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AN ALGEBRAIC CLASSIFICATION OF SOME LINKS OF CODIMENSION TWO

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ABSTRACT. For $q > 2$, J. Levine proved that two simple $(2q-1)$-knots are isotopic if and only if their Seifert matrices are equivalent. In this paper, we will prove the analogue of Levine's result for simple boundary $(2q-1)$-links; we will show that: "For $q \geqslant 3$, two simple boundary $(2q-1)$-links are isotopic if and only if their Seifert matrices are l-equivalent (defined by some algebraic moves)."

An n-link of multiplicity m, denoted by $L = K_1 \cup \cdots \cup K_m$ is an embedding of m disjoint copies of the n-sphere (or homotopy spheres) K_i into the $(n+2)$-sphere S^{n+2}. L is called boundary if it extends to an embedding of m disjoint orientable compact $(n+1)$-manifolds M_i, called the Seifert manifolds, with $\partial M_i = K_i$. Let X denote the link complement. Gutiérrez **[1]** showed that an n-link of multiplicity m is boundary if and only if there is an epimorphism from $\pi_1(X)$ onto F_m, the free group in m generators, sending meridians to generators. An $(2q-1)$-link L is called simple if $\pi_i(X) = \pi_i(\bigvee_m S^1)$ for $i < q$; in case L is a boundary link, we require that the meridians be sent to generators.

For $q \geqslant 2$, Levine **[5]** proved that two simple $(2q-1)$-knots are isotopic if and only if their Seifert matrices are "equivalent" (defined by certain algebraic "moves" in **[5]**, also called S-equivalent in [7]). In this paper, we will prove the analogue of Levine's Theorems 1–3 for simple boundary $(2q-1)$-links, $q \geqslant 3$: two simple boundary $(2q-1)$-links are isotopic if and only if their "Seifert matrices" are related by certain algebraic "moves".

Since our proofs are almost the same as those of **[4]** and **[5]**, we will only give the outlines here.

1. For simplicity, we will consider only the $(2q-1)$-link of multiplicity 2. Everything considered here is in the smooth category.

Let $L = K_1 \cup K_2$ be a boundary $(2q-1)$-link. According to **[1]**, there exist two disjoint $2q$-dimensional Seifert manifolds M_1 and M_2 for L, that is, $\partial M_1 = K_1$ and $\partial M_2 = K_2$. Let A_1 be the corresponding Seifert matrix for the

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knot K_1 (in S^{2q+1}) with respect to the basis $\{a_1, \ldots, a_n\}$ of the torsion-free part of $H_q(M_1)$, and A_2 the Seifert matrix for K_2 with respect to the basis $\{b_1, \ldots, b_m\}$ of $H_q(M_2)/\text{Torsion}$.

A linking from θ: $(H_q(M_1) \oplus H_q(M_2)) \otimes (H_q(M_1) \oplus H_q(M_2)) \to Z$ is defined by letting $\theta(\alpha \otimes \beta)$ be the linking number $L(z_1, z_2)$ (in S^{2q+1}), where z_1, a cycle in M_1 (or M_2), represents α and z_2 represents $i_+ \beta$, the translate in the positive normal direction off M_1 (or M_2) into $S^{2q+1} - M_1 - M_2$ of a cycle representing β. With respect to the basis $\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$ of the torsion-free part of $H_q(M_1) \oplus H_q(M_2)$, the matrix θ has the following form:

$$D = \begin{pmatrix} A_1 & P \\ -\varepsilon P' & A_2 \end{pmatrix},$$

also written as $D = [A_1, A_2, P]$, where $\varepsilon = (-1)^q$ and P' denotes the transpose of P. We call D a Seifert matrix for the boundary link L. It is obvious that $D + \varepsilon D'$ is unimodular. Algebraically, we will call $D = [A_1, A_2, P]$ a Seifert matrix of type 2 if $A_1 + \varepsilon A_1'$, $A_2 + \varepsilon A_2'$ and $D + \varepsilon D'$ are unimodular. Here A' denotes the transpose of A.

Actually, D is a Seifert matrix for the link L corresponding to the manifold $M_1 \# M_2$ with $\partial(M_1 \# M_2) = K_1 \cup K_2$ in the sense of [6, Theorem 3.2]. The $(n \times m)$-matrix $P = (p_{ij})$ in D can be obtained as follows: let $\{c_1, \ldots, c_n\}$ be a basis for $H_q(S^{2q+1} - M_1)/\text{Torsion}$, which is the Alexander dual of $\{a_i\}$, that is, $L(a_i, c_j) = \delta_{ij}$. In $S^{2q+1} - M_1$, we have $b_j = \sum p_{kj} c_k$, hence

$$L(a_i, i_+ b_j) = L(a_i, b_j) = \sum_j L(a_i, c_k) p_{kj} = p_{ij}.$$

Following [5], we now define certain algebraic "moves" for Seifert matrices of type 2. Let $D = [A_1, A_2, P]$ be one. Then any matrix of the form (which is again a Seifert matrix of type 2):

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & x \\ 0 & \lambda & A_1 & P \\ 0 & -\varepsilon x' & -\varepsilon P' & A_2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & y & x \\ 0 & 0 & A_1 & P \\ 0 & -\varepsilon x' & -\varepsilon P' & A_2 \end{bmatrix},$$

$$\begin{bmatrix} A_1 & P & x' & 0 \\ -\varepsilon P' & A_2 & \tau & 0 \\ -\varepsilon x & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} A_1 & P & x' & 0 \\ -\varepsilon P' & A_2 & 0 & 0 \\ -\varepsilon x & y & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where x, y are row vectors, and λ, τ are column vectors, will be called an elementary l-enlargement of D; D is an elementary l-reduction. Let C be a unimodular matrix having the same dimension as A_1, and E a unimodular matrix having the same dimension as A_2. Then each of the operations below will be called an l-congruence:

$$D \to \begin{pmatrix} C & 0 \\ 0 & I_m \end{pmatrix} D \begin{pmatrix} C' & 0 \\ 0 & I_m \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} I_n & 0 \\ 0 & E \end{pmatrix} D \begin{pmatrix} I_n & 0 \\ 0 & E' \end{pmatrix}.$$

Two Seifert matrices of type 2 are called l-equivalent if they can be connected by a chain of elementary l-enlargements, l-reductions, and l-congruences (with C or E having the appropriate dimension).

2. We first prove the analogue of [**5**, Theorem 1].

THEOREM 1. *Seifert matrices of isotopic boundary* $(2q - 1)$-*links are* l-*equivalent.*

PROOF. Suppose $L_1 = K_1 \cup K_2$ and $L_2 = J_1 \cup J_2$ are isotopic boundary links with Seifert manifolds M_1, M_2 and N_1, N_2, respectively. Then the argument in [**5**, p. 186] gives us two disjoint $(2q + 1)$-dimensional manifolds V_i ($i = 1$ or 2) in $S^{2q+1} \times I$ meeting $S^{2q+1} \times 0$ along M_i and $S^{2q+1} \times 1$ along N_i, with $\partial V_i = M_i \cup X_i \cup N_i = Y_i$.

After rearranging the level of the critical points for the "height" functions Φ_i: $V_i \to I$ as in [**5**, p. 187], we need only consider the case where Φ_1 has only one critical point and Φ_2 has none. Then we use the argument in [**5**, pp. 187–188] to conclude that the Seifert matrix D for $L_2 = J_1 \cup J_2$ with respect to an appropriate basis has the following form:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & u & v & x \\ 0 & \lambda & A_1 & P \\ 0 & -\varepsilon x' & -\varepsilon P' & A_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & u & v & x \\ 0 & \lambda & A_1 & P \\ 0 & -\varepsilon x' & -\varepsilon P' & A_2 \end{bmatrix}$$

where $[A_1, A_2, P]$ is the Seifert matrix for $L_1 = K_1 \cup K_2$ associated with M_1 and M_2. D is l-congruent to an elementary l-enlargement of $[A_1, A_2, P]$ as in [**5**, Theorem 1]. Q.E.D.

3. Let q denote an integer and recall that $\varepsilon = (-1)^q$.

THEOREM 2. *Let* $q \geqslant 3$, *and* $D = [A_1, A_2, P]$ *a square integral matrix such that* $A_1 + \varepsilon A_1'$, $A_2 + \varepsilon A_2'$, *and* $D + \varepsilon D'$ *are unimodular. Then there is a simple boundary* $(2q - 1)$-*link* $L = K_1 \cup K_2$ *with* D, A_1, A_2 *the Seifert matrices of* L, K_1, K_2, *respectively.*

PROOF. Let B_1, B_2 denote two disjoint $(2q + 1)$-balls in S^{2q+1}. We know from [**2**, pp. 255–257] that there exist two handlebodies $M_1 = D^{2q} \cup h_1 \cup \cdots \cup h_n$, $M_2 = D^{2q} \cup h_1' \cup \cdots \cup h_m'$, where each h_i, h_i' is a handle of index q; and two embeddings g_i: $M_i \to B_i \subseteq S^{2q+1}$ such that $g_i(\partial M_i) = J_i$ represents a simple knot with Seifert matrix A_i. Let $\{a_1, \ldots, a_n\}$ be a basis for $H_q(g_1(M_1))$ and $\{b_1, \ldots, b_m\}$ a basis for $H_q(g_2(M_2))$; each represents the core of a handle in M_1 or M_2. As in [**2**, p. 257] we may choose n q-spheres $S_1, \ldots, S_n$ in B_2 such that $L(S_i, b_j) = \delta_{ij}$ and $L(S_i, S_j) = 0$. Since $S_i \subseteq B_2$, $L(a_i, S_j) = 0$. Then we define a new embedding f for M_1 such that $f = g_1$ near D^{2q}, $f(\gamma_i) = g_1(\gamma_i) \# \Sigma p_{ij} S_j$, where $P = (p_{ij})$ and γ_i denotes the core of the handle h_i. Let $c_i \in H_q(f(M_1))$ represent $f(\gamma_i)$. Since $f(M_1)$ and $g_2(M_2)$ are $(q-1)$-connected, the link $L = K_1 \cup K_2$, where $K_1 = \partial f(M_1)$ and $K_2 = \partial g_2(M_2)$, is a simple boundary link [**1**]. Furthermore, with respect to the basis $\{c_1, \ldots, c_n, b_1, \ldots, b_m\}$, the Seifert matrices of L, K_1, K_2 are D, A_1, A_2, respectively. Q.E.D.

4. A Seifert matrix of a simple boundary $(2q-1)$-link L obtained from two disjoint $(q-1)$-connected Seifert manifolds will be called special.

LEMMA 1. *Let $L = K_1 \cup K_2$ be a simple boundary $(2q-1)$-link with a special Seifert matrix $D = [A_1, A_2, P]$. If E is an elementary l-enlargement of D, then E is also a special Seifert matrix of L.*

PROOF. The proof is essentially the same as [**5**, Lemma 2].

LEMMA 2. *For $q \geqslant 3$, two simple boundary $(2q-1)$-links admitting identical special Seifert matrices are isotopic.*

PROOF. Let $L_1 = K_1 \cup K_2$ and $L_2 = J_1 \cup J_2$ be two simple boundary $(2q-1)$-links bounding $(q-1)$-connected Seifert manifolds M_1, M_2 and N_1, N_2, respectively, with $M_1 \cap M_2 = \varnothing = N_1 \cap N_2$. Suppose also that there exists an isomorphism Φ: $H_q(M_1 \cup M_2) \to H_q(N_1 \cup N_2)$ preserving the linking form with $\Phi | H_q(M_i) \to H_q(N_i)$ an isomorphism.

Lemma 3 of [**5**] showed that M_1 and N_1 are isotopic submanifolds of S^{2q+1}. Hence we may assume that $M_1 = N_1$. According to [**8**], M_1, M_2 and N_2 have handle decompositions:

$$M_1 = D_0^{2q} \cup \alpha_1 \cup \cdots \cup \alpha_n, \quad M_2 = D^{2q} \cup \beta_1 \cup \cdots \cup \beta_m,$$
$$N_2 = D^{2q} \cup \gamma_1 \cup \cdots \cup \gamma_m,$$

where each α_i, β_i, γ_i is a handle of index q. By a further isotopy keeping M_1 fixed, we may assume that the base disks D^{2q} in the handle decompositions of M_2 and N_2 coincide as imbedded in S^{2q+1}.

We connect the boundaries of D_0^{2q} and D^{2q} with a path τ and then thickening τ to $\tau \times I^{2q-1} = Q$ avoiding all handles, and meeting D and D_0 transversely in two $(2q-1)$-disks. But $M_1 \cup Q \cup M_2$, with appropriate orientation, is just $M_1 \natural M_2$, the boundary connected sum of M_1 and M_2 [**3**]. Moreover, $M_1 \natural M_2$ is a Seifert manifold for the $(2q-1)$-knot $K_1 \# K_2$. Similarly, $M_1 \natural N_2$ is a Seifert manifold for $K_1 \# J_2$. The special Seifert matrix for L_1 and L_2 is just a special Seifert matrix for both $K_1 \# K_2$ and $K_1 \# J_2$. Let $D_1 = D_0^{2q} \natural D^{2q} = D_0 \cup Q \cup D$. Then $M_1 \natural M_2$ and $M_1 \natural N_2$ have the following handle decompositions:

$$M_1 \natural M_2 = D_1 \cup \alpha_1 \cdots \cup \alpha_n \cup \beta_1 \cdots \cup \beta_m,$$
$$M_1 \natural N_2 = D_1 \cup \alpha_1 \cdots \cup \alpha_n \cup \gamma_1 \cdots \cup \gamma_m.$$

According to [**5**, p. 192], we can move one handle β_i (onto γ_i) at a time by an isotopy in $S^{2q+1} - (D_1 \cup \alpha_1 \cdots \cup \alpha_n \cup \beta_1 \cup \cdots \cup \beta_{i-1})$. Thus we can map $M_1 \natural M_2$ diffeomorphically onto $M_1 \natural N_2$ by an isotopy in $S^{2q+1} -$

$(D_1 \cup \alpha_1 \cdots \cup \alpha_n)$. Since the thickened path $Q \subseteq D_1$, we see that $L_1 = K_1 \cup K_2$ is isotopic to $L_2 = J_1 \cup J_2$. Q.E.D.

The next theorem follows from Lemmas 1 and 2 exactly as in [**5**, p. 189].

THEOREM 3. *Let $L_1 = K_1 \cup K_2$ and $L_2 = J_1 \cup J_2$ be two simple boundary $(2q - 1)$-links, $q \geqslant 3$, with l-equivalent Seifert matrices. Then L_1 is isotopic to L_2.*

5. A $(2q - 1)$-link $L = K_1 \cup K_2$ in S^{2q+1} is splittable if there exist two disjoint $(2q + 1)$-balls B_1 and B_2 in S^{2q+1} such that $K_1 \subseteq B_1$ and $K_2 \subseteq B_2$ [**6**, p. 110]. The next theorem follows immediately from Theorems 1–3.

THEOREM 4. *A simple boundary $(2q - 1)$-link $L = K_1 \cup K_2$, $q \geqslant 3$, is splittable if and only if it has a Seifert matrix of the form $[A_1, A_2, 0]$.*

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