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An Algorithm for Computing Logarithms and Arctangents

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Abstract. An iterative algorithm with fast convergence can be used to compute logarithms, inverse circular functions, or inverse hyperbolic functions according to the choice of initial conditions. Only rational operations and square roots are required. The method consists in adding an auxiliary recurrence relation to Borchardt's algorithm to speed the convergence.

1. Introduction. Logarithms, inverse circular functions, and inverse hyperbolic functions are computed in practice by a variety of methods, including infinite series, continued fractions, Chebyshev approximations, and rational approximations [4]. A method known as Borchardt's algorithm [2, Eq. (2.3)], [3] employs very simple recurrence relations containing a square root but is seldom used, because the error is reduced by a factor of only four per cycle. Thus, 10D accuracy typically requires fifteen to twenty square roots. A related algorithm, due to Thacher [6], has a factor of sixteen per cycle but loses some significant figures through cancellation.

In the present paper, we introduce an auxiliary recurrence relation to speed the convergence of Borchardt's algorithm so that only three or four square roots are required for 10D accuracy. (We assume here that the range of the independent variable has been reduced by familiar devices such as $\arctan(1/x) = \pi/2 - \arctan x$. More cycles are required outside the reduced range.) The method amounts to repeated application of a well-known technique [5, pp. 86–87] for improving convergence by extrapolation. The recurrence relations are the same for all the functions considered, there are no constants to be stored, and no serious cancellation occurs. The precision is limited only by the number of cycles performed, and the rate of convergence gradually accelerates. Although still not as fast as some other methods because of the square roots, the algorithm might well be preferable when economy of storage space is important. It might be useful also in verifying the accuracy of faster algorithms. The method is being extended to computation of elliptic integrals, but only elementary functions are discussed in the present paper.

2. Statement of the Algorithm. The same recurrence relations will be used for computing a logarithm, an inverse circular function, or an inverse hyperbolic function, but the initial values and the final step will depend on which function is being computed. For definiteness, we state first the complete algorithms for computing natural logarithms and arctangents and later give the changes necessary for computing other functions.

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Algorithm for logarithms. If $x > 0$, compute

$$a_0 = \tfrac{1}{2}(1 + x), \qquad g_0 = x^{1/2}, \tag{2.1}$$

$$a_{n+1} = \tfrac{1}{2}(a_n + g_n), \qquad g_{n+1} = (a_{n+1}g_n)^{1/2}, \qquad n = 0, 1, 2, \cdots, \tag{2.2}$$

$$d(0, n) = a_n, \qquad n = 0, 1, 2, \cdots, \tag{2.3}$$

$$d(k, n) = \frac{d(k-1, n) - 2^{-2k}\, d(k-1, n-1)}{1 - 2^{-2k}}, \qquad k = 1, 2, \cdots, n. \tag{2.4}$$

Then,

$$\log x = \frac{x - 1}{d(n, n)}(1 + \epsilon_n), \tag{2.5}$$

where the fractional error ϵ_n is given approximately by

$$\epsilon_n \cong 2^{-n^2-3n-1}\left(\frac{\log x}{\pi}\right)^{2n+2}. \tag{2.6}$$

Algorithm for arctangents. If x is real, let

$$a_0 = 1, \qquad g_0 = (1 + x^2)^{1/2}, \tag{2.7}$$

and compute $d(n, n)$ according to (2.2), (2.3), and (2.4). Then,

$$\arctan x = \frac{x}{d(n, n)}(1 + \epsilon_n), \tag{2.8}$$

where the fractional error is given approximately by

$$\epsilon_n \cong (-1)^{n+1} 2^{-n^2-n+1}\left(\frac{\arctan x}{\pi}\right)^{2n+2}. \tag{2.9}$$

To save computation time, the denominator of (2.4) may be deleted if the right-hand sides of (2.5) and (2.8) are multiplied by $(1-1/4)(1-1/16)(1-1/64)\cdots(1-2^{-2n})$. Then, (2.4) requires only a shift in binary arithmetic and a subtraction. To show that the subtraction never leads to loss of significant figures, we observe first that for $k = 1$ the numerator of (2.4) is $a_n - \frac{1}{4}a_{n-1}$. By (2.2), the first term is more than twice the second, and hence, serious cancellation cannot occur. As k increases, the amount of cancellation decreases, for it is easy to show by induction that the ratio of the first to the second term in the numerator of (2.4) exceeds 2^k.

In some circumstances, one might also save computation time by taking g_n as a first approximation to g_{n+1}, when extracting the square root in (2.2) by the iterative method [4, pp. 90–91].

The functions arcsin x and arccos x can be computed directly by starting from the initial values a_0 and g_0 shown in Table I and ending the algorithm according to the last column of the table. The inverse hyperbolic functions, which are expressible in terms of logarithms, can be computed in the same way. We have simplified the initial values by using the fact that $d(n, n)$ is homogeneous of first degree in a_0 and g_0. In the case of the inverse circular functions, the approximate error is given by (2.9), with arctan x replaced by the function $f(x)$ in the first column of the table. For the inverse hyperbolic functions, it is given by (2.6), with log x replaced by $2f(x)$, the factor 2 coming from the left side of

$$2 \operatorname{arctanh} x = \log (1 + x)/(1 - x). \tag{2.10}$$

TABLE I. Initial and final data for computing inverse circular and inverse hyperbolic functions. The first two rows correspond to (2.1), (2.5), (2.7), and (2.8). Although $\arccos x$ can be computed if $-1 < x < 0$, loss of significant figures occurs near -1.

$f(x)$	Range of x	a_0	g_0	$\frac{f(x)d(n,n)}{1+\epsilon_n}$
$\log x$	$x > 0$	$\frac{1}{2}(1 + x)$	$x^{1/2}$	$x - 1$
$\arctan x$	$-\infty < x < \infty$	1	$(1 + x^2)^{1/2}$	x
$\arcsin x$	$-1 \leqq x \leqq 1$	$(1 - x^2)^{1/2}$	1	x
$\arccos x$	$0 \leqq x \leqq 1$	x	1	$(1 - x^2)^{1/2}$
$\operatorname{arctanh} x$	$-1 < x < 1$	1	$(1 - x^2)^{1/2}$	x
$\operatorname{arcsinh} x$	$-\infty < x < \infty$	$(1 + x^2)^{1/2}$	1	x
$\operatorname{arccosh} x$	$x \geqq 1$	x	1	$(x^2 - 1)^{1/2}$

3. **Numerical Examples.** If $y > 0$, we can find an integer m such that $y = 2^m x$, where x lies in the reduced range $2^{-1/2} < x \leqq 2^{1/2}$. Since $\log y = \log x + m \log 2$, it suffices to compute logarithms in the reduced range, and the worst case is $x = 2^{1/2}$ because the rate of convergence improves as $|\log x|$ decreases, according to (2.6). Therefore, we take as an example the computation of $\log x$ for $x = 1.41421\ 35623\ 731$ and list the values of g_n, $a_n = d(0, n)$, and $d(k, n)$, all rounded to 10D:

n	g_n	a_n	$d(1, n)$	$d(2, n)$
0	1.18920 71150	1.20710 67812		
1	1.19367 36437	1.19815 69481	1.19517 36704	
2	1.19479 39440	1.19591 52959	1.19516 80785	1.19516 77057
3		1.19535 46200	1.19516 77280	1.19516 77046

$$d(3, 3) = 1.19516\ 77046$$

$$\log x = \frac{x - 1}{d(3, 3)} = \frac{0.41421\ 35624}{1.19516\ 77046} = 0.34657\ 35903$$

This result is the correct value of $\frac{1}{2} \log 2$ to 10D. Because g_3 is not needed to find $d(3, 3)$, only three square roots were extracted. Although g_n, a_n, and $d(n, n)$ approach the same limit, a_3 and $d(3, 3)$ agree to only 3D. Since Borchardt's algorithm would use a_3 in place of $d(3, 3)$, the auxiliary relation (2.4) produces a dramatic improvement in accuracy without additional square roots.

The fractional error ϵ_n computed from (2.5) can be compared with the estimate (2.6). For $n = 0$, the actual value is $\epsilon_0 = 0.010$ and the estimate is 0.006. For $n > 0$, the actual errors are $\epsilon_1 = 5 \times 10^{-6}$, $\epsilon_2 = 9 \times 10^{-10}$, $\epsilon_3 = 4 \times 10^{-14}$, and the estimates agree to one significant figure. These errors show the gradual acceleration of convergence expected from the exponent $-n^2$ in (2.6).

Outside the reduced range, the fractional errors are larger and the estimates are less accurate. For example, if $x = 1024$, some actual errors are $\epsilon_0 = 2.5$, $\epsilon_2 = 0.02$, $\epsilon_4 = 2 \times 10^{-6}$, $\epsilon_6 = 6 \times 10^{-13}$, and, for $n > 0$, the estimated errors are approximately three times the actual errors.

In computing arctangents, the relation $\arctan(1/x) = \pi/2 - \arctan x$ makes it sufficient to consider the reduced range $-1 \leqq x \leqq 1$. We take as an example the worst case, $x = 1$:

n	g_n	a_n	$d(1, n)$	$d(2, n)$
0	1.41421 35624	1.00000 00000		
1	1.30656 29649	1.20710 67812	1.27614 23749	
2	1.28145 77239	1.25683 48730	1.27341 09036	1.27322 88056
3	1.27528 71547	1.26914 62985	1.27325 01069	1.27323 93871
4		1.27221 67266	1.27324 02026	1.27323 95423

n	$d(3, n)$	$d(4, n)$
3	1.27323 95551	
4	1.27323 95448	1.27323 95447

$$\arctan x = \frac{x}{d(4, 4)} = \frac{1}{1.27323\ 95447} = 0.78539\ 81634$$

This result is the correct value of $\pi/4$ to 10D. Only four square roots were extracted. Since a_4 agrees with $d(4, 4)$ to only 2D, the auxiliary relation (2.4) has supplied eight additional decimal places.

The actual fractional errors calculated from (2.8) are $\epsilon_0 = -0.2$, $\epsilon_1 = 0.002$, $\epsilon_2 = -8 \times 10^{-6}$, $\epsilon_3 = 8 \times 10^{-9}$, $\epsilon_4 = -2 \times 10^{-12}$, and the errors estimated by (2.9) agree to within one unit in the first significant figure. The gradual acceleration of convergence expected from (2.9) is again discernible.

4. Proof of the Algorithm.

The solution of the recurrence relations (2.2) with initial conditions (2.1) is easily verified to be

$$(4.1) \quad a_n = 2^{-n-1}(x - 1)\coth(2^{-n-1} \log x), \qquad g_n = 2^{-n-1}(x - 1)\operatorname{csch}(2^{-n-1} \log x),$$

from which it follows that

$$(4.2) \qquad \lim a_n = \lim g_n = (x - 1)/(\log x)$$

as $n \to \infty$. The solution with initial conditions (2.7) is

$$(4.3) \qquad a_n = 2^{-n}x \cot(2^{-n} \arctan x), \qquad g_n = 2^{-n}x \csc(2^{-n} \arctan x),$$

which implies

$$(4.4) \qquad \lim a_n = \lim g_n = x/(\arctan x).$$

Both solutions can be deduced directly from the usual form of Borchardt's algorithm [2].

Consider (4.1) and define for $x > 0$ and $t > 0$,

$$(4.5) \qquad L = (x - 1)/(\log x), \qquad \tau^{1/2} = \tfrac{1}{2} \log x, \qquad f(t) = Lt^{1/2} \coth t^{1/2},$$

with $f(0) = L$. If $\theta = \frac{1}{4}$, then

$$(4.6) \qquad a_n = f(\theta^n \tau).$$

Assume for the moment that $|\tau| < \pi^2$, so that $\tau^{1/2} \coth \tau^{1/2}$ can be represented by a power series in τ. Then,

$$(4.7) \qquad \begin{aligned} a_0 &= f(\tau) = c_0 + c_1\tau + c_2\tau^2 + c_3\tau^3 + \cdots, \\ a_1 &= f(\theta\tau) = c_0 + \theta c_1\tau + \theta^2 c_2\tau^2 + \theta^3 c_3\tau^3 + \cdots, \\ a_2 &= f(\theta^2\tau) = c_0 + \theta^2 c_1\tau + \theta^4 c_2\tau^2 + \theta^6 c_3\tau^3 + \cdots, \end{aligned}$$

and so on, where c_n depends on the Bernoulli number B_{2n}:

$$(4.8) \qquad c_n = L2^{2n}B_{2n}/(2n)!.$$

The problem is to estimate $c_0 = f(0) = \lim a_n$, knowing the values of a finite number of the a_n. We can eliminate the terms linear in τ by forming

$$(4.9) \qquad \begin{aligned} \frac{a_1 - \theta a_0}{1 - \theta} &= c_0 - \theta c_2\tau^2 - \theta(1 + \theta)c_3\tau^3 - \cdots, \\ \frac{a_2 - \theta a_1}{1 - \theta} &= c_0 - \theta^3 c_2\tau^2 - \theta^4(1 + \theta)c_3\tau^3 - \cdots, \end{aligned}$$

and so on. The left side of the last equation gives an improved estimate of c_0. If the first equation of (4.9) is multiplied by θ^2 and subtracted from the second, the terms in τ^2 are removed, and the procedure can be continued to any order in τ. The same result, an extrapolation from $a_0, a_1, \cdots, a_n$ to $\lim a_n$, is achieved by the following lemma, which does not require convergence of the power series.

Lemma 1. *Let $f(t)$ and its first $n + 1$ derivatives be defined and continuous on the interval $0 \leqq t \leqq \tau$. Let θ be a fixed number, $0 < \theta < 1$, and suppose that $f(\theta^m\tau)$ is known for $m = 0, 1, \cdots, n$, and $f(0)$ is to be estimated. Define*

$$(4.10) \qquad d(0, m) = f(\theta^m\tau), \qquad m = 0, 1, \cdots, n,$$

$$(4.11) \qquad d(k, m) = \frac{d(k - 1, m) - \theta^k\, d(k - 1, m - 1)}{1 - \theta^k}, \qquad k = 1, 2, \cdots, m.$$

Then,

$$(4.12) \qquad f(0) = d(n, n) + \theta^{n(n+1)/2} \frac{(-\tau)^{n+1}}{(n + 1)!} f^{(n+1)}(x_n),$$

where $0 < x_n < \tau$.

Proof. Let $t_m = \theta^m\tau$ and define divided differences

$$(4.13) \qquad \begin{aligned} f[0, t_m] &= \frac{f(0) - f(t_m)}{0 - t_m}, \\ f[0, t_m, t_{m-1}, \cdots, t_{m-k}] &= \frac{f[0, t_m, t_{m-1}, \cdots, t_{m-k+1}] - f[0, t_{m-1}, t_{m-2}, \cdots, t_{m-k}]}{t_m - t_{m-k}}. \end{aligned}$$

The solution of the recurrence relations (4.11) with initial conditions (4.10) is easily verified to be

$$d(k, m) = f(0) + (-1)^k t_m t_{m-1} \cdots t_{m-k} f[0, t_m, t_{m-1}, \cdots, t_{m-k}]. \tag{4.14}$$

Solving for $f(0)$, we find

$$\begin{aligned} f(0) &= d(k, m) + \theta^N (-\tau)^{k+1} f[0, t_m, t_{m-1}, \cdots, t_{m-k}] \\ &= d(k, m) + \theta^N \frac{(-\tau)^{k+1}}{(k+1)!} f^{(k+1)}(x_{km}), \end{aligned} \tag{4.15}$$

where $0 < x_{km} < \theta^{m-k}\tau$ and

$$N = \sum_{r=m-k}^{m} r = \tfrac{1}{2}(2m - k)(k + 1). \tag{4.16}$$

The case $k = m = n$ is (4.12).

To apply Lemma 1 to the computation of logarithms, we choose f as in (4.5) and $\theta = \frac{1}{4}$, so that $d(0, m) = a_m = f(\theta^m \tau)$. Then, (4.12) becomes

$$L = d(n, n) - L\epsilon_n, \tag{4.17}$$

where

$$L\epsilon_n = (-1)^n 2^{-n(n+1)} \left(\frac{\log x}{2}\right)^{2n+2} \frac{f^{(n+1)}(x_n)}{(n+1)!}. \tag{4.18}$$

To estimate ϵ_n, we replace x_n by 0 and note that

$$\frac{f^{(n+1)}(0)}{(n+1)!} = c_{n+1} = \frac{L 2^{2n+2} B_{2n+2}}{(2n+2)!} \cong \frac{(-1)^n 2L}{\pi^{2n+2}}, \tag{4.19}$$

where the last approximation follows from [1, Eq. (23.1.15)]. Thus,

$$\epsilon_n \cong 2^{-n^2-n+1} \left(\frac{\log x}{2\pi}\right)^{2n+2}, \tag{4.20}$$

in agreement with (2.6), while (4.17) and (4.5) imply (2.5).

The analysis for arctangents is similar, except that (4.5) is replaced by

$$L = x/(\arctan x), \qquad \tau^{1/2} = \arctan x, \qquad f(t) = Lt^{1/2} \cot t^{1/2}, \tag{4.21}$$

and the right side of (4.8) is multiplied by $(-1)^n$.

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