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An algorithm for evolutionary surfaces

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Summary. An Algorithm is presented which allows to split the calculation of the mean curvature flow of surfaces with or without boundary into a series of Poisson problems on a series of surfaces. This gives a new method to solve Plateau's problem for H-surfaces.

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1 Introduction

The evolutionary motion of surfaces by their mean curvature has been studied by several authors from different points of view. A measure theoretic approach is studied by Brakke [B], the classical parametric problem by G. Huisken [H1]. Non-parametric mean curvature mean curvature evolution with boundary condition is treated in [H2]. For a detailed bibliography see [H1].

In this note we want to present a numerical algorithm which can be used to calculate evolutionary surfaces in $\mathbb{R}^3$ practically without any parametrization. As a consequence this algorithm allows us to calculate surfaces with prescribed mean curvature, for example stable solutions of Plateau's problem. Numerically, parametric minimal surfaces have been studied by several authors. References can be found in Wohlrab's paper [W].

Since we want to present just the idea how to calculate surfaces which are moved into the direction of the mean curvature vector, we restrict most of our considerations to surfaces without boundary. Other problems, such as the evolution of surfaces with boundary can be treated similarly.

With respect to parametric evolutionary surfaces without boundary there is an existence proof by G. Huisken [H1] where the initial surface is assumed to be a smooth convex surface. For surfaces with boundary the non-parametric problem has been solved in [H2].

I thank G. Huisken for fruitful discussions. The calculations were done on an Iris workstation at SFB 256, Bonn. The graphical representation was done with the help of A. Schmidt and K. Steinberger.

2 The basic equations

Consider a compact twodimensional surface $S(0)$ without boundary smoothly imbedded in $\mathbb{R}^3$ which is represented locally by a diffeomorphism $u_0: \Omega \to \mathbb{R}^3$ where $\Omega \subset \mathbb{R}^2$, $u_0(\Omega) \subset S(0)$. Then one looks for maps $u(.,t): \Omega \to \mathbb{R}^3$, $(t>0)$ such that for some $T>0$

$$(1) \qquad \frac{\partial u}{\partial t} - \Delta_{S(t)} u = 0 \quad \text{in } \Omega \times (0,T)$$

and

$$u(.,0) = u_0.$$

Here $\Delta_{S(t)}$ represents the Beltrami operator on $S(t)$ where $S(t)$ is given by $u(\Omega, t) \subset S(t)$. Since

$$u(.,t) = id_{S(t)} \quad \text{on } S(t),$$

we have

$$(2) \qquad -\Delta_{S(t)} u(.,t) = 2H(.,t)\, n(.,t)$$

and (1) reduces to

$$\frac{\partial u}{\partial t} = -2Hn$$

where H is the mean curvature of $S(t)$ and n is the outer unit normal.

The most simple example for a solution of (1) is the shrinking sphere $S(t) = \{x \in \mathbb{R}^3 \mid |x| = R(t)\}$ with radius $R(t) = (R(0)^2 - 4t)^{1/2}$ and initial radius $R(0)$.

If one looks at non parametric surfaces $u = u(x_1, x_2, t)$, then (1) becomes

$$(3) \qquad \frac{\partial u}{\partial t} - (1+|\nabla u|^2)^{\frac{1}{2}} \nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \quad \text{in } \Omega \subset \mathbb{R}^2.$$

In [H1] it was proved that for uniformly convex initial surface $S(0)$ the system (1) possesses a smooth solution on a finite time interval $(0, T)$ and $S(t)$ converges "spherelike" to a single point as $t \uparrow T$.

In the non-parametric case (3) $u(.,t)$ converges to a solution of the minimal surface equation as $t \to \infty$ if one imposes a Dirichlet boundary condition $u = u_0$ on $\partial\Omega$, $\partial\Omega$ having non-negative mean curvature, and $u(.,0) = u_0 \in C^{2,\alpha}(\bar{\Omega})$. For the proof we refer to [H2].

This motivates us to study numerically evolutionary surfaces with fixed boundary in the parametric case too. Then we have to solve

$$(4) \qquad \begin{aligned} \frac{\partial u}{\partial t} - \Delta_{S(t)} u &= 0 \qquad \text{in } \Omega \times (0,T) \\ u(.,0) &= u_0 \qquad \text{in } \Omega \\ u(.,t) &= u_0 \qquad \text{on } \partial\Omega \text{ (for short).} \end{aligned}$$

If we arrive at a stationary solution of (4) then we have solved Plateau's problem for minimal surfaces with given boundary curve $\Gamma = \partial S(0)$. Since we calculate experimentally nothing prevents us from solving Plateau's problem for surfaces of prescribed mean curvature numerically. Let $H:\mathbb{R}^3\to\mathbb{R}$ be a given function. The basic system now reads

(5) $$\frac{\partial u}{\partial t}-\Delta_{S(t)}\,u=2H(u)\,n.$$

For this problem no analytic result is known yet.

3 A FEM for the Beltrami operator

In [D] the author proved asymptotic convergence for a Finite Element method for elliptic equations on smooth twodimensional surfaces in $\mathbb{R}^3$. The problem

$$-\Delta_S u=f\quad\text{on } S,\qquad u=g\quad\text{on }\partial S$$

is solved by linear elements. The smooth surface S is approximated by a polyhedron S_h with vertices on S. The discrete problem then is to find $u_h\in X_h$, u_h = interpolation of g on ∂S_h such that

(6) $$\int_{S_h}\nabla_{S_h}u_h\,\nabla_{S_h}\varphi_h=\int_{S_h}f\,\varphi_h\qquad(\varphi_h\in X_h,\ \varphi_h=0\quad\text{on }\partial S_h),$$

where ∇_{S_h} represents the tangential gradient on the Lipschitz surface S_h and $X_h=X_h(S_h)=\{v_h:S_h\to\mathbb{R}\,|\,v_h|_{T_h}$ is a linear polynomial on each triangle $T_h\subset S_h$, $v_h\in C^0(\bar{S}_h)\}\subset H^1(S_h)$. Thus (6) is a linear system of equations for $u_h\in X_h$. The numerical scheme is the same as for a plane twodimensional elliptic problem. The only difference is that the computer has to memorize threedimensional nodes instead of twodimensional ones.

4 The algorithm for evolutionary surfaces

We shall use the method described in Sect. 3 to solve system (1) or (4) representively (5). Therefore let $\tau>0$ be a discrete time step and let us write $u^m(x)$ for $u(x,m\tau)$ $(m\in\mathbb{N}\cup\{0\})$. For shortness we assume that we have global parametrization over $\Omega\subset\mathbb{R}^2$.

An implicit time discretization of (5) for $\partial S(t)=\emptyset$ is given by

(7) $$\frac{u^{m+1}-u^m}{\tau}-\Delta_{S^m}u^{m+1}=2H(u^m)\,n_{S^m}\quad\text{on }\Omega$$
$$u^0=u_0.$$

If $\partial S(t)\neq\emptyset$, we add

$$u^{m+1}=u_0\quad\text{on }\partial\Omega.$$

Here $S^m=u_m(\Omega)$, $\quad m=0,1,\ldots$.

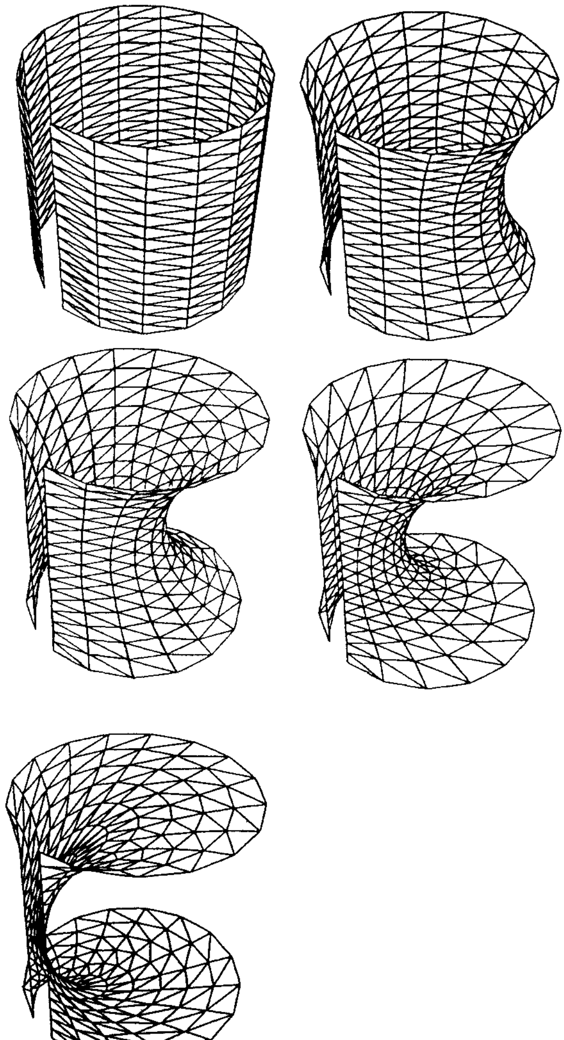

Fig. 1. The *standard example*. Evolution of a cylindrical surface with boundary into a catenoid. $H \equiv 0$

Δ_{S^m} is a nasty operator (numerically) if u is viewed at as a mapping from $\Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$. We avoid this parametrization if we write

$$(8) \qquad \frac{1}{\tau} u^{m+1}(x) - \Delta_{S^m} u^{m+1}(x) = \frac{1}{\tau} x + 2H(x)\, n_{S^m}(x) \qquad (x \in S^m),$$
$$S^{m+1} = \{u^{m+1}(x) \mid x \in S^m\},$$
$$S^0 = u_0(\Omega),$$

and, if necessary

$$u^{m+1}(x) = x \qquad (x \in \partial S^m).$$

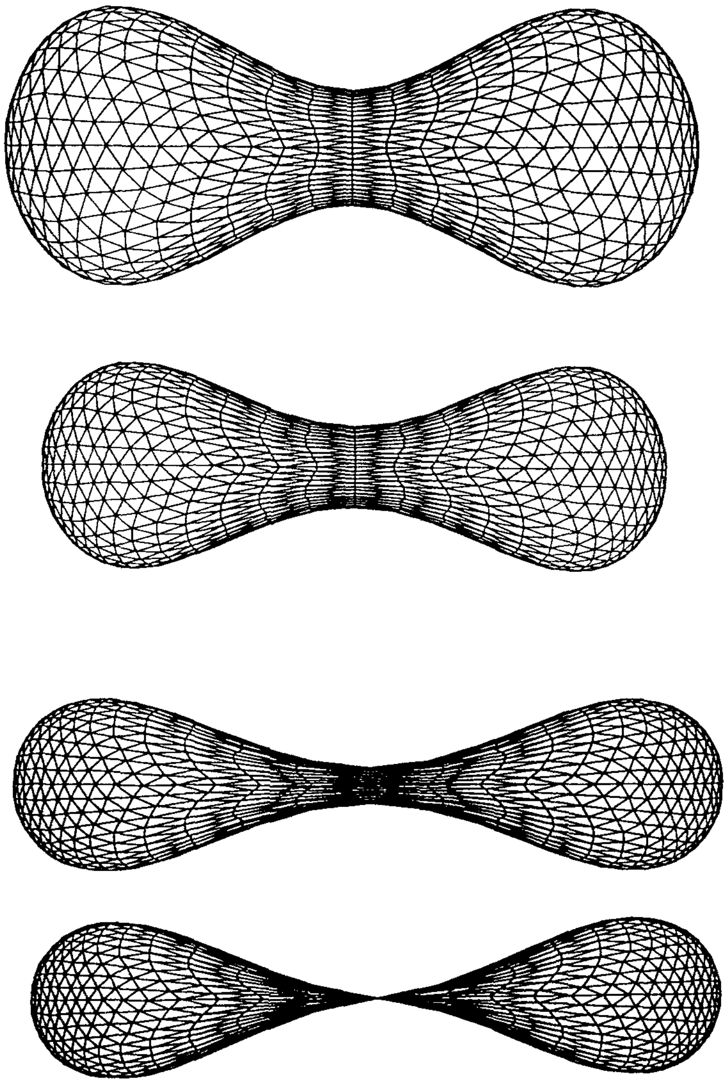

Fig. 2. Evolution of a *singularity*. The initial surface is non-convex without boundary. The surface shrinks during evolution but is rescaled graphically in order to show the singularity, ($H \equiv 0$, variable time-step, 12 time steps have been calculated)

This means that we parametrize S^{m+1} over S^m. Now we are ready to formulate our algorithm in discrete form. Please note the difference between u^{m+1}: $\Omega \subset \mathbb{R}^2 \to S^{m+1}$ in (7) and $u^{m+1}: S^m \to S^{m+1}$ in (8).

Algorithm EVO

1. Let S_h^0 be a polyhedron approximating S^0,

For $m = 0, 1, 2, \ldots$

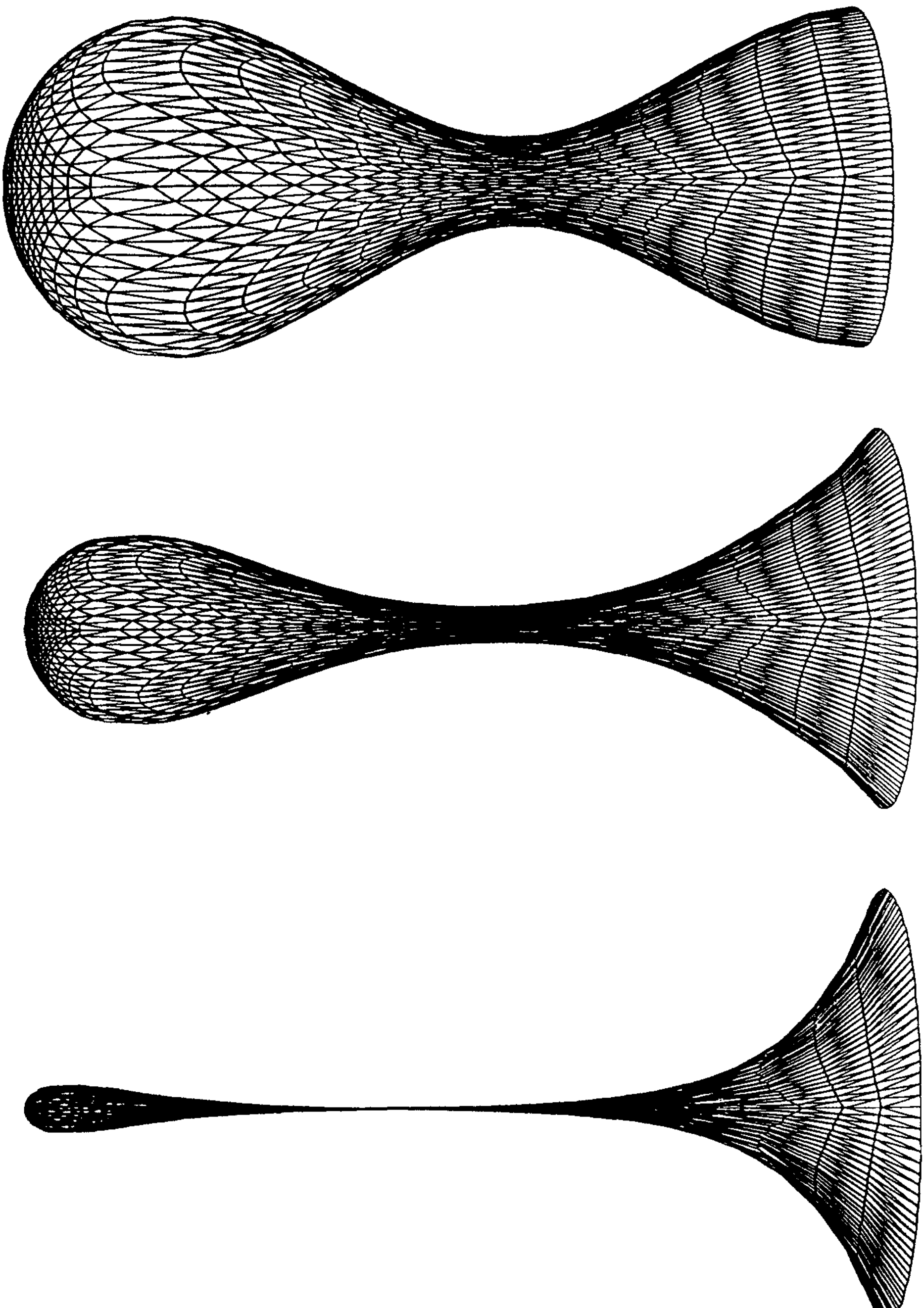

Fig. 3. The same problem as in Fig. 2 but with fixed boundary curve

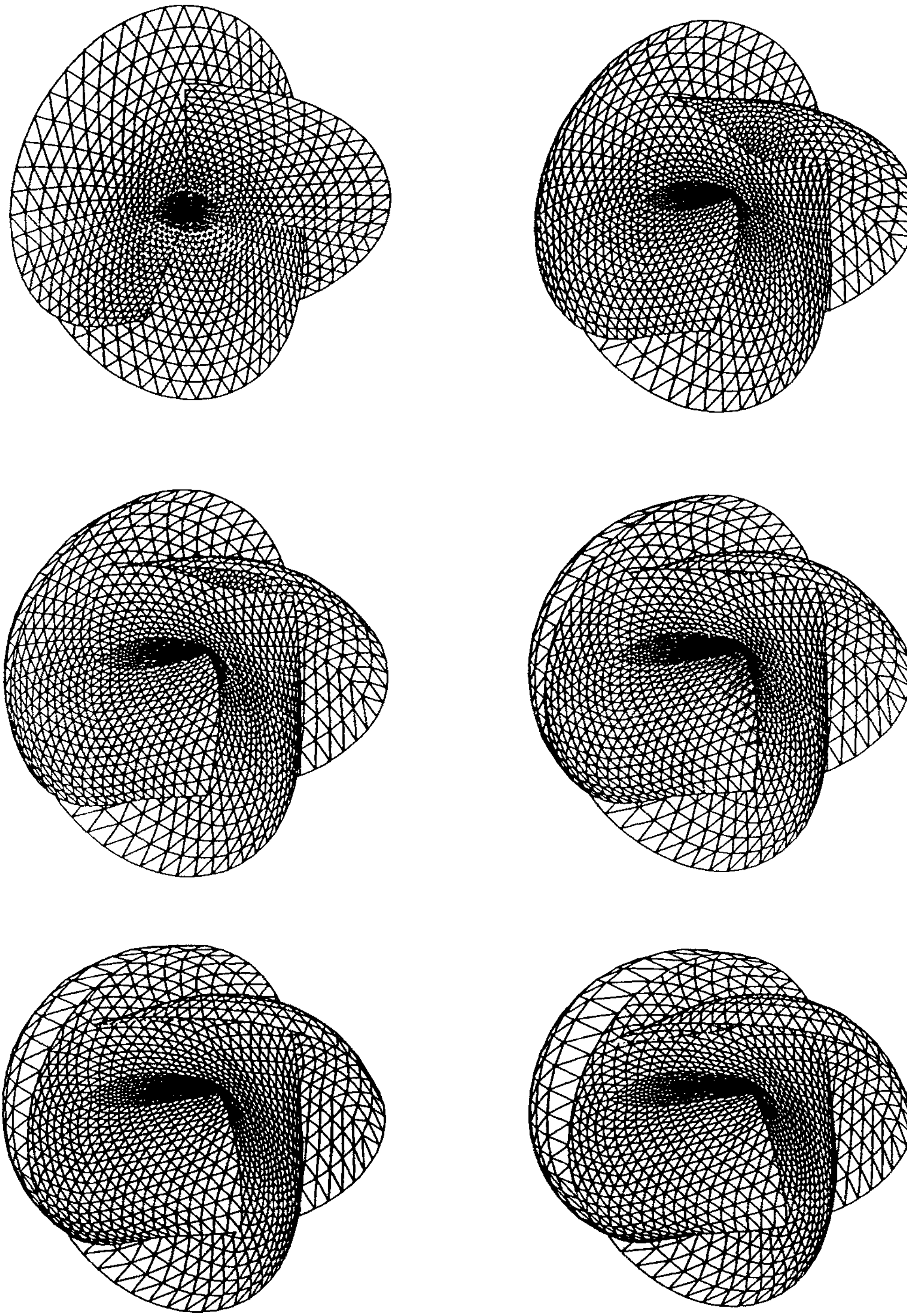

Fig. 4. Mean curvature flow of a surface which is bounded by the clover knot. The surface blows up as time grows because H is chosen to be a big constant

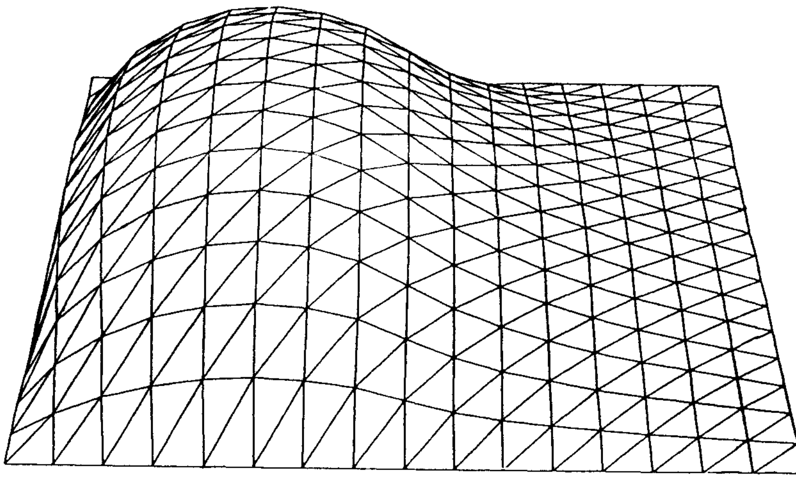

Fig. 4. A surface of prescribed mean curvature $H(x)=1(x_1<0.5)$, $H(x)=0(x_1\geqq 0.5)$ with $\partial((0,1)\times(0,1))$ as fixed boundary curve, calculated with EVO

2. solve the system

$$\int_{S_h^m} \nabla_{S_h^m} u_h^{m+1} \nabla_{S_h^m} \varphi_h + \frac{1}{\tau} u_h^{m+1} \varphi_h = \int_{S_h^m} \left(\frac{1}{\tau} id_{S_h^m} + 2H n_{S_h^m}\right) \varphi_h \tag{9}$$

for all $\varphi_h \in X_h(S_h^m)^3$ with $u_h^{m+1} \in X_h(S_h^m)^3$.

2a. If S_h^0 has a boundary which shall remain fixed during the evolution, then modify 2. as follows

$$u_h^{m+1}(x)=x, \qquad \varphi_h(x)=0 \qquad \text{for } x\in\partial S_h^m.$$

3. Generate the new surface

$$S_h^{m+1}=\{u_h^{m+1}(x)\,|\,x\in S_h^m\}.$$

Let us remark that the system (9) decouples into three equations for u_{hj}^{m+1} $(j=1,2,3)$ and that the stiffness matrix is the same for these three problems. If one wishes to calculate shrinking surfaces or surfaces which develop singularities during the evolution then of course $\tau=\tau_m$ and $h=h_m$ where τ_m has to be chosen suitably and the gridsize h_m is a result of the evolution.

References

[B] Brakke, K.A.: The motion of a surface by its mean curvature. Math. Notes, Princeton Univ. Press, Princeton N.J., 1978

[D] Dziuk, G.: Finite elements for the Beltrami operator on arbitrary surfaces. In: S. Hildebrandt, R. Leis (ed.) Partial differential equations and calculus of variations. Lect. Notes Math Vol. 1357, pp. 142–155, Berlin Heidelberg New York: Springer 1988

[H1] Huisken, G.: Flow by mean curvature of convex surfaces into spheres. J. Diff. Geom. **20**, 237–266 (1984)

[H2] Huisken, G.: Non-parametric mean curvature evolution with boundary conditions. CMA Research Report 44 (1987)

[H3] Huisken, G.: Asymptotic behaviour for singularities of the mean curvature flow. CMA Research Report 49 (1987)

[W] Wohlrab, O.: Zur numerischen Behandlung von parametrischen Minimalflächen mit halbfreien Rändern. Dissertation Bonn 1989