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AN ALGEBRAIC THEORY OF LANDAU-KOLMOGOROV INEQUALITIES*

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1. Introduction. This paper is concerned with the so-called Landau-Kolmogorov (or Hardy-Littlewood) inequalities

$$(1.1) \qquad \| T^k u \| \leqq M_{n,k} \| T^n u \|^{k/n} \| u \|^{1-k/n} \qquad (0 < k < n) ,$$

for linear *dissipative* operators T in a Hilbert space $\mathscr{H}$. (T is dissipative if $\mathrm{Re}\,(Tu, u) \leqq 0$ for all $u \in \mathscr{D}(T)$ (domain of T). See Chernoff [1] for a survey of the inequalities for more general operators.) In [1] it was shown that the constants $M_{n,k}$ for the special operator $T = D = d/dt$ in $\mathscr{H} = L^2(0, \infty)$ are universal, strengthening older results due to Ljubič [2], Kupcov [3], and Kato [4]. A similar result was recently published by Kwong and Zettl [5]. For related results under somewhat different assumptions, see Protter [6].

Chernoff's proof of (1.1) is extremely simple and elegant, but it is transcendental in the sense that a large "model space" is used. The proof by Kwong-Zettl is relatively elementary but appears more complicated. Here we present a "finite" proof based on an elementary polynomial identity. A merit of this method is that it leads to a simple necessary and sufficient condition for the equality to hold in (1.1), generalizing a condition given in [4] (which is in turn a generalization of the one due to Hardy and Littlewood [7]). It is also shown that the constants $M_{n,k}$ have interesting algebraic properties; they are algebraic units except for certain simple factors, a well-known fact for small values of n (see [5]).

Our main results are summarized in

THEOREM. *Let n, k be integers such that $0 < k < n$. There exist real algebraic integers c, a_j ($j = 1, 2, \cdots, n-1$), and $a_{ij} = a_{ji}$ ($i, j = 0, 1, \cdots, n-1$), depending on n and k, with the following properties.*

(i) *c is an algebraic unit, with $0 < c < c_0 = (k/n)^{-k/n}(1 - k/n)^{k/n-1}$.*

(ii) *All the zeros of the polynomial $1 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n$ have negative real part (so that $a_j \geqq 0$).*

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(iii) *The* $n \times n$ *symmetric matrix* (a_{ij}) *is positive semi-definite, but not strictly positive-definite.*

(iv) *For any linear dissipative operator* T *in any Hilbert space* $\mathscr{H}$, *one has*

$$\| T^k u \| \leqq (c_0/c)^{1/2} \| T^n u \|^{k/n} \| u \|^{1-k/n} \quad \text{for} \quad u \in \mathscr{D}(T) .$$

(v) *Equality holds in* (iv) *if and only if there is a real number* $s > 0$ *such that*

$$u + a_1 s T u + \cdots + a_{n-1} s^{n-1} T^{n-1} u + s^n T^n u = 0 ,$$
$$\sum_{i,j=0}^{n-1} a_{ij} s^{i+j} ((T^{i+1}u, T^j u) + (T^i u, T^{j+1} u)) = 0 .$$

(vi) *The factor* $(c_0/c)^{1/2}$ *in* (iv) *is the best possible, with the equality attained by the differential operator* $T = D = d/dt$ *in* $\mathscr{H} = L^2(0, \infty)$ *for certain* $u \in \mathscr{S}[0, \infty)$ *(the Schwartz space).*

2. The inequality. In this section, we prove the theorem except for the algebraic properties of the numbers c, a_j, a_{ij}.

In what follows n and k are fixed. We introduce a polynomial

$$p_c(x, y) = 1 - c x^k y^k + x^n y^n , \tag{2.1}$$

where c is a real parameter and x, y are noncommuting indeterminates.

LEMMA 2.1. *If* $c < c_0$ *(see the Theorem), there is a unique real polynomial* $f_c(x)$ *such that (the* a_j *depend on* c*)*

$$f_c(x) = 1 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n , \tag{2.2}$$

(2.3) *all the zeros of* f_c *have negative real part* ,

$$p_c(x, -x) = f_c(x) f_c(-x) . \tag{2.4}$$

PROOF. It is easy to see that if $c < c_0$, $p_c(x, -x)$ has no zeros on the imaginary axis. Since these zeros are symmetrically distributed with respect to the real and imaginary axes, $p_c(x, -x)$ admits a unique factorization of the form (2.4) with all the zeros of f_c having negative real part.

LEMMA 2.2. *Set*

$$g_c(x, y) = f_c(x) f_c(y) - p_c(x, y) . \tag{2.5}$$

Then there is a real symmetric matrix (a_{ij}), $i, j = 0, \cdots, n-1$, *depending on* c, *such that*

$$g_c(x, y) = \sum_{i,j=0}^{n-1} a_{ij} x^i (x + y) y^j . \tag{2.6}$$

PROOF. In the proof one may assume that x and y commute, since x's stand to the left of y's in each term in (2.5) and (2.6). Then (2.6) follows by long division by $x + y$ because $g_c(x, -x) = 0$ by (2.4). The symmetry of (a_{ij}) follows from that of $g_c(x, y)$ in x, y.

LEMMA 2.3. *Let $\mathscr{H}$ be a Hilbert space. Given any $n + 1$ vectors $u_0, u_1, \cdots, u_n$ of $\mathscr{H}$, one has*

$$\|u_0\|^2 - c\|u_k\|^2 + \|u_n\|^2 = \|u_0 + a_1u_1 + \cdots + a_{n-1}u_{n-1} + u_n\|^2 - \sum_{i,j=0}^{n-1} a_{ij}((u_{i+1}, u_j) + (u_i, u_{j+1})) . \tag{2.7}$$

PROOF. One may assume, without loss of generality, that $\mathscr{H}$ has dimension $n + 1$ and $u_0, \cdots, u_n$ form a basis of $\mathscr{H}$. Define a linear operator T on $\mathscr{H}$ by $Tu_j = u_{j+1}$ for $j = 0, 1, \cdots, n - 1$ and $Tu_n = 0$, so that $T^ju_0 = u_j$, $0 \leqq j \leqq n$. Then (2.7) may be written

$$(p_c(T^*, T)u_0, u_0) = (f_c(T^*)f_c(T)u_0, u_0) - (g_c(T^*, T)u_0, u_0) .$$

But this is true because of the identity (2.5).

LEMMA 2.4. *For any $u \in H^n(0, \infty)$ (the Sobolev space), one has*

$$\|u\|^2 - c\|D^ku\|^2 + \|D^nu\|^2 = \|f_c(D)u\|^2 + \sum_{i,j=0}^{n-1} a_{ij}D^iu(0)\overline{D^ju(0)} , \tag{2.8}$$

where $D = d/dt$ and $\| \quad \|$ denotes the $L^2(0, \infty)$-norm.

PROOF. Apply Lemma 2.3 with $\mathscr{H} = L^2(0, \infty)$, $u_j = D^ju$, noting that

$$(D^{i+1}u, D^ju) + (D^iu, D^{j+1}u) = -D^iu(0)\overline{D^ju(0)} .$$

LEMMA 2.5. *Suppose the matrix (a_{ij}) is positive semi-definite. For any dissipative operator T in any Hilbert space, one has*

$$c\|T^ku\|^2 \leqq \|u\|^2 + \|T^nu\|^2 , \tag{2.9}$$

$$c\|T^ku\|^2 \leqq c_0\|T^nu\|^{2k/n}\|u\|^{2(1-k/n)} \qquad \text{for} \quad u \in \mathscr{D}(T^n) . \tag{2.10}$$

PROOF. If T is dissipative, the (Hermitian) matrix with elements $(T^{i+1}u, T^ju) + (T^iu, T^{j+1}u)$ is negative semi-definite. Thus we see that the right member of (2.7) is nonnegative if $u_j = T^ju$. (The second term in (2.7) is nonnegative, being the trace of the product of two positive semi-definite matrices.) This proves (2.9). Then (2.10) follows by replacing T with sT and optimizing in $s > 0$.

LEMMA 2.6. *Suppose (a_{ij}) is not (strictly) positive-definite. Then there is $u \in \mathscr{S}[0, \infty)$, $u \neq 0$, such that*

$$c\|D^ku\|^2 \geqq \|u\|^2 + \|D^nu\|^2 . \tag{2.11}$$

Note that D is dissipative in $L^2(0, \infty)$.

PROOF. There is a nontrivial real n-vector $(s_0, \cdots, s_{n-1})$ such that $\sum a_{ij}s_is_j \leqq 0$. Solve the n-th order differential equation $f_c(D)u = 0$ on $[0, \infty)$, with the initial conditions $D^ju(0) = s_j$, $j = 0, \cdots, n-1$. The solution u exists, is nontrivial, and belongs to $\mathscr{S}[0, \infty)$ because all the zeros of f_c have negative real part. Thus (2.11) follows from (2.8), of which the right member is nonpositive.

LEMMA 2.7. *There is a unique positive number $\gamma < c_0$ such that (a_{ij}) is (strictly) positive-definite if and only if $c < \gamma$. (a_{ij}) is positive semi-definite for $c = \gamma$.*

PROOF. Let Γ be the set of all $c < c_0$ such that (a_{ij}) is positive definite. Γ is not empty, since Lemma 2.6 shows that $c = 0$ belongs to Γ. In view of Lemmas 2.5, 2.6, it is obvious that Γ is an open interval of the form $(-\infty, \gamma)$. It remains to show that $\gamma < c_0$. Otherwise, one would have, on letting $c \to c_0$ in (2.10),

$$\| T^k u \|^2 \leqq \| T^n u \|^{2k/n} \| u \|^{2(1-k/n)} \qquad (u \in \mathscr{D}(T^n)) ,$$

for any dissipative operator T in any Hilbert space $\mathscr{H}$. But this is not true, as is seen from the example

$$\mathscr{H} = \boldsymbol{C}^2 , \quad T = -\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} , \quad u = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad \| T^j u \|^2 = 1 + 4j^2 ,$$

because $1 + 4k^2 > (1 + 4n^2)^{k/n}$.

PROOF OF THE THEOREM (up to the algebraic properties of c, a_i, a_{ij}). It suffices to set $c = \gamma$ and take the corresponding values of a_j and a_{ij}.

3. The integrality.

In this section, we prove that the a_i, a_{ij} determined above are algebraic integers and c is an algebraic unit. We put $a_0 = a_n = 1$ and $a_i = 0$ for $i < 0$ or $i > n$. From (2.1), (2.2), (2.4), one obtains

$$1 - (-1)^k c x^{2k} + (-1)^n x^{2n} = \left(\sum_{i=0}^{n} a_i x^i \right) \left(\sum_{i=0}^{n} (-1)^i a_i x^i \right) ,$$

which gives the relation:

$$\sum_{i=0}^{n} (-1)^i a_i a_{j-i} = \begin{cases} 1 & \text{if } j = 0 \\ (-1)^{k+1} c & \text{if } j = 2k \\ (-1)^n & \text{if } j = 2n \\ 0 & \text{otherwise} . \end{cases} \tag{3.1}$$

For $j = 2l$ $(1 \leqq l \leqq n-1)$, this can be rewritten as

$$\delta_{k,l}c + a_l^2 = 2\sum_{i=1}^{\infty}(-1)^{i+1}a_{l-i}a_{l+i} . \tag{3.2}$$

(All other relations in (3.1) are trivial.) From (2.1) $\sim$ (2.5), one has

$$\sum_{i,j=0}^{n-1} a_{ij}x^i(x+y)y^j = \sum_{1\leqq i+j\leqq 2n-1} a_i a_j x^i y^j + cx^k y^k ,$$

which gives, for $0 \leqq i \leqq n-1$,

$$a_{i,0} = a_{0,i} = a_{i+1} , \qquad a_{i,n-1} = a_{n-1,i} = a_i$$

and, for $i, j = 1, \cdots, n-1$,

$$a_{i-1,j} + a_{i,j-1} = \begin{cases} a_k^2 + c & \text{if} \quad i = j = k \\ a_i a_j & \text{otherwise} . \end{cases} \tag{3.3}$$

Let A denote the $n \times n$ matrix (a_{ij}) and set

$$\boldsymbol{a}_i = \begin{bmatrix} a_{1-i} \\ \vdots \\ a_{n-i} \end{bmatrix} \quad (i \in \boldsymbol{Z}) , \qquad A_j = \begin{bmatrix} a_{0,j-1} \\ \vdots \\ a_{n-1,j-1} \end{bmatrix} \quad (1 \leqq j \leqq n) ,$$

($\boldsymbol{a}_i = 0$ for $i > n$ or $i < 1-n$); A_j is the j-th column vector of the matrix A.

LEMMA 3.1. *One has*

$$\sum_{i=0}^{n}(-1)^i a_i \boldsymbol{a}_{i-j} = \boldsymbol{e}_{-j} + (-1)^{k+1}c\boldsymbol{e}_{2k-j} + (-1)^n \boldsymbol{e}_{2n-j} , \tag{3.4}$$

where $\boldsymbol{e}_l$ *denotes the* l*-th (standard) unit vector in* $\boldsymbol{R}^n$ *and we set* $\boldsymbol{e}_l = 0$ *if* $l \leqq 0$ *or* $l > n$.

This follows from (3.1).

LEMMA 3.2. *One has*

$$A_j = \begin{cases} \displaystyle\sum_{i=0}^{j-1}(-1)^{i-j+1}a_i\boldsymbol{a}_{i-j+1} & \textit{for} \quad 1 \leqq j \leqq k , \\ \displaystyle\sum_{i=j}^{n}(-1)^{i-j}a_i\boldsymbol{a}_{i-j+1} & \textit{for} \quad k+1 \leqq j \leqq n . \end{cases} \tag{3.5}$$

PROOF. Set

$$P = \begin{bmatrix} 0 & \cdot & & & 0 \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ 0 & & & 1 & 0 \end{bmatrix}, \qquad P^* = \begin{bmatrix} 0 & 1 & & & 0 \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ 0 & & & & 0 \end{bmatrix}.$$

Then, by the definition, one has $\boldsymbol{a}_{i+1} = P^i\boldsymbol{a}_1$, $\boldsymbol{a}_{-i} = P^{*i}\boldsymbol{a}_0$ for $i > 0$. From (3.3) one has

$$A_j + PA_{j+1} = a_j\boldsymbol{a}_1 \ , \quad P^*A_j + A_{j+1} = a_j\boldsymbol{a}_0$$

for $1 \leqq j \leqq n-1$, $j \neq k$. We prove (3.5) for $1 \leqq j \leqq k$ by induction on j. For $j = 1$, $A_1 = \boldsymbol{a}_0$ is trivial. Assuming the validity of (3.5) for j $(<k)$, one has

$$A_{j+1} = a_j\boldsymbol{a}_0 - P^*A_j = a_j\boldsymbol{a}_0 - \sum_{i=0}^{j-1}(-1)^{i-j+1}a_i\boldsymbol{a}_{i-j} = \sum_{i=0}^{j}(-1)^{i-j}a_i\boldsymbol{a}_{i-j} \ ,$$

which proves (3.5) for $j+1$. The case $k+1 \leqq j \leqq n$ can be treated similarly, starting from the case $j = n$, where (3.5) reduces to $A_n = \boldsymbol{a}_1$. q.e.d.

(In view of Lemma 3.1, we see that both expressions in (3.5) are valid for *all* j, $1 \leqq j \leqq n$, if one replaces $a_k\boldsymbol{a}_{k-j+1}$ by $a_k\boldsymbol{a}_{k-j+1} + c\boldsymbol{e}_{2k-j+1}$.)

Now from our choice of c (Lemma 2.7) we have

$$\det(A) = 0 \ . \tag{3.6}$$

We will show that the relations (3.2), (3.4), (3.6) imply the integrality of the a_i's. It is known that, under these conditions, the a_i's are algebraic. (See the remark below.) Let K be an algebraic number field of finite degree containing all a_i $(1 \leqq i \leqq n-1)$ and let $\mathfrak{p}$ be any prime ideal in K. Put

$$\nu_0 = \operatorname*{Min}_{0 \leqq i \leqq n} \nu_{\mathfrak{p}}(a_i) \ ,$$

where $\nu_{\mathfrak{p}}$ denotes the (exponential) valuation defined by $\mathfrak{p}$.

LEMMA 3.3. *If $\nu_0 < 0$, one has $\nu_{\mathfrak{p}}(a_k) = \nu_0$ and $\nu_{\mathfrak{p}}(a_l) > \nu_0$ for $l \neq k$.*

PROOF. Let $I_0 = \{i \mid \nu_{\mathfrak{p}}(a_i) = \nu_0, i \neq k\}$. Suppose $\nu_0 < 0$ and $I_0 \neq \varnothing$. Then there exists either a maximal element l in I with $k < l < n$ or a minimal element l in I_0 with $0 < l < k$. In (3.2) one has for any $i > 0$

$$\nu_{\mathfrak{p}}(2a_{l-i}a_{l+i}) = \nu_{\mathfrak{p}}(2) + \nu_{\mathfrak{p}}(a_{l-i}) + \nu_{\mathfrak{p}}(a_{l+i}) \geqq 2\nu_0 = \nu_{\mathfrak{p}}(a_l^2) \ .$$

Hence there must be at least one $i > 0$ such that $\nu_{\mathfrak{p}}(a_{l-i}) = \nu_{\mathfrak{p}}(a_{l+i}) = \nu_0$. If $l > k$ (resp. $< k$), then $l + i$ (resp. $l - i$) $\in I_0$, which is absurd. q.e.d.

Now we prove that $a_1, \cdots, a_{n-1}$ are algebraic integers and c is an algebraic unit. By (3.6) the vectors $A_1, \cdots, A_n$ are linearly dependent. By Lemma 3.2, this is equivalent to saying that the $\boldsymbol{a}_i$ $(1-k \leqq i \leqq n-k)$ are linearly dependent. Thus one has

$$
\Delta = \begin{vmatrix} a_k & \cdots & a_1 & 1 & & & 0 \\ \vdots & \ddots & \vdots & a_1 & \ddots & & \\ \vdots & & a_k & \vdots & & \ddots & \\ a_{n-1} & & \vdots & a_k & & & 1 \\ 1 & \ddots & \vdots & \vdots & \ddots & & a_1 \\ & \ddots & a_{n-1} & \vdots & & \ddots & \vdots \\ 0 & & 1 & a_{n-1} & \cdots & \cdots & a_k \end{vmatrix} = 0 . \tag{3.7}
$$

Suppose $\nu_0 < 0$. Then, by Lemma 3.3, one has

$$
a_k^{-n}\Delta \equiv \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{vmatrix} \equiv 1 \pmod{\mathfrak{p}} ,
$$

which is absurd. Thus one should have $\nu_0 \geqq 0$ for all prime ideals $\mathfrak{p}$ in K. This proves that $a_1, \cdots, a_{n-1}$ are integral. By (3.1), (3.5), c and a_{ij} are also integral.

REMARK. By a similar argument, one can show that, for any generalized valuation ϕ (with values in a linearly ordered abelian group) of any field containing $a_1, \cdots, a_{n-1}$, one has $\phi(a_i) \geqq 0$. This proves that the a_i are algebraic.

Next, we prove that c is a unit. Since the constant c is unchanged if we replace k by $n-k$, we may assume that $k \leqq n/2$. By Lemma 3.1, one has for $1 \leqq j \leqq k$

$$
\sum_{i=0}^{n} (-1)^i a_i \boldsymbol{a}_{j-k+i} = (-1)^{k+1} c \boldsymbol{e}_{j+k} , \tag{$*$}
$$

$$
\sum_{i=1}^{j} (-1)^{n-j+i} a_{n-j+i} \boldsymbol{a}_{-n+i} = (-1)^n \boldsymbol{e}_j . \tag{$**$}
$$

Applying $(-1)^{n-k}cP^k$ on $(**)$ and adding it to $(*)$, one obtains

$$
\sum_{i=0}^{n-j} (-1)^i a_i \boldsymbol{a}_{j-k+i} + \sum_{i=1}^{j} (-1)^{n-j+i} a_{n-j+i} \boldsymbol{a}'_{n-k+i} = 0 \quad (1 \leqq j \leqq k) , \tag{3.8}
$$

where $\boldsymbol{a}'_{n-k+i} = \boldsymbol{a}_{n-k+i} + (-1)^{n-k} cP^k \boldsymbol{a}_{-n+i}$. Since $\boldsymbol{a}_{1-k}, \cdots, \boldsymbol{a}_{n-k}$ are linearly dependent, this implies that $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{n-k}, \boldsymbol{a}'_{n-k+1}, \cdots, \boldsymbol{a}'_n$ are also linearly dependent. From $|\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{n-k}, \boldsymbol{a}'_{n-k+1}, \cdots, \boldsymbol{a}'_n| = 0$, one obtains a relation of the form

$$
cg(c, a_1, \cdots, a_{n-1}) + 1 = 0 ,
$$

where g is a polynomial with coefficients in $\boldsymbol{Z}$. Hence c^{-1} is integral, and so c is an algebraic unit.

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