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AN ALGORITHM FOR FINDING AN OPITIMAL "INDEPENDENT ASSIGNMENT"

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Abstract

The ordinary assignment problem on a bipartite graph with weighted arcs is extended to the case where both of the two sets of vertices of the graph are given matroidal structures, and a practical solution algorithm is presented to this extended problem.

Introduction

A number of fundamental problems in electrical network theory were recently shown to be adequately formulated in terms of matroids and thus be solved [1] ~ [5]. In connection with the problem of the "order of complexity" of a network, in particular, there has arisen an extension of the assignment problem famous in Operations Research, extension in that both of the two sets of vertices of the underlying bipartite graph are given respective matroidal structures and that the end vertices of a matching are required to form an independent set in each of the vertex sets [5]. In the field of Operations Research, the ordinary assignment problem is converted into a kind of independent assignment problem when the restriction of "system-of-distinct-representatives" type is imposed upon "persons" as well as upon "jobs". (However, since the relevant matroidal structures are of transversal type in this kind of problem, it can better be treated through the network-flow formulation.)*

In the present paper, we shall define in rigorous terms this extended assignment problem ---- which we shall call the "independent assignment problem" ---- and shall present a practically efficient algorithm for it. We shall propose also a duality formalism for a kind of mathematical programming problems involving 0-1 variables under combinatorial constraints as an auxiliary tool in proving the optimality of a solution. The approach we shall present in this paper will open a new vista for the algorithmic approach to matroid theory itself.

As for graphs and networks, we follow largely the standard notation and terminology --- those adopted in [12] in particular.

1. Preliminaries from Matroid Theory

According to the standard literature on matroid theory [6], [7], we shall say that a finite set V is given the structure of a matroid when a (nonempty) family $\mathcal{V}$ of subsets of V is specified in such a way that

(I) if $I \in \mathcal{V}$ and $I' \subseteq I$ then $I' \in \mathcal{V}$

and

(II) if $I, I' \in \mathcal{V}$ and $|I| > |I'|$ then there is in $I - I'$ an element v such that $I' \cup \{v\} \in \mathcal{V}$.

The set V with the matroidal structure specified by $\mathcal{V}$ will be denoted by $M(V, \mathcal{V})$. We call an element of $\mathcal{V}$ an "independent set", and an element of $2^V - \mathcal{V}$ a "dependent set". A maximal independent set is called a "base" of the matroid, and a minimal dependent set a "circuit". We define the "closure" $\mathrm{cl}(U)$ of a subset U of V by

* The authors became aware after submitting the manuscript of this paper that E. L. Lawler is going to publish a book on "Combinatorial Optimization" in which he formulates the problem of finding a spanning arborescence (directed tree) on a directed graph as the problem of finding an optimal independent assignment where the two sets of vertices of the underlying bipartite graph are two replicas of the set of branches of the given directed graph, the branches of the bipartite graph connect the corresponding vertices in the two sets, one of the two vertex sets is provided with the same matroidal structure as that defined on the set of branches of the given directed graph (with trees as independent sets) and the other vertex set is provided with the matroidal structure such that the vertices corresponding to those branches in the given graph which end at one and the same vertex are dependent on one another. This example by Lawler will be a first nontrivial example which evidences the significance of the independent assignment problem in Operations Research. (added in proof)

(1.1) $\mathrm{cl}(U) = U \cup \{v \mid \text{there is a circuit containing } v \text{ in } U \cup \{v\}\}$

and the "rank" $\rho(U)$ of U by

(1.2) $\rho(U) = \max\{|I| \mid I \in \mathcal{V},\ I \subseteq U\}$.

we shall make use of the following fundamental properties of the matroid without proof:

(III) For a subset U of V, all the maximal independent subsets of U have the same cardinality;

(IV) If $U \subseteq U'(\subseteq V)$ then $\mathrm{cl}(U) \subseteq \mathrm{cl}(U')$;

(V) For an independent set $I(\in \mathcal{V})$ and an element $v \notin I$, we have $v \notin \mathrm{cl}(I)$ if and only if $I \cup \{v\} \in \mathcal{V}$;

(VI) If $I \in \mathcal{V}$ and $v \in \mathrm{cl}(I)-I$ then

(1.3) $\{u \mid (I \cup \{v\})-\{u\} \in \mathcal{V}\} = \{u \mid v \notin \mathrm{cl}(I-\{u\})\} \cup \{v\}$

is the unique circuit in $\{v\} \cup I$, and we have

(1.4) $\mathrm{cl}(I) = \mathrm{cl}\big((I \cup \{v\})-\{u\}\big)$ for every u ($\neq v$) in the circuit.

(VII) If C_1 and C_2 are distinct circuits having an element v in common, then there is a circuit C such that $C \subseteq (C_1 \cup C_2)-\{v\}$.

(VIII) For any subset U of V, we have

(1.5) $\rho(U) = \rho\big(\mathrm{cl}(U)\big)$.

2. Description of the Problem

Let us consider a (finite) bipartite graph $G(V_1, V_2; A)$ with vertex sets V_1 and V_2 and arc set $A(\subseteq V_1 \times V_2$; arcs are assumed to have initial vertices in V_1 and terminal vertices in V_2), where each arc $a(\in A)$ is given a real number $w(a)$ (called the "weight" of a) and where V_1 (resp. V_2) is given the structure of a matroid specified by $\mathcal{V}_1$ (resp. $\mathcal{V}_2$). A matching or an assignment on G is a subset $B(\subseteq A)$ of arcs such that no two arcs in B have a vertex in common either in V_1 or in V_2. If we denote the set of the end vertices of B in V_1 (resp. V_2) by $\partial_1 B$ (resp. $\partial_2 B$), we may characterize a matching B by the equation:

(2.1) $|B| = |\partial_1 B| = |\partial_2 B|$.

Following D. J. A. Welsh [8], we shall call a matching B on G an independent matching if $\partial_1 B \in \mathcal{V}_1$ and $\partial_2 B \in \mathcal{V}_2$, and an independent matching of the largest cardinality a maximum independent matching. The principal problem we shall deal with in the present paper is:

to find a maximum independent matching $\hat{B}$ of which the sum of the weights of the arcs is as small as possible.

The solutions $\hat{B}$ of the problem will be called optimal independent assignments on the given bipartite graph $G(V_1, V_2; A)$ with regard to the weight function w and to the matroids $M_1(V_1, \mathcal{V}_1)$ and $M_2(V_2, \mathcal{V}_2)$.

As in the case of the ordinary assignment problem, we may assume, without loss of generality, that all the weights of arcs are nonnegative.

3. Incidence Function associated with an Independent set of a Matroid

For an independent set $I(\in \mathcal{V})$ of a matroid $M(V, \mathcal{V})$, we define an incidence function $D(\cdot, \cdot|I): I\times(\mathrm{cl}(I)-I) \to \{0, 1\}$ such that

(3.1) $$D(u, v|I) = \begin{cases} 1 & \text{if } v \notin \mathrm{cl}(I-\{u\}), \\ 0 & \text{if } v \in \mathrm{cl}(I-\{u\}). \end{cases}$$

From the definition of D and the property (VI) of §1 it follows that

(3.2) $$\{u|D(u, v|I) = 1\} = [\text{the circuit in } \{v\} \cup I]-\{v\}$$

for a $v \in \mathrm{cl}(I)-I$, and that

(3.3) $$\{v|D(u, v|I) = 1\} = \mathrm{cl}(I)-\mathrm{cl}(I-\{u\})-\{u\}$$

for a $u \notin I$.

The following lemmas are useful for the subsequent discussions.

Lemma 1. If $I \subseteq I'$ $(\in \mathcal{V})$ and $v \in \mathrm{cl}(I)-I$, then we have

(3.4) $$D(u, v|I) = D(u, v|I') \quad \text{for every } u \in I.$$

Proof: If $v \in \mathrm{cl}(I)-I$, then $v \in \mathrm{cl}(I')-I'$ by virtue of (IV) and (V). Moreover, it is seen from (VI) that the circuit in $I \cup \{v\}$ coincides with that in $I' \cup \{v\}$. (3.4) then follows from (3.2).

Lemma 2. If, for an independent set $I(\in \mathcal{V})$ of a matroid $M(V, \mathcal{V})$, $2q$ distinct elements $u_1, \cdots, u_q$ $(\in I)$ and $v_1, \cdots, v_q$ $(\in \mathrm{cl}(I)-I)$ satisfy the relations:

(3.5) $$D(u_j, v_j|I) = 1 \quad \text{for } j = 1, \cdots, q$$

and

(3.6) $$D(u_j, v_i|I) = 0 \quad \text{for every } i \text{ and every } j \text{ such that } 1 \le i < j \le q,$$

then

(3.7) $$I' = (I-\{u_1, \cdots, u_q\}) \cup \{v_1, \cdots, v_q\}$$

is also an independent set and we have

(3.8) $$\mathrm{cl}(I) = \mathrm{cl}(I').$$

Proof: It is obvious that the lemma with $q = 1$ is no other than the property (VI) of the matroid. Let us then suppose that the lemma holds for $q = p$ and prove that it holds also for $q = p+1$. Namely, let us suppose that $2(p+1)$ elements $u_1, \cdots, u_{p+1} \in I$ and $v_1, \cdots, v_{p+1} \in \mathrm{cl}(I)-I$ are given which satisfy (3.5) and (3.6) for an I with $q = p+1$. It then follows directly from (3.5) (for $j = p+1$) and (VI) that

(3.9) $$I'' = (I-\{u_{p+1}\}) \cup \{v_{p+1}\}$$

is an independent set and that

(3.10) $$\mathrm{cl}(I) = \mathrm{cl}(I'').$$

Since

(3.11) $$v_i \in \mathrm{cl}(I-\{u_{p+1}\}) \quad \text{for } i = 1, \cdots, p$$

by (3.6), we have from Lemma 1

(3.12) $$D(u_j, v_i|I) = D(u_j, v_i|I-\{u_{p+1}\}) = D(u_j, v_i|I'')$$

for every i and j (= 1, $\cdots$, p). Therefore, all the relations (3.5) and (3.6) hold with q put equal to p and I replaced by I''. Hence, by the induction hypothesis, we can conclude that

(3.13) $$I' = (I''-\{u_1, \cdots, u_p\}) \cup \{v_1, \cdots, v_p\}$$
$$= (I-\{u_1, \cdots, u_p, u_{p+1}\}) \cup \{v_1, \cdots, v_p, v_{p+1}\}$$

is an independent set and

(3.14) $$\mathrm{cl}(I') = \mathrm{cl}(I'') = \mathrm{cl}(I),$$

which was to be demonstrated.

Lemma 3. Let I and I' be independent sets of a matroid $M(V, \mathcal{Y})$ such that

(3.15) $$I' = (I-\{u\}) \cup \{v\},$$

where

(3.16) $$\begin{cases} u \in I, \quad v \in \mathrm{cl}(I)-I, \\ v \in I', \quad u \in \mathrm{cl}(I')-I' \\ \text{and} \\ \mathrm{cl}(I) = \mathrm{cl}(I'). \end{cases}$$

Moreover, let u' and v' be elements such that

(3.17) $$u' \in I \cap I' = I-\{u\} = I'-\{v\}$$

and

(3.18) $$v' \in \mathrm{cl}(I)-I-\{v\} = \mathrm{cl}(I')-I'-\{u\}.$$

Then, we have

(3.19) $$D(v, u|I') = D(u, v|I) = 1,$$

(3.20) $$D(u', u|I') = D(u', v|I),$$

(3.21) $$D(v, v'|I') = D(u, v'|I)$$

and

(3.22) $D(u', v'|I') = D(u', v'|I)$

if either $D(u', v|I) = 0$ or $D(u, v'|I) = 0$.

Proof: First, we note that the circuit $C_1 \subseteq I \cup \{v\}$ and the circuit $C_2 \subseteq I' \cup \{u\}$ should coincide with each other, for, otherwise, by virtue of the property (VII) of §1, there would be a third circuit C_3 in $C_1 \cup C_2 - \{u\} \subseteq I'$, which is a contradiction. Therefore, we have (3.19) and (3.20).

Let us next suppose that the right-hand side of (3.21) were not equal to the left-hand side, for example, that

(3.23) $D(v, v'|I') = 0$ and $D(u, v'|I) = 1$.

Then, the circuit $C_1 \subseteq I \cup \{v'\}$ would contain u and v' but not v, whereas the circuit $C_2 \subseteq I' \cup \{v'\}$ would contain v' but not u nor v. Therefore, there would be a third circuit $C_3 \subseteqq C_1 \cup C_2 - \{v'\} \subseteqq I \cup I' - \{v\} = I$, which is a contradiction.

If $D(u, v'|I) = 0$, there is a circuit $C_1 \subseteq I \cup \{v'\}$ containing v' but not u nor v. If the circuit $C_2 \subseteq I' \cup \{v'\}$ (containing v' but not u) did not coincide with C_1, there would be a third circuit $C_3 \subseteq C_1 \cup C_2 - \{v'\} \subseteq I \cup I' - \{u\} = I'$, which is a contradiction. Therefore, (3.22) holds if $D(u, v'|I) = 0$.

If $D(u, v'|I) = 1$ and $D(u', v|I) = 0$, then, by (3.20) and (3.21), $D(v, v'|I') = 1$ and $D(u', u|I') = 0$. Since the circuit $C_1 \subseteq I \cup \{v\}$ contains u and v but not u' and the circuit $C_2 \subseteq I' \cup \{v'\}$ contains v and v' but not u, there is a third circuit

(3.24) $C_3 \subseteq C_1 \cup C_2 - \{v\} \subseteq I \cup \{v'\}$.

(3.24) indicates that, if u' is contained in C_3, so is it in C_2 (for C_1 does not contain u'), i.e., that if $D(u', v'|I) = 1$ then $D(u', v'|I') = 1$. Interchanging the roles of I and I', we can conclude also that if $D(u' v'|I') = 1$ then $D(u', v'|I) = 1$. Thus, (3.22) has been proved.

Lemma 4. Let I, I', $u_1, \cdots, u_q (\in I)$ and $v_1, \cdots, v_q$ $(\in \mathrm{cl}(I)-I)$ be the same as those in Lemma 2. Then we have

(3.25) $D(v_j, u_j|I') = 1$ for $j = 1, \cdots, q$

and

(3.26) $D(v_j, u_i|I') = 0$ for every i and every j such that $1 \le i < j \le q$.

Proof: Let us put

$$I_p = (I - \{u_1, \cdots, u_p\}) \cup \{v_1, \cdots, v_p\} \quad (p = 1, \cdots, q)$$

and

$$I_0 = I.$$

Then we have

$$I' = I_q.$$

By virtue of Lemma 2, I_p's are independent sets and

$$\mathrm{cl}(I_p) = \mathrm{cl}(I) \quad (p = 0, \cdots, q).$$

We shall prove that

$$(3.27) \qquad \begin{aligned} D(v_j, u_j | I_p) &= 1 \quad \text{for } j = 1, \cdots, p, \\ D(u_j, v_j | I_p) &= 1 \quad \text{for } j = p+1, \cdots, q \end{aligned}$$

and

$$(3.28) \qquad \begin{aligned} D(v_j, u_i | I_p) &= 0 \quad \text{if } 1 \le i < j \le p, \\ D(u_j, u_i | I_p) &= 0 \quad \text{if } 1 \le i \le p < j \le q, \\ D(u_j, v_i | I_p) &= 0 \quad \text{if } p \le i < j \le q. \end{aligned}$$

(3.27) and (3.28) are obvious for $p = 0$ by the assumption of the lemma. Let us suppose that they hold valid for a p ($< q$). Then, their validity for p+1 follows from Lemma 3, where I_p, I_{p+1}, u_{p+1} and v_{p+1} are taken, respectively, for I, I', u and v.

4. Auxiliary Graph associated with an Independent Matching

Let B be an independent matching on $G(V_1, V_2; A)$ with regard to the matroids $M_1(V_1, \mathcal{V}_1)$ and $M_2(V_2, \mathcal{V}_2)$. Then, $I_1 \equiv \partial_1 B$ (resp. $I_2 \equiv \partial_2 B$) is an independent set of M_1 (resp. M_2). The <u>auxiliary graph</u> $\tilde{G}_B(\tilde{V}, \tilde{A})$ associated with the independent matching B is defined as follows. The vertex set $\tilde{V}$ is the same as that of G, i.e.

$$(4.1) \qquad \tilde{V} = V_1 \cup V_2.$$

The arc set $\tilde{A}$ is composed of four disjoint parts:

$$(4.2) \qquad \tilde{A} = A \cup B^* \cup A_1 \cup A_2.$$

A is the arc set of the original bipartite graph G; B^* consists of the arcs obtained from those of B by reversing the direction (i.e. they are directed from V_2 to V_1); A_1 consists of the arcs which connect the pairs of vertices

(u, v)'s (from u to v) such that

(4.3) $\quad u \in I_1,\ v \in \mathrm{cl}_1(I_1)-I_1,\ D_1(u, v| I_1) = 1;$

and A_2 consists of the arcs which connect the pairs of vertices (v, u)'s (from v to u) such that

(4.4) $\quad u \in I_2,\ v \in \mathrm{cl}_2(I_2)-I_2,\ D_2(u, v| I_2) = 1,$

where cl_1 and cl_2 are the closure functions, and D_1 and D_2 are the incidence functions for M_1 and M_2, respectively, defined in §3. Furthermore, we call the vertices of the set

(4.5) $\quad S_1 \equiv V_1-\mathrm{cl}_1(I_1)$

the <u>entrances</u> (or <u>sources</u>) of $\tilde{G}_B$, and the vertices of

(4.6) $\quad S_2 \equiv V_2-\mathrm{cl}_2(I_2)$

the <u>exits</u> (or <u>sinks</u>) of $\tilde{G}_B$.

5. A Duality Formalism for a Kind of Mathematical Programs with 0-1 Variables under Combinatorial Constraints

A number of duality formalisms, together with duality-gap arguments, have been proposed for the mathematical programming problems with integer variables and/or (0,1)-variables (see, e.g., [15]). We shall present in the following a formalism which is most convenient for application to the problem of our concern.

Let J, K and L be (finite) index sets, a_{jk}'s, b_{jl}'s, c_j's and e_k's be given (real) constants ($j \in J$, $k \in K$, $l \in L$), and $\mathcal{L}$ be a family of subsets of L. The "primal" problem with variables ξ_k's ($k \in K$) and η_l's ($l \in L$) to be considered below is:

to minimize the objective function:

(5.1) $$f(\xi) = \sum_{k\in K} e_k\xi_k$$

subject to the constraints:

(5.2) $\quad \xi_k \geq 0 \quad$ for all $k \in K$,

(5.3) $\quad \eta_l = 0$ or $1 \quad$ for all $l \in L$,

(5.4) $$\sum_{k\in K} a_{jk}\xi_k - \sum_{l\in L} b_{jl}\eta_l = c_j \quad \text{for all } j \in J$$

and

(5.5) $L(\eta) \equiv \{l \mid \eta_l = 1\} \in L.$

It should be noted that η_l's are 0-1 variables on which the combinatorial constraints (5.5) are imposed.

As the "dual" problem we introduce the following problem (5.6), (5.7) with variables ζ_j's ($j \in J$):

to maximize the objective function:

(5.6) $g(\zeta) = \sum_{j \in J} c_j \zeta_j + \min_{I \in L}[\sum_{l \in I}(\sum_{j \in J} b_{jl}\zeta_j)]$

subject to the constraints:

(5.7) $\sum_{j \in J} a_{jk}\zeta_j \leq e_k$ for all $k \in K$.

From the definition of the pair of primal and dual problems follows readily a kind of duality theorem, i.e. we have

Theorem 1. For an arbitrary feasible solution $\{\xi_k, \eta_l\}$ (satisfying (5.2) ~ (5.5)) of the primal problem and an arbitrary feasible solution $\{\zeta_j\}$ (satisfying (5.7)) of the dual, the corresponding objective functions satisfy the inequality:

(5.8) $f(\xi) \geq g(\zeta)$.

In addition, if the equality holds in (5.8), the solutions are optimal both for the primal and for the dual problem. (Note that we do not claim the validity of the converse of the proposition!) The equality holds in (5.8) if and only if the following "complementary slackness conditions" are satisfied:

(5.9) $(\sum_{j \in J} a_{jk}\zeta_j - e_k)\xi_k = 0$ for all $k \in K$

and

(5.10) $\min_{I \in L}[\sum_{l \in I}(\sum_{j \in J} b_{jl}\zeta_j)] = \sum_{l \in L(\eta)}(\sum_{j \in J} b_{jl}\zeta_j).$

Proof: By virtue of the feasibility of the solutions we have

(5.11)
$$
\begin{aligned}
f(\xi) &= \sum_{k \in K} e_k \xi_k \\
&\geq \sum_{k \in K}(\sum_{j \in J} a_{jk}\zeta_j)\xi_j \\
&= \sum_{j \in J}(\sum_{k \in K} a_{jk}\xi_k)\zeta_j
\end{aligned}
$$

$$= \sum_{j\in J} c_j\zeta_j + \sum_{j\in J}(\sum_{l\in L} b_{jl}\eta_l)\zeta_j$$
$$= \sum_{j\in J} c_j\zeta_j + \sum_{l\in L(\eta)}(\sum_{j\in J} b_{jl}\zeta_j)$$
$$\geq \sum_{j\in J} c_j\zeta_j + \min_{I\in L}[\sum_{l\in I}(\sum_{j\in J} b_{jl}\zeta_j)]$$
$$= g(\zeta),$$

where the first inequality is reduced to the equality if and only if (5.9) holds and the second inequality is reduced to the equality if and only if (5.10) holds.

The relation

(5.12) $$\min_{(5.2)\sim(5.5)} f(\xi) \geq \max_{(5.7)} g(\zeta)$$

is an immediate consequence of (5.8), so that, if the equality holds for a primal feasible solution $\{\hat{\xi}_k, \hat{\eta}_l\}$ and a dual feasible solution $\{\hat{\zeta}_j\}$, then we have

$$f(\hat{\xi}) = \min_{(5.2)\sim(5.5)} f(\xi)$$

and

$$g(\hat{\zeta}) = \max_{(5.7)} g(\zeta),$$

i.e. $\{\hat{\xi}_k, \hat{\eta}_l\}$ is an optimal solution for the primal problem and $\{\hat{\zeta}_j\}$ is an optiaml solution for the dual.

6. Algorithm for Finding an Optimal Independent Assignment

The algorithm starts from the trivial independent matching $B^{(0)} = \phi$ on G and increases the size of matching one by one to get a maximum matching. Every matching $B^{(r)}$ appearing in the course is of the minimum total weight among the independent matchings of the same size as $B^{(r)}$.

The independent matching $B^{(r)}$ obtained in the r-th stage of the algorithm is enlarged to $B^{(r+1)}$ according to the following rules:

(i) Construct the auxiliary graph $\tilde{G}_{B^{(r)}}(\tilde{V}, \tilde{A}^{(r)})$ associated with $B^{(r)}$ (see §4).

(ii) Find the shortest path $P^{(r)}$ in $\tilde{G}_{B^{(r)}}$ from an entrance to an exit, where the lengths $l^+(a)$ and $l^-(a)$ of the arcs of $\tilde{G}_{B^{(r)}}$ are taken, in this case, as shown in the following table. (If there are more than one such path, choose one which consists of the fewest arcs.)

	length $l^+(a)$ in the positive direction	length $l^-(a)$ in the negative direction
arc a in A	$w(a)$	∞
arc a^* in $B^{(r)*}$	$-w(a)$	∞
arc a in $A_1^{(r)}$	0	∞
arc a in $A_2^{(r)}$	0	∞

(iii-a) If no path of finite length is found, then $B^{(r)}$ is an optimal assignment. (The "length of a path" is the sum of the lengths of the arcs on that path.)

(iii-b) If the length of $P^{(r)}$ is finite, then transform $B^{(r)}$ into $B^{(r+1)}$ by adding to $B^{(r)}$ those arcs of A which belong to $P^{(r)}$ and removing from it those arcs which correspond to the arcs of $B^{(r)*}$ belonging to $P^{(r)}$.

Remark 1. In the case where $M_1(V_1, \mathcal{V}_1)$ and $M_2(V_2, \mathcal{V}_2)$ are matroids determined by matrices, the number of computations required for modifying $\tilde{G}_{B^{(r)}}$ into $\tilde{G}_{B^{(r+1)}}$ is at most proportional to $\max\{|V_1|^3, |V_2|^3\}$. The search for the shortest path in $\tilde{G}_{B^{(r)}}$ is carried out with computations of the number at most proportional to $|V_1 \cup V_2|^3 = |V|^3$ by any known method. Thus, the total number of computations required for attaining an optiaml independent assignment is at most proportional to $|V_1 \cup V_2|^3 \times \max\{|V_1|, |V_2|\}$. As is seen from this estimation of the amount of necessary computational labor, the algorithm presented in the above is efficient enough to be applicable to practical large-scale problems. If $M_1(V_1, \mathcal{V}_1)$ and $M_2(V_2, \mathcal{V}_2)$ are matroids determined by graphs, then the modification of $\tilde{G}_{B^{(r)}}$ into $\tilde{G}_{B^{(r+1)}}$ can be effected with computations of the number proportional to $\max\{|V_1|^2, |V_2|^2\}$. Therefore, if the variant of the Dijkstra method [9], [10] is adopted in searching for the shortest path, the total number of computations may be reduced to $|V_1 \cup V_2|^2 \times \max\{|V_1|, |V_2|\}$.

Remark 2. In case both $M_1(V_1, \mathcal{Y}_1)$ and $M_2(V_2, \mathcal{Y}_2)$ are free matroids, i.e. if $\mathcal{Y}_1 = 2^{V_1}$ and $\mathcal{Y}_2 = 2^{V_2}$, then the arc sets $A_1^{(r)}$ and $A_2^{(r)}$ of every $\tilde{G}_{B^{(r)}}$ are empty. Therefore, the auxiliary graph $\tilde{G}_{B^{(r)}}$ is essentially the same as the original bipartite graph G augmented with the "reverse" arcs $B^{(r)*}$ corresponding to the arcs of matchings, and the entrances as well as the exits are "non-saturated" vertices. In this case the algorithm presented above coincides with the ordinary "assignment algorithm" well known in network-flow theory [11], [12], where $P^{(r)}$ is an "alternating path" with regard to matching $B^{(r)}$.

7. Proof of the Validity of the Algorithm

The ultimate goal of this section is to prove that the $B^{(r)}$ obtained by the algorithm presented in §6 is the independent matching on G which has the minimum total weight among the independent matchings of cardinality r and that the $B^{(r)}$ is a maximum independent matching when the algorithm terminates at step (iii-a) (Theorems 2, 3 and 4).

We begin with the definition of the potential function and its compatibility and admissibility.

Definition 1. Let $G(V, A)$ be a graph to each of whose arcs $a(\in A)$ the length $l^+(a)$ in the positive direction as well as that $l^-(a)$ in the negative direction is associated, where we assume $l^+(a) + l^-(a) \geq 0$. We call a function ζ which makes correspond a real value to every vertex of V a potential function, and the value $\zeta(v)$ at vertex v the potential at v. In particular, a potential function ζ is said to be compatible with the arc lengths $l^{\pm}(a)$'s if

(7.1) $\quad l^+(a) \geq \zeta(\partial^+ a) - \zeta(\partial^- a) \geq -l^-(a)$

for every arc $a \in A$, where $\partial^+ a$ and $\partial^- a$ are, respectively, the initial and the termianl vertex of arc a.

Definition 2. Let $G(V, A)$ be a graph of the kind mentioned in Definition 1, in which, furthermore, a subset $S_1(\subseteq V)$ of vertices and another $S_2(\subseteq V)$ are designated, respectively, as the set of entrances and that of exits. A potential function ζ is said to be admissible if it is compatible with the arc lengths and if there is a value ζ_S such that

$$(7.2)\quad \begin{cases} \zeta(v) = 0 & \text{for every } v \in S_2, \\ \zeta(v) = \zeta_S & \text{for every } v \in S_1, \\ \text{and} & \\ \zeta_S \geq \zeta(v) \geq 0 & \text{for every } v \in V - (S_1 \cup S_2). \end{cases}$$

Definition 3. Let $G(V, A)$ be a graph of the kind mentioned in Definition 1. We define the length of a path P as the sum of the lengths of the arcs lying on P, where the length in the positive or the negative direction is taken according as the arc lies on P in the positive or the negative direction. For a subset $U_1 (\subseteq V)$ and another $U_2 (\subseteq V)$ of vertices, we define a shortest path from U_1 to U_2 as a path having the minimum length among those paths starting from a vertex of U_1 and ending at a vertex of U_2.

Lemma 5. On a graph of the kind mentioned in Definition 1, the following three conditions are equivalent to one another.

<a> There is a potential function compatible with the arc lengths.

<b> The length of any closed path (or cycle, circuit, loop, ···) is nonnegative.

<c> There is a shortest path (whose length may be finite or infinite) from any subset of V to any subset of V.

Proof: This fact is fundamental in network-flow theory and is well known (see, e.g., [12]).

Lemma 6. Let ζ be a potential function on a graph of the kind mentioned in Definition 2 such that

$$(7.3)\quad \begin{aligned} \zeta(v) &= \zeta_1 \quad \text{for every } v \in S_1, \\ \zeta(v) &= \zeta_2 \quad \text{for every } v \in S_2. \end{aligned}$$

Then, a shortest path from S_1 to S_2 with regard to the arc lengths $l^{\pm}(a)$'s is also the shortest with regard to the arc lengths $\bar{l}^{\pm}(a)$'s defined as follows:

$$(7.4)\quad \begin{aligned} \bar{l}^{+}(a) &= l^{+}(a) - [\zeta(\partial^{+}a) - \zeta(\partial^{-}a)], \\ \bar{l}^{-}(a) &= l^{-}(a) - [\zeta(\partial^{-}a) - \zeta(\partial^{+}a)]. \end{aligned}$$

Furthermore, if $\bar{\zeta}$ is compatible with $\bar{l}^{\pm}(a)$'s then $\zeta+\bar{\zeta}$ is compatible with $l^{\pm}(a)$'s.

Proof: The length of a path from S_1 to S_2 with regard to $\bar{l}^{\pm}(a)$'s is equal to the length with regard to $l^{\pm}(a)$'s minus $\zeta_1-\zeta_2$ (since the potentials at the intermediate vertices cancel one another), so that the relative magnitude of the lengths of paths from S_1 to S_2 remains unchanged.

From the compatibility of $\bar{\zeta}$ with $\bar{l}^{\pm}(a)$'s i.e. from the relations:

$$\bar{l}^{+}(a) \geq \bar{\zeta}(\partial^{+}a) - \bar{\zeta}(\partial^{-}a) \geq -\bar{l}^{-}(a) \quad \text{for every arc } a(\in A) \tag{7.5}$$

we have the relations:

$$l^{+}(a) \geq [\zeta(\partial^{+}a) + \bar{\zeta}(\partial^{+}a)] - [\zeta(\partial^{-}a) + \bar{\zeta}(\partial^{-}a)] \geq -l^{-}(a) \tag{7.6}$$

for every arc $a(\in A)$,

which are exactly the conditions for the compatibility of $\zeta+\bar{\zeta}$ with $l^{\pm}(a)$'s.

Lemma 7. On a graph of the kind mentioned in Definition 2, if all the arc lengths $l^{\pm}(a)$'s are nonnegative, there is an admissible potential function ζ.

Proof: Since the arc lengths are nonnegative, there is no closed path of negative length, and hence, by Lemma 5, the shortest paths exist from any vertex subset to any other. Let us define the potential function $\tilde{\zeta}$ by

$\tilde{\zeta}(v)$ = the length of a shortest path from vertex v to the exits S_2,

and denote by $\hat{v}$ an entrance ($\in S_1$) such that

$$\tilde{\zeta}(\hat{v}) = \min_{v \in S_1} \tilde{\zeta}(v) \equiv \zeta_S. \tag{7.7}$$

Then, we can show that the potential function ζ defined by

$$\zeta(v) = \min\{\tilde{\zeta}(v), \zeta_S\} \tag{7.8}$$

is admissible. In fact, because of the nonnegativity of arc lengths, all the $\zeta(v)$'s as well as $\tilde{\zeta}(v)$'s are nonnegative. In particular, since

$$\tilde{\zeta}(v) = 0 \quad \text{for every } v \in S_2,$$

we have

$$\zeta(v) = 0 \quad \text{for every } v \in S_2.$$

Since

$$\tilde{\zeta}(v) \geq \zeta_S \quad \text{for every } v \in S_1$$

by definition (7.7), we have

$$\zeta(v) = \zeta_S \quad \text{for every } v \in S_1.$$

It also follows from (7.8) that

$$\zeta_S \geq \zeta(v) \ (\geq 0) \quad \text{for every } v \in V.$$

Furthermore, for an arbitrary arc a, if $\tilde{\zeta}(\partial^{-}a) \geq \zeta_S$ then we have

$$\zeta(\partial^{+}a) - \zeta(\partial^{-}a) = \zeta(\partial^{+}a) - \zeta_S \leq 0$$

so that

$$l^{+}(a) \geq \zeta(\partial^{+}a) - \zeta(\partial^{-}a).$$

If $\tilde{\zeta}(\partial^- a) < \zeta_S$ then we have

$$\zeta(\partial^+ a) - \zeta(\partial^- a) \le \tilde{\zeta}(\partial^+ a) - \tilde{\zeta}(\partial^- a).$$

Therefore, if we had

$$\zeta(\partial^+ a) - \zeta(\partial^- a) > l^+(a)$$

for an arc a then we would have

$$\tilde{\zeta}(\partial^+ a) > \tilde{\zeta}(\partial^- a) + l^+(a). \tag{7.9}$$

However, (7.9) would contradict the definition of $\tilde{\zeta}(\partial^+ a)$ as the length of a shortest path from $\partial^+ a$ to S_2, since the concatenation of the arc a (placed in the positive direction) to a path from $\partial^- a$ to S_2 of length $\tilde{\zeta}(\partial^- a)$ would be a path from $\partial^+ a$ to S_2 of length $l^+(a) + \tilde{\zeta}(\partial^- a)$ smaller than $\tilde{\zeta}(\partial^+ a)$. Thus, the relations

$$l^+(a) \ge \zeta(\partial^+ a) - \zeta(\partial^- a)$$

should hold for all the arcs. By similar arguments we can show that

$$l^-(a) \ge \zeta(\partial^- a) - \zeta(\partial^+ a)$$

should hold for all the arcs.

Lemma 8. On a graph of the kind mentioned in Definition 2, if ζ' is an admissible potential function with regard to arc lengths $l^{\pm}(a)$'s and if ζ'' is an admissible potential function determined as in the proof of Lemma 7 with regard to the arc lengths $\bar{l}^{\pm}(a)$'s defined as

$$\begin{aligned} \bar{l}^+(a) &= l^+(a) - (\zeta'(\partial^+ a) - \zeta'(\partial^- a)) \ge 0, \\ \bar{l}^-(a) &= l^-(a) - (\zeta'(\partial^- a) - \zeta'(\partial^+ a)) \ge 0, \end{aligned} \tag{7.10}$$

then the potential function $\zeta = \zeta' + \zeta''$ is also admissible with regard to $l^{\pm}(a)$'s and

$$\zeta(v) = \zeta'(v) + \zeta''(v) \ge \zeta'(v) \quad \text{for every } v \in V. \tag{7.11}$$

Furthermore, if an arc a lies in the positive (resp. negative) direction on a shortest path from S_1 to S_2, then we have

$$l^+(a) = \zeta(\partial^+ a) - \zeta(\partial^- a) \tag{7.12}$$

$$(\text{resp. } l^-(a) = \zeta(\partial^- a) - \zeta(\partial^+ a)).$$

Proof: The compatibility of ζ with $\bar{l}^{\pm}(a)$'s is assured by Lemma 6. We can prove the admissibility of ζ from that of ζ' and ζ'' as follows.

The equation $\zeta(v) = 0$ for $v \in S_2$ is a consequence of $\zeta'(v) = 0$ and $\zeta''(v) = 0$ for $v \in S_2$. Likewise, for $v \in S_1$, $\zeta(v) = \zeta_S \equiv \zeta'_S + \zeta''_S$ follows from $\zeta'(v) = \zeta'_S$ and $\zeta''(v) = \zeta''_S$. The inequality $\zeta_S \ge \zeta(v) \ge 0$ for every $v \in V$ also follows from the corresponding inequalities $\zeta'_S \ge \zeta'(v) \ge 0$ and $\zeta''_S \ge \zeta''(v) \ge 0$.

In order to show the latter statement of the lemma, we may sum up the inequalities:

$$l^{+}(a) \geq \zeta(\partial^{+}a) - \zeta(\partial^{-}a),$$
$$l^{-}(a) \geq \zeta(\partial^{-}a) - \zeta(\partial^{+}a)$$

over the arcs lying on a shortest path P from S_1 to S_2, where the inequality of the former (resp. latter) form is adopted if an arc lies on P in the positive (resp. negative) direction. Then, the left-hand sum gives the length of P, whereas the right-hand sum is equal to $\zeta_S - 0$ (since the potentials at the intermediate vertices of P cancel one another) which is again equal to the length of P by the definition of ζ_S. Therefore, all the inequalities taken into the sum must be equalities.

Theorem 2. The $B^{(r)}$ obtained by the algorithm of §6 is an independent matching of cardinality r on $G(V_1, V_2; A)$ with regard to matroids $M_1(V_1, \mathcal{V}_1)$ and $M_2(V_2, \mathcal{V}_2)$, and an admissible potential function $\zeta^{(r)}$ can be defined on $\tilde{G}_{B^{(r)}}$.

Note:- The existence of a shortest path from the entrances $S_1^{(r)}$ to the exits $S_2^{(r)}$ of $\tilde{G}_{B^{(r)}}$ is guaranteed by Lemma 5 and this theorem.

Proof by induction:

<1: Base of Induction> Following the notation adopted in §4 and §6, we have $B^{(0)} = I_1^{(0)} (= \partial_1 B^{(0)}) = I_2^{(0)} (= \partial_2 B^{(0)}) = \phi$, so that $A_1^{(0)} = A_2^{(0)} = B^{(0)*} = \phi$ and $S_1^{(0)} = V_1 - \mathrm{cl}_1(\phi)$, $S_2^{(0)} = V_2 - \mathrm{cl}_2(\phi)$. Furthermore, the lengths of the arcs of $\tilde{G}_{B^{(0)}}$ are either equal to the weights (assumed to be nonnegative) or to infinity. Therefore, $B^{(0)}$ is trivially an independent matching and the $\zeta^{(0)}$ determined by

(7.13) $\qquad \zeta^{(0)}(v) = 0 \quad$ for all $v \in V_1 \cup V_2$

is an admissible potential function on $\tilde{G}_{B^{(0)}}$.

<2: Induction hypothesis> We assume that $B^{(r)}$ is an independent matching of cardinality r on $G(V_1, V_2; A)$, that $\zeta^{(r)}$ is an admissible potential on $\tilde{G}_{B^{(r)}}$, and that there is a shortest path $P^{(r)}$ from $S_1^{(r)}$ to $S_2^{(r)}$ of finite length.

<3: Induction step> Starting form the arc lenghts $l^{\pm(r)}(a)$'s and

the admissibel potential function $\zeta^{(r)}$ on $\tilde{G}_{B^{(r)}}$, we determine a new potential function $\zeta^{(r+1)}$ in the same way as we determined ζ from ζ' and $l^{\pm}(a)$'s in Lemma 8. The $\zeta^{(r+1)}$ is admissible on $\tilde{G}_{B^{(r)}}$. Then, let us prove that the $\zeta^{(r+1)}$ is admissible also on $\tilde{G}_{B^{(r+1)}}$, and that the $B^{(r+1)}$ is an independent matching on $G(V_1, V_2; A)$.

<3.1: Independence of the matching $B^{(r+1)}$> We begin with the preliminary investigation into the details concerning the way of transformation of $I_1^{(r)} = \partial_1 B^{(r)}$ and $I_2^{(r)} = \partial_2 B^{(r)}$ into $I_1^{(r+1)} = \partial_1 B^{(r+1)}$ and $I_2^{(r+1)} = \partial_2 B^{(r+1)}$. Let $v_0 (\in S_1^{(r)} = V_1 - \mathrm{cl}_1(I_1^{(r)}))$ be the initial vertex of $P^{(r)}$, and u_i and v_i $(i = 1, \cdots, q)$ be the initial and the terminal vertices of the arcs a_i in $A_1^{(r)}$ lying on $P^{(r)}$. Then, we have

$$(7.14) \qquad \zeta^{(r+1)}(u_i) = \zeta^{(r+1)}(v_i) \quad \text{for } i = 1, \cdots, q$$

by Lemma 8, and

$$(7.15) \qquad \zeta^{(r+1)}(v_0) \geq \zeta^{(r+1)}(u_i) = \zeta^{(r+1)}(v_i) \quad (i = 1, \cdots, q)$$

by the admissibility of $\zeta^{(r+1)}$ in $\tilde{G}_{B^{(r)}}$. Furthermore, we have

$$(7.16) \qquad I_1^{(r+1)} = (I_1^{(r)} - \{u_1, \cdots, u_q\}) \cup \{v_0, v_1, \cdots, v_q\},$$

as is obvious from the way of transforming $B^{(r)}$ into $B^{(r+1)}$ by the algorithm. Without loss in generality, we may assume that the u_i's and v_i's (as well as a_i's) are numbered in such a way that

$$(7.17) \qquad \zeta^{(r+1)}(u_i) = \zeta^{(r+1)}(v_i) \geq \zeta^{(r+1)}(u_j) = \zeta^{(r+1)}(v_j)$$

if $1 \leq i < j \leq q$, and that if $\zeta^{(r+1)}(u_i) = \zeta^{(r+1)}(v_i) = \zeta^{(r+1)}(u_j) = \zeta^{(r+1)}(v_j)$ $(i < j)$ then a_i lies nearer v_0 than a_j along $P^{(r)}$. Under these assumptions it is seen that no arc in $A_1^{(r)}$ connects u_i to v_j if $1 \leq i < j \leq q$ by virtue of the compatibility of $\zeta^{(r+1)}$ in $\tilde{G}_{B^{(r)}}$ and the definition of $P^{(r)}$ (recall the requirement stated in the parentheses of step (ii) of the algorithm of §6). Thus, we have

$$(7.18)\quad \begin{aligned} &D_1(u_i, v_i | I_1^{(r)}) = 1 \quad \text{for } i = 1, \cdots, q, \\ &D_1(u_i, v_j | I_1^{(r)}) = 0 \quad \text{for } 1 \le i < j \le q. \end{aligned}$$

Hence, by Lemma 2 (with the order of 1, $\cdots$, q reversed),

$$(7.19)\quad I_1' = (I_1^{(r)} - \{u_1, \cdots, u_q\}) \cup \{v_1, \cdots, v_q\}$$

is an independent set of $M_1(V_1, Y_1)$ and

$$(7.20)\quad \mathrm{cl}_1(I_1') = \mathrm{cl}_1(I_1^{(r)}) = V_1 - S_1^{(r)} \not\ni v_0.$$

Then, by the property (V) of the matroid (§1),

$$(7.21)\quad I_1^{(r+1)} = I_1' \cup \{v_0\}$$

is also an independent set of $M_1(V_1, Y_1)$. We can apply similar arguments to prove the independence of $I_2^{(r+1)}$ in $M_2(V_2, Y_2)$, thus proving that $B^{(r+1)}$ is an independent matching of cardinality r+1 on $G(V_1, V_2; A)$.

<3.2: Compatibility of $\zeta^{(r+1)}$> In order to prove that $\zeta^{(r+1)}$ is compatible on $\tilde{G}_{B^{(r+1)}}$, it suffices to examine if the conditions of compatibility (7.1) are satisfied at those arcs which are in $\tilde{G}_{B^{(r+1)}}$ but not in $\tilde{G}_{B^{(r)}}$. The difference between the arc set $A^{(r+1)}$ of $\tilde{G}_{B^{(r+1)}}$ and the $A^{(r)}$ of $\tilde{G}_{B^{(r)}}$ takes place possibly in $A_1^{(r+1)}$ and $A_1^{(r)}$, $A_2^{(r+1)}$ and $A_2^{(r)}$, and $B^{(r+1)*}$ and $B^{(r)*}$. An arc $a*$ in $B^{(r+1)*} - B^{(r)*}$ is the reverse arc corresponding to an arc a in A lying on the path $P^{(r)}$, so that its lengths are

$$(7.22)\quad \begin{aligned} &l^{+(r+1)}(a*) = -l^{+(r)}(a) = -w(a), \\ &l^{-(r+1)}(a*) = \infty, \end{aligned}$$

and we have, by Lemma 8,

$$(7.23)\quad l^{+(r)}(a) = \zeta^{(r+1)}(\partial^+ a) - \zeta^{(r+1)}(\partial^- a).$$

Since $\partial^+ a* = \partial^- a$ and $\partial^- a* = \partial^+ a$, we therefore have

$$(7.24)\quad l^{+(r+1)}(a*) = \zeta^{(r+1)}(\partial^+ a*) - \zeta^{(r+1)}(\partial^- a*) > -l^{-(r+1)}(a*),$$

which is the condition of compatibility at arc $a*$.

The compatibility of $\zeta^{(r+1)}$ at the arcs in $A_1^{(r+1)} - A_1^{(r)}$ and in $A_2^{(r+1)} - A_2^{(r)}$ can be shown as follows.

We have shown in the proof of Lemma 2 that the transformation of $I_1^{(r)}$ into I_1' can be decomposed into the series of simpler ones, i.e. if we put

$$
(7.25)\quad \begin{cases} I_{1,p} = (I_1^{(r)} - \{u_1, \cdots, u_p\}) \cup \{v_1, \cdots, v_p\} \quad (p = 1, \cdots, q) \\ \text{and} \\ I_{1,0} = I_1^{(r)}, \end{cases}
$$

then $I_{1,p}$'s are independent sets and we have

$$
(7.26)\quad \begin{aligned} I_{1,p} &= (I_{1,p-1} - \{u_p\}) \cup \{v_p\}, \\ I_{1,q} &= I'_1 \end{aligned}
$$

and

$$
(7.27)\quad \mathrm{cl}_1(I_{1,p}) = \mathrm{cl}_1(I_1^{(r)}) = \mathrm{cl}_1(I'_1).
$$

Let us then consider the set $A_{1,p}$ consisting of those arcs which connect a vertex u ($\in I_{1,p}$) to a vertex v ($\in \mathrm{cl}_1(I_{1,p}) - I_{1,p}$) such that

$$
(7.28)\quad D_1(u, v | I_{1,p}) = 1,
$$

and prove inductively from $p = 0$ to $p = q$ that the conditions of compatibility hold for $\zeta^{(r+1)}$ at all the arcs of $A_{1,p}$. Since $A_{1,0} = A_1^{(r)}$, the conditions of compatibility hold for $\zeta^{(r+1)}$ at all the arcs of $A_{1,0}$. The arcs of $A_{1,p} - A_{1,p-1}$ are possibly classified into four types, i.e.

(a) the arc connecting v_p to u_p,

(b) arcs connecting a vertex u ($\in I_{1,p} \cap I_{1,p-1}$) to u_p,

(c) arcs connecting v_p to a vertex v ($\in \mathrm{cl}_1(I_1^{(r)}) - (I_{1,p} \cup I_{1,p-1})$)

and

(d) arcs connecting u ($\in I_{1,p} \cap I_{1,p-1}$) to v ($\in \mathrm{cl}_1(I_1^{(r)}) - (I_{1,p} \cup I_{1,p-1})$).

The compatibility conditions for $\zeta^{(r+1)}$ are satisfied at the arc of type (a) by virtue of (7.14). An arc a of type (b) exists only when there is an arc a' from u to v_p in $A_{1,p-1}$ (Lemma 3), in which case the compatibility of $\zeta^{(r+1)}$ at arc a follows from the compatibility at a' since

$$
(7.29)\quad \zeta^{(r+1)}(u) \leq \zeta^{(r+1)}(v_p) = \zeta^{(r+1)}(u_p).
$$

Similarly, an arc a of type (c) exists only when there is an arc a' from u_p to v in $A_{1,p-1}$ (Lemma 3), in which case the compatibility at arc a follows from the compatibility at arc a'. An arc a of type (d) exists only when there are in $A_{1,p-1}$ both an arc a' from u_p to v and an arc a'' from u to v_p

(Lemma 3). In this case, the conditions of compatibility at arcs a' and a'' are

$$\text{(7.30)}\qquad \begin{aligned} &\zeta^{(r+1)}(u_p) \le \zeta^{(r+1)}(v), \\ &\zeta^{(r+1)}(u) \le \zeta^{(r+1)}(v_p), \end{aligned}$$

from which we readily have the compatibility at arc a, i.e.

$$\text{(7.31)}\qquad \zeta^{(r+1)}(u) \le \zeta^{(r+1)}(v_p) = \zeta^{(r+1)}(u_p) \le \zeta^{(r+1)}(v).$$

Thus the compatibility of the potential funciton $\zeta^{(r+1)}$ for $A_{1,q}$ has been proved.

The arcs in $A_1^{(r+1)} - A_{1,q}$ are classified into two types by Lemma 1 as follows:

(e) arcs connecting a vertex $u \in I_1'$ to a vertex $v \in S_1^{(r)} - S_1^{(r+1)} - \{v_0\}$;

(f) arcs connecting v_0 to a vertex $v \in S_1^{(r)} - S_1^{(r+1)} - \{v_0\}$.

The compatibility of $\zeta^{(r+1)}$ at arcs of type (e) follows from its admissibility in $\tilde{G}_{B^{(r)}}$, since we have, from the admissibility in $\tilde{G}_{B^{(r)}}$,

$$\text{(7.32)}\qquad \zeta^{(r+1)}(u) \le \zeta^{(r+1)}(v).$$

Likewise, we have the compatibility at arcs of type (f):

$$\text{(7.33)}\qquad \zeta^{(r+1)}(v_0) = \zeta^{(r+1)}(v)$$

from the admissibility of $\zeta^{(r+1)}$ in $\tilde{G}_{B^{(r)}}$.

We can prove the compatibility of $\zeta^{(r+1)}$ at the arcs of $A_2^{(r+1)} - A_2^{(r)}$ in a similar way.

<3.3: Admissibility of $\zeta^{(r+1)}$> Finally, since $\zeta^{(r+1)}$ is admissible in $\tilde{G}_{B^{(r)}}$ and

$$\text{(7.34)}\qquad \begin{aligned} &S_1^{(r+1)} \subsetneqq S_1^{(r)}, \\ &S_2^{(r+1)} \subsetneqq S_2^{(r)}, \end{aligned}$$

it is admissible a fortiori in $\tilde{G}_{B^{(r+1)}}$.

<u>Theorem 3.</u> The $B^{(r)}$ has the smallest total weight among all the independent matchings of cardinality r on $G(V_1, V_2; A)$.

<u>Proof:</u> Let us consider the following mathematical-programming problem involving 0-1 valiables, which is slightly more general than the problem of finding the independent matching of cardinality r and of minimum total weight.

We deal with two kinds of variables $\xi(a)$'s and $\eta(v)$'s, of which the former correspond to the arcs a's ($\in A$) of $G(V_1, V_2; A)$ and have real values, whereas the latter correspond to the vertices v's ($\in V_1 \cup V_2$) of G and have values 0 and 1. Moreover, we put

(7.35) $$L = \{I_1 \cup I_2 \mid I_1 \in \mathcal{Y}_1,\ I_2 \in \mathcal{Y}_2,\ |I_1| = |I_2| = r\}$$

where $M_1(V_1, \mathcal{Y}_1)$ and $M_2(V_2, \mathcal{Y}_2)$ are the matroids attached to the vertex sets V_1 and V_2 of G. Then, we define the problem of minimizing the function

(7.36) $$f(\xi) = \sum_{a\in A} w(a)\xi(a)$$

under the constraints:

(7.37) $$\begin{aligned} &\xi(a) \geq 0 \quad \text{for every } a \in A,\\ &\sum_{\partial_1 a=u} \xi(a) - \eta(u) = 0 \quad \text{for every } u \in V_1,\\ &-\sum_{\partial_2 a=v} \xi(a) + \eta(v) = 0 \quad \text{for every } v \in V_2 \end{aligned}$$

and

(7.38) $$\begin{aligned} &\eta(v) = 0 \text{ or } 1 \quad \text{for every } v \in V_1 \cup V_2,\\ &L(\eta) \equiv \{v \mid \eta(v) = 1,\ v \in V_1 \cup V_2\} \in L, \end{aligned}$$

where $\partial_1 a$ (resp. $\partial_2 a$) is the end vertex of arc a in V_1 (resp. V_2).

If $\xi(a)$'s are further constrained to 0 and 1 and if we take $\{a \mid \xi(a) = 1\}$ ($\subseteq A$) for $B^{(r)}$, then the above-defined problem is exactly the problem of finding an independent matching of cardinality r and of minimum total weight, where $L(\eta) \cap V_1 = \partial_1 B^{(r)}$ and $L(\eta) \cap V_2 = \partial_2 B^{(r)}$. However, it will eventually be seen that the $\xi(a)$'s determined from the $B^{(r)}$ obtained by the algorithm of §6, i.e.

(7.39) $$\begin{aligned} \xi(a) &= 1 \quad \text{if } a \in B^{(r)},\\ &= 0 \quad \text{otherwise}, \end{aligned}$$

constitute an optimal solution of the more general problem (7.36) ~ (7.38), so that the $B^{(r)}$ is shown to have tha desired properties.

The problem (7.36) ~ (7.38) is of the kind considered in §5. The dual problem is, then, to maximize

(7.40) $$\begin{aligned} g(\zeta) &= \min_{I_1 \cup I_2 \in L} \Big[\sum_{u\in I_1} \zeta(u) - \sum_{v\in I_2} \zeta(v)\Big]\\ &= \min_{\substack{I_1\in\mathcal{Y}_1\\ |I_1|=r}} \sum_{u\in I_1} \zeta(u) - \max_{\substack{I_2\in\mathcal{Y}_2\\ |I_2|=r}} \sum_{v\in I_2} \zeta(v) \end{aligned}$$

under the constraints:

(7.41) $\quad w(a) \geq \zeta(\partial_1 a) - \zeta(\partial_2 a)$ for every $a \in A$,

where $\zeta(v)$'s are real-valued variables which correspond to the vertices v's ($\in V_1 \cup V_2$) of G.

The "complementary slackness" conditions (5.9) and (5.10) take the form:

(7.42) $\quad w(a) = \zeta(\partial_1 a) - \zeta(\partial_2 a) \quad$ if $\xi(a) > 0$

and

(7.43)
$$\min_{\substack{I_1 \in \mathcal{Y}_1 \\ |I_1|=r}} \sum_{u \in I_1} \zeta(u) = \sum_{\substack{u \in V_1 \\ \eta(u)=1}} \zeta(u),$$
$$\max_{\substack{I_2 \in \mathcal{Y}_2 \\ |I_1|=r}} \sum_{v \in I_2} \zeta(v) = \sum_{\substack{v \in V_2 \\ \eta(v)=1}} \zeta(v),$$

which will be shown to be satisfied by the $\xi(a)$'s defined in (7.39) and the admissible potentials $\zeta^{(r)}(v)$'s in $\tilde{G}_{B^{(r)}}$, where we put

(7.44)
$$\eta(v) = 1 \quad \text{if } v \in I_1^{(r)} \cup I_2^{(r)}$$
$$= 0 \quad \text{otherwise.}$$

First, we observe that the $\xi(a)$'s and $\eta(v)$'s defined in (7.39) and (7.44) obviously satisfy (7.37) and (7.38) since the conditions (7.37) and (7.38) are expressed as

(7.45) $\quad I_1^{(r)} = \partial_1 B^{(r)} \in \mathcal{Y}_1$ and $I_2^{(r)} = \partial_2 B^{(r)} \in \mathcal{Y}_2$

in terms of (7.39) and (7.44), so that they are feasible in the primal problem. Similarly, the conditions (7.41) are part of the conditions of compatibility of $\zeta^{(r)}$ in $\tilde{G}_{B^{(r)}}$ and hence are satisfied, so that $\zeta^{(r)}(v)$'s are feasible in the dual problem.

The conditions (7.42) are satisfied since we have

$$-w(a) \geq \zeta^{(r)}(\partial_2 a) - \zeta^{(r)}(\partial_1 a)$$

or

(7.46) $\quad w(a) \leq \zeta^{(r)}(\partial_1 a) - \zeta^{(r)}(\partial_2 a)$

at the arcs a's of $B^{(r)}$ by virtue of the compatibility of $\zeta^{(r)}$ at the arcs of $B^{(r)*}$ in $\tilde{G}_{B^{(r)}}$.

In order to prove (7.43) we make use of Kruskal's theorem on "Minimum Spanning Trees"[13]. An obvious extension of his theorem (cf., e.g., [14]) says, "In order to find an independent set I of cardinality r of a matroid

$M(V, \mathcal{V})$ (to the elements of which "weights" are associated) such that the sum of the weights over I is the smallest among all the independent sets of cardinality r, one may start from $J_0 = \phi$ ($\in \mathcal{V}$), then augment J_i to $J_{i+1} = J_i \cup \{v_i\}$ with the element $v_i (\in V)$ which has the samllest weight among the elements of $V - \mathrm{cl}(J_i)$ ($i = 0, 1, \cdots, r-1$), and finally put $I = J_r$."

If we arrange the elements of $I_1^{(r)} = \partial_1 B^{(r)} = \{u_1, \cdots, u_r\}$ in such a way that

$$(7.47) \qquad \zeta^{(r)}(u_1) \leq \zeta^{(r)}(u_2) \leq \cdots \leq \zeta^{(r)}(u_r),$$

we can prove, as shown just below, that

$$(7.48) \qquad \zeta^{(r)}(v) \geq \zeta^{(r)}(u_p) \quad \text{for any } v \in V_1 - \mathrm{cl}_1(\{u_1, \cdots, u_{p-1}\})$$

for $p = 1, 2, \cdots, r$. Then, by Kruskal's theorem (as extended above), we have

$$(7.49) \qquad \min_{\substack{I_1 \in \mathcal{V}_1 \\ |I_1| = r}} \sum_{u \in I_1} \zeta^{(r)}(u) = \sum_{u \in I_1^{(r)}} \zeta^{(r)}(u),$$

which is equivalent to the first part of (7.43). (The second part is proved in a similar manner.) Thus, by Theorem 1, we are led to the desired result.

To prove (7.48) we resort to the induction on p. Since we have

$$(7.50) \qquad \zeta^{(r)}(v) \geq \zeta^{(r)}(u) \quad \text{for every } v \in V_1 - \mathrm{cl}_1(I_1^{(r)}) \text{ and every } u \in \mathrm{cl}_1(I_1^{(r)})$$

from the admissibility of $\zeta^{(r)}$ in $\tilde{G}_{B^{(r)}}$, it suffices to prove (7.48) for $v \in \mathrm{cl}_1(I_1^{(r)})$. For $p = 1$ (7.48) is true, because every vertex $v \in \mathrm{cl}_1(I_1^{(r)}) - I_1^{(r)}$ is related to a vertex $u_i \in I_1^{(r)}$ by $D_1(u_i, v | I_1^{(r)}) = 1$ and hence the compatibility of $\zeta^{(r)}$, together with (7.47), guarantees

$$(7.51) \qquad \zeta^{(r)}(v) \geq \zeta^{(r)}(u_i) \geq \zeta^{(r)}(u_1).$$

Assuming that (7.48) is true for some p, let us take a vertex $v \in \mathrm{cl}_1(I_1^{(r)}) - \mathrm{cl}_1(\{u_1, \cdots, u_p\})$ and the element $u_i \in I_1^{(r)}$ which has the largest index i among the elements of the circuit in $\{v\} \cup I_1^{(r)}$. By the definition of v, the index i must be greater than or equal to $p+1$ and we have

$$(7.52) \qquad D_1(u_i, v | I_1^{(r)}) = 1.$$

Therefore, we have

(7.53) $\zeta^{(r)}(v) \geq \zeta^{(r)}(u_i) \geq \zeta^{(r)}(u_{p+1})$,

i.e. (7.48) is true for p+1.

Theorem 4 [4]. If there is no path of finite length from $S_1^{(r)}$ to $S_2^{(r)}$ in $\tilde{G}_{B^{(r)}}$, then the $B^{(r)}$ is a maximum independent matching.

Proof: Let us denote by S the set of those vertices of $V_1 \cup V_2$ to which there is a path of finite length from the entrances $S_1^{(r)}$, and put

(7.54) $$T_1 = V_1 - S,$$
$$T_2 = V_2 \cap S.$$

From the definition of S, T_1, T_2 and $\tilde{G}_{B^{(r)}}$ it follows that

(7.55) $$T_1 \subseteqq V_1 - S_1^{(r)} = \mathrm{cl}_1(I_1^{(r)}),$$
$$T_2 \subseteqq V_2 - S_2^{(r)} = \mathrm{cl}_2(I_2^{(r)})$$

and

(7.56) $$T_1 \subseteqq \mathrm{cl}_1(T_1 \cap I_1^{(r)}),$$
$$T_2 \subseteqq \mathrm{cl}_2(T_2 \cap I_2^{(r)}),$$

and that

(7.57) if $a \in A$ then either $\partial_1 a \in T_1$ or $\partial_2 a \in T_2$

and

(7.58) if $a \in B^{(r)}$ then either $\partial_1 a \notin T_1$ or $\partial_2 a \notin T_2$.

Therefore, for an arbitrary independent matching B, we have from (7.57),

(7.59) $$|B| \leq |T_1 \cap \partial_1 B| + |T_2 \cap \partial_2 B|$$
$$\leq \rho_1(T_1) + \rho_2(T_2),$$

where ρ_1 and ρ_2 are the rank functions of $M_1(V_1, \mathcal{V}_1)$ and $M_2(V_2, \mathcal{V}_2)$, respectively. However, for the $B^{(r)}$, we have, from (7.56) and property (VIII) of the matroid (§1),

(7.60) $$\rho_1(T_1) \leq \rho_1(T_1 \cap I_1^{(r)}) = |I_1^{(r)} \cap T_1|,$$
$$\rho_2(T_2) \leq \rho_2(T_2 \cap I_2^{(r)}) = |I_2^{(r)} \cap T_2|$$

on the one hand, and, on the other,

(7.61) $$|I_1^{(r)} \cap T_1| + |I_2^{(r)} \cap T_2| \leq |B^{(r)}|$$

from (7.58). (7.59) (with $B^{(r)}$ substituted for B), (7.60) and (7.61) are combined into

$$|B^{(r)}| \le \rho_1(T_1) + \rho_2(T_2) \le |B^{(r)}|$$

or

(7.62) $$|B^{(r)}| = \rho_1(T_1) + \rho_2(T_2).$$

Comparison of (7.62) with (7.59) will yield the relation

(7.63) $$|B| \le |B^{(r)}|,$$

which asserts the maximality of $B^{(r)}$.

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