of $T(v_0)$ we derive from the Peirce multiplication rules $T(v_0)(tc + y_2 + y_0) = 2v_0 \circ (v_0 \circ y_2) + 2\langle v_0, v_0\rangle y_0 + 4\langle v_0, y_0\rangle v_0$, hence $\operatorname{trace} T(v_0) = 2 \operatorname{trace}(T(v_0, c) | V_2)^2 + 2\langle v_0, v_0\rangle \dim V_0 + 4\langle v_0, v_0\rangle = 2 \operatorname{trace}(T(v_0, c) | V_2)^2 + \langle v_0, v_0\rangle(2m_2 + 2 + 4)$. Therefore (1) holds for $i = 0$ iff $\operatorname{trace}(T(v_0, c) | V_2)^2 = 2m_1\langle v_0, v_0\rangle$, which is equivalent to (b) by Lemma 3.7.c.

Finally, we have for $T(v_2)$: $T(v_2)(tc + y_2 + y_0) = 2\langle v_2, v_2\rangle tc + (v_2 \circ v_2)_0 + 2\langle v_2, v_2\rangle y_2 + 4\langle v_2, y_2\rangle v_2 - 2v_2 \circ (v_2 \circ y_2)_0 - y_2 \circ (v_2 \circ v_2)_0 + \{v_2 v_2 y_2\}_0 + \langle v_2 \circ y_0, v_2\rangle c + 2[v_2 \circ (v_2 \circ y_0)]_0 + \{v_2 v_2 y_0\}_2$. Hence $\operatorname{trace} T(v_2) = 2\langle v_2, v_2\rangle + 2\langle v_2, v_2\rangle \dim V_2 + 4\langle v_2, v_2\rangle - 2 \operatorname{trace}(T(v_2, c)\pi_0 T(v_2, c)\pi_2) - \operatorname{trace}(T(c, (v_2 \circ v_2)_0)\pi_2) + 2 \operatorname{trace}(\pi_0(T(v_2, c)^2)\pi_0)$, where π_i, $i = 0, 2$, is the orthogonal projection of V onto V_i. In this sum $\operatorname{trace}(T(c, (v_2 \circ v_2)_0)\pi_2) = \operatorname{trace} T(c, (v_2 \circ v_2)_0)$ vanishes by Lemma 3.9. Moreover, $2\langle v_2, v_2\rangle[1 + \dim V_2 + 2] = 2\langle v_2, v_2\rangle[3 + 2m_1 + m_2]$. Therefore, in case $i = 2$, (1) is equivalent to $\operatorname{trace}(T(v_2, c)\pi_0 T(v_2, c)\pi_2) = \operatorname{trace}(\pi_0 T(v_2, c)^2\pi_0)$. Using orthonormal bases for V_0 resp. V_2 it is straightforward to show $\operatorname{trace} T(v_2, c)\pi_0 T(v_2, c)\pi_2 = \operatorname{trace} \pi_0 T(v_2, c)^2\pi_0$.

Putting together the previous lemmas, we can prove the following characterization of isoparametric triple systems:

THEOREM 3.11. *Let c be a minimal tripotent of the symmetric triple system V. Then V is isoparametric iff the following conditions hold*:

(a) *Every* $f \in V_0(c)$ *with* $\langle f, f\rangle = 1$ *is a minimal tripotent of* V.

(b) $\langle \{v_2 v_2 v_2\}, v_2 \circ v_2\rangle = 0$, $v_2 \in V_2$.

(c) $\langle \mathscr{M}(v_2), v_2\rangle = 0$, $v_2 \in V_2$.

(d) *There exist positive integers* m_1, m_2 *such that*

(1) $\dim V_0 = m_2 + 1$, $\dim V_2 = 2m_1 + m_2$

(2) $\operatorname{trace}(T(f_0, c) | V_2)^2 = 2m_1$ *for some* $f_0 \in V_0$ *with* $\langle f_0, f_0\rangle = 1$.

REMARK 3.12. By Lemmas 3.4 and 3.8 the conditions (a) and (c) of Theorem 3.11 hold iff for all $v_i \in V_i$ we have (a.1) $T(v_0, c)^3 v_2 = \langle v_0, v_0\rangle T(v_0, c)v_2$,

(a.2) $\langle v_0 \circ v_2, T(v_2)v_0\rangle = 0$,

(a.3) $3\langle T(v_2)^2 v_0, v_0\rangle + 2\langle T(v_0)v_2, T(v_2)v_2\rangle - 3\langle v_0, v_0\rangle\langle v_2, T(v_2)v_2\rangle - 18\langle v_2, v_2\rangle\langle v_2, T(v_0)v_2\rangle + 18\langle v_0, v_0\rangle\langle v_2, v_2\rangle^2 = 0$,

(c) $\langle T(x_2)^2 x_2, x_2\rangle - 9\langle x_2, x_2\rangle\langle x_2, T(x_2)x_2\rangle + 18\langle x_2, x_2\rangle^3 = 0$.

PROOF OF THEOREM 3.11. We have to show that (ISO 3) and (ISO 4) hold iff (a) to (d) are fulfilled. By Lemmas 3.5 and 3.10 the theorem follows if we can prove that (a) to (d) imply for all $v_i \in V_i$

(1) $\langle\{v_2v_2v_2\}, \{v_2v_2v_0\}\rangle = 6\langle v_2, v_2\rangle\langle\{v_2v_2v_2\}, v_0\rangle$ and

(2) trace $T(v_0, v_2) = 0$.

Here the second assertion follows from Lemma 3.8 in view of Remark 3.12. To prove (1) we derive $\langle\{v_2v_2v_2\}, \{v_2v_2v_0\}\rangle = 6\langle v_2, v_2\rangle\langle v_2, \{v_2v_2v_0\}\rangle - 3\langle v_2 \circ (v_2 \circ v_2)_0, \{v_2v_2v_0\}\rangle + 2\langle\{v_2v_2v_2\}_0, v_2 \circ (v_2 \circ v_0)\rangle$, hence (1) is equivalent to

(1)′ $3\langle v_2 \circ (v_2 \circ v_2)_0, \{v_2v_2v_0\}\rangle = 2\langle\{v_2v_2v_2\}_0, v_2 \circ (v_2 \circ v_0)\rangle$.

To prove (1)′ we linearize the identity (a.2) of Remark 3.12 in v_0 and identity (b) in v_2. We get

(2)′ $\langle v_0 \circ v_2, T(v_2)w_0\rangle + \langle w_0 \circ v_2, T(v_2)v_0\rangle = 0$

(b)′ $3\langle\{v_2v_2w_2\}_0, v_2 \circ v_2\rangle + 2\langle\{v_2v_2v_2\}_0, v_2 \circ w_2\rangle = 0$.

We now pick an orthonormal basis (w_0^α) of V_0. Then the left hand side of (1)′ becomes using (2)′ and (b)′ $3\sum_\alpha \langle v_2 \circ v_2, w_0^\alpha\rangle\langle v_2 \circ w_0^\alpha, \{v_2v_2v_0\}\rangle = -3\sum_\alpha \langle v_2 \circ v_2, w_0^\alpha\rangle\langle v_2 \circ v_0, \{v_2v_2w_0^\alpha\}\rangle = -3\langle T(v_2)(v_2 \circ v_0), \sum_\alpha \langle v_2 \circ v_2, w_0^\alpha\rangle w_0^\alpha\rangle = -3\langle T(v_2)(v_2 \circ v_0), (v_2 \circ v_2)_0\rangle = 2\langle\{v_2v_2v_2\}_0, v_2 \circ (v_2 \circ v_0)\rangle$.

REMARK. We point out that there always exists a minimal tripotent in an isoparametric triple system (see Corollary 4.9). Therefore Theorem 3.11 can be applied to describe all isoparametric triple systems.

3.3 In this subsection we dualize the results obtained in §3.2, i.e., we state the results in terms of maximal tripotents. This will allow us to make a comparison with [10].

THEOREM 3.13. *Let V be a symmetric triple system and e a maximal tripotent of V. Then V is isoparametric iff the following conditions are satisfied:*

(a) *Every $f \in V_3(e)$ with $\langle f, f\rangle = 1$ is a maximal tripotent.*

(b) $\langle\{v_1v_1v_1\}, T(v_1)e\rangle = 0$ *for all* $v_1 \in V_1$.

(c) $\langle \mathscr{M}(v_1), v_1\rangle = 0$ *for all* $v_1 \in V_1$.

(d) *There exist positive integers m_1, m_2 such that*

(1) $\dim V_3 = m_1 + 1$, $\dim V_1 = m_1 + 2m_2$

(2) trace $(T(f_3, e)|V_1)^2 = 2m_2$ *for some* $f_3 \in V_3(e)$ *with* $\langle f_3, f_3\rangle = 1$.

PROOF. It is enough to show that the conditions (a) to (d) of

Theorem 3.11 for the minimal tripotent $e \in V'$ are equivalent to (a), (b), (c) and (d) of the theorem, which is straightforward to check.

As in Remark 3.12, we may express (a) and (c) of Theorem 3.13 equivalently in terms of identities yielding the following theorem which is proved in the same way as Theorem 3.13.

THEOREM 3.14. *Let V be a symmetric triple and e a maximal tripotent of V. Then V is isoparametric iff the following conditions hold:*

(a) $T(v_3, e)^3 v_1 = \langle v_3, v_3\rangle T(v_3, e)v_1$.

(b) $\langle T(v_3, e)v_1, T(v_1)v_3\rangle = 0$.

(c) $3\langle T(v_1)^2 v_3, v_3\rangle + 2\langle T(v_3)v_1, T(v_1)v_1\rangle - 3\langle v_3, v_3\rangle\langle v_1, T(v_1)v_1\rangle - 18\langle v_1, v_1\rangle\langle v_1, T(v_3)v_1\rangle + 18\langle v_3, v_3\rangle\langle v_1, v_1\rangle^2 = 0$.

(d) $\langle T(v_1)e, T(v_1)v_1\rangle = 0$.

(e) $\langle T(v_1)^2 v_1, v_1\rangle - 9\langle v_1, v_1\rangle\langle T(v_1)v_1, v_1\rangle + 18\langle v_1, v_1\rangle^2 = 0$.

(f) *There exist positive integers m_1, m_2 such that*

(1) $\dim V_3 = m_1 + 1$, $\dim V_1 = m_1 + 2m_2$

(2) $\operatorname{trace}(T(f_3, e)|V_1)^2 = 2m_2$ *for some* $f_3 \in V_3(e)$ *with* $\langle f_3, f_3\rangle = 1$.

REMARK 3.15. We recall from Remark 3.3 that Ozeki and Takeuchi characterized isoparametric triple systems in [10, Theorem 1] by conditions on the quadratic forms (p_α) and the cubic forms (q_α). Theorem 3.14 above is an improved version of their result. Indeed, using the identifications of §3.1 it is easy to check the following dictionary for the formulas in Theorem 3.14:

(a) $\Leftrightarrow$ [10] (3-2), (3-3) and first equation of (3-1)

(b) $\Leftrightarrow$ [10] (3-4) and (3-5)

(c) $\Leftrightarrow$ [10] (3-9) and (3-10)

(d) $\Leftrightarrow$ [10] (3-7)

(e) $\Leftrightarrow$ [10] (3-8)

(f.1) is part of the assumptions in [10]

(f.2) $\Leftrightarrow$ [10] part of the third condition in [10] (3-1).

We point out that the second equation of [10] (3-1) and [10] (3-6) are superfluous and the third condition in [10] (3-1) is only needed for one α.

4. **Orthogonal tripotents.** In this section we introduce the notion of orthogonal tripotents and show that for isoparametric triple systems orthogonal tripotents always exist. In §4.1 and §4.2, *V is always a symmetric triple system.*

4.1. A subspace U of a triple system V with the property $\{UUU\} \subset U$ is called a *subsystem* of V. The following lemma in particular implies that in an isoparametric triple system the subsystem generated by x is at most 2-dimensional.

LEMMA 4.1. *Let V be symmetric triple system and $x \in V$. We put $y := \{xxx\}$. If $\mathscr{M}(x) = \mathscr{M}(x; y) = \mathscr{M}(x; y, y) = 0$, then the vector space $V_x = \boldsymbol{R}x + \boldsymbol{R}y$ is a subsystem of V.*

PROOF. We have to show $\{xxy\}$, $\{xyy\}$ and $\{yyy\}$ lie in V_x. First, $\{xxy\} \in V_x$ because $\mathscr{M}(x) = 0$. Thus $T(x)$ leaves V_x invariant. By (1.11) we know $\mathscr{M}(x; y) = 0$ iff (1.6) applied to y holds. This implies $\{xyy\} = T(x, y)y \in V_x$. Since $\mathscr{M}(x; y, y) = 0$ we can use (1.7) for $u = y$ and apply it to y. Because we already know that $T(x)$ and $T(x, y)$ leave V_x invariant, it easily follows that $\{yyy\} \in V_x$.

THEOREM 4.2. *Let V be a symmetric triple and assume for a fixed $x \in V$ that $\mathscr{M}(x; u, v) = 0$ for all $u, v \in V$. Then*

$$[T(z_1, z_2), T(z_3, z_4)] = 0 \quad \textit{for all} \quad z_i \in V_x = \boldsymbol{R}x + \boldsymbol{R}\{xxx\} .$$

PROOF. We first define the odd powers of x by $x^{2n+1} = T(x)^n x$, $n \geqq 1$ and put $y = x^3$. We set $\alpha := 6\langle x, x\rangle$ and $\beta := 3\langle x^3, x\rangle - 18\langle x, x\rangle^2$. Then $\mathscr{M}(x) = 0$ just means $(*)$ $x^5 = \alpha y + \beta x$. To prove the theorem it is enough to show (a) $[T(x), T(x, y)] = 0$, (b) $[T(x), T(x, y)] = 0$ and (c) $[T(x, y)T(y)] = 0$.

(a) Using (1.7) the assertion (a) is equivalent to $(**)$ $[T(x), xy^* + yx^* - \alpha xx^*] = 0$ that is easily verified.

(b) Using $(*)$ and (1.8) with $u = y = x^3$ we see that (b) is fulfilled if and only if $T(x)$ commutes with $-2x^3x^{3*} - 3(xx^{5*} + x^5x^*) + 12\langle x, x^3\rangle xx^* + \alpha(xx^{3*} + x^3x^*) = -2\alpha(xy^* + yx^* - \alpha xx^*) - 2\beta xx^* - 2yy^*$. By $(**)$ this follows readily.

(c) By (1.7) for x and (b) we know $0 = [T(y), 2T(x, y) - 12(xy^* + yx^*) + 12\alpha xx^*]$. Hence (c) is equivalent to $0 = [T(y), \alpha xx^* - (xy^* - yx^*)]$. But $T(y)x = (\alpha^2 + \beta)y + \alpha\beta x$ and $T(y)y = \alpha(\alpha^2 + 2\beta)y + \beta(\alpha^2 + \beta)x$; a short computation gives the assertion.

Two tripotents e_1, e_2 are called *orthogonal* if $T(e_1)e_2 = 0 = T(e_2)e_1$. This is equivalent to the conditions $e_2 \in V_0(e_1)$ and $e_1 \in V_0(e_2)$. We add some

REMARKS. (a) If (e_1, e_2) are orthogonal, then e_1 and e_2 are minimal tripotents by Theorem 2.2. Moreover, the elements $e = \lambda(e_1 + e_2)$ and $\hat{e} = \lambda(e_1 - e_2)$ where $\lambda = (\sqrt{2})^{-1}$ satisfy $\langle e, e\rangle = 1$, $\{eee\} = 3e$, $\mathscr{M}(e; u, v) = 0$ for all $u, v \in V$ and $\langle \hat{e}, \hat{e}\rangle = 1$, $\{\hat{e}\hat{e}\hat{e}\} = 3\hat{e}$, $\mathscr{M}(\hat{e}; u, v) = 0$ for all $u, v \in V$. Therefore they are in general "nearly" maximal tripotents. Of course, e and $\hat{e}$ are maximal tripotents in case V is isoparametric.

(b) In the example $V = \mathrm{Mat}\,(2, r; \boldsymbol{F})$ considered in Remark 2.4.c, two tripotents E_{ij} and E_{kl} are orthogonal as soon as $\{i, j\} \cap \{k, l\} = \varnothing$.

Orthogonal tripotents always have a "common Peirce decomposition" which will be studied in the next section. The reason for this is

COROLLARY 4.4. *If* (e_1, e_2) *are orthogonal tripotents in a symmetric triple* V, *then*

$$[T(e_1), T(e_2)] = [T(e_1), T(e_1, e_2)] = [T(e_2), T(e_1, e_2)] = 0 .$$

PROOF. We put $x = e_1 + 2e_2$. Then the corollary follows from Theorem 4.2.

The following lemma shows that $e_2 \in V_0(e_1)$ or $e_1 \in V_0(e_2)$ for tripotents e_i already implies that e_1 and e_2 are orthogonal.

LEMMA 4.5. *Let* V *be a symmetric triple system.*

(a) *Assume* c *and* f *are minimal tripotents. Then, for* $\mu = 0, 2$,

$$f \in V_\mu(c) \text{ iff } c \in V_\mu(f) .$$

(b) *Assume* c *and* d *are minimal tripotents. Then for* $\kappa = 3, 1$

$$d \in V_\kappa(e) \text{ iff } e \in V_\kappa(d) .$$

PROOF. (a) Assume $f \in V_\mu(c)$. Then $\langle c, f\rangle = 0$ and therefore c has a decomposition $c = c_2 + c_0$ with $c_j \in V_j(f)$, $j = 0, 2$. Thus $\langle T(f)c, c\rangle = \langle 2c_2, c_2 + c_0\rangle = 2\langle c_2, c_2\rangle$. On the other hand $\langle T(f)c, c\rangle = \langle T(c)f, f\rangle = \mu$. Hence $\langle c_2, c_2\rangle = \mu/2$. From this the assertion follows easily. (b) is a consequence of (a) by dualization.

4.2. In this subsection we study symmetric triple systems V satisfying (ISO 3), i.e., $\mathscr{M}(x) = 0$ for all $x \in V$.

Let $0 \neq x \in V$. Then Lemma 4.1 implies that V_x, the subsystem of V generated by x, is either one- or two-dimensional. In the first case, x is a scalar multiple of a minimal tripotent by Lemma 2.1.b. In the second case x is called *regular*. We note that for homogeneous isoparametric triple systems x is regular in our sense iff it is regular in the sense of [12] § 3. We have the following characterization of regular elements:

THEOREM 4.6. *Let* V *be a symmetric triple system satisfying* (ISO 3). *Then* $x \in V$ *is regular if and only if there are orthogonal tripotents* $e_1, e_2 \in V$ *and* $\alpha_i \in \boldsymbol{R}$ *with* $0 < \alpha_1 < \alpha_2$ *such that* $x = \alpha_1 e_1 + \alpha_2 e_2$. *In this case* (e_1, e_2) *and* (α_1, α_2) *are uniquely determined by* x: *if* $c_1, c_2 \in V$ *are orthogonal tripotents and* $\beta_1, \beta_2 \in \boldsymbol{R}$ *with* $0 < \beta_1 < \beta_2$ *such that* $x = \beta_1 c_1 + \beta_2 c_2$, *then* $c_i = e_i$ *and* $\alpha_i = \beta_i$ *for* $i = 1, 2$.

PROOF. Let x be regular. By Theorem 4.2 we know that $\{T(u, v) | V_x; u, v \in V_x\}$ is a set of commuting self-adjoint endomorphisms of V_x. Therefore there exists an orthonormal basis (e_1, e_2) of V_x such that $T(u)e_i \in \boldsymbol{R}e_i$ for every $u \in V_x$. In particular, $T(e_i)e_i \in \boldsymbol{R}e_i$ implies that each e_i is a tripotent (Lemma 2.1.b). Further, $T(e_1, e_1)e_2 = T(e_1, e_2)e_1 \in \boldsymbol{R}e_1 \cap \boldsymbol{R}e_2$ shows $T(e_1)e_2 = 0$. Since also $T(e_2)e_1 = 0$, it follows that e_1, e_2 are orthogonal tripotents. We thus have $x = \alpha_1 e_1 + \alpha_2 e_2$ with orthogonal tripotents e_1, e_2 and real numbers α_1, α_2. The rest of the proof is straightforward.

COROLLARY 4.7. *A symmetric triple system satisfying* (ISO 3) *contains no nilpotent elements, i.e.,* $T(x)^k x = 0$ *for some* $k \in \boldsymbol{N}$ *implies* $x = 0$.

4.3. In this subsection we deal exclusively with isoparametric triple systems.

THEOREM 4.8. *The set of regular elements of an isoparametric triple system V is open and dense.*

PROOF. Assume the contrary. Then $\{xxx\} = \kappa\langle x, x\rangle x$ with a fixed κ for all $x \in V$. This leads to a contradiction to (ISO 4).

COROLLARY 4.9. *There exist orthogonal tripotents in V.*

A *minimal decomposition* of $x \in V$ is a representation $x = \alpha_1 e_1 + \alpha_2 e_2$ with orthogonal tripotents e_1, e_2 and $\alpha_1, \alpha_2 \in \boldsymbol{R}$ with $\alpha_i \geqq 0$. We already proved that every regular $x \in V$ has a minimal decomposition (Theorem 4.6), and obviously every scalar multiple of a minimal tripotent has a minimal decomposition too. We even have

THEOREM 4.10. *Every element of an isoparametric triple system has a minimal decomposition.*

PROOF. Let $e \in V$ be a maximal tripotent. By Theorem 2.2 there exists an element $\hat{e} \in V_3(e)$ with $\langle \hat{e}, \hat{e}\rangle = 1$. Then (2.13) implies that $\hat{e}$ is a maximal tripotent. Furthermore, $e \in V_3(\hat{e})$ by Lemma 4.5.b. A straightforward coputation now shows that $e_1 := \lambda(e + \hat{e})$ and $e_2 := \lambda(e - \hat{e})$, $\lambda = 2^{-1/2}$ are orthogonal tripotents with $e = \lambda(e_1 + e_2)$. This finishes the proof of the theorem.

5. Peirce decomposition relative to orthogonal tripotents. In this section we establish the Peirce decomposition of a symmetric triple system relative to two orthogonal tripotents. We finish this section by proving the main theorem of this paper characterizing isoparametric triple systems by Peirce decompositions. *Unless stated otherwise V will always be a symmetric triple system.*

5.1. We consider a symmetric triple system V with two orthogonal tripotents (e_1, e_2). By Corollary 4.4 we know $[T(e_1), T(e_2)] = 0$ and

$[T(e_1, e_2), T(e_i)] = 0$, $i = 1, 2$. Hence the various Peirce spaces are invariant under $T(e_i)$ and $T(e_1, e_2)$.

LEMMA 5.1. *Let (e_1, e_2) be orthogonal tripotents of the symmetric triple system V. Then*

(a) $V_0(e_1) \cap V_0(e_2) = 0$

(b) $T(e_1, e_2)^2 v = v$ *for* $v \in V_2(e_1) \cap V_2(e_2)$

(c) $T(e_1, e_2) | (V_2(e_1) \cap V_2(e_2))^\perp = 0$.

PROOF. (a) Assume $y \in V_0(e_1)$, $\langle y, e_2 \rangle = 0$. Then $T(e_2)y = 2y$ by (2.6). This implies (a).

(b) Since $e_2 \in V_0(e_1)$ it follows from (2.7) applied to $c = e_1$ that $2v = T(e_2)v = 2e_2 \circ (e_2 \circ v) = 2T(e_1, e_2)^2 v$.

(c) We note $(V_2(e_1) \cap V_2(e_2))^\perp = V_2(e_1)^\perp + V_2(e_2)^\perp = V_0(e_1) + V_0(e_2)$. But $T(e_1, e_2) V_0(e_1) = 0$ by (2.3) for $c = e_1$. Similarly, $T(e_1, e_2) V_0(e_2) = 0$.

We now define

$$V_{12}(e_1, e_2) := V_2(e_1) \cap V_2(e_2) .$$

Since $T(e_1, e_2)^2 | V_{12}(e_1, e_2) = \mathrm{Id}$ we have an ortogonal decomposition

$$V_{12}(e_1, e_2) = V_{12}^+(e_1, e_2) \oplus V_{12}^-(e_1, e_2)$$

where $V_{12}^\pm(e_1, e_2) = \{x \in V_{12}(e_1, e_2);\ T(e_1, e_2)x = \pm x\}$. It is sometimes convenient to use the abbreviation

$$\bar{y}_{12} := T(e_1, e_2) y_{12} \quad \text{for} \quad y_{12} \in V_{12} .$$

We obviously have $\bar{\bar{y}}_{12} = y_{12}$, $\bar{y}_{12}^\varepsilon = \varepsilon y_{12}^\varepsilon$ for $y_{12} \in V_{12}$, $y_{12}^\varepsilon \in V_{12}^\varepsilon$, $\varepsilon = \pm$. By Lemma 5.1.c the space $V_{12}(e_1, e_2)^\perp$ can be split up using only $T(e_1)$ and $T(e_2)$. We further define

$$V_{ii}^-(e_1, e_2) := V_2^0(e_i) \quad \text{for} \quad i = 1, 2 .$$
$$V_{ii}(e_1, e_2) := V_2^K(e_i) = \boldsymbol{R} e_i \oplus V_2^0(e_i) \quad \text{for} \quad i = 1, 2 .$$

We note that Lemma 2.7.b implies $V_2^K(e_1) \subset V_0(e_2)$ and $V_2^K(e_2) \subset V_0(e_1)$, in particular $V_{12}(e_1, e_2) \oplus V_{ii}^-(e_1, e_2)$ is an orthogonal sum. Its orthogonal complement in $V_2(e_i)$ is

$$V_{i0}(e_1, e_2) := V_2(e_i) \ominus (V_{12}(e_1, e_2) \oplus V_{ii}^-(e_1, e_2)) , \qquad i = 1, 2 .$$

The spaces defined above are called *Peirce spaces* relative to (e_1, e_2). If it is clear which pair of orthogonal tripotents is referred to we simply write V_{ij} instead of $V_{ij}(e_1, e_2)$. By construction we have

COROLLARY 5.2. *The following sums are orthogonal:*

$$V = \boldsymbol{R} e_1 \oplus V_{11}^- \oplus V_{10} \oplus V_{12} \oplus \boldsymbol{R} e_2 \oplus V_{22}^- \oplus V_{20} ,$$

$$V_2(e_i) = V_{ii}^- \oplus V_{i0} \oplus V_{12}\,, \qquad i = 1, 2\,,$$
$$V_0(e_1) = V_{22} \oplus V_{20}\,, \qquad V_0(e_2) = V_{11} \oplus V_{10}\,.$$

REMARK 5.3. (a) In the example $V = \mathrm{Mat}\,(2, r; \boldsymbol{F})$ (see also Remark 2.4.c, (2.19) and Remark 4.3.b) we may choose (E_{11}, E_{22}) as orthogonal tripotents. Their Peirce spaces are $V_{10} = \bigoplus_{3\leq k\leq r} \boldsymbol{F}E_{1k}$, $V_{20} = \bigoplus_{3\leq k\leq r} \boldsymbol{F}E_{2k}$, $V_{12}^+ = \boldsymbol{F}(E_{12} + E_{21})$, $V_{12}^- = \boldsymbol{F}(E_{12} - E_{21})$, $V_{ii}^- = \boldsymbol{F}^- E_{ii}$, $i = 1, 2$, where as usual $\boldsymbol{F}^-$ denotes the orthogonal complement of 1 in $\boldsymbol{F}$.

(b) We often write V_{12}^ε if we want to treat the cases V_{12}^+ and V_{12}^- simultaneously. We denote by x_{ij} the component of x in V_{ij}, $x_{12}^\pm$ is defined similarly.

(c) Peirce decompositions of the dual triple relative to the elements $(e = \lambda(e_1 + e_2), \hat{e} = \lambda(e_1 - e_2))$—see Remark 4.3.a—are considered in §5.5.

(d) The notation for the Peirce spaces is adapted from the analogous decomposition of Jordan triple systems. In the case $V = \mathrm{Mat}\,(2, r; \boldsymbol{F})$ it is known that V also carries the structure of a Jordan triple system. In both structures—isoparametric triple system and Jorden triple system—the notations for orthogonal tripotents and Peirce decompositions coincide.

By Theorem 2.2.b and Remark 4.3.a we know that $V = \boldsymbol{R}e \oplus V_3(e) \oplus V_1(e)$ for $e = \lambda(e_1 + e_2)$ and $V = \boldsymbol{R}\hat{e} \oplus V_3(\hat{e}) \oplus V_1(\hat{e})$ for $\hat{e} = \lambda(e_1 - e_2)$. The following lemma expresses these eigenspaces by the Peirce spaces of (e_1, e_2):

LEMMA 5.4. $V_3(e) = \boldsymbol{R}\hat{e} \oplus V_{12}^+$, $V_1(e) = V_{11}^- \oplus V_{10} \oplus V_{12}^- \oplus V_{22}^- \oplus V_{20}$,
$V_3(\hat{e}) = \boldsymbol{R}e \oplus V_{12}^-$, $V_1(\hat{e}) = V_{11}^- \oplus V_{10} \oplus V_{12}^+ \oplus V_{22}^- \oplus V_{20}$.

PROOF. For $x = \alpha e_1 + x_{11}^- + x_{10} + x_{12}^+ + x_{12}^- + \beta e_2 + x_{22}^- + x_{20}$ we compute $T(e)x = 3\alpha e_1 + x_{11}^- + x_{10} + 3x_{12}^+ + x_{12}^- + 3\beta e_2 + x_{22}^- + x_{20}$ which establishes the claim for the Peirce spaces relative to e. The assertion for $V_\kappa(\hat{e})$ follows analogously.

From Theorem 2.2.b we obtain

COROLLARY 5.5. *If (e_1, e_2) are orthogonal tripotents of the isoparametric triple system V, then*

$$\dim V_{12}^+ = \dim V_{12}^- = m_1 > 0 \text{ and } \dim (V_{11}^- \oplus V_{10}) = \dim (V_{22}^- \oplus V_{20}) = m_2 > 0\,.$$

5.2. We impose now more conditions on the triple system V which are satisfied in case V is isoparametric. This will enable us to establish multiplication rules between Peirce spaces of orthogonal tripotents.

LEMMA 5.6. *Let (e_1, e_2) be orthogonal tripotents of the symmetric triple system V and assume that each element of V_{11}, V_{10}, V_{22} and V_{20} is a scalar multiple of a minimal tripotent.*

(a) *For $i = 1, 2$ and $u \in V_{ii}$ with $\langle u, u \rangle = 1$ we have*

$$\dim V_\mu(u) = \dim V_\mu(e_i) , \qquad \mu = 0, 2 .$$

(b) *For $x_i \in V_{ii}$, $i = 1, 2$, and $x_{12} \in V_{12}$ the following formulas hold*

$$T(x_i)x_{12} = 2\langle x_i, x_i \rangle x_{12} \tag{5.1}$$

$$T(x_1, x_2)^2 x_{12} = \langle x_1, x_1 \rangle \langle x_2, x_2 \rangle x_{12} \tag{5.2}$$

$$T(x_1, x_2) \mid V_{12}^{\perp} = 0 . \tag{5.3}$$

PROOF. (a) Let $u \in V_{11}$, $\langle u, u \rangle = 1$. Then (u, e_2) are orthogonal, in particular $[T(u), T(e_2)] = 0$. Moreover, (2.6) for $c = e_2$ implies $V_0(e_2) = V_{11} \oplus V_{10} \subset \boldsymbol{R}u \oplus V_2(u)$. Therefore $V_2(u) = (V_{11} \ominus \boldsymbol{R}u) \oplus V_{10} \oplus V_2(e_2) \cap V_2(u)$, and it suffices to show that $\dim (V_2(e_2) \cap V_2(u)) = \dim V_2(e_2) \cap V_2(e_1)$.

We know $V_2(e_2) = (V_2(e_2) \cap V_2(u)) \oplus (V_2(e_2) \cap V_0(u))$ where $V_2(e) \cap V_2(u) = \{x \in V_2(e_2),\ T(u, e_2)^2 x = x\}$ by (2.7). For $x \in V_2(e_2)$ we get $x \in V_2(e_2) \cap V_0(u)$ iff $T(u, e_2)^2 x = 0$, which is equivalent to $T(u, e_2)x = 0$. Because $T(u, e_2) V_2(e_2) \subset V_2(e_2)$ we conclude that $H: V_{11} \to \operatorname{End} V_2(e_2)$, $H(v) = T(v, e_2) | V_2(e_2)$ induces a cubic space on $V_2(e_2)$. Therefore $\dim V_2(e_2) \cap V_2(u) = \operatorname{rank} H(u) = \operatorname{trace} H(u)^2$ is constant by Lemma 3.7.c.

(b) Because of (a) we can apply Lemma 2.7.d for every $x_i \in V_{ii}$, $\langle x_i, x_i \rangle = 1$ and get $V_2(x_1) \cap V_2(x_2) = V_{12}(e_1, e_2)$. In particular (5.1) follows. The formulas (5.2) and (5.3) are implied by Lemma 5.1.

It is convenient to denote the algebra products induced by e_1 and e_2 in the following manner:

$$x \circ y = \{x e_1 y\} , \quad x, y \in V , \qquad x * y = \{x e_2 y\} , \quad x, y \in V .$$

THEOREM 5.7. *Let (e, e_2) be orthogonal tripotents of the symmetric triple system V such that each element of the Peirce spaces V_{11}, V_{10}, V_{22} and V_{20} relative to (e_1, e_2) is a scalar multiple of a minimal tripotent. Then we have for all $x_{ij}, y_{ij} \in V_{ij}$*

$$x_{11}^- \circ y_{11}^- = 2\langle x_{11}^-, y_{11}^- \rangle e_1 \qquad x_{22}^- * y_{22}^- = 2\langle x_{22}^-, y_{22}^- \rangle e_2 \tag{5.4}$$

$$x_{11}^- \circ (V_{10} + V_{12} + V_{22} + V_{20}) = 0 \qquad x_{22}^- * (V_{20} + V_{12} + V_{11} + V_{10}) = 0 \tag{5.5}$$

$$x_{10} \circ y_{10} = 2\langle x_{10}, y_{10} \rangle e_1 \qquad x_{20} * y_{20} = 2\langle x_{20}, y_{20} \rangle e_2 \tag{5.6}$$

$$x_{10} \circ y_{12} \in V_{20} \qquad x_{20} * y_{12} \in V_{10} \tag{5.7}$$

$$x_{10} \circ y_{22} = 0 \qquad x_{20} * y_{11} = 0 \tag{5.8}$$

$$x_{10} \circ y_{20} \in V_{12} \qquad x_{10} * y_{20} \in V_{12} \tag{5.9}$$

$$x_{12}^\varepsilon \circ y_{12}^\varepsilon = \langle x_{12}^\varepsilon, y_{12}^\varepsilon \rangle (2e_1 + \varepsilon e_2) \qquad x_{12}^\varepsilon * y_{12}^\varepsilon = \langle x_{12}^\varepsilon, y_{12}^\varepsilon \rangle (\varepsilon e_1 + 2e_2) \tag{5.10}$$

(5.11) $x_{12}^+ \circ y_{12}^- \in V_{22}^- \oplus V_{20}$ $\quad x_{12}^+ * y_{12}^- \in V_{11}^- \oplus V_{10}$

(5.12) $x_{12}^\varepsilon \circ y_{22}^- \in V_{12}^{-\varepsilon}$ $\quad x_{12}^\varepsilon * y_{11}^- \in V_{12}^{-\varepsilon}$

(5.13) $x_{12}^\varepsilon \circ y_{20} \in V_{12}^{-\varepsilon} \oplus V_{10}$ $\quad x_{12}^\varepsilon * y_{10} \in V_{12}^{-\varepsilon} \oplus V_{20}$

(5.14) $(V_{22} + V_{20}) \circ (V_{22} + V_{20}) = 0$ $\quad (V_{11} + V_{10}) * (V_{11} + V_{10}) = 0$

(5.15) $e_2 \circ (V_{11} + V_{10} + V_{22} + V_{20}) = 0$ $\quad e_1 * (V_{11} + V_{10} + V_{22} + V_{20}) = 0 .$

PROOF. By symmetry we only have to prove the multiplication rules for the algebra "$\circ$". Obviously, (5.4) follows from (2.6), (5.5) from Lemma 2.7.a, (5.6) again from (2.6), (5.7) from (2.5) and (5.3), (5.8) also from (5.3) and (5.9) from (2.4), (5.5) and (5.6).

To prove (5.10) we recall $x_{12}^+ \in V_3(e) \cap V_1(\hat{e})$ from Lemma 5.4. Because e, $\hat{e}$ satisfy $\mathscr{M}(e; u, v) = 0 = \mathscr{M}(\hat{e}; u, v)$ for all $u, v \in V$ we can apply (2.10) to derive $\{x_{12}^+ e y_{12}^+\} = 3\langle x_{12}^+, y_{12}^+\rangle e$, and (2.12) to derive $\{x_{12}^+ \hat{e} y_{12}^+\} = \langle x_{12}^+, y_{12}^+\rangle \hat{e} + z$ for some $z \in V_3(\hat{e}) = \boldsymbol{R}e \oplus V_{12}^-$. Because $\langle \{x_{12}^+ \hat{e} y_{12}^+\}, e\rangle = \langle x_{12}^+, \{e\hat{e}y_{12}^+\}\rangle = 0$ we actually have $z \in V_{12}^-$. Since $e_1 = \lambda(e + \hat{e})$ we have proved $x_{12}^+ \circ y_{12}^+ = \langle x_{12}^+, y_{12}^+\rangle(3\lambda e + \lambda\hat{e}) + z = \langle x_{12}^+, y_{12}^+\rangle(2e_1 + e_2) + z$ for some $z \in V_{12}^-$. But $x_{12}^+ \circ y_{12}^+ \in \boldsymbol{R}e_1 + V_0(e_1)$ by (2.5). Therefore $z = 0$. Similarly, $x_{12}^+ \circ y_{12}^- = \langle x_{12}^-, y_{12}^-\rangle(2e_1 - e_2)$ follows.

Finally, (5.11) is implied by (2.5), (5.12) by (5.3) and (5.10), (5.13) by (2.4), (5.5) and (5.10), (5.14) is just (2.4) and (5.15) follows from Lemma 5.1.c.

REMARK. (a) In the case of the (homogeneous) isoparametric examples mentioned in §1.5 one gets sharper results only for the formulas (5.11) and (5.13). More precisely, for these we get

(5.11)′ $x_{12}^+ \circ y_{12}^- \in V_{22}^-$, $\quad x_{12}^+ * y_{12}^- \in V_{11}^-$,

(5.13)′ $x_{12}^\varepsilon \circ y_{20} \in V_{10}$, $\quad x_{12}^\varepsilon * y_{10} \in V_{20}$.

These examples are closely related to Jordan triple systems. We therefore say that an isoparametric triple satisfying (5.11)′ and (5.13)′ for each (resp. the) pair of orthogonal tripotents (e_1, e_2) is a triple with *Jordan composition* (relative to (e_1, e_2)). For short, V is a triple of *JC-type* (relative to (e_1, e_2)).

Another subclass of isoparametric triples are the triples of *algebra type*. By definition, an isoparametric triple is said to be of algebra type if there exist orthogonal tripotents e_1, e_2 such that $V_{10}(e_1, e_2) = 0$ and $V_{20}(e_1, e_2) = 0$. The triples of algebra type will be classified in a subsequent paper [3].

(b) If the symmetric triple system satisfies (ISO 3), then every element of $V_0(c)$, c a minimal tripotent, is a scalar multiple of a minimal tripotent, which follows from (2.6). In particular, the assumptions of Theorem 5.7 are satisfied in this case.

We prove some more identities involving the algebra $\circ$ and $*$.

LEMMA 5.8. *With the assumptions of Theorem* 5.7 *we have*

$$x_{10} \circ y_{20} = T(e_1, e_2)x_{10} * y_{20} , \tag{5.16}$$

$$\langle x_{10} \circ y_{20}, u_{10} \circ v_{20}\rangle = \langle x_{10} * y_{20}, u_{10} * v_{20}\rangle \tag{5.17}$$

$$x_{10} \circ (u_{10} \circ v_{20}) = [x_{10} * (u_{10} * v_{20})]_{20} \tag{5.18}$$

$$[y_{20} \circ (v_{20} \circ u_{10})]_{20} = y_{20} * (v_{20} * u_{10}) \tag{5.19}$$

$$\langle x_{12}^{\varepsilon} \circ y_{12}^{-\varepsilon}, x_{12}^{\varepsilon} \circ z_{12}^{-\varepsilon}\rangle = \langle x_{12}^{\varepsilon} * y_{12}^{-\varepsilon}, x_{12}^{\varepsilon} * z_{12}^{-\varepsilon}\rangle \tag{5.20}$$

$$\langle\{x_{12}^{+}x_{12}^{+}y_{12}^{-}\}, z_{12}^{-}\rangle = 3\langle x_{12}^{+}, x_{12}^{+}\rangle\langle y_{12}^{-}, z_{12}^{-}\rangle - 2\langle x_{12}^{+} * y_{12}^{-}, x_{12}^{+} * z_{12}^{-}\rangle . \tag{5.21}$$

PROOF. To prove (5.16) we put $c = e_2$ and have e, $x_{10} \in V_0(c)$, $y_{20} \in V_2(c)$. Using (2.7) we get $x_{10} \circ y_{20} = e_1 * (x_{10} * y_{20}) + x_{10} * (e_1 * y_{20})$, but $e_1 * y_{20} = 0$ by (5.15). This proves (5.16). We remark that (5.17) follows from (5.16) by the fact that $T(e_1, e_2)$ is orthogonal on V_{12}. By (5.17) we know $\langle x_{10} \circ (u_{10} \circ v_{20}), y_{20}\rangle = \langle x_{10} * (u_{10} * v_{20}), y_{20}\rangle$. Hence $[x_{10} \circ (u_{10} \circ v_{20})]_{20} = [x_{10} * (u_{10} * v_{20})]_{20}$. But (5.7) and (5.9) show that $x_{10} \circ (u_{10} \circ v_{20})$ already lies in V_{20}, whence (5.18). Interchanging "1" and "2" in (5.18) proves (5.19). To prove (5.20) and (5.21) we first recall $V_{12} = V_2(e_1) \cap V_2(e_2)$. Hence we may use (2.9) to expand $\{x_{12}^{+}, x_{12}^{+}, y_{12}^{-}\}$ for $c = e_1$ and also for $c = e_2$. With (5.10) and (5.11) we get $\{x_{12}^{+}, x_{12}^{+}, y_{12}^{-}\} - 2\langle x_{12}^{+}, x_{12}^{+}\rangle y_{12}^{-} = -2x_{12}^{+} \circ (x_{12}^{+} \circ y_{12}^{-})_{0(e)_1} - y_{12}^{-} \circ (x_{12}^{+} \circ x_{12}^{+})_{0(e_1)} + a_0(e_1) = -2x_{12}^{+} \circ (x_{12}^{+} \circ y_{12}^{-}) + \langle x_{12}^{+}, x_{12}^{+}\rangle y_{12}^{-} + a_0(e_1)$. Similarly we get $\{x_{12}^{+}, x_{12}^{+}, y_{12}^{-}\} - 2\langle x_{12}^{+}, x_{12}^{+}\rangle y_{12}^{-} = -2x_{12}^{+} * (x_{12}^{+} * y_{12}^{-}) + \langle x_{12}^{+}, x_{12}^{+}\rangle y_{12}^{-} + a_0(e_2)$. We now consider $\langle\{x_{12}^{+}x_{12}^{+}y_{12}^{-}\}, z_{12}^{-}\rangle$ and easily derive (5.20) and (5.21).

5.3. Under the assumptions used in § 5.2 we prove in this subsection that V_{10} vanishes if and only if V_{20} vanishes. This in particular implies that an isoparametric triple system is of algebra type relative to (e_1, e_2) if $V_{10}(e_1, e_2) = 0$ or $V_{20}(e_1, e_2) = 0$.

THEOREM 5.9. *Let (e_1, e_2) be orthogonal tripotents of the symmetric triple system V such that each element of the Peirce spaces V_{11}, V_{10}, V_{22} and V_{20} is a scalar multiple of a minimal tripotent. Let $(i, j) = (1, 2)$ or $(2, 1)$ and assume $e_{i0} \in V_{i0}$, $e_{i0} \neq 0$. Then $e_{i0} \circ V_{j0} \neq 0$ and $e_{i0} * V_{j0} \neq 0$.*

PROOF. It suffices to consider $(i, j) = (2, 1)$. By definition, $e_{20} \in V_2(e_2)$, $e_{20} \notin V_{22}^{-} = \{y \in V_2(e_2); y * V_0(e_2) = 0\}$. Hence there exists $x = x_{11} + x_{10} \in V_0(e_2)$ satisfying $x * e_{20} \neq 0$. But $x_{11}^{-} * e_{20} = 0$ by (5.8) and $x_{10} * e_{20} \neq 0$ follows. To prove $x_{10} \circ e_{20} \neq 0$ we apply (5.16) and use the fact that $T(e_1, e_2)$ is bijective on V_{12}.

COROLLARY 5.10. *Under the assumptions of Theorem* 5.9 *we have*

$$V_{10} \neq 0 \Leftrightarrow V_{10} \circ V_{20} \neq 0 \Leftrightarrow V_{10} * V_{20} \neq 0 \Leftrightarrow V_{20} \neq 0 .$$

PROOF. By (5.16) we have $V_{10} \circ V_{20} \neq 0$ iff $V_{10} * V_{20} \neq 0$ and from Theorem 5.11 we derive $V_{10} \neq 0 \Leftrightarrow V_{10} \circ V_{20} \neq 0$ and $V_{20} \neq 0 \Leftrightarrow V_{10} * V_{20} \neq 0$.

We are now in a position to identify the Peirce spaces relative to (x_{11}, x_{22}):

THEOREM 5.11. *Let (e_1, e_2) be orthogonal tripotents of the symmetric triple V such that each element of the Peirce spaces V_{11}, V_{10}, V_{22} and V_{20} is a scalar multiple of a minimal tripotent. Then the Peirce spaces relative to the orthogonal tripotents (x_1, x_2) where $x_i \in V_{ii}$, $\langle x_i, x_i\rangle = 1$, are*

$$V_{ii}(x_1, x_2) = V_{ii}(e_1, e_2)\ , \quad i = 1, 2\ , \qquad V_{i0}(x_1, x_2) = V_{i0}(e_1, e_2)\ , \quad i = 1, 2\ ,$$
$$V_{12}(x_1, x_2) = V_{12}(e_1, e_2)\ .$$

PROOF. The Peirce spaces of (e_1, e_2) are denoted by V_{ij}. Because of Lemma 5.6.a we can apply Lemma 2.7.d and conclude $V_0(x_i) = V_0(e_i)$, $V_2(x_i) = (V_{ii} \ominus \boldsymbol{R}x_i) \oplus V_{i0} \oplus V_{12}$. This in particular implies $V_{12}(x_1, x_2) = V_{12}$ and $V_{12}^0(x_i) \subset V_{ii} \oplus V_{i0}$. Therefore it suffices to show $V_{ii}(x_1, x_2) = V_{ii}$, i.e., $V_2^0(x_i) = V_{ii} \ominus \boldsymbol{R}(x_i)$. Without loss of generality we consider $i = 1$. Let $u = u_{11} + u_{10} \in V_2^0(x_i)$. By definition of $V_2^0(x_i)$ we have $0 = \{ux_1v_{20}\}$ for every $v_{20} \in V_{20}$. But $\{ux_1v_{20}\} = u * (x_1 * v_{20}) + x_1 * (u * v_{20})$ by (2.7) and the first summand vanishes by (5.3), moreover $u_1 * v_{20} = 0$. Thus $0 = x_1 * (u_{10} * v_{20})$ where $u_{10} * v_{20} \in V_{12}$ by (5.9). Again (5.3) implies $u_{10} * v_{20} = 0$. Since this is valid for all $v_{20} \in V_{20}$ we conclude from Theorem 5.9 that $u_{10} = 0$, whence $u = u_{11} \in V_2^0(x_1) \cap V_{11} \subset V_{11} \ominus \boldsymbol{R}x_1$. On the other hand we have for every $y_1 \in V_{11}$ and $v_{22} \in V_{22}$ that $\{y_1x_1v_{22}\} = 0$ by (5.3) and for every $v_{20} \in V_{20}$ that $\{y_1x_1v_{20}\} = y_1 * (x_1 * v_{20}) + x_1 * (y_1 * v_{20}) = 0$, again by (5.3). This implies $V_{ii} \ominus \boldsymbol{R}x_i \subset V_2^0(x_i)$ and therefore we have equality.

REMARK 5.12. We point out that by Theorem 5.11 we now have at our disposal the multiplication rules of Theorem 5.7 where e_i is replaced by suitable x_i. This will be of great importance in the proof of Theorem 5.20.

As a consequence of Theorem 5.11 we have the following characterization of $V_2^0(c)$ in the case of isoparametric triple systems:

THEOREM 5.13. *Let V be an isoparametric triple system and c, u minimal tripotents of V. Then*

$$u \in V_2^K(c) \Leftrightarrow V_2^K(c) = V_2^K(u) \Leftrightarrow V_0(c) = V_0(u) \Leftrightarrow c \in V_2^K(u) .$$

PROOF. By symmetry it suffices to show $u \in V_2^K(c) \Leftrightarrow V_2^K(c) = V_2^K(u) \Leftrightarrow V_0(c) = V_0(u)$. Here $V_2^K(c) = V_2^K(u) \Rightarrow u \in V_2^K(c)$ is trivial, and $u \in V_2^K(c) \Rightarrow V_0(u) = V_0(c)$ follows from Corollary 2.8.b. So it remains to show

(a) $u \in V_2^K(c) \Rightarrow V_2^K(c) = V_2^K(u)$ and (b) $V_0(u) = V_0(c) \Rightarrow u \in V_2^K(c)$.

We put $e_1 := c$ and choose a tripotent which is orthogonal to e_1. This is possible because $\dim V_0(e_1) > 0$. Now (a) follows from Theorem 5.11.

We assume $V_0(u) = V_0(e_1)$. Then $u \in V_0(e_2) = V_{11} \oplus V_{10}$ by Lemma 4.5.a where V_{ij} denotes the Peirce spaces relative to (e_1, e_2). Moreover, $0 = T(u)v_{20} = 2u * (u * v_{20})$ and thus $u * v_{20} = 0$ since $T(u, e_2)$ is symmetric. But $u_{11} * v_{20} = 0$ by (5.3) and so $u_{10} * v_{20} = 0$. Because this holds for all $v_{20} \in V_{20}$ we conclude $u_{10} = 0$ from Theorem 5.9, i.e., $u = u_{11} \in V_{11}$.

5.4 Although the results of this subsection are true in a slightly more general situation, we restrict ourselves to simplify notation to the case of a symmetric triple system V satisfying (ISO 3). Let (e_1, e_2) be orthogonal tripotents of V. Then, with $\lambda = 2^{-1/2}$ the elements $e = \lambda(e_1 + e_2)$, $\hat{e} := \lambda(e_1 - e_2)$ are maximal tripotents of V. We have $e \in V_3(\hat{e})$ and $\hat{e} \in V_3(e)$. From Theorem 2.2 we derive that $(e, \hat{e})$ is a pair of minimal orthogonal tripotents in the dual system V' defined in Lemma 1.2. In this subsection we discuss the connections between the Peirce decomposition of V' relative to $(e, \hat{e})$ and the Peirce decomposition of V relative to (e_1, e_2). It will be convenient to use the algebras

$$x \square y := \{xey\} , \quad x, y \in V , \qquad v \,\hat{\square}\, y := \{x\hat{e}y\} , \quad x, y \in V .$$

The Peirce spaces relative to (e_1, e_2) will be denoted by V_{ij} as usual. The Peirce spaces relative to $(e, \hat{e})$ will be denoted by V'_{ij}. Where necessary we write $V_{ij}(e_1, e_2)$ and $V'_{ij}(e, \hat{e})$ or $(V')_{ij}(e, \hat{e})$. We also use $V'_i(e)$ and $V'_i(\hat{e})$.

LEMMA 5.14. *Assume the symmetric triple system V satisfies* (ISO 3), *i.e., $\mathscr{M}(x) = 0$ for all $x \in V$. Then*

(a) $V'_0(e) = V_3(e) = \boldsymbol{R}\hat{e} \oplus V_{12}^{+}$,

(b) $V'_0(\hat{e}) = V_3(\hat{e}) = \boldsymbol{R}e \oplus V_{12}^{-}$,

(c) $V'_{12} = V_1(e) \cap V_1(\hat{e}) = V_{11}^{-} \oplus V_{10} \oplus V_{22}^{-} \oplus V_{20}$,

(d) $(V')_{12}^{+} = V_{22}^{-} \oplus V_{20}$,

(e) $(V')_{12}^{-} = V_{11}^{-} \oplus V_{10}$.

PROOF. (a), (b) and (c) follow immediately from Theorem 2.2 and Lemma 5.4. For (d) and (e) we compute: $\{e\hat{e}x'_{12}\}' = -\{e, \hat{e}, a_{11}^{-} + a_{10} + a_{22}^{-} + a_{20}\} = -(a_{11}^{-} + a_{10}) + a_{22}^{-} + a_{20}$ where we put $x'_{12} = a_{11}^{-} + a_{10} + a_{22}^{-} + a_{20}$ according to (c). This implies (d) and (e).

LEMMA 5.15. *Assume V satisfies* (ISO 3). *Then*

(a) $(V')_2^0(e) = \{x \in V_{12}^-; x \square V_{12}^+ = 0\}$.

(b) *For $y_{12}^- \in V_{12}^-$ the following conditions are equivalent*:

(1) $y_{12}^- \square V_{12}^+ = 0$

(2) $y_{12}^- \circ V_{12}^+ = 0$

(3) $y_{12}^- * V_{12}^+ = 0$

(4) $y_{12}^- * V_{11}^- = 0 = y_{12}^- \circ V_{12}^-$ *and* $y_{12}^- * V_{10} \subset V_{20}$, $y_{12}^- \circ V_{20} \subset V_{10}$.

PROOF. (a) By definition, $(V')_2^0(e) = \{x \in V_2'(e); \{x, e, V_0'(e)\}' = 0\} = \{x \in V_1(e); x \square V_3(e) = 0\}$ where we have used Theorem 2.2. We know $V_1(e) = V_{11}^- \oplus V_{12}^- \oplus V_{22}^- \oplus V_{10} \oplus V_{20}$, $\langle (V')_2^0, V_{12}' \rangle = 0$ and $V_3(e) = \boldsymbol{R}\hat{e} \oplus V_{12}^+$. Hence $(V')_2^0 = \{x \in V_{12}^-; x \square V_3(e) = 0\}$. But $x_{12}^- \square \hat{e} = 0$ is always satisfied, hence the assertion.

(b) Since $y_{12}^- \square x_{12}^+ = \lambda(y_{12}^- \circ x_{12}^+ + y_{12}^- * x_{12}^+)$ and $y_{12}^- \circ x_{12}^+ \in V_{22}^- \oplus V_{20}$, $y_{12}^- * x_{12}^+ \in V_{11}^- + V_{10}$ by (5.11) we see that (1) implies (2) and also (3). Now (5.20) implies that (1), (2) and (3) are equivalent.

By (5.11) we always have $y_{12}^- \square V_{12}^+ \subset V_{11}^- + V_{10} + V_{22}^- + V_{20}$. Therefore $y_{12}^- \square V_{12}^+ = 0$ iff $\langle V_{12}^+, y_{12}^- \square (V_{11}^- + V_{10} + V_{22}^- + V_{20})\rangle = 0$. The multiplication rules (5.5), (5.7), (5.12) and (5.13) show that this equation is equivalent to (4).

Interchanging e and $\hat{e}$ we get

LEMMA 5.16. *Let V satisfy* (ISO 3).

(a) $(V')_2^0(\hat{e}) = \{x \in V_{12}^+; x \hat{\square} V_{12}^- = 0\}$.

(b) *For $y_{12}^+ \in V_{12}^+$ the following conditions are equivalent*:

(1) $y_{12}^+ \hat{\square} V_{12}^- = 0$,

(2) $y_{12}^+ \circ V_{12}^- = 0$,

(3) $y_{12}^+ * V_{12}^- = 0$,

(4) $y_{12}^+ * V_{11}^- = 0 = y_{12}^+ \circ V_{22}^-$ *and* $y_{12}^+ * V_{10} \subset V_{20}$, $y_{12}^+ \circ V_{20} \subset V_{10}$.

COROLLARY 5.17. *Let V satisfy* (ISO 3). *If $V_{11}^- \neq 0$ or $V_{22}^- \neq 0$ then*

$$(V')_{11}^- = 0 \quad and \quad (V')_{22}^- = 0\,.$$

PROOF. Assume $V_{11}^- \neq 0$. Choose $x_{11}^- \neq 0$, $x_{11}^- \in V_{11}^-$. Then $T(x_{11}^-, e_2)$ is injective on V_{12}. Therefore $x_{11}^- * y_{12}^- = 0$ for $y_{12}^- \in (V')_{11}^-$ and $x_{11}^- * y_{12}^+ = 0$ for $y_{12}^+ \in (V')_{22}^-$ imply $y_{12}^- = 0 = y_{12}^+$ and the assertion follows. If $V_{22}^- \neq 0$ a similar argument establishes the claim.

COROLLARY 5.18. *Let V satisfy* (ISO 3).

(a) $(V')_{11}^- = \{x \in V_{12}^-; x \circ V_{12}^+ = 0\}$.

(b) $V_{10}' = V_{12}^- \ominus (V')_{11}^-$.

(c) $(V')_{22}^- = \{x \in V_{12}^+; x \circ V_{12}^- = 0\}$.

(d) $V_{20}' = V_{12}^+ \ominus (V')_{22}^-$.

REMARK. One can also dualize the results of §5.2 and §5.3, but since one has as yet no precise description of V'_{i0} this dualization produces few new results. We mention only the following immediate consequence of (5.16):

LEMMA 5.19. *Let V satisfy* (ISO 3). *Then*

$$V_{10} \square V_{20} \subset V_{12}^{+} \quad \textit{and} \quad V_{10} \hat{\square} V_{20} \subset V_{12}^{-} .$$

5.5. In this final section we characterize isoparametric triple systems among the symmetric triple systems by Peirce decompositions:

THEOREM 5.20. *Let (e_1, e_2) be orthogonal tripotents of the symmetric triple system V. The Peirce spaces of (e_1, e_2) are denoted by V_{ij}. Then the following conditions are equivalent*:

(a) *V is isoparametric.*

(b) (1) *Every element of V_{11}, V_{10}, V_{22} and V_{20} is a scalar multiple of a minimal tripotent.*

(2) *Every element of V_{12}^{+} and V_{12}^{-} is a scalar multiple of a maximal tripotent.*

(3) *For all $x_{ij} \in V_{ij}$ the following identity holds*:

$$\langle\{x_{11}x_{22}x_{12}^{+}\}, \{x_{10}x_{20}x_{12}^{-}\}\rangle + \langle\{x_{11}x_{22}x_{12}^{-}\}, \{x_{10}x_{20}x_{20}^{+}\}\rangle$$
$$+ \langle\{x_{11}x_{20}x_{12}^{+}\}, \{x_{10}x_{22}x_{12}^{-}\}\rangle + \langle\{x_{11}x_{20}x_{12}^{-}\}, \{x_{10}x_{22}x_{12}^{+}\}\rangle = 0 .$$

(4) *There exist positive integers m_1, m_2 such that*

$$\dim (V_{11}^{-} \oplus V_{10}) = m_2 = \dim (V_{22}^{-} \oplus V_{20}) \quad \textit{and} \quad \dim V_{12} = 2m_1 .$$

PROOF. By definition, V is isoparametric iff (ISO 3) and (ISO 4) hold.

First we transform (ISO 3), i.e., $\mathscr{M}(x)=0$ for all $x \in V$. We introduce the six-linear form $h\colon V^6 \to \boldsymbol{R}$, $h(x, y, z, u, v, w) = \langle \mathscr{M}(x, y, z, u, v), w\rangle$ which is totally symmetric by Lemma 1.5.a. Clearly, $\mathscr{M} \equiv 0$ iff $h \equiv 0$. Also, $h = 0$ iff $h(x, y, z, u, v, w) = 0$, where (x, y, z, u, v, w) has at least degree two for one Peirce space, and $h(x_{11}, x_{10}, x_{12}^{+}, x_{12}^{-}, x_{22}, x_{20}) = 0$ for all $x_{ij} \in V_{ij}$. Using Theorem 2.3 and Theorem 2.5 it is easy to see that the first condition is equivalent to (b1) and (b2). Moreover, by (1.15), the second condition holds if (1.10) for $x = x_{11}$, $u = x_{12}^{-}$, $v = x_{22}$, $w = x_{20}$ and applied to x_{10} vanishes in the scalar product with x_{12}^{+}, i.e.,

$$\begin{aligned} 0 = {} & \langle\{x_{20}x_{12}^{-}x_{12}^{+}\}, \{x_{11}x_{22}x_{10}\}\rangle + \langle\{x_{11}x_{12}^{-}x_{12}^{+}\}, \{x_{20}x_{22}x_{10}\}\rangle + \langle\{x_{11}x_{12}^{-}x_{10}\}, \{x_{20}x_{22}x_{12}^{+}\}\rangle \\ & + \langle\{x_{20}x_{12}^{-}x_{10}\}, \{x_{11}x_{22}x_{12}^{-}\}\rangle + \langle\{x_{12}^{-}x_{22}x_{12}^{+}\}, \{x_{11}x_{20}x_{10}\}\rangle + \langle\{x_{11}x_{20}x_{12}^{+}\}, \{x_{12}x_{22}x_{10}\}\rangle \\ & + \langle\{x_{12}^{-}x_{20}x_{11}\}, \{x_{22}x_{10}x_{12}^{-}\}\rangle + \langle\{x_{11}x_{12}^{-}x_{22}\}, \{x_{20}x_{10}x_{12}^{+}\}\rangle + \langle\{x_{20}x_{12}^{-}x_{22}\}, \{x_{11}x_{10}x_{12}^{+}\}\rangle \\ & + \langle\{x_{11}x_{20}x_{22}\}, \{x_{12}^{-}x_{10}x_{12}^{+}\}\rangle =: R . \end{aligned}$$

Because of (b1) and (b2) we can apply Lemma 5.6. In particular (5.3) implies $\{x_{11}x_{22}x_{i0}\} = 0$, $i = 1, 2$, whence the first and last term of R vanish. Moreover, by Theorem 5.11 we can use all the multiplication rules derived in Theorem 5.7 for (x_1, x_2) since without restriction $\langle x_{ii}, x_{ii}\rangle = 1$, $i = 1, 2$. We thus get $\langle\{x_{11}x_{12}^{-}x_{12}^{+}\}, V_{12}\rangle = 0$ and $\{x_{20}x_{22}x_{10}\} \in V_{12}$, which forces the second term to be zero. Similarly the fifth term vanishes. Moreover $\{x_{11}x_{12}x_{10}\} \in V_{20}$, $\{x_{20}x_{22}x_{12}^{-}\} \in V_{10}$, whence the third and ninth term is zero. What remains from R is just (b3).

It remains to prove that under the validity of (ISO 3) the condition (b4) is equivalent to (ISO 4). We first remark that (ISO 3) in particular implies that every $f \in V_0(e_1)$ with $\langle f, f\rangle = 1$ is a minimal tripotent and thus, by Lemma 3.4.a,

$$(*) \qquad T(v_0, e_1)^3 v_2 = \langle v_0, v_0\rangle T(v_0, e_1)v_2$$

for all $v_0 \in V_{22} \oplus V_{20}$ and $v_2 \in V_{11}^{-} \oplus V_{10} \oplus V_{12}$. Hence we can apply Lemma 3.10 for $c = e_1$. Putting $f_0 = e_2$ it remains then to show that trace $T(x_0, x_2) = 0$ for $x_0 \in V_{22} \oplus V_{20}$, $x_2 \in V_2(e_1)$. Without restriction we assume $\langle x_0, x_0\rangle = 1$. Then x_0 is a minimal tripotent and, by Lemma 3.9, it follows that trace $T(x_0, x_2) = 0$ as soon as we know that $\dim V_0(x_0) \geqq 2$. The same argument as used in the proof of Lemma 5.6.a shows $\dim V_0(x_0) = \dim V_0(e_2)$, which is at least two by (b4).

We point out that Theorem 5.20 is a very convenient tool for the investigation of isoparametric triple systems. It replaces a lengthy identity by the shorter Peirce multiplication rules. Moreover, it shows that (ISO 4) is actually only a condition on the dimensions of three subspaces.

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