Journal of the Operations Research
Society of Japan
Vol. 20, No. 1, March, 1977

AN ALGORITHM FOR FINDING AN OPTIMAL INDEPENDENT LINKAGE

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(Received August 3, 1976; Revised December 14, 1976)

Abstract. This paper considers the weighted independent linkage problem which is a natural extension of the independent assignment problem recently treated by M. Iri and N. Tomizawa. Given a directed graph with two specified vertex subsets V_1 and V_2 on which matroidal structures are defined respectively, an independent linkage is a set of pairwise-arc-disjoint paths from V_1 to V_2 such that the set of the initial vertices (resp. terminal vertices) of those paths is an independent set on V_1 (resp. V_2). The problem is to find an optimal independent linkage, i.e., a maximum independent linkage having the smallest total weight among all maximum independent linkages, where a weight is given to each arc. We present an algorithm for finding an optimal independent linkage.

1. Introduction

The independent assignment problem has recently been considered by M. Iri and N. Tomizawa [4], where a primal-dual-type algorithm for finding an optimal independent assignment is presented. By an approach different from that adopted in [4], a primal-type algorithm is proposed by the author [3]. Also E. L. Lawler has considered a related problem called the weighted matroid intersection problem in [5].

In the present paper we shall consider the weighted independent linkage problem and present a primal-dual-type algorithm for finding an optimal independent linkage. Given a directed graph with two specified vertex subsets V_1 and V_2 on which matroidal structures are defined respectively, an independent linkage is a set of pairwise-arc-disjoint paths from V_1 to V_2 such that the set of the initial vertices (resp. the terminal vertices) of those paths is an independent set on V_1 (resp. V_2). The weighted independent linkage problem is to find an optimal independent linkage, i.e., a maximum independent linkage from V_1 to V_2 which has the smallest total weight among all maximum independent linkages, where a weight is given to each arc. The precise formulation of the problem will be given in Section 2.

The weighted independent linkage problem includes as a special case the independent assignment problem considered by M. Iri and N. Tomizawa [4] as well as the weighted matroid intersection problem by E. L. Lawler [5] with an obvious modification. The weighted independent linkage problem may be regarded as a kind of minimum-cost flow problem for a network with matroidal constraints on "entrances" and "exits".*)

2. Definitions and Problem Formulation

For a finite set X and a nonempty family F of subsets of X, $M(X,F)$ is called a *matroid* if F satisfies

(i) if $I \in F$ and $I' \subseteq I$, then $I' \in F$

and

(ii) if $I, I' \in F$ and $|I| > |I'|$, then there exists an element x, in $I - I'$, such that $I' \cup \{x\} \in F$.

An element of F is called an *independent set* and an element of $2^X - F$ a *dependent set*, where 2^X is a family of all subsets of X. A minimal dependent set is called a *circuit*. The *closure function* $\mathrm{cl} : 2^X \to 2^X$ is defined in terms of circuits as follows: for any subset Y of X,

$$\mathrm{cl}(Y) = Y \cup \{x \mid x \in X-Y \text{ and there exists a circuit containing } x \text{ in } Y \cup \{x\}\}.$$

We assume a familiarity with fundamental properties of a matroid as described in [9,10].

Consider a directed finite graph $G(V,A)$ with a vertex set V and an arc set A. For each arc a in A, we denote the initial vertex (resp. the terminal vertex) of a by $\partial^+ a$ (resp. $\partial^- a$). A *directed path* on $G(V,A)$ is a sequence $P = (v_0, a_1, v_1, \ldots, v_{k-1}, a_k, v_k)$ of distinct vertices v_i and arcs a_i, where k is a positive integer, $v_i \in V$ $(i=0,1,\ldots,k)$, $a_i \in A$ $(i=1,2,\ldots,k)$ and $\partial^+ a_i = v_{i-1}$, $\partial^- a_i = v_i$ $(i=1,2,\ldots,k)$. Here, v_0 is called the *initial vertex of* P and v_k the *terminal vertex of* P. Similarly, a *directed cycle* on $G(V,A)$ is a sequence $C = (v_0, a_1, v_1, \ldots, v_{k-1}, a_k, v_k, a_0, v_0)$

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*) After having submitted the present paper, the author became aware that the weighted independent linkage problem can be further extended to a minimum-cost flow problem on a network with *polymatroidal* constraints on its entrances and exits (cf. [6]). This further extension will be discussed in a subsequent paper.

of distinct vertices v_i (except for v_0) and arcs a_i, where k is a nonnegative integer, $\partial^+ a_i = v_{i-1}$, $\partial^- a_i = v_i$ $(i=1,2,\ldots,k)$ and $\partial^+ a_0 = v_k$, $\partial^- a_0 = v_0$. Two paths P and Q are *pairwise-arc-disjoint* if P and Q have no common arcs on them. We shall consider only directed paths and directed cycles, so that the term "directed" will be suppressed in the following. If, for vertices u, v in V, there exists one and only one arc from u to v in $G(V,A)$, then we denote the arc by (u,v). Moreover, if for each succeeding two vertices u, v on a path (or a cycle) there exists one and only one arc (u,v) in the graph, then we express the path (or the cycle) in terms of a sequence of vertices only.

We shall denote by $G(V,A;V_1,V_2)$ a directed finite graph $G(V,A)$ with two specified vertex subsets V_1 and V_2 of V, where we assume $V_1 \cap V_2 = \emptyset$ for simplicity. If, for a path P, its initial vertex is in V_1 and its terminal vertex is in V_2, then P is called a *path from* V_1 *to* V_2. A *linkage from* V_1 *to* V_2 is a set L of pairwise-arc-disjoint paths from V_1 to V_2 such that

$$|L| = |\partial_1 L| = |\partial_2 L|,$$

where $\partial_1 L$ (resp. $\partial_2 L$) denotes the set of the initial vertices (resp. the terminal vertices) of the paths which belong to L. That is to say, a linkage L determines a one-to-one correspondence between the sets $\partial_1 L$ $(\subseteq V_1)$ and $\partial_2 L$ $(\subseteq V_2)$ through the paths in L.

Now, we assume that two matroids $M_1(V_1,F_1)$ and $M_2(V_2,F_2)$ are defined on V_1 and V_2, respectively. An *independent linkage* from V_1 to V_2 with regard to matroids $M_i(V_i,F_i)$ $(i=1,2)$ is a linkage L from V_1 to V_2 such that

$$\partial_i L \in F_i\,, \quad i = 1,\ 2,$$

and it will be simply called an independent linkage. A *maximum independent linkage* is an independent linkage containing the largest number of paths from V_1 to V_2.

Moreover, let a real weight function w be defined on the arc set A. The *weighted independent linkage problem* considered in the present paper is to find a maximum independent linkage L which has the smallest total weight:

$$\sum_{a \in \tilde{L}} w(a)$$

among all maximum independent linkages, where $\tilde{L}$ is the set of the arcs lying on L. A solution of the problem will be called an *optimal independent* *linkage*.

The *length of a path* P is defined as the sum of the weights of the arcs lying on P, and, similarly, the *length of a cycle*. A cycle having a negative length is called a *negative cycle*.

We assume that there is no negative cycle on $G(V,A;V_1,V_2)$.

3. Auxiliary Graph Associated with an Independent Linkage

Given an independent linkage L from V_1 to V_2 on $G(V,A;V_1,V_2)$ with regard to matroids $M_i(V_i,F_i)$ $(i=1,2)$, we define an *auxiliary graph* $\overline{G}_L(\overline{V},\overline{A})$ associated with the independent linkage L as a directed graph with a vertex set $\overline{V}$ and an arc set $\overline{A}$. Here, the vertex set $\overline{V}$ is given by

(3.1) $\quad \overline{V} = V\cup\{s,t\}$,

where s and t are two added vertices; and the arc set $\overline{A}$ is the union of six disjoint arc sets:

(3.2) $\quad A_0 = A - \tilde{L}$,

(3.3) $\quad L^* =$ the set of the arcs obtained by reversing the direction of the arcs in $\tilde{L}$,

(3.4) $\quad A_1 = \{(u,v)\,|\,u\in\partial_1 L, v\in \mathrm{cl}_1(\partial_1 L)-\partial_1 L, v\notin \mathrm{cl}_1(\partial_1 L-\{u\})\}$,

(3.5) $\quad A_2 = \{(u,v)\,|\,v\in\partial_2 L, u\in \mathrm{cl}_2(\partial_2 L)-\partial_2 L, u\notin \mathrm{cl}_2(\partial_2 L-\{v\})\}$,

(3.6) $\quad S_1 = \{(s,v)\,|\,v\in V_1-\mathrm{cl}_1(\partial_1 L)\}\cup\{(v,s)\,|\,v\in\partial_1 L\}$,

(3.7) $\quad S_2 = \{(v,t)\,|\,v\in V_2-\mathrm{cl}_2(\partial_2 L)\}\cup\{(t,v)\,|\,v\in\partial_2 L\}$,

where $\tilde{L}$ is a set of the arcs lying on L and cl_i $(i=1,2)$ are the closure functions associated with matroids $M_i(i=1,2)$. It should be noted that L^* defined by (3.3) is an arc subset of $\overline{A}$ satisfying $L^*\cap A = \emptyset$ and that, if $\tilde{L}$ is an m-element arc set $\{a_1,a_2,\ldots,a_m\}$, then L^* is also an m-element arc set $\{a_1^*,a_2^*,\ldots,a_m^*\}$ such that $\partial^+a_i^* = \partial^-a_i$ and $\partial^-a_i^* = \partial^+a_i$ $(i=1,2,\ldots,m)$. When a_i^* $(\in L^*)$ and a_i $(\in \tilde{L})$ are such arcs, we say that a_i^* corresponds to a_i and vice versa; and we assume the underlying one-to-one correspondence between L^* and $\tilde{L}$ whenever L^* is defined.

Furthermore, we define a weight function $\overline{w}$ on the arc set $\overline{A}$ as follows:

(3.9) $\quad \overline{w}(a) = w(a) \qquad \text{if } a\in A_0,$

$= -w(a')$ if $a \in L^*$, where a' is an arc (on L) which corresponds to a,

$= 0$ otherwise.

4. Fundamental Theorems

We shall show fundamental theorems which will give a basis for obtaining a primal-dual-type algorithm for finding an optimal independent linkage.

We shall make use of the lemmas due to N. Tomizawa and M. Iri [8]. The lemmas are concerned with the transformation of independent sets and play an important role in this section. In the following lemmas, L is an independent linkage.

Lemma 1. Let $\{(u_i,v_i)\mid i=1,2,\ldots,p\}$ be a subset of the arc set A_1 defined by (3.4). If there is no arc in A_1 such that

(4.1) $\quad (u_i,v_j), \quad i < j, \quad i, j = 1, 2,\ldots, p,$

then we have

(4.2) $\quad I_1 \equiv (\partial_1 L - \{u_1,u_2,\ldots,u_p\}) \cup \{v_1,v_2,\ldots,v_p\} \in F_1$

and

(4.3) $\quad \mathrm{cl}_1(\partial_1 L) = \mathrm{cl}_1(I_1).$

Lemma 2. Let $\{(u_i,v_i)\mid i=1,2,\ldots,p\}$ be a subset of the arc set A_2 defined by (3.5). If there is no arc in A_2 such that

(4.4) $\quad (u_i,v_j), \quad i < j, \quad i, j = 1, 2,\ldots, p,$

then we have

(4.5) $\quad I_2 \equiv (\partial_2 L - \{v_1,v_2,\ldots,v_p\}) \cup \{u_1,u_2,\ldots,u_p\} \in F_2$

and

(4.6) $\quad \mathrm{cl}_2(\partial_2 L) = \mathrm{cl}_2(I_2).$

For a nonnegative integer r, an (independent) linkage consisting of r pairwise-arc-disjoint paths from V_1 to V_2 will be called an *r-(independent) linkage*.

Theorem 1. Suppose that L is an r-independent linkage. If L has the smallest total weight among all r-independent linkages, then there is no negative cycle on the auxiliary graph $\bar{G}_L(\bar{V},\bar{A})$.

Proof: Suppose that there is a negative cycle on $\bar{G}_L(\bar{V},\bar{A})$, and let C

be a negative cycle, on $\overline{G}_L(\overline{V},\overline{A})$, having the smallest number of arcs.

Now, let us define:

(4.7) $\mathcal{A}_1$ = the set of the arcs, in A_1, lying on C,

(4.8) $\mathcal{A}_2$ = the set of the arcs, in A_2, lying on C.

Moreover, let $\mathcal{A}_i$ $(i=1,2)$ be expressed as

(4.9) $\mathcal{A}_1 = \{(a_i^1,a_i^2)\,|\,i=1,2,\ldots,k_1\}$,

(4.10) $\mathcal{A}_2 = \{(b_i^1,b_i^2)\,|\,i=1,2,\ldots,k_2\}$.

We define a directed graph $\mathcal{G}(\mathcal{A}_1)$ (resp. $\mathcal{G}(\mathcal{A}_2)$) with a "vertex" set $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) as follows.

(4.11) "there exists an arc from (a_p^1,a_p^2) to (a_q^1,a_q^2) (resp. from (b_r^1,b_r^2) to (b_s^1,b_s^2)) on $\mathcal{G}(\mathcal{A}_1)$ (resp. $\mathcal{G}(\mathcal{A}_2)$) if and only if there exists an arc (a_p^1,a_q^2) (resp. (b_r^1,b_s^2)) in A_1 (resp. A_2).

We first show that there is no cycle on $\mathcal{G}(\mathcal{A}_1)$ and $\mathcal{G}(\mathcal{A}_2)$. Assume that there exists a cycle on $\mathcal{G}(\mathcal{A}_1)$ which is given by

(4.12) $((\bar{a}_1^1,\bar{a}_1^2),\ldots,(\bar{a}_\ell^1,\bar{a}_\ell^2),(\bar{a}_1^1,\bar{a}_1^2))$ $\quad (\ell \leqq k_1)$.

By the definition (4.11), there exist arcs

(4.13) $(\bar{a}_i^1,\bar{a}_{i+1}^2)$, $\quad i = 1, 2,\ldots, \ell$

in A_1, where $\bar{a}_{\ell+1}^2 \equiv \bar{a}_1^2$. For $i = 1, 2,\ldots, \ell$, let us define:

(4.14) C_i = the cycle obtained from C by removing the path on C from vertex $\bar{a}_i^1$ to vertex $\bar{a}_{i+1}^2$ and by adding the arc $(\bar{a}_i^1,\bar{a}_{i+1}^2)$,

(4.15) w_i = the length of the cycle C_i (with regard to the weight function $\bar{w}$ defined on $\overline{G}_L(\overline{V},\overline{A})$).

Since the length (or weight) of each arc in A_1 is equal to zero, w_i defined by (4.15) is equal to the length of the path on C from vertex $\bar{a}_{i+1}^2$ to vertex $\bar{a}_i^2$ for each $i = 1, 2,\ldots, \ell$, where $\bar{a}_{\ell+1}^2 \equiv \bar{a}_1^2$. Therefore, we have

(4.16) $\sum_{i=1}^{\ell} w_i = \ell' W < 0$,

where W (< 0) is the length of C and ℓ' is a positive integer. Consequently, from (4.16), for some $i_0 \in \{1,2,\ldots,\ell\}$

(4.17) $w_{i_0} < 0$.

This means that there exists a negative cycle C_{i_0}, on $\overline{G}_L(\overline{V},\overline{A})$, having a smaller number of arcs than C, which contradicts the definition of C. Therefore, there is no cycle on $\tilde{G}(\tilde{A}_1)$.

Similarly, we can show that there is no cycle on $\tilde{G}(\tilde{A}_2)$.

From Lemmas 1 and 2, we thus have

(4.18) $E_1 \equiv (\partial_1 L - \{a_1^1, a_2^1, \ldots, a_{k_1}^1\}) \cup \{a_1^2, a_2^2, \ldots, a_{k_1}^2\} \in F_1,$

(4.19) $\mathrm{cl}_1(E_1) = \mathrm{cl}_1(\partial_1 L),$

(4.20) $E_2 \equiv (\partial_2 L - \{b_1^2, b_2^2, \ldots, b_{k_2}^2\}) \cup \{b_1^1, b_2^1, \ldots, b_{k_2}^1\} \in F_2,$

(4.21) $\mathrm{cl}_2(E_2) = \mathrm{cl}_2(\partial_2 L).$

Denote by $\tilde{L}$ the set of the arcs lying on L, by $\tilde{C}_0$ the set of the arcs, in A_0, lying on C and by $\tilde{C}^*$ the set of the arcs, in A, corresponding to the arcs, in L^*, lying on C. Let an arc set $\tilde{L}'$ be given by

(4.22) $\tilde{L}' = (\tilde{L} - \tilde{C}^*) \cup \tilde{C}_0.$

It can be easily shown that there exists an r-linkage L'' and a set of pairwise-arc-disjoint cycles $\overline{C}_i$ $(i \in I)$ on $G(V,A;V_1,V_2)$ such that the arc set $\tilde{L}'$ of (4.22) is the union of the two disjoint arc sets: (1) the set of the arcs lying on the r-linkage L'' and (2) the set of the arcs lying on the cycles $\overline{C}_i$ $(i \in I)$, where I is a finite index set. Furthermore, by the definition of C, for each $i = 1, 2$, $\partial_i L''$ is obtained by adding to E_i at most one vertex in $V_i - \mathrm{cl}_i(\partial_i L)$ and by removing from it at most one vertex in $\partial_i L$ (also in E_i). Therefore, from (4.19) and (4.21) we have

$$\partial_i L'' \in F_i, \qquad i = 1, 2$$

and L'' is an r-independent linkage.

Moreover, from (4.22) we see that

(4.23) $$(\text{the total weight of } L'') + \sum_{i \in I} (\text{the length of } \overline{C}_i) - (\text{the total weight of } L) = (\text{the length of } C),$$

where the weights and the length in the left-hand side of (4.23) are given with respect to the weight function w, while the length of C in the right-hand side is with respect to $\overline{w}$ defined on $\overline{G}_L(\overline{V},\overline{A})$. Since by the assumptions

$$(\text{the length of } C) < 0,$$

$$(\text{the length of } \overline{C}_i) \geqq 0, \quad i \in I,$$

it follows from (4.23) that

$$\text{(the total weight of } L''\text{)} < \text{(the total weight of } L\text{)}.$$

Consequently, there exists an r-independent linkage having a smaller total weight than L, which is a contradiction. The theorem thus follows. Q.E.D.

Theorem 2. Let L_r be an r-independent linkage having the smallest total weight among all r-independent linkages. Suppose there exists a path from vertex s to vertex t on the auxiliary graph $\bar{G}_{L_r}(\bar{V},\bar{A})$ associated with L_r. Let P be the shortest path from vertex s to vertex t on the auxiliary graph. (If more than one such path exists, let P be the one which consists of the fewest arcs.) Also define a set $\tilde{L}$ of arcs by

$$(4.24) \qquad \tilde{L} = (\tilde{L}_r - \tilde{P}^*) \cup \tilde{P}_0$$

where $\tilde{L}_r$ is the set of the arcs on L_r, $\tilde{P}_0$ the set of the arcs on P which belong to A_0 and $\tilde{P}^*$ the set of the arcs (in A) corresponding to the arcs on P which belong to L_r^*. Then $\tilde{L}$ is the union of two disjoint arc sets: (i) the nonempty set $\tilde{L}_{r+1}$ of the arcs which lie on an $(r+1)$-independent linkage L_{r+1} and (ii) the (possibly empty) set $\tilde{A}$ of the arcs which lie on pairwise-arc-disjoint cycles. Here, A_0 and L_r^* are those defined by (3.2) and (3.3) with L replaced by L_r, respectively.

Proof: It can be easily shown from the definition (4.24) that the $\tilde{L}$ is the union of two disjoint arc sets: (i) the nonempty set of the arcs which lie on an $(r+1)$-linkage L_{r+1} and (ii) the (possibly empty) set of the arcs which lie on pairwise-arc-disjoint cycles. Therefore, the only thing we have to show is that the L_{r+1} is an $(r+1)$-*independent* linkage.

Let us define:

$$(4.25) \qquad \tilde{A}_i = \text{the set of the arcs, in } A_i\text{, lying on } P \quad (i=1,2),$$

where A_1 and A_2 are those defined by (3.4) and (3.5) with L replaced by L_r. Suppose that $\tilde{A}_i$ $(i=1,2)$ are expressed as

$$(4.26)_1 \qquad \tilde{A}_1 = \{(a_i^1, a_i^2) \mid i=1,2,\ldots,m_1\},$$

$$(4.26)_2 \qquad \tilde{A}_2 = \{(b_i^1, b_i^2) \mid i=1,2,\ldots,m_2\}.$$

In the same manner as (4.11), for each $i = 1, 2$ we define a directed graph $\tilde{G}(\tilde{A}_i)$ with "vertex" set $\tilde{A}_i$.

Suppose there is a cycle, on $\tilde{G}(\tilde{A}_1)$, given by

$$(4.27) \qquad ((\bar{a}_1^1, \bar{a}_1^2), \ldots, (\bar{a}_n^1, \bar{a}_n^2), (\bar{a}_1^1, \bar{a}_1^2)) \qquad (n \leqq m_1).$$

From the definition of $\tilde{G}(\tilde{A}_1)$, there are arcs:

(4.28) $(\bar{a}_i^1, \bar{a}_{i+1}^2)$, $i = 1, 2, \ldots, n$

in A_1, where $\bar{a}_{n+1}^2 \equiv \bar{a}_1^2$. Here, the arc $(\bar{a}_i^1, \bar{a}_{i+1}^2)$ is assumed to be

(4.29) in the direction of the path P if $i \in I$, and
in the direction opposite to that of P if $i \in J$,

where $I \cup J = \{1, 2, \ldots, n\}$ and $I \cap J = \emptyset$. Note that both I and J are nonempty.

Let C be the cycle obtained by adding to P an arc from its terminal vertex t to the initial vertex s of zero length. Also define:

$\tilde{C}_i \equiv$ the cycle obtained from C by removing the path on C from vertex $\bar{a}_i^1$ to vertex $\bar{a}_{i+1}^2$ and by adding to it the arc $(\bar{a}_i^1, \bar{a}_{i+1}^2)$,

(4.30) w_i = the length of the path on C from $\bar{a}_i^1$ to $\bar{a}_{i+1}^2$,

(4.31) $\tilde{w}_i$ = the length of $\tilde{C}_i$ $(= W - w_i)$,

where W is the length of P (or the length of C).

Since P is the shortest path from s to t (consisting of the fewest arcs) and since the length of the arcs of (4.28) is equal to zero, we have from (4.29) and (4.30)

(4.32) $w_i < 0$ if $i \in I$.

On the other hand, similarly as in the proof of Theorem 1 we have

(4.33) $\sum_{i\in I} \tilde{w}_i + \sum_{i\in J} \tilde{w}_i = n'W,$

where n' is a positive integer less than n.

We can easily see from (4.29) that the positive integer n' appearing in (4.33) is given by

(4.34) $n' = |I|$.

Moreover, from (4.31)

(4.35) $\sum_{i\in I} \tilde{w}_i = |I|W - \sum_{i\in I} w_i.$

It follows from (4.32)-(4.35) that

(4.36) $\sum_{i\in J} \tilde{w}_i = \sum_{i\in J} w_i < 0.$

Therefore, for some $i_0 \in J$

$$\tilde{w}_{i_0} < 0.$$

This means that there exists a negative cycle $\tilde{C}_{i_0}$ on the auxiliary graph $\overline{G}_{L_r}(\overline{V},\overline{A})$, which contradicts the assumption that L_r is an r-independent linkage having the smallest total weight among all r-independent linkages (see Theorem 1). Consequently, there is no cycle on $\tilde{G}(\tilde{A}_1)$.

Similarly, we can show that there is no cycle on $\tilde{G}(\tilde{A}_2)$.

Since there is no cycle either on $\tilde{G}(\tilde{A}_1)$ or on $\tilde{G}(\tilde{A}_2)$, we have from Lemmas 1 and 2

$$(4.37)_1 \quad I_1 \equiv (\partial_1 L_r - \{a_1^1, a_2^1, \ldots, a_{n_1}^1\}) \cup \{a_1^2, a_2^2, \ldots, a_{n_1}^2\} \in F_1,$$

$$(4.37)_2 \quad I_2 \equiv (\partial_2 L_r - \{b_1^2, b_2^2, \ldots, b_{n_2}^2\}) \cup \{b_1^1, b_2^1, \ldots, b_{n_2}^1\} \in F_2,$$

and

$$(4.38) \qquad \mathrm{cl}_1(I_1) = \mathrm{cl}_1(\partial_1 L_r), \qquad \mathrm{cl}_2(I_2) = \mathrm{cl}_2(\partial_2 L_r).$$

Since by the assumption P is the shortest path (consisting of the fewest arcs), $\partial_i L_{r+1}$ is obtained by adding to I_i a vertex in $V_i - \mathrm{cl}_i(\partial_i L_r)$ for each $i = 1, 2$. Therefore, from (4.38)

$$\partial_i L_{r+1} \in F_i, \qquad i = 1, 2. \qquad \text{Q.E.D.}$$

Theorem 3. Under the assumption of Theorem 2, the $(r+1)$-independent linkage L_{r+1} of Theorem 2 has the smallest total weight among all $(r+1)$-independent linkages. Moreover, the total weight of the set $\tilde{A}$ of the arcs which lie on the pairwise-arc-disjoint cycles of Theorem 2 is equal to zero.

Proof: First, since P defined in Theorem 2 is the shortest path (consisting of the fewest arcs) and since L_r is the smallest-total-weight r-independent linkages, the total weight of the arc set $\tilde{A}$ is equal to zero. Therefore,

$$(4.39) \qquad \text{(the total weight of } L_{r+1}) - \text{(the total weight of } L_r) = \text{(the length of } P),$$

where the weights of L_{r+1} and L_r are given with respect to the weight function w and the length of P is with respect to $\overline{w}$ defined on $\overline{G}_{L_r}(\overline{V},\overline{A})$.

Next, we show that the $(r+1)$-independent linkage L_{r+1} of Theorem 2 has the smallest total weight among all $(r+1)$-independent linkages.

Let $\overline{L}_{r+1}$ be an arbitrary $(r+1)$-independent linkage. In the following, we shall consider a subgraph $G(\hat{L})$ of $G(V,A;V_1,V_2)$ which consists of the vertices and the arcs lying on $\hat{L} = L_r \cup \bar{L}_{r+1}$. For each $i = 1, 2$ we restrict the matroid $M_i(V_i, F_i)$ on the vertex set $\partial_i L_r \cup \partial_i \bar{L}_{r+1}$. Let the restriction matroid be given by $\bar{M}_i(\bar{V}_i, \bar{F}_i)$ with the associated closure function $\overline{\mathrm{cl}}_i$ for each $i = 1, 2$.

The vertex set $\partial_i L_r \cup \partial_i \bar{L}_{r+1}$ is the union of four disjoint sets:

$$\left.\begin{aligned} V_{i1} &= \partial_i L_r \cup \partial_i \bar{L}_{r+1} - \overline{\mathrm{cl}}_i(\partial_i L_r), \\ V_{i2} &= \partial_i L_r \cap \partial_i \bar{L}_{r+1}, \\ V_{i3} &= \overline{\mathrm{cl}}_i(\partial_i L_r) - \partial_i L_r, \\ V_{i4} &= \partial_i L_r - \partial_i L_r \cap \partial_i \bar{L}_{r+1}, \end{aligned}\right\} \quad i = 1, 2.$$

We denote by $\bar{G}_{L_r}(\hat{L})$ the auxiliary graph associated with the r-independent linkage L_r on $G(\hat{L})$. We also denote by $\bar{A}_1$ the set of the arcs from V_{14} to V_{13} on $\bar{G}_{L_r}(\hat{L})$ and by $\bar{A}_2$ the set of the arcs from V_{23} to V_{24} on $\bar{G}_{L_r}(\hat{L})$. Then we can show that for each $i = 1, 2$ there exists a complete matching A_i^o on a bipartite graph with vertex set $V_{i3} \cup V_{i4}$ and arc set $\bar{A}_i$, where

(the set of the terminal vertices of the arcs in A_1^o) $= V_{13}$,

(the set of the initial vertices of the arcs in A_2^o) $= V_{23}$

(cf. the proof of the theorem in [3]).

Now, let us remove from $\bar{G}_{L_r}(\hat{L})$ the arcs belonging to:

(i) $(A_1 - A_1^o) \cup (A_2 - A_2^o)$,

(ii) $\{(v,s) \mid v$ is the initial vertex of an arc in $A_1^o\}$,

(iii) $\{(t,v) \mid v$ is the terminal vertex of an arc in $A_2^o\}$,

(iv) the set of the arcs, in L_r^*, corresponding to the common arcs lying on L_r and $\bar{L}_{r+1}$,

(v) $\{(v,s) \mid v \in \partial_1 L_r \cap \partial_1 \bar{L}_{r+1}\} \cup \{(t,v) \mid v \in \partial_2 L_r \cap \partial_2 \bar{L}_{r+1}\}$

and further remove from it isolated vertices if they exist. Denote the resultant graph by $\hat{G}$.

From the way of constructing the graph $\hat{G}$, we can see that

(i) the positive degree (resp. negative degree) of vertex s (resp. t) is larger by 1 than its negative degree (resp. positive degree);

(ii) for every vertex of $\hat{G}$ except for vertices s and t its positive degree is equal to its negative degree.

Here, the positive degree (resp. negative degree) of a vertex v is the number of the arcs which have the vertex v as their initial vertex (resp. terminal vertex). Therefore, $\hat{G}$ can be covered by

(1) a path from vertex s to vertex t and

(2) some pairwise-arc-disjoint cycles.

Let us denote the path of (1) by P' and the cycles of (2) by C_i $(i \in I)$. Then we have

(4.40) (the total weight of $\overline{L}_{r+1}$) - (the total weight of L_r)

$$= (\text{the length of } P') + \sum_{i \in I} (\text{the length of } C_i),$$

where the weights of $\overline{L}_{r+1}$ and L_r are given with respect to the weight function w and the length of P' and C_i's are with respect to $\overline{w}$ defined on $\overline{G}_{L_r}(\hat{L})$.

From Theorem 1 there holds

(4.41) (the length of C_i) $\geqq 0$, $i \in I$

and from the definition of P

(4.42) (the length of P') $\geqq$ (the length of P).

It follows from (4.39)-(4.42) that

(the total weight of $\overline{L}_{r+1}$) $\geqq$ (the total weight of L_{r+1}).

Q.E.D.

Remark 1. The argument from below (4.39) till (4.40) is based only on the assumptions that L_r is an r-independent linkage and that there exists an $(r+1)$-independent linkage. Therefore, we have already shown the following theorem.

Theorem 4. Let L be an independent linkage. If there is no path from vertex s to vertex t on the auxiliary graph $\overline{G}_L(\overline{V},\overline{A})$, then L is a maximum independent linkage.

Remark 2. Define a *cut* on $G(V,A;V_1,V_2)$ as an ordered triple (U_1,B,U_2), where, for each $i = 1, 2$, U_i is a vertex subset of V_i and B is an arc subset of A such that for every path P from V_1 to V_2 on $G(V,A;V_1,V_2)$ there holds at least one of the following three: (1) the initial vertex of P is in U_1; (2) the terminal vertex of P is in U_2; and (3) there is an arc, on P, which belongs to B. By this definition, for an arbitrary cut (U_1,B,U_2) and an arbitrary independent linkage L, there holds

(4.43) $\quad |L| \leqq r_1(U_1) + |B| + r_2(U_2),$

where r_i $(i=1,2)$ are the rank functions associated with matroids $M_i(V_i,F_i)$. Moreover, let $\overline{L}$ be a maximum independent linkage and $\overline{U}_1$, $\overline{U}_2$, $\overline{B}$ be those defined by

$$\overline{U}_1 = V_1 - S,$$
$$\overline{U}_2 = V_2 \cap S,$$
$$\overline{B} = \text{the set of the arcs, in } A, \text{ from } S \text{ to } V - S,$$

where S is the set of the vertices to which there are paths, on $\overline{G}_{\overline{L}}(\overline{V},\overline{A})$, from vertex s. Then, in a similar manner as in the proof of the theorem (in [4,8]) concerning a maximum independent matching, we can show that

(4.44) $\quad |\overline{L}| = r_1(\overline{U}_1) + |\overline{B}| + r_2(\overline{U}_2).$

Consequently, from (4.43) and (4.44) we have a kind of max-min formula:

$$\max\{|L| \mid L \text{ is an independent linkage}\}$$
$$= \min\{r_1(U_1)+|B|+r_2(U_2) \mid (U_1,B,U_2) \text{ is a cut}\}$$

(also cf. [1]).

5. Algorithm for Finding an Optimal Independent Linkage

We shall present an algorithm for finding an optimal independent linkage based on Theorems 1 - 4 of the preceeding section.

Algorithm for finding an optimal independent linkage

0° Set $r \leftarrow 0$ and $L_0 \leftarrow \emptyset$. Go to 1°.

1° Construct the auxiliary graph $\overline{G}_{L_r}(\overline{V},\overline{A})$ associated with L_r as described in Section 3. Go to 2°.

2° If there is a path from s to t on $\overline{G}_{L_r}(\overline{V},\overline{A})$, then go to 3°, or else go to 4°.

3° Find a shortest path P from s to t on $\overline{G}_{L_r}(\overline{V},\overline{A})$. (If more than one such shortest path exists, we should take P as the shortest path which consists of the fewest arcs.) Let $\mathcal{P}_0$ be the set of the arcs, on P, belonging to A and $\mathcal{P}^*$ be the set of the arcs (in A) which correspond to the arcs, on P, belonging to $L_r{}^*$.

Set

$$\mathcal{L} \leftarrow (\mathcal{L}_r - \mathcal{P}^*) \cup \mathcal{P}_0 \;,$$

where $\tilde{L}_r$ is the set of the arcs lying on L_r.

Then find a maximal independent linkage (i.e., $(r+1)$-independent linkage) L_{r+1} from V_1 to V_2 such that the set of the arcs lying on L_{r+1} is a subset of $\tilde{L}$.

Set $r \leftarrow r+1$ and go to 1°.

4° L_r is an optimal independent linkage and the algorithm terminates.

Remark 3. The validity of the above algorithm is clear from Theorems 1 - 4 of the preceeding section. The cardinality of the independent linkage L_r is equal to r and increases by 1 every time we go through step 3°.

We can show that, when matroids $M_i(V_i,F_i)$ $(i=1,2)$ are determined by matrices or graphs, the number of computations required is at most proportional to $|V|^3 \cdot \min\{|V_1|,|V_2|\}$. More precisely, we can reach an optimal independent linkage after finding at most $\min\{|V_1|,|V_2|\}$ shortest paths through step 3°, where the number of computations required for finding a shortest path on an auxiliary graph is at most proportional to $|V|^3$ by any existing methods, while the number of computations required for modifying an auxiliary graph is at most proportional to $\max\{|V_1|^3,|V_2|^3\}$ for matroids $M_i(V_i,F_i)$ $(i=1,2)$ determined by matrices and to $\max\{|V_1|^2,|V_2|^2\}$ for matroids determined by graphs. Therefore, when matroids $M_i(V_i,F_i)$ $(i=1,2)$ are determined by graphs, the total number of computations may be reduced and proportional to $|V|^2 \cdot \min\{|V_1|,|V_2|\}$ if we employ the technique developed in [7] for finding shortest paths on auxiliary graphs. (Also see the discussion on the computational complexity in [4].)

Remark 4. A maximal independent linkage L_{r+1} in step 3° can be easily found as follows. Let I_1 and I_2 be subsets of V_1 and V_2, respectively, given by

$$I_1 = (\partial_1 L_r - \{a_1^1, a_2^1, \ldots, a_{\ell_1}^1\}) \cup \{a_1^2, a_2^2, \ldots, a_{\ell_1}^2\} \cup \{e_1\},$$
$$I_2 = (\partial_2 L_r - \{b_1^2, b_2^2, \ldots, b_{\ell_2}^2\}) \cup \{b_1^1, b_2^1, \ldots, b_{\ell_2}^1\} \cup \{e_2\},$$

where $\{(a_i^1, a_i^2) \mid i=1,2,\ldots,\ell_1\}$ and $\{(b_i^1, b_i^2) \mid i=1,2,\ldots,\ell_2\}$ are, respectively, the sets of the arcs, on P, belonging to A_1 and A_2 defined by (3.4) and (3.5) with L replaced by L_r and e_i $(i=1,2)$ are the vertices, on P, belonging to $V_i - \mathrm{cl}_i(\partial_i L_r)$ $(i=1,2)$, respectively. Then any set of pairwise-arc-disjoint paths (having the set of the arcs contained in $\tilde{L}$) such that I_1 and I_2 are, respectively, the sets of the initial vertices and the terminal vertices of those paths is the desired L_{r+1} to be found.

Remark 5. The auxilairy graph $\overline{G}_L(\overline{V},\overline{A})$ associated with an independent linkage L can be determined by $\partial_i L$ ($\subsetneq V_i$) ($i=1,2$) and $\tilde{L}$, the set of the arcs lying on L, instead of L. Therefore, if we define an auxiliary graph in this way, we do not need to find a maximal independent linkage from $\tilde{L}$ in every step 3° but we should find it only once in step 4° for obtaining an optimal independent linakge.

Remark 6. By a similar approach as adopted in [3] we can show a primal-type algorithm which starts from a maximum independent linkage and gives us maximum independent linkages having smaller total weights than the old ones as the computation proceeds. The modification of the algorithm presented in [3] may be straightforward.

Remark 7. It should be noted that the existence of the arcs of

$$\{(v,s)\,|\,v\varepsilon\partial_1 L\}\cup\{(t,v)\,|\,v\varepsilon\partial_2 L\} \tag{5.1}$$

(see (3.6) and (3.7)) on the auxiliary graph $\overline{G}_L(\overline{V},\overline{A})$ makes no difference in finding an optimal independent linkage by the algorithm presented above. Therefore, the arcs of (5.1) may be removed from the auxiliary graph $\overline{G}_L(\overline{V},\overline{A})$ for implementing the algorithm. However, the existence of the arcs of (5.1) is crutial for the primal-type algorithm touched upon in Remark 6 (cf. [3]).

Remark 8. Though we defined a linkage in terms of pairwise-arc-disjoint paths, we can also define it as a set of *pairwise-vertex-disjoint* paths from V_1 to V_2. An algorithm for finding an optimal independent linkage defined in terms of pairwise-vertex-disjoint paths is given by the above algorithm with a slight modification: the modification may be easily made because a set of pairwise-vertex-disjoint paths can be found by imposing a vertex capacity equal to 1 on each vertex (cf. [2]).

Remark 9. If the directed graph $G(V,A;V_1,V_2)$ is a bipartite graph with its end-vertex sets V_1 and V_2, then the algorithm presented above coincides with the one by M. Iri and N. Tomizawa for the independent assignment problem [4]. This means that we have given an alternative proof of the validity of the Iri-Tomizawa optimal-independent-assignment algorithm.

Acknowledgments

The author would like to express his sincere thanks to Professor Masao Iri of the University of Tokyo for his helpful suggestions and valuable discussions; in fact, he suggested the author the possible extension of the independent assignment problem to the weighted independent linkage problem. Thanks are also due to the referees for their useful comments.

This work was supported by the Grant in Aid for Scientific Research of the Ministry of Education, Science and Culture of Japan under Grant: Cooperative Research (A) 135017 (1976).

The author is also supported by the Sakkokai Foundation.

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