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Luis Alvarez, Rafael Sendra

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Instituto Universitario de Ciencias y Tecnologías Cibernéticas
Universidad de Las Palmas de Gran Canaria
Campus de Tafira
35017 Las Palmas, España
http://www.iuctc.ulpgc.es

An Algebraic Approach to Lens Distortion by Line Rectification

Luis Alvarez* and J. Rafael Sendra†

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Abstract

A very important property of the usual pinhole model for camera projection is that $3D$ lines in the scene are projected in $2D$ lines. Unfortunately, wide-angle lenses (specially low-cost lenses) may introduce a strong barrel distortion which makes the usual pinhole model fail. Lens distortion models try to correct such distortion. In this paper, we propose an algebraic approach to the estimation of the lens distortion parameters based on the rectification of lines in the image. Using the proposed method, the lens distortion parameters are obtained by minimizing a 4 total-degree polynomial in several variables. We perform numerical experiments using a lens distortion calibration pattern to show the performance of the proposed method.

1 Introduction

Typically, wide angle lenses tend to suffer from barrel distortion and tele lenses from pincushion distortion. Both effects tend to be stronger at the extreme ends of zoom lenses, especially on low-cost compact cameras, web-cam, etc.

Lens distortion correction is an important issue in camera calibration where the pinhole model is used (see for instance [8], [9] or [10]) The basic standard model for barrel and pincushion distortion compensation (see for instance [1] , [7] or [5]) is a radial distortion model given by the following expression:

$$\begin{pmatrix} \hat{x} - x_c \\ \hat{y} - y_c \end{pmatrix} = L(r) \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} \tag{1}$$

where (x, y) are the original point coordinates (distorted), $(\hat{x}, \hat{y})$ are the corrected (undistorted) point coordinates, (x_c, y_c) is the center of the camera distortion model (usually the center of the image), $r = \sqrt{(x - x_c)^2 + (y - y_c)^2}$ and $L(r)$ is the function which defines the shape of the distortion model. Usually, $L(r)$ is approximated by a Taylor expansion, that is

$$L(r) = k_0 + k_1 r + k_2 r^2 + k_3 r^3 + ...$$

*Departamento de Informática y Sistemas. Universidad de Las Palmas de Gran Canaria. Campus de Tafira, 35017 Las Palmas.

†Departamento de Matemáticas. Universidad Alcalá. 28871 Alcalá de Henares. Madrid

where the set $\{k_i\}_{i=0,.N_k}$ are the distortion parameters. The complexity of the model is given by the number of terms of the Taylor expansion we use to approximate $L(r)$.

In this paper, we use the general approach to determine $L(r)$ by imposing the requirement that the projection of $3D$ lines in the image has to be $2D$ straight lines. This approach has been used in [4], where authors use as measure of distortion a least square approximation of edges that should be a projection of $3D$ lines, they take for the distortion error the sum of squares of the distances from the point to the line. In this paper we propose a new distortion error model inspired in the residual variance obtained when edges are approached by using an standard linear regression model. The main advantage of our formulation is that it yields to a general 4 degree polynomial in the distortion parameters k_i, that can be minimized using powerful techniques of computer algebra. In particular, we directly obtain solutions of the minimization problem without any kind of initialization of the distortion parameters, which is one of the main drawback in the usual bundle adjustment schemes because of the existence of a lot of local minima where the solution can be trapped. In fact, the solution provided by our method could be used as an initialization for the bundle adjustment schemes.

The paper is organized as follows: In section 2 we introduce the measure of the distortion error we propose based on a linear regression analysis. In section 3 we present the algebraic analysis of the proposed measure of the distortion error. In section 4 we present the numerical aspects of the algorithm we have implemented to estimate the distortion parameters. In section 5 we present the numerical experiments we have performed and finally, in section 6 we present some conclusions.

2 Measure of the distortion error.

Let $\{(x_i, y_i)\}_{i=1,..,N}$ be the projection of a set of $3D$ aligned points in the $2D$ image, $\{(\hat{x}_i, \hat{y}_i)\}_{i=1,..,N}$ the corrected (undistorted) points using the distortion model (1) and $\mathbf{k} = (k_0, k_1, ..., k_{N_k})^T$ the distortion parameters. A linear regression analysis, to study the relation between variables $\hat{x}_i$ and $\hat{y}_i$, yields to the least square minimization problem

$$R(m,n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{y}_i - m\hat{x}_i - n)^2$$

It is well-known that this minimization problem attaints its minimum at

$$m = \frac{\hat{S}_{xy}}{\hat{S}_{xx}} \qquad n = \overline{\hat{y}_i} - \frac{\hat{S}_{xy}}{\hat{S}_{xx}}\overline{\hat{x}_i}$$

where $\overline{\hat{y}_i}$ and $\overline{\hat{x}_i}$ are the average of the respective variables and

$$\hat{S}_{xx} = \frac{1}{N}\sum_{i=1}^{N}(\hat{x}_i - \overline{\hat{x}_i})^2 \qquad \hat{S}_{yy} = \frac{1}{N}\sum_{i=1}^{N}(\hat{y}_i - \overline{\hat{y}_i})^2 \qquad \hat{S}_{xy} = \frac{1}{N}\sum_{i=1}^{N}(\hat{y}_i - \overline{\hat{y}_i})(\hat{x}_i - \overline{\hat{x}_i}).$$

Moreover, the residual variance of the variable $z_i = \hat{y}_i - m\hat{x}_i - n$ is given by

$$V_r = \frac{\hat{S}_{xx}\hat{S}_{yy} - \hat{S}_{xy}^2}{\hat{S}_{xx}}.$$

On the other hand, if we change the role of x and y in the regression analysis the associated residual variance V_r changes by replacing $\hat{S}_{xx}$ by $\hat{S}_{yy}$ in the denominator of the above expression. Therefore the minimal residual variance we obtain by using linear regression is given by

$$V_{min} = \frac{\hat{S}_{xx}\hat{S}_{yy} - \hat{S}_{xy}^2}{\max\{\hat{S}_{xx}, \hat{S}_{yy}\}} \tag{2}$$

In particular we obtain that $\hat{S}_{xx}\hat{S}_{yy} - \hat{S}_{xy}^2 \geq 0$, and the points $\{(\hat{x}_i, \hat{y}_i)\}_{i=1,..,N}$ lie in a straight line if and only if $\hat{S}_{xx}\hat{S}_{yy} - \hat{S}_{xy}^2 = 0$. On the other, the denominator of the above expression, $\max\{\hat{S}_{xx}, \hat{S}_{yy}\}$, does not usually changes a lot between the distorted and undistorted set of points. Based on this analysis of the residual variance V_r, we propose to use as measure of the distortion error the value

$$\hat{E}(\mathbf{k}) = \hat{S}_{xx}\hat{S}_{yy} - \hat{S}_{xy}^2.$$

Using the distortion model (1) we obtain:

$$\begin{aligned}
\hat{S}_{xx} &= \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{j=0}^{N_k} k_j \left(x_i\,(r_i)^j - \overline{x_i\,(r_i)^j}\right)\right)^2 = \mathbf{k}^T A\mathbf{k} \\
\hat{S}_{yy} &= \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{j=0}^{N_k} k_j \left(y_i\,(r_i)^j - \overline{y_i\,(r_i)^j}\right)\right)^2 = \mathbf{k}^T B\mathbf{k} \\
\hat{S}_{xy} &= \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{j=0}^{N_k} k_j \left(x_i\,(r_i)^j - \overline{x_i\,(r_i)^j}\right)\right)\left(\sum_{j=0}^{N_k} k_j \left(y_i\,(r_i)^j - \overline{y_i\,(r_i)^j}\right)\right) = \mathbf{k}^T C\mathbf{k}
\end{aligned}$$

where r_i=$\sqrt{(x_i - x_c)^2 + (y_i - y_c)^2}$, and A, B, C are $(N_k + 1) \times (N_k + 1)$ matrix given by

$$\begin{aligned}
A_{m,n} &= \frac{1}{N}\sum_{i=1}^{N}((r_i)^m x_i - \overline{(r_i)^m x_i})((r_i)^n x_i - \overline{(r_i)^n x_i}) \\
B_{m,n} &= \frac{1}{N}\sum_{i=1}^{N}((r_i)^m y_i - \overline{(r_i)^m y_i})((r_i)^n y_i - \overline{(r_i)^n y_i}) \\
C_{m,n} &= \frac{1}{N}\sum_{i=1}^{N}((r_i)^m x_i - \overline{(r_i)^m x_i})((r_i)^n y_i - \overline{(r_i)^n y_i}).
\end{aligned}$$

Therefore, the distortion error measure $\hat{E}(\mathbf{k})$ can be expressed as

$$\hat{E}(\mathbf{k}) = \mathbf{k}^T A\mathbf{k}\mathbf{k}^{\mathbf{T}}\mathbf{B}\mathbf{k} - \mathbf{k}^T C\mathbf{k}\mathbf{k}^T C\mathbf{k} \tag{3}$$

which is a 4 degree polynomial in the variables $\mathbf{k}$. In the general case, where we use several edge segments to fit the distortion parameters, we simply add the above expression for all the edge segments.

Of course, the global minimum of $\hat{E}(\mathbf{k})$ corresponds to the trivial solution $\mathbf{k} \equiv \mathbf{0}$. To avoid this problem, usually k_0 is fitted to one ($k_0 = 1$). As it is explained in section 4, in this paper we use another approach, we fit k_0 by minimizing the sum of the square distance between the distorted and undistorted points

3 Algebraic analysis of the distortion error measure.

In this section, we see how to approach the problem by means of computer algebra techniques. For simplicity in the exposition, we present the results for polynomials with real coefficients, but it must be said that they are valid over more general polynomials rings; for further details on this topic we refer the reader to [6] or [2].

As mentioned in section 2, one needs to minimize the distortion error measure function $\hat{E}(\mathbf{k})$, which is real polynomial in the variables $\mathbf{k}$. Minimizing a polynomial in several variables can be reduced to compute the solutions of an algebraic system of equations, namely the one generated by its gradient. In our case:

$$\mathcal{S} := \left\{ \frac{\partial \hat{E}(\mathbf{k})}{\partial k_i} = 0 \right\}_{i=0,\ldots,N_k}$$

The case where only one variable is considered, say k_p, is easy and it just requires to approximate the real roots on the univariate polynomial

$$\frac{\partial \hat{E}(k_p)}{\partial k_p}.$$

However, when more than one variable appear the problem is not so trivial. In order to approach this new situation, one can apply computer algebra techniques to prepare symbolically the algebraic system $\mathcal{S}$ before numerical methods are executed. In addition, even though computer algebra machinery is applied, the problem can be more handle depending on the number of variables. In fact, the case of two variable can be treated by means of symbolic linear algebra techniques while the case of more than two variable requires abstract algebra techniques. In both cases, the underlining theory comes from algebraic geometry and commutative algebra. To be more precise, we first describe in detail how to approach the problem when two variable are considered, and afterward we give a brief description on how to proceed in the general case.

So, let us assume that we are working with two variables, say k_p, k_q. Observe that this is the case when working with two distortion parameters, and that the system $\mathcal{S}$ turns to be

$$\mathcal{S} := \left\{ \frac{\partial \hat{E}(k_p, k_q)}{\partial k_p} = 0, \ \frac{\partial \hat{E}(k_p, k_q)}{\partial k_q} = 0 \right\}.$$

In order to compute the solutions of $\mathcal{S}$ we apply the so called resultant-based method. Let us describe this method. For this purpose, let $G_1(k_p, k_q)$ and $G_2(k_p, k_q)$ be two bivariate polynomials with real coefficients. Choosing one variable, say k_q, as a main variable, we can write G_1 and G_2 as

$$G_1(k_p, k_q) = a_n(k_p)k_q^n + \cdots + a_1(k_p)k_q + a_0(k_p),$$

$$G_2(k_p, k_q) = b_m(k_p)k_q^m + \cdots + b_1(k_p)k_q + b_0(k_p),$$

where $a_i(k_p)$ and $b_i(k_p)$ are univariate polynomials with real coefficients, and $a_n(k_p), b_m(k_p)$ are not identically zero, with $n > 0$ and $m > 0$. In this situation, the resultant of G_1 and G_2 with respect to the variable k_q (we denote it by $\mathrm{Res}_{k_q}(G_1, G_2)$) is defined as the determinant of the $(n+m)\times(n+m)$ matrix

$$\left(\begin{array}{ccccccc} a_n(k_p) & a_{n-1}(k_p) & \cdots & a_0(k_p) & 0 & \cdots & 0 \\ 0 & a_n(k_p) & a_{n-1}(k_p) & \cdots & a_0(k_p) & \cdots & 0 \\ \vdots & & \ddots & & & \ddots & \vdots \\ 0 & 0 & \cdots & a_n(k_p) & a_{n-1}(k_p) & \cdots & a_0(k_p) \\ b_m(k_p) & b_{m-1}(k_p) & \cdots & b_0(k_p) & 0 & \cdots & 0 \\ 0 & b_m(k_p) & b_{m-1}(k_p) & \cdots & b_0(k_p) & \cdots & 0 \\ \vdots & & \ddots & & & \ddots & \vdots \\ 0 & 0 & \cdots & b_m(k_p) & b_{m-1}(k_p) & \cdots & b_0(k_p) \end{array}\begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\ \vdots\\a\end{array}}\right\} m \\ \left.\vphantom{\begin{array}{c}a\\a\\ \vdots\\a\end{array}}\right\} n \end{array}\right).$$

Observe that $\mathrm{Res}_{k_q}(G_1, G_2)$ is a real univariate polynomial in the variable k_p. Therefore, the variable k_q has been eliminated. For our purposes, the main applicable properties on resultants are the following.

Theorem 1 *Let $G_1(k_p, k_q), G_2(k_p, k_q)$ as above, and let $G(k_p) = \mathrm{Res}_{k_q}(G_1, G_2)$. Then, it holds that*

1. *$G(k_p)$ is identically zero if and only if G_1 and G_2 have a common non–constant factor.*
2. *If $(\lambda, \mu) \in \mathbb{C}^2$ is a common root G_1 and G_2 then $G(\lambda) = 0$.*
3. *If $G(\lambda) = 0$ then one of the following statements holds*
   - *3.1. $a_n(\lambda) = b_m(\lambda) = 0$,*
   - *3.2. $\exists \mu \in \mathbb{C}$ such that (λ, μ) is a common root of G_1 and G_2.*

Proof: see Theorem 4.3.3, pp. 98, in [6]. □

The geometrical meaning of Theorem 1 is as follows. Let G_1, G_2 and G be as above. Then we can see G_1 and G_2 as curves in the k_pk_q-coordinate plane $\mathbb{R}^2$. In this situation, the real roots of G are the k_p-coordinates of the real intersection points of the two curves (see Figure 1).

In order to apply Theorem 1, first, note that in the construction of $\mathrm{Res}_{k_q}(G_1, G_2)$, we have required that $\deg_{k_q}(G_1) > 0$ and $\deg_{k_q}(G_2) > 0$. Let us see that this assumption is not a loos of generality for our purposes. Indeed, if $\deg_{k_q}(G_1) = 0$ (similarly if $\deg_{k_q}(G_2) = 0$), then G_1 only depends on k_p. Then, if $\deg_{k_q}(G_2) = 0$, then G_2 is also univariate and the real solutions of $\{G_1(k_p) = G_2(k_p) = 0\}$ are the real roots of the greatest common divisor of both polynomials. On the other hand, if $\deg_{k_q}(G_2) > 0$, for each real root α of the univariate polynomial $G_1(k_p)$, one has to determine the real roots of the univariate polynomial $G_2(\alpha, k_q)$. That is, if $\deg_{k_q}(G_1) = 0, \deg_{k_q}(G_2) > 0$, the real solutions of the system $\{G_1(k_p) = G_2(k_p, k_q) = 0\}$ are

$$\{(\alpha, \beta_\alpha) \in \mathbb{R}^2 \,|\, G_1(\alpha) = 0, G_2(\alpha, \beta_\alpha) = 0\}.$$

Moreover, if the following conditions are satisfied:

(i) the conditions on the degree are fulfilled (i.e. $\deg_{k_q}(G_1) > 0$ and $\deg_{k_q}(G_2) > 0$),

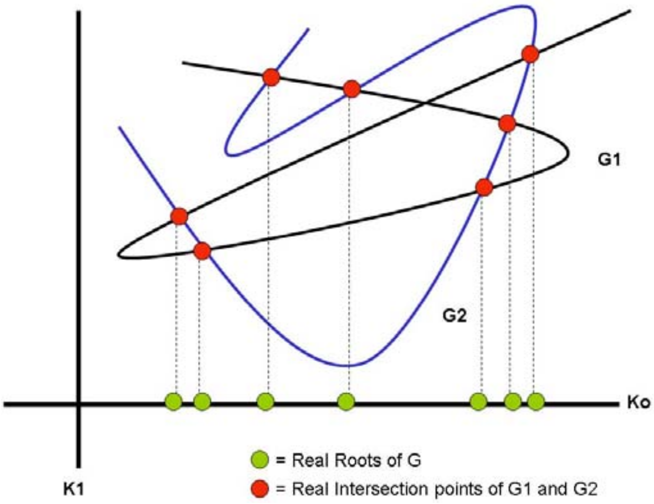

Figure 1: Geometric interpretation of the resultant $G = \mathrm{Res}_{k_q}(G_1, G_2)$.

(ii) $\gcd(G_1, G_2) = 1$ (i.e. the greatest common divisor of both polynomials is 1),

(iii) and either $a_n(k_p)$ or $b_m(k_p)$ is a constant polynomial (note that a_n and b_m are, by definition, not identically zero),

then Theorem 1 implies that all the solutions (in the particular the real ones) of the system $\{G_1(k_p, k_q) = 0, G_2(k_p, k_q) = 0\}$ can be obtained from the roots of $G(k_p)$; this process is known as the lifting process.

We have already seen that hypothesis (i) (see above) can be assumed w.l.o.g. Let us see how to proceed in general with hypotheses (ii) and (iii).

- Hypothesis (ii). If $\gcd(G_1, G_2) = D \neq 1$, dividing G_1, G_2 by D one gets two new polynomials, say G_1^* and G_2^*, fulfilling the *gcd* condition, and the solutions of $\{G_1 = 0, G_2 = 0\}$ are the solutions of $D = 0$ union the finitely many solutions of $\{G_1^* = 0, G_2^* = 0\}$. Moreover, note that since in our case the polynomials come from empirical data the most expectable situation is that the polynomials are coprime, i.e. its gcd is 1.

- Hypothesis (iii). If none of the polynomials $a_n(k_p), b_m(k_p)$ is constant, one can check whether taking k_p as a main variable the property holds. If for none of the variables k_p and k_q the requirement hods, then one can always apply a linear change of coordinates such that the new polynomials verifies the property; note that applying the inverse of the linear change of coordinates to the solutions of the new system one gets the solutions of the initial one. In order to deterministically choose this linear change of coordinate, one reasons as follows. We express one of the polynomials, say G_1, as a sum of homogenous polynomials (recall that a bivariate polynomial $H(k_p, k_q)$ is homogeneous of degree r is $H(tk_p, tk_q) = t^r H(k_p, k_q)$ where t is a new variable):
$$G_1(k_p, k_q) = H_r(k_p, k_q) + \cdots + H_1(k_p, k_q) + H_0(k_p, k_q),$$

where H_i is homogeneous of degree r. So, H_i collects all terms in G_1 of total degree i; or equivalently H_i is the i-degree part of the Taylor expansion of G_i around $(0,0)$. In this situation, if $(1,b) \in \mathbb{R}^2$ is such that $H_r(1,b) \neq 0$ then

$$G_1(k_p + bk_q, k_q) = H_r(1,b)k_q^r + \text{terms of lower degree}$$

and therefore the requirement is achieved.

The next proposition shows that, in our case, hypothesis (iii) always holds.

Proposition 2 *If the edge points $\{(x_i, y_i)\}_{i=1,..,N}$ are not aligned then hypothesis (iii) always holds.*

Proof: From (3), in the particular case of the distortion model, one has that

$$\frac{\partial \hat{E}(k_p, k_q)}{\partial k_q} = b_3(k_p)k_q^3 + b_2(k_p)k_q^2 + b_1(k_p)k_q + b_0(k_q)$$

where

$$b_3(k_p) = 4(A_{pp}B_{pp} - C_{pp}^2).$$

Therefore $b_3(k_p)$ is constant, and $b_3(k_p) = 0$ if and only if the points $((r_i)^p\, x_i - \overline{(r_i)^p\, x_i}, (r_i)^p\, y_i - \overline{(r_i)^p\, y_i})$ lie on a line. In particular there exist a, b such that, for each i,

$$a((r_i)^p\, x_i - \overline{(r_i)^p\, x_i}) + b((r_i)^p\, y_i - \overline{(r_i)^p\, y_i}) = 0$$

Dividing the above expression by $(r_i)^p$, (we assume that $(r_i)^p \neq 0$, because otherwise the above equation is trivial) we obtain that, for every i,

$$a(x_i - \overline{x_i}) + b(y_i - \overline{y_i}) = 0.$$

So, in particular the original points (x_i, y_i) lie on a line, which is a trivial case because no model distortion is needed. Therefore we conclude that, except for the trivial case where the initial distorted points are aligned, the leading polynomial coefficient $b_3(k_p)$ is constant and different from 0. Thus, hypothesis (iii) holds. $\square$

Summarizing, one can derive the following algorithm to compute the real solutions of

$$\mathcal{S} := \left\{ \frac{\partial \hat{E}(k_p, k_q)}{\partial k_p} = 0, \ \frac{\partial \hat{E}(k_p, k_q)}{\partial k_q} = 0 \right\},$$

where we assume w.l.o.g. that hypotheses (i),(ii), and (iii) hold. Note that, once these solutions are known, minimizing the distortion error measure function $\hat{E}(k_p, k_q)$, in the compact set of analysis, is trivial.

1. Determine $G_1 := \frac{\partial \hat{E}}{\partial k_p}$ and $G_2 := \frac{\partial \hat{E}}{\partial k_q}$.
2. Determine $G(k_p) := \text{Res}_{k_q}(G_1, G_2)$ and approximate the real roots of $G(k_p)$. Let $\mathcal{R} = \{\alpha_1, \ldots, \alpha_s\}$ be the set of real roots of G.

3. For each $\alpha \in \mathcal{R}$ approximate the common real roots of the univariate polynomials $G_1(\alpha, k_q)$ and $G_2(\alpha, k_2)$. Let $\mathcal{R}_\alpha$ be the set of this real common roots.

4. The real solutions of $\mathcal{S}$ are $\{(\alpha, \beta_\alpha)/\alpha \in \mathcal{R}$ and $\beta_\alpha \in \mathcal{R}_\alpha\}$.

In the general case, i.e. when working with $s > 2$ variables, say $k_{p_1}, \ldots, k_{p_s}$, the problem cannot be approached so directly by means of resultants. Nevertheless, one can apply *Gröbner basis* techniques or multivariate-resultants (see [3] and [6] for further information). The basic idea of Gröbner basis, as a tool for solving algebraic systems, is to provide a new algebraic system of equations equivalent to $\mathcal{S}$ (i.e. with the same solutions) but much simpler, and such that it has a suitable structure ("triangular") to compute the solutions. Roughly speaking, Gröbner basis can be seen as a generalization of the gaussian elimination when the equations are not linear.

For instance, let us consider the algebraic system of equations

$$\begin{cases} G_1(k_{p_1}, k_{p_2}, k_{p_3}) = k_{p_2}^2 - k_{p_1}^2 - 1 = 0 \\ G_2(k_{p_1}, k_{p_2}, k_{p_3}) = k_{p_1}^2 + k_{p_3}^2 - 4 = 0 \\ G_3(k_{p_1}, k_{p_2}, k_{p_3}) = k_{p_1}^3 - 2k_{p_1} - 2 + k_{p_2}^2 + k_{p_1}k_{p_3}^2k_{p_2}^2 - p_{p_1}^3k_{p_2}^2 - k_{p_1}k_{p_3}^2 = 0 \end{cases}$$

Applying *Gröbner* basis one gets the following equivalent system

$$\begin{cases} G_1^*(k_{p_1}, k_{p_2}, k_{p_3}) = k_{p_1}^2 + k_{p_3}^2 - 4 = 0 \\ G_2^*(k_{p_1}, k_{p_2}) = k_{p_2}^2 - k_{p_1}^2 - 1 = 0 \\ G_3^*(k_{p_1}) = (k_{p_1} - 1)(k_{p_1}^2 - 2) = 0 \end{cases}$$

from where one deduces that the original system has 12 solutions, namely:

$$(1, \sqrt{2}, \sqrt{3}), (1, \sqrt{2}, -\sqrt{3}), (1, -\sqrt{2}, \sqrt{3}), (1, -\sqrt{2}, -\sqrt{3}),$$

$$(\sqrt{2}, \sqrt{3}, \sqrt{2}), (\sqrt{2}, \sqrt{3}, -\sqrt{2}), (\sqrt{2}, -\sqrt{3}, \sqrt{2}), (\sqrt{2}, -\sqrt{3}, -\sqrt{2}),$$

$$(-\sqrt{2}, \sqrt{3}, \sqrt{2}), (-\sqrt{2}, \sqrt{3}, -\sqrt{2}), (-\sqrt{2}, -\sqrt{3}, \sqrt{2}), (-\sqrt{2}, -\sqrt{3}, -\sqrt{2}).$$

Note that, geometrically these 12 solutions correspond to the 12 intersection points of two cylinders and three planes (see Figure 2)

Nevertheless, the computational complexity of the Gröbner basis method is doble exponential in the number of the polynomial variables, i.e. s, while the resultant-based method is polynomial in time and more stable numerically. We leave, as future research work, the applications of the Gröbner basis method to the current problem.

4 The algorithm.

We will use an iterative scheme to estimate the distortion parameters $\mathbf{k}$. We note by $\mathbf{k}^n$ the distortion parameter at step n, and by $(\hat{x}_i^n, \hat{y}_i^n)$ the corrected (undistorted) points at step n. We will analyse separately the update of k_0^n and k_i^n for $i > 0$. The initial values for the distortion parameters are $\mathbf{k}^0 = (1, 0, ..., 0)$. In each step, first we select the distortion parameters we want to update, it can be 1 or 2 distortion parameters. The case to update more of 2 distortion parameters simultaneously

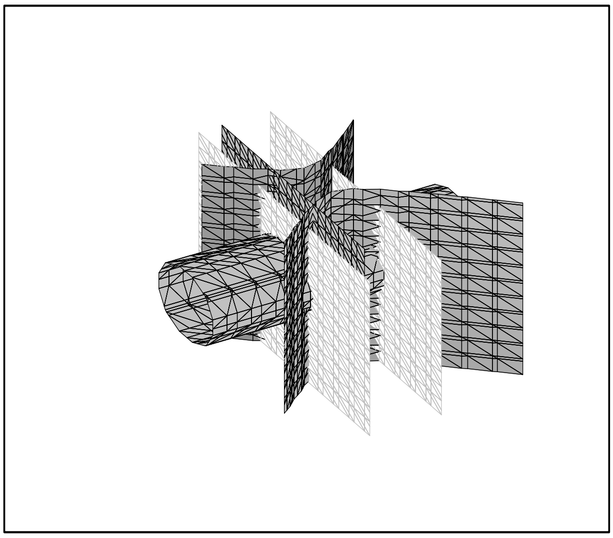

Figure 2: Geometric interpretation of the solutions.

is much more complex and it requires, in general, the use of Gröbner basis techniques. We do not consider here this situation and we leave it as future research work.

Let us consider, for instance, that at step n we want to update coefficients k_p^n, k_q^n with $1 \leq p, q \leq N_k$. We express $k_p^{n+1} = k_p^n + \epsilon_p$, $k_q^{n+1} = k_q^n + \epsilon_q$ and $(\hat{x}_i^{n+1}, \hat{y}_i^{n+1})$ as

$$\begin{pmatrix} \hat{x}_i^{n+1} - x_c \\ \hat{y}_i^{n+1} - y_c \end{pmatrix} = \left(\sum_{j=0}^{N_k} k_j^n \, (r_i)^j + \epsilon_p \, (r_i)^p + \epsilon_q \, (r_i)^q \right) \begin{pmatrix} x_i - x_c \\ y_i - y_c \end{pmatrix}.$$

To find ϵ_p, ϵ_q we minimize the distortion measure error with respect to ϵ_p and ϵ_q using the algebraic approach presented in the above section.

In order to update k_0^n we use a different approach. Usually k_0 is fitted to 1, in this paper we fit k_0^n in order to minimize the sum of the square distance between the distorted and the corrected (undistorted) points. i.e. we minimize :

$$H(k_0^n) = \sum_{i=1}^{N} \left(\hat{x}_i^n - x_i\right)^2 + \left(\hat{y}_i^n - y_i\right)^2$$

a straightforward computation leads to

$$H(k_0^n) = \sum_{i=1}^{N} \left(\left(k_0^n - 1\right) r_i + \sum_{j=1}^{N_k} k_j^n \, (r_i)^{j+1} \right)^2$$

and the minimum of $H(k_0^n)$ is given by

$$k_0^n = 1 - \frac{\sum_{j=1}^{N_k} k_j^n \, (r_i)^{j+2}}{\sum_{j=1}^{N_k} (r_i)^2} \tag{4}$$

An interesting advantage of this approach is that the resolution of the undistorted image is similar to the resolution of the original (distorted) image. This is a very useful property if we need to generate the undistorted image from the original distorted one. We notice that k_0 plays the role of a zoom factor to fit the undistorted and distorted image as much as possible.

We point out that, in order to simplify the notation, we always state the algorithms for a single edge segment. To generalize it to a collection of edge segments is straightforward just by adding the influence of it segment, i.e. we build a single polynomial to minimize just by adding the polynomial associated to each edge segment.

Therefore the derived algorithm for performing the numerical experiments can be structured in the following steps :

1. We compute the edges of the image using an edge detection algorithm with subpixel precision.
2. We select some collections of edge points, that will be used to fit the distortion parameters.
3. We initialize $\mathbf{k}^0 = (1, 0, ..., 0)$ and do until convergence:
   (a) We select the distortion coefficients we want to update at scale n
   (b) We minimize $\hat{E}(\mathbf{k}^n)$ with respect to such coefficients
   (c) We update k_0^n using (4).

Remark: Point coordinates normalization. It is well known that when we deal with algebraic methods (see for instance [7]) it is usually better to normalize the point coordinates before computing the algebraic solution of the problem. Following this strategy, at a fist step, we normalize the edge points (x_i, y_i) using the transformation

$$x_i' = \frac{(x_i - x_c)}{A} \qquad y_i' = \frac{(y_i - y_c)}{A}$$

where A is given by

$$A = \sqrt{\frac{\sum_{i=1}^{N}(x_i - x_c)^2 + (y_i - y_c)^2}{2N}}$$

we compute the distortion parameters k_i' for the normalized edge points $\{(x_i', y_i')\}_{i=1}^{N}$ Finally, in order to recover the distortion parameters k_i for the original edge points we have just to take into account that following the above expressions and (1) we have that

$$k_i = \frac{k_i'}{A^i}$$

4.0.1 Inversion of the radial distortion model

For some applications we need to invert the radial distortion model. For instance, to build the undistorted version of the image it is usually better to use the inverted of the radial distortion model. So we look for a radial function $G(\hat{r})$ such that

$$\begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} = G(\hat{r}) \begin{pmatrix} \hat{x} - x_c \\ \hat{y} - y_c \end{pmatrix}$$

where

$$\hat{r} = \sqrt{(\hat{x} - x_c)^2 + (\hat{y} - y_c)^2}.$$

From the above expression we obtain that

$$r = G(\hat{r})\hat{r}.$$

On the other hand we have

$$\begin{pmatrix} \hat{x} - x_c \\ \hat{y} - y_c \end{pmatrix} = L(r) \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix}$$

and therefore

$$\hat{r} = L\left(G(\hat{r})\hat{r}\right) G(\hat{r})\hat{r}$$

So we conclude that $G(\hat{r})$ is a root of the polynomial

$$P(\lambda) = 1 - L(\lambda\hat{r})\lambda = 1 - \sum_{i=0}^{N_k} k_i \hat{r}^i \lambda^{i+1}.$$

In order to minimize the distance between the undistorted point $(\hat{x}, \hat{y})$ we choose, among all possible real roots of $P(\lambda)$, the one nearest to 1.

5 Numerical Experiments.

Throughout this section, we will assume that the distortion center (x_c, y_c) is known. Moreover, in the presented numerical experiments, we always take as distortion center the center of the image.

To perform the experiments we will use the simple planar lens distortion calibration pattern that we have created (see figure 3). We have printed this calibration pattern and we have taken photos of the printed image with a low cost wide-angle lens camera. The advantage of this calibration pattern is that we can easily identify the rectangles presented in the image, and automatically select the edge segments and points we will use for the estimation of the distortion model parameters. In figure 4 we illustrate the image obtained by the camera and the automatically selected edge points.

We analyze the performance of the method using 1 or 2 distortion parameters, we will also compare the results obtained by using $k_0 = 1$ and k_0 computed using (4).

The achieved quantitative results are presented in table 1. In the first line we present the residual variance (given by (2)) for the original distorted image. In the next 8 lines we present the results

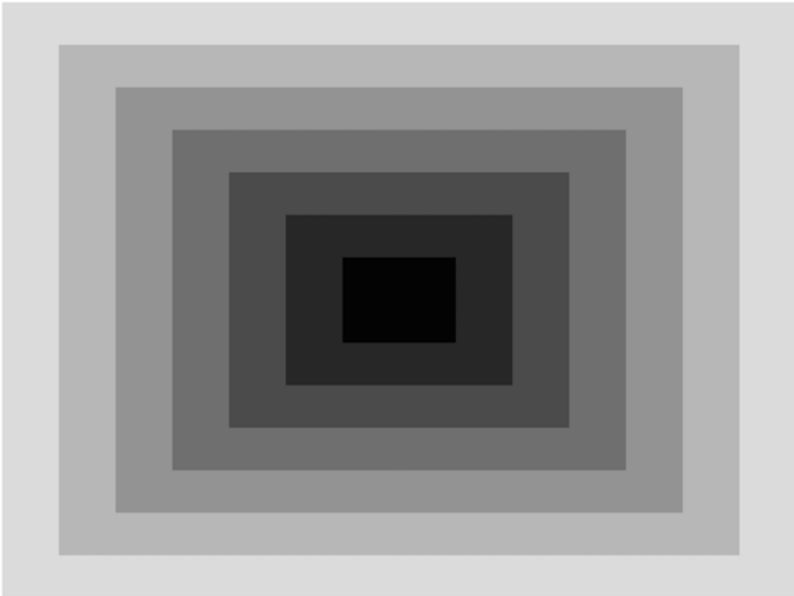

Figure 3: Lens distortion calibration pattern used in the experiments.

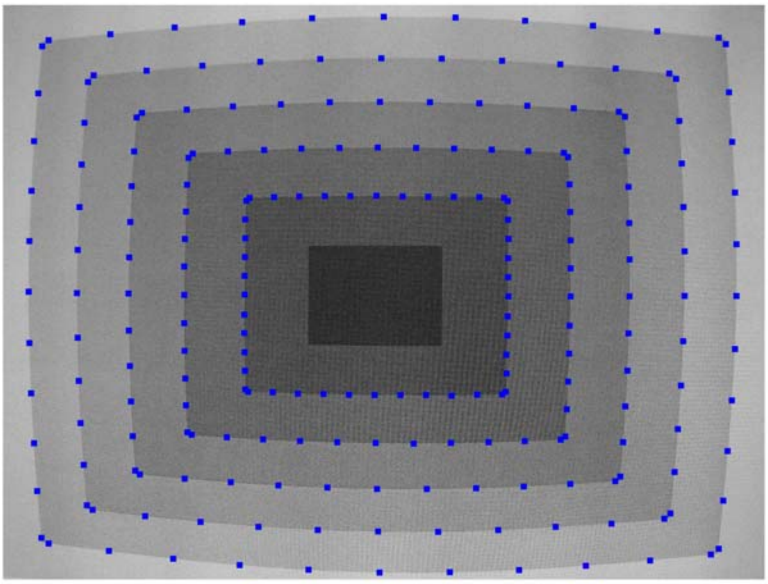

Figure 4: A photo taken with a wide-lens camera on a printed version of image in figure 3. The small squares represent the location of the edge points we use to estimate the distorsion parameters.

$iter$	k_0	k_1	k_2	k_3	k_4	V_{min}
1	1	0	0	0	0	1.321e+02
1	1	2.6e-04	0	0	0	2.891e+00
1	7.7e-01	2.0e-04	0	0	0	1.734e+00
1	1	0	1.2e-07	0	0	2.095e+00
1	9.0e-01	0	1.1e-07	0	0	1.704e+00
1	1	0	0	8.0e-11	0	8.736e+00
1	9.4e-01	0	0	7.6e-11	0	7.706e+00
1	1	0	0	0	5.8e-14	1.736e+01
1	9.6e-01	0	0	0	5.5e-14	1.590e+01
1	1	1.4e-04	5.9e-08	0	0	1.021e+00
1	8.3e-01	1.2e-04	5.0e-08	0	0	7.097e-01
1	1	2.0e-04	0	2.1e-11	0	1.310e+00
1	8.1e-01	1.6e-04	0	1.7e-11	0	8.676e-01
1	1	2.2e-04	0	0	1.1e-14	1.560e+00
1	8.0e-01	1.8e-04	0	0	8.6e-15	1.007e+00
1	1	0	2.3e-07	-6.7e-11	0	6.143e-01
1	8.7e-01	0	2.0e-07	-5.9e-11	0	4.676e-01
1	1	0	1.8e-07	0	-2.8e-14	5.337e-01
1	8.8e-01	0	1.6e-07	0	-2.5e-14	4.115e-01
1	1	0	0	2.6e-10	-1.3e-13	5.971e-01
1	9.0e-01	0	0	2.3e-10	-1.2e-13	4.863e-01
1000	8.8e-01	4.3e-05	1.1e-07	0	-1.1e-14	5.360e-01
1000	9.0e-01	-2.3e-05	1.7e-07	1.1e-11	-3.3e-14	3.981e-01

Table 1: Estimated distortion parameters using the proposed method. On the left of each line we present the number of iterations we perform, next, we present the obtained distortion parameters using different choices for the distortion parameters to be estimated and on the right we present the residual variance given by (2).

obtained using 1 distortion parameter and $k_0 = 1$ and k_0 computed using (4). We observe that the best result (minimal residual variance) is obtained for $k_0 = 9.1e-01$ and $k_2 = 1.1e-07$. In the next 12 lines we present the results obtained using 2 distortion parameter and $k_0 = 1$ and k_0 computed using (4). We observe that the best result (minimal residual variance) is obtained for $k_0 = 8.9e-01$, $k_2 = 1.6e-07$ and $k_4 = -2.5e-14$. Finally in the last 2 lines we presents the results obtained by iterations of the algorithm (1000 iterations), updating each time 1 or 2 distortion parameters. First we present the results by iterations of the 1 distortion parameter estimation model (residual variance $= 5.360e-01$) and next we present results obtained by iterations of the 2 distortion parameter estimation model (residual variance $= 3.981e-01$)

First, we notice that, when we apply a single iteration of the algorithm, the best results are obtained using even distortion parameters (i.e. $k_2, k_4, k_6.....$) which is coherent with the fact that the even distortion parameters are more relevant than odd distortion parameters (see for instance [1]). On the other hand we can appreciate the improvement in the residual variance distortion

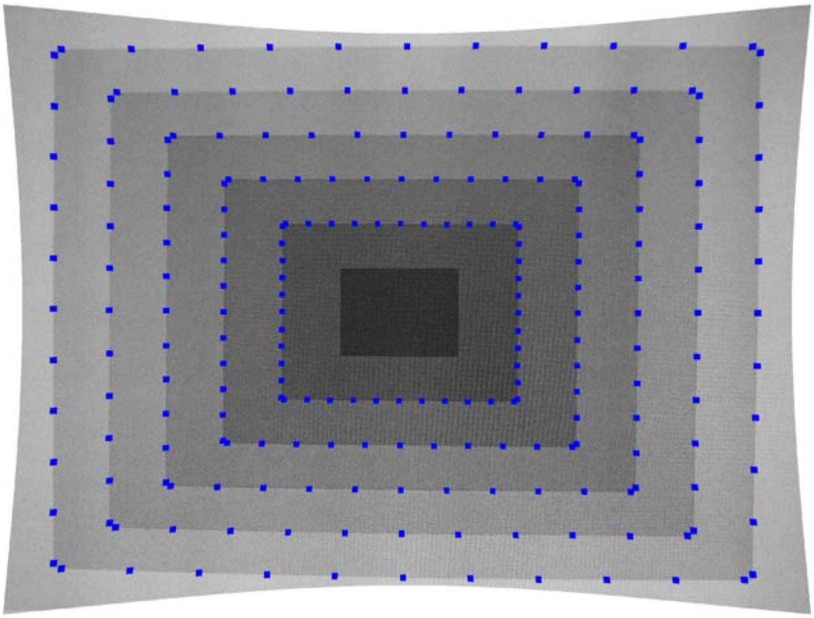

Figure 5: Distortion correction obtained from image in figure 4 by applying the proposed method.

error when we use the original distorted image ($V_{min} = 1.3e + 02$), when we use the distortion parameters k_0 and k_2 ($V_{min} = 1.9e + 00$) and when we use the distortion parameters k_0, k_2 and k_4 ($V_{min} = 4.2e{-}01$). Finally we observe that when we use multiple iterations of the algorithm updating each time different lens distortion parameters the results are improved, however we observe that we can get better results using a single iteration of the 2 parameter distortion model than using 1000 iterations of the 1 parameter distortion model.

To illustrate the visual effect of the obtained undistorted image we present in figure 5 the undistorted image using the estimated lens distortion model with the lower residual variance given by $k_0 = 9.0e - 01$, $k_1 = -2.3e - 05$, $k_2 = 1.7e - 07$, $k_3 = 1.1e - 11$ and $k_4 = -3.3e - 14$.

To illustrate also de edge displacement between the distorted and undistorted image when we fit k_0 using (4), we present in figure 6 the location of edges in the distorted and undistorted image.

6 Conclusions

In this paper we present an algebraic approach to radial lens distortion parameter estimation based on edge line rectification. We propose a distortion error measure based on a linear regression analysis. We present an algebraic analysis of the distortion error measure. We have implemented the proposed method for 1 or 2 distortion parameter models. We also propose to estimate the distortion parameter k_0 by minimizing the square distance between the distorted and undistorted edge points.

The numerical experiences we have presented are very promising. We have implemented an algorithm which allow to update by iterations all the distortion parameters of the distortion model.

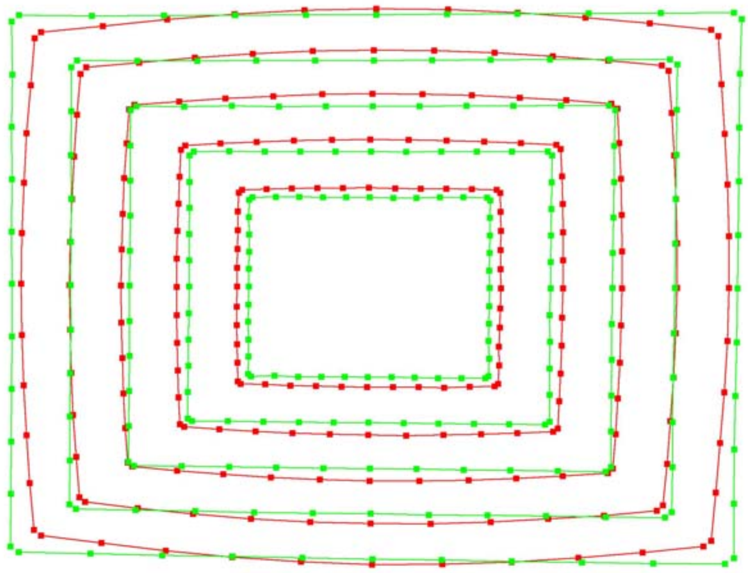

Figure 6: Illustration of the edge points location displacement after the distortion correction of the image in figure 4.

The residual variance in the linear regression approximation of the edge lines is strongly reduced in the undistorted image, and the barrel distortion is properly removed. An important advantage of our method is that it does not require initialization for the distortion parameter k_i. In particular it can be used as initialization of the distortion parameter in bundle adjustment calibration techniques.

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Instituto Universitario de Ciencias y Tecnologías Cibernéticas
Universidad de Las Palmas de Gran Canaria
Campus de Tafira
35017 Las Palmas, España
http://www.iuctc.ulpgc.es