An algorithm for approximating convex functions by means of first degree splines

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An algorithm for determining minimax approximations to strictly convex functions by means of first degree splines with free knots is proposed. The method may be used in one of two modes. Either the minimax approximation having a prescribed number of knots can be found, or the minimax approximation with the smallest number of knots whose maximum error does not exceed a given error can be determined. The method, while possessing quadratic convergence, does not require a knowledge of derivatives higher than the first.

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1. Introduction

In a number of fields, including the design of diode function generators for analogue computers and the reduction of non-linear programming problems to approximately equivalent linear programming problems, it is sometimes necessary to approximate functions by means of first degree splines. The numerical construction of these splines for a certain class of functions is considered in this paper and algorithms are proposed for their determination.

For a strictly increasing sequence of real numbers $a = u_0, u_1, u_2, \ldots, u_{n-1}, u_n = b$, a first degree spline $s(x)$ with interior knots $u_1, u_2, \ldots, u_{n-1}$ is a function possessing the following two properties:

1. In the rth sub-interval $[u_{r-1}, u_r]$, $r = 1, 2, \ldots, n$, $s(x)$ is a linear function of x.
2. $s(x)$ is continuous everywhere in (a, b).

A non-redundant representation of $s(x)$ is

$$s(x) = q_0 + \sum_{r=1}^{n} q_r(x-u_{r-1})_+ \, ,$$

where

$$x_+ = \begin{cases} x, & x \geqslant 0 \\ 0, & x < 0 \end{cases} .$$

However, it is more convenient to employ here the following redundant representation

$$s(x) = \begin{cases} s_1(x) = c_1 + d_1(x - u_0), & x \in [u_0, u_1] \\ s_2(x) = c_2 + d_2(x - u_1), & x \in [u_1, u_2] \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ s_n(x) = c_n + d_n(x - u_{n-1}), & x \in [u_{n-1}, u_n]. \end{cases} \quad (1)$$

For continuity of $s(x)$ we require

$$s_r(u_r) = s_{r+1}(u_r), \; r = 1, 2, \ldots, n-1,$$

and hence the relationships

$$d_r(u_r - u_{r-1}) = c_{r+1} - c_r, \; r = 1, 2, \ldots, n-1,$$

exist among the parameters of $s(x)$. $s_r(x)$ is referred to as the rth segment of $s(x)$.

For a function $f(x)$, strictly convex in (a, b), the following problems are considered:

P1: Given $\varepsilon > 0$, determine parameters $n, c_r, d_r, r = 1, 2, \ldots, n$, and $u_r, r = 1, 2, \ldots, n-1$, of $s(x)$ such that n is the smallest integer for which $\max_{x \in [a,b]} | s(x) - f(x) | \leqslant \varepsilon$.

P2: Given an integer $n > 0$, determine parameters $\varepsilon, c_r, d_r, r = 1, 2, \ldots, n$, and $u_r, r = 1, 2, \ldots, n-1$, of $s(x)$ such that $\varepsilon = \max_{x \in [a,b]} | s(x) - f(x) |$ is minimised.

We shall show that in general the solution to P1 is not unique. A natural extension of P1 is the following problem:

P3: Of all solutions to P1 determine a solution $s(x)$ such that $\max_{x \in [a,b]} | s(x) - f(x) |$ is minimised.

It is apparent that once the minimising value of n in P1 has been determined, the solution of P2, which we shall show is unique, then solves P3.

Phillips (1968) gives a direct method for solving P1 and suggests an iterative method for P2 for that class of strictly convex functions $f(x) \in C^2(a, b)$. Here $C^p(a, b)$ represents the class of functions that is p times differentiable in (a, b). We also develop a direct technique for P1 and iterative techniques for P2 (and P3) that hold for the less restricted class of strictly convex functions $f(x) \in C^1(a, b)$. Phillips' method for solving P2, although undoubtedly numerically stable, being based upon the method of bisections for determining the minimax error ε^*, is resultingly very slow. It is shown here how a recurrence relation existing between the knot derivatives $du_r/d\varepsilon$ may be utilised to yield a quadratically convergent method for the solution of P2. Moreover, knowledge of the knot derivatives gives an excellent indication of the sensitivity of the positions of the knots with respect to small perturbations about ε^*.

Throughout this paper the dependence of s and other functions upon ε is only indicated explicitly when such emphasis is required or when ambiguity is likely to arise.

2. Approximation with a prescribed maximum error

We solve P1 in a manner similar to Phillips but show that the method holds for $f(x) \in C^1(a, b)$. Given ε we construct sub-intervals $[u_{r-1}, u_r]$, $r = 1, 2, \ldots,$ such that the resulting approximation is continuous and over each interval is minimax. The rth such sub-interval is characterised by the following four equations (Davis, 1963, p. 151),

$$s(u_{r-1}) - f(u_{r-1}) + \varepsilon = 0, \quad (2)$$

$$s(v_r) - f(v_r) - \varepsilon = 0, \quad (3)$$

$$s'(v_r) - f'(v_r) = 0, \quad (4)$$

$$s(u_r) - f(u_r) + \varepsilon = 0, \quad (5)$$

Here $v_r \in (u_{r-1}, u_r)$. It is to be noted that in deriving (2), (3), (4) and (5) Davis considers the case $f(x) \in C^2(a, b)$, but in fact it is sufficient that $f(x) \in C^1(a, b)$. Then (1), (2) and (4) give

$$c_r = f(u_{r-1}) - \varepsilon \quad (6)$$

and

$$d_r = f'(v_r). \quad (7)$$

Thus, if sub-intervals 1 to $r - 1$ have already been determined, u_{r-1} is known and hence c_r can be evaluated immediately from (6). It is now shown how v_r, d_r and u_r can be found. From (1), (3), (6) and (7),

$$f(v_r) - f(u_{r-1}) - (v_r - u_{r-1})f'(v_r) + 2\varepsilon = 0. \quad (8)$$

Only v_r is unknown in (8). Let

$$F_r(x) \equiv f(x) - f(u_{r-1}) - (x - u_{r-1})f'(x) + 2\varepsilon. \quad (9)$$

Then the following theorem may be stated.

Theorem 1:
$F_r(x)$ has at most one zero in (u_{r-1}, b).

Proof:
Consider any two values of x, viz. x_1 and x_2, such that $u_{r-1} < x_1 < x_2 < b$. Then, using the mean value theorem,

$$\begin{aligned} &F_r(x_2) - F_r(x_1) \\ &= f(x_2) - f(x_1) - (x_2 - u_{r-1})f'(x_2) + (x_1 - u_{r-1})f'(x_1) \\ &= f(x_2) - f(x_1) - (x_2 - x_1)f'(x_2) \\ &\qquad - (x_1 - u_{r-1})\{f'(x_2) - f'(x_1)\} \\ &= -(x_2 - x_1)\{f'(x_2) - f'(x_m)\} \\ &\qquad - (x_1 - u_{r-1})\{f'(x_2) - f'(x_1)\}, \end{aligned}$$

where $x_1 < x_m < x_2$. Hence, since $f(x)$ is convex for $x \in (a, b)$, $F_r(x_2) - F_r(x_1) < 0$. It follows that $F_r(x)$ is a decreasing function for $u_{r-1} < x < b$. Therefore $F_r(x)$ has at most one zero in (u_{r-1}, b).

But $F_r(u_{r-1}) = 2\varepsilon > 0$. So if $F_r(b) < 0$ the equation $F_r(x) = 0$ has a unique root v_r in (u_{r-1}, b). This root may be found using, for example, an iterative technique that brackets the root at each stage. v_r having been obtained, d_r is calculated from (7). The rth sub-interval is completely determined if u_r can be found. From (5), (6) and (7)

$$(u_r - u_{r-1})f'(v_r) - \{f(u_r) - f(u_{r-1})\} = 0. \quad (10)$$

Only u_r is unknown in (10). Let

$$G_r(x) \equiv (x - u_{r-1})f'(v_r) - \{f(x) - f(u_{r-1})\}. \quad (11)$$

Then we have the following theorem.

Theorem 2:
$G_r(x)$ has at most one zero in (v_r, b).

Proof:
From (11),

$$G_r'(x) = f'(v_r) - f'(x).$$

Hence, since $f(x)$ is strictly convex in (a, b), it follows that $G_r'(x) < 0$ for $v_r < x < b$. So $G_r(x)$ is a decreasing function and hence possesses at most one zero in (v_r, b).

From (8) and (11), $G_r(v_r) = 2\varepsilon > 0$. Therefore if $G_r(b) < 0$ the equation $G_r(x) = 0$ may be solved for the unique root u_r, again using for example a bracketing technique.

Thus, given ε, the values c_r, v_r, d_r and u_r can be determined for $r = 1, 2, \ldots$. In particular, for any given value of ε, the sequence $u_r(\varepsilon)$, $r = 1, 2, \ldots$, can be generated. In the following section we show that at the same time the values of the knot derivatives $du_r/d\varepsilon$ can also be determined with little extra effort.

3. Knot derivatives

Elimination of $f'(v_r)$ from (8) and (10) gives

$$\begin{aligned} &(u_r - u_{r-1})\{f(v_r) - f(u_{r-1}) + 2\varepsilon\} \\ &\qquad - (v_r - u_{r-1})\{f(u_r) - f(u_{r-1})\} = 0. \quad (12) \end{aligned}$$

An incremental change $\delta\varepsilon$ in ε in (12) gives

$$\begin{aligned} &(\delta u_r - \delta u_{r-1})\{f(v_r) - f(u_{r-1}) + 2\varepsilon\} \\ &\quad + (u_r - u_{r-1})\{f'(v_r)\delta v_r - f'(u_{r-1})\,\delta u_{r-1} + 2\delta\varepsilon\} \\ &\qquad - (\delta v_r - \delta u_{r-1})\{f(u_r) - f(u_{r-1})\} \\ &\quad - (v_r - u_{r-1})\{f'(u_r)\,\delta u_r - f'(u_{r-1})\,\delta u_{r-1}\} = 0, \quad (13) \end{aligned}$$

ignoring terms of higher order. Simplification of (13) together with further use of (8) and (10) yields

$$\begin{aligned} &(v_r - u_{r-1})\{f'(u_r) - f'(v_r)\}\,\delta u_r = \\ &\quad (u_r - v_r)\{f'(v_r) - f'(u_{r-1})\}\,\delta u_{r-1} + 2(u_r - u_{r-1})\,\delta\varepsilon, \quad (14) \end{aligned}$$

the coefficient of δv_r being identically zero. After dividing (14) by $(v_r - u_{r-1})(u_r - v_r)\,\delta\varepsilon$ and taking the limit as $\delta\varepsilon \to 0$ we obtain the result

$$A_r \frac{du_r}{d\varepsilon} = B_r \frac{du_{r-1}}{d\varepsilon} + C_r\,, \quad (15)$$

where

$$A_r = \frac{f'(u_r) - f'(v_r)}{u_r - v_r}\,,$$

$$B_r = \frac{f'(v_r) - f'(u_{r-1})}{v_r - u_{r-1}}\,,$$

$$C_r = \frac{2(u_r - u_{r-1})}{(u_r - v_r)(v_r - u_{r-1})}\,,$$

which is a two-point recurrence relation for $du_r/d\varepsilon$.

We observe that for the class of functions $f(x) \in C^1(a, b)$ considered here, $dv_r/d\varepsilon$ does not in general exist. Hence (15) could not have been obtained by differentiating (12) with respect to ε and simplifying by means of (8) and (10), since the existence of $dv_r/d\varepsilon$ is required. However, for the more restricted class $f(x) \in C^2(a, b)$, $dv_r/d\varepsilon$ does exist and, since

$$\frac{d}{d\varepsilon} f'(v_r) = f''(v_r)\frac{dv_r}{d\varepsilon}\,,$$

is identically equal to $[df'(v_r)/d\varepsilon]/f''(v_r)$. Note that $f''(v_r) \neq 0$ since f is strictly convex.

Since $u_0 = a$ = a constant, $du_0/d\varepsilon = 0$ and (15) may therefore be used to generate the values of $du_r/d\varepsilon$ for $r = 1, 2, \ldots$.

We are now in a position to derive some results relating u_r and v_r to ε.

Theorem 3:
$u_r(\varepsilon)$ is a strictly increasing function.

Proof:
In (15), $C_r > 0$ and from the convexity of f, A_r, $B_r > 0$. Hence if $du_{r-1}/d\varepsilon \geqslant 0$ it follows that $du_r/d\varepsilon > 0$. But $du_0/d\varepsilon = 0$. Hence by induction $du_r/d\varepsilon > 0$ for $r = 1, 2, \ldots$. It follows that $u_r(\varepsilon)$ is a strictly increasing function.

Theorem 4:
$v_r(\varepsilon)$ is a strictly increasing function.

Proof:
Consider the sets $\{u_r \mid r = 1, 2, \ldots\}$ and $\{v_r \mid r = 1, 2, \ldots\}$ corresponding to the maximum error ε and the sets $\{\tilde{u}_r \mid r = 1, 2, \ldots\}$ and $\{\tilde{v}_r \mid r = 1, 2, \ldots\}$ corresponding to the maximum error $\tilde{\varepsilon} > \varepsilon$. It follows from Theorem 3 that $\tilde{u}_{r-1} > u_{r-1}$. We show this implies that $\tilde{v}_r > v_r$. From (8),

$$f(\tilde{v}_r) - f(\tilde{u}_{r-1}) - (\tilde{v}_r - \tilde{u}_{r-1})f'(\tilde{v}_r) + 2\tilde{\varepsilon} = 0. \quad (16)$$

Subtraction of (8) from (16) gives

$$\begin{aligned} &f(\tilde{v}_r) - f(v_r) - \{f(\tilde{u}_{r-1}) - f(u_{r-1})\} \\ &\quad - (\tilde{v}_r - \tilde{u}_{r-1})f'(\tilde{v}_r) + (v_r - u_{r-1})f'(v_r) + 2(\tilde{\varepsilon} - \varepsilon) = 0. \quad (17) \end{aligned}$$

We now assume that $\tilde{v}_r \leqslant v_r$ and show that this assumption leads to a contradiction. From (17), it follows on using the mean value theorem that

$$\begin{aligned} &(\tilde{v}_r - \tilde{u}_{r-1})\{f'(\xi_r) - f'(\tilde{v}_r)\} + (v_r - \tilde{u}_{r-1})\{f'(v_r) - f'(\xi_r)\} \\ &\quad + (\tilde{u}_{r-1} - u_{r-1})\{f'(v_r) - f'(\eta_r)\} + 2(\tilde{\varepsilon} - \varepsilon) = 0, \quad (18) \end{aligned}$$

where $u_{r-1} < \eta_r < \tilde{u}_{r-1} < \tilde{v}_r \leqslant \xi_r \leqslant v_r$. But since the first two terms on the left hand side of (18) are non-negative and the remaining two are positive this is a contradiction. Hence the assumption that $\tilde{v}_r \leqslant v_r$ is incorrect. It follows that $\tilde{v}_r > v_r$ and therefore that $v_r(\varepsilon)$ is a strictly increasing function.

Definition: We say that the rth segment is defined if $F_r(b) < 0$; i.e. the value of d_r can be determined. This is always the case if $\varepsilon < \varepsilon^*$.

A solution to P1 may be found by successively generating subintervals $[u_{r-1}, u_r]$, $r = 1, 2, \ldots$, until a value of r, $= n$, say, is reached for which either $|s(b) - f(b)| \leqslant \varepsilon$ or the nth segment is not defined. In the latter case we can follow Phillips (1968) and choose as the nth segment the straight line

$$s_n(x) = f(u_{n-1}) - \varepsilon + \frac{f(b) - f(u_{n-1})}{b - u_{n-1}} (x - u_{n-1}) .$$

Alternatively, any straight line of the form

$$s_n(x) = f(u_{n-1}) - \varepsilon + K(x - u_{n-1}),$$

where K is such that the maximum error committed is not greater than ε for $x \in [u_{n-1}, b]$, could be selected. In any case the resulting approximation $s(x)$ is a solution to P1 and is in general not unique.

4. Approximation with a prescribed number of knots

Suppose ε is such that the last (nth) segment is defined. The error of the approximation at $x = b$ is $s(b) - f(b)$. If $\varepsilon = \varepsilon^*$ then the corresponding value of the function

$$H(\varepsilon) = s(b) - f(b) + \varepsilon \qquad (19)$$

is zero. So if $H(\varepsilon)$ possesses a single zero, it is identical to the required minimax error ε^*. We demonstrate that this is indeed the case after first determining an expression for $dH/d\varepsilon$. Equation (19), together with (1), (6) and (7), gives

$$\begin{aligned} H(\varepsilon) &= c_n + d_n(b - u_{n-1}) - f(b) + \varepsilon \\ &= (b - u_{n-1})f'(v_n) - \{f(b) - f(u_{n-1})\} . \end{aligned} \qquad (20)$$

Using (8), (20) becomes

$$\begin{aligned} (v_n - u_{n-1})\, H(\varepsilon) = (b - u_{n-1})\, &\{f(v_n) - f(u_{n-1}) + 2\varepsilon\} \\ &- (v_n - u_{n-1})\, \{f(b) - f(u_{n-1})\} . \end{aligned} \qquad (21)$$

An incremental change $\delta\varepsilon$ in ε in (21) gives

$$\begin{aligned} &(\delta v_n - \delta u_{n-1})\, H(\varepsilon) + (v_n - u_{n-1})\, \delta H = \\ &\quad - \delta u_{n-1}\{f(v_n) - f(u_{n-1}) + 2\varepsilon\} \\ &\quad\quad + (b - u_{n-1})\, \{f'(v_n)\, \delta v_n - f'(u_{n-1})\, \delta u_{n-1} + 2\delta\varepsilon\} \\ &\quad - (\delta v_n - \delta u_{n-1})\, \{f(b) - f(u_{n-1})\} \\ &\quad\quad + (v_n - u_{n-1})f'(u_{n-1})\, \delta u_{n-1}, \end{aligned} \qquad (22)$$

ignoring terms of higher order. After substituting for $H(\varepsilon)$ from (20), simplifying by means of (8), dividing by $\delta\varepsilon$ and taking the limit as $\delta\varepsilon \to 0$, the following expression for $dH/d\varepsilon$ is obtained,

$$\begin{aligned} &(v_n - u_{n-1})\, dH/d\varepsilon = \\ &(b - v_n)\, \{f'(v_n) - f'(u_{n-1})\}\, du_{n-1}/d\varepsilon + 2(b - u_{n-1}), \end{aligned} \qquad (23)$$

the calculation of which requires only values already determined.

Theorem 5:
$H(\varepsilon)$ has at most one zero.

Proof:
(23) is of the form

$$R dH/d\varepsilon = S du_{n-1}/d\varepsilon + T,$$

where R, S and T are positive. But from Theorem 3 $du_{n-1}/d\varepsilon \geqslant 0$. Hence $dH/d\varepsilon > 0$ and therefore $H(\varepsilon)$ has at most one zero. Moreover it follows that the solution to P2 is unique.

A further consequence of Theorem 3 is that the number of knots needed to obtain a first degree spline approximation of $f(x)$ having maximum error ε is a non-decreasing function of ε. We now consider a means of obtaining a minimax approximation to $f(x)$ over $[a, b]$ having exactly $n - 1$ knots, i.e. n segments.

A trial value of ε is first employed. If the nth segment is defined and the resulting value of $H(\varepsilon) < 0$ then it follows from Theorem 5 that ε is a lower bound on the minimax error ε^*. If, on the other hand, for some value of $r \leqslant n$ the rth segment is not defined, or if the nth segment is defined but $H(\varepsilon) > 0$, then ε is an upper bound on ε^*. Thus the trial value of ε furnishes either a lower or upper bound on ε^*. The following strategy is then suggested. If $H(\varepsilon) < 0$ then a new trial value of ε is obtained by replacing ε by $K_1\varepsilon$. If, on the other hand, the nth segment is not defined or if $H(\varepsilon) > 0$ then ε is replaced by $K_2\varepsilon$. Here $K_1 > 1$, $0 < K_2 < 1$ (e.g. $K_1 = 2$, $K_2 = \frac{1}{2}$). This process is repeated until lower and upper bounds ε_1 and ε_2, respectively, are obtained. At each stage a bound is updated in an obvious manner. It may happen that corresponding to the error ε_2 the nth segment is not defined, in which case repeated bisection of the interval $[\varepsilon_1, \varepsilon_2]$ is carried out until the nth segment is defined. Ultimately a stage is reached where $\varepsilon^* \in [\varepsilon_1, \varepsilon_2]$ and for all $\varepsilon \in [\varepsilon_1, \varepsilon_2]$, $H(\varepsilon)$ and $dH/d\varepsilon$ can be evaluated. It is now possible to employ an iterative scheme of higher order to determine the unique zero ε^* of $H(\varepsilon)$. One such technique based on successive linear interpolations and extrapolations, together with a strategy to avoid possible divergence, is given by van Wijngaarden, Zonneveld, Dijkstra and Dekker (1963) and described by Peters and Wilkinson (1969), who also give an ALGOL procedure for the method. The method uses function values only and the ultimate convergence rate is superlinear. Another technique (Cox, 1970) is based on successive interpolation by rational functions. Both function values and derivatives are employed and the method possesses quadratic convergence. An ALGOL procedure is also given for this method. Since derivatives are readily available the latter method, because of its higher convergence rate, is to be preferred in this context.

The solution of P2 as described above may be accelerated by making use of the results of Theorems 3 and 4. Let the solution values of u_r and v_r be u_r^* and v_r^* and $\underline{u}_r$ and $\bar{u}_r$ be lower and upper bounds on u_r^*, and $\underline{v}_r$ and $\bar{v}_r$ be lower and upper bounds on v_r^* for $r = 1, 2, \ldots, n$. Initial values for these bounds are $\underline{u}_r = \underline{v}_r = a$ and $\bar{u}_r = \bar{v}_r = b$. If, for a specific value of ε, $H(\varepsilon) < 0$ then the resulting values of u_r and v_r replace $\underline{u}_r$ and $\underline{v}_r$, otherwise they replace $\bar{u}_r$ and $\bar{v}_r$. Thus as the calculation proceeds increasingly tighter bounds on u_r and v_r are obtained.

We turn now to the manner in which (10) and (8) may be solved for u_r and v_r. From (11) we find that

$$G_r'(x) = f'(v_r) - f'(x) .$$

Hence the derivative $G_r'(x)$ is readily evaluated and we may again employ the quadratically convergent method of Cox (1970) to determine the zero u_r of $G_r(x)$ using $\underline{u}_r$ and $\bar{u}_r$ as initial bounds on the solution. Turning to (9) we see that the derivative of $F_r(x)$ is not available, since it requires knowledge of $f''(x)$. However, we can employ the method described by Peters and Wilkinson (1969), which requires function values only, in order to determine v_r, using v_r and $\bar{v}_r$ as initial bounds on the solution.

The approach of determining the minimax solution by way of finding the zero of $H(\varepsilon)$ compares favourably with the method of Phillips (1968). He states that it is clear that the number of segments is a non-increasing function, $J(\varepsilon)$, say. In fact $J(\varepsilon)$ is a step function and the minimax error is identified by the property that $J(\varepsilon^*-) = n + 1$ and $J(\varepsilon^*+) = n$. Hence initial bounds on ε^* can be established in a manner similar to that described here, but by utilising the function $n - J(\varepsilon)$ instead of $H(\varepsilon)$. The method of bisections is then applied repeatedly to this function until the desired accuracy is obtained. Since $J(\varepsilon)$ is a step function it is necessary to employ a low order 'go–no go' process such as bisections, which merely utilises the signs of the function values. Higher order

Table 1 **Progress of the algorithm for solving P3**

ITERATION	ε	$H(\varepsilon)$	$dH/d\varepsilon$
1	0·01	—	—
2	0·005	−0·175863	164·676
3	0·0075	—	—
4	0·00625	−0·027503	76·034
5	0·006875	0·007615	36·008
6	0·00669179	$-9·92 \times 10^{-5}$	48·052
7	0·00669385	10^{-9}	47·920
8	0·00669385	-10^{-9}	47·920

Table 2 Parameters of the minimax approximation of erfc(x).

r	u_r	$du_r/d\varepsilon$	c_r	d_r
1	0·46524	24·538	0·99331	−1·05199
2	0·80149	47·934	0·50388	−0·75408
3	1·15804	85·631	0·25032	−0·43621
4	1·64573	187·859	0·09479	−0·16719
5	4·00000	—	0·01325	−0·00847

processes requiring continuity and perhaps differentiability of $J(\varepsilon)$ can obviously not be applied.

5. Numerical examples

The first example we consider is that given by Phillips. Consider the minimax approximation of $f(x) = e^x$ over the range $7 \leqslant x \leqslant 1$ by means of a first degree spline with three interior knots. Starting with $\varepsilon = 0·01$, Phillips' method requires 22 iterations to determine ε^* to an accuracy of eight decimal places, whereas the method described here requires only six iterations.

As a second example we consider the minimax approximation of the complementary error function

$$f(x) \equiv \text{erfc}\,(x) = -\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt$$

over the range $0 \leqslant x \leqslant 4$ by a first degree spline whose maximum error is not greater than 0·01. For this case we give detailed results. The first iteration of the algorithm indicates that five segments suffice to obtain the required accuracy. The progress of the algorithm is given in **Table 1**. After five iterations lower and upper defined bounds on ε^* have been established. Three further iterations achieve the minimax solution to full machine accuracy (nine significant decimal places), the last iteration merely serving as a check on convergence. **Table 2** gives the values of u_r, $du_r/d\varepsilon$, c_r and d_r at the minimax solution. The minimax error is $\varepsilon^* = 0·00669385$. The values of the knot spacings $u_{r+1} - u_r$, viz., 0·46524, 0·33625, 0·35655, 0·48769, 2·35427 are particularly interesting because of their non-uniformity.

6. Conclusions

An algorithm for determining minimax approximations to a class of functions $f(x)$ by means of first degree splines with free knots has been given. The results of Phillips (1968) have been extended to the class of strictly convex functions $f(x) \in C^1(a, b)$. The method converges quadratically to the solution, which is shown to be unique.

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