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An Algorithm for the Second Immanant

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Abstract. Let χ be an irreducible character of the symmetric group S_n. For $A = (a_{ij})$ an n-by-n matrix, define the immanant of A corresponding to χ by

$$d(A) = \sum_{\sigma \in S_n} \chi(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)}.$$

The article contains an algorithm for computing $d(A)$ when χ corresponds to the partition $(2, 1^{n-2})$.

Introduction. Denote by χ_k the (irreducible, characteristic zero) character of the symmetric group S_n corresponding to the partition $(k, 1^{n-k})$, for $k = 1, 2, \ldots, n$. If $A = (a_{ij})$ is an n-by-n matrix, define

$$d_k(A) = \sum_{\sigma \in S_n} \chi_k(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)}.$$

Then, for example, $d_1(A) = \det(A)$ and $d_n(A) = \operatorname{per}(A)$, the permanent of A. In general, d_k is known as an *immanant* or a generalized matrix function. (An immanant is a generalized matrix function based on S_n.)

Suppose G is a (simple) graph on n vertices. Denote by $L(G)$ the *Laplacian* matrix corresponding to some labeling of the vertices of G, i.e., $L(G)$ is an n-by-n matrix, the (i, j) entry of which is the degree of vertex i when $i = j$, -1 if $i \neq j$ but vertex i is adjacent to vertex j, and zero otherwise. It is shown in [5] that the number of Hamiltonian circuits in G is given by the formula

$$h(G) = \frac{1}{2n} \sum_{k=2}^{n} (-1)^k d_k(L(G)). \tag{1}$$

While there is an immense literature on generalized matrix functions, Eq. (1) is already sufficient motivation to seek "fast" algorithms for their actual computation. The main result of this note is an algorithm for computing d_2. (See the next section.)

It seems that d_2 may be especially appropriate for the study of Laplacian matrices for the following reason: If G is a graph on n vertices, then $L(G)$ is positive semidefinite symmetric and singular. Moreover, G is connected if and only if $\operatorname{rank} L(G) = n - 1$. For arbitrary positive semidefinite symmetric matrices without a zero row, it was established in [3, Corollaries 5 and 6] that $d_2(A) \geqslant 0$ with equality if and only if $\operatorname{rank}(A) < n - 1$.

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Two graphs, G_1 and G_2 are isomorphic if and only if $L(G_1)$ is permutation similar to $L(G_2)$. Thus, any permutation-similarity invariant of $L(G)$ is actually a property of the underlying graph G. This observation has motivated the use of determinants in the study of graphs. (See, e.g. [1].) The trouble with determinants is that they are preserved under arbitrary similarities. It was suggested in [6] that permanents might be substituted for determinants. The trouble with permanents is their computational intractability. (See [7]. It is already evident from (1) that the computation of at least some immanants is as hard as the Hamiltonian circuit problem.) It occurred to us that d_2 might afford a reasonable compromise, and this turns out to be the case.

The key to fast algorithms for d_1 lies in the observation that the determinant of a matrix with two equal rows is zero. If A has $k+1$ equal rows. it turns out [3, Corollary 4] that $d_k(A) = 0$. Although our algorithm does not make explicit use of this observation, it provides some evidence (corroborated by Werner Hartmann [2]) that $d_k(A)$ should become easier to compute as k decreases. On the other hand, there may be something unique about d_2. It was shown in [4] that d_2, but not d_k for $k > 2$, is linked in an interesting way with the determinant, namely $d_2(A)\det(A^{-1}) = d_2(A^{-1})\det(A)$, for all invertible matrices A.

An Algorithm for d_2. If $\sigma \in S_n$, then $\chi_2(\sigma) = \varepsilon(\sigma)(F(\sigma) - 1)$, where ε is the alternating (signum) character and $F(\sigma)$ is the number of fixed points of σ. It follows from this fact that $d_2(A) = -\det(A)$ for any matrix A with main diagonal consisting entirely of zeros. Our algorithm is based on this observation. We proceed with its description.

1. *Scale A.* Denote by D the n-by-n diagonal matrix; the (i, i) entry of which is 1 if $a_{ii} = 0$ and $1/a_{ii}$ if $a_{ii} \neq 0$. Let $A_1 = D^{-1}A$. Then $d_2(A) = \det(D)d_2(A_1)$.

2. *Permutation Similarity.* (While this step is unnecessary, it makes the subsequent discussion easier to follow.) If A has r main diagonal elements equal to zero, let P be a permutation matrix such that the *first* r main diagonal elements of $A_2 = P^tA_1P$ are zero. Then $d_2(A_1) = d_2(A_2)$.

3. *Polynomial Coefficients.* At this point, we are dealing with a matrix, A_2, whose main diagonal begins with r (possibly $r = 0$) zero entries. The remaining $n - r$ entries are ones. Imagine replacing each 1 with an indeterminate λ. Call the resulting matrix Λ. Then $\Lambda = \operatorname{diag}(0,\ldots,0,\lambda,\ldots,\lambda) + E$, where the (i, j) entry of E is equal to the (i, j) entry of A_2 provided $i \neq j$; and each diagonal entry of E is zero. Denote the polynomial $\det(\Lambda)$ by $c_r\lambda^{n-r} + \cdots + c_{n-1}\lambda + c_n$. Consider the coefficient c_k. For $k = 0$, c_0 is 0 or 1; it is 1 if and only if $r = 0$, i.e., if and only if $\Lambda = \lambda I_n + E$. Observe that c_1 is always 0, for no permutation of S_n has exactly $n - 1$ fixed points. For $k > 1$, c_k is the sum of the determinants of all k-by-k principal submatrices of E *which contain the leading r-by-r principal submatrix*. We may formalize this statement as follows: For $r \leqslant k \leqslant n$, let $Q^r_{k,n}$ denote the set of strictly increasing functions α: $\{1,2,\ldots,k\} \to \{1,2,\ldots,n\}$ such that $\alpha(i) = i$ for $i = 1,2,\ldots,r$. Then (for $k > 1$),

$$c_k = \sum_{\alpha \in Q^r_{k,n}} \det(E[\alpha]), \tag{2}$$

where $E[\alpha]$ is the k-by-k principal submatrix of E corresponding to rows and columns α, i.e., the (i, j) entry of $E[\alpha]$ is the $(\alpha(i), \alpha(j))$ entry of E.

Now, the indeterminate λ is merely a device to help us organize together those diagonal products of A_2 which involve exactly $n-k$ fixed points, none of which corresponds to a main diagonal zero. That is,

$$c_k = \sum \varepsilon(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)}^{(2)}, \tag{3}$$

where $A_2 = (a_{ij}^{(2)})$, and where the summation is over all permutations $\sigma \in S_n$ which fix exactly $n-k$ of the integers $\{r+1, r+2, \ldots, n\}$ and none of the integers $\{1, 2, \ldots, r\}$. To achieve $d_2(A_2)$, it remains to multiply each diagonal product in (3) by $F(\sigma) - 1 = n - k - 1$, and sum on k. Thus,

$$d_2(A) = \det(D) \sum_{k=0}^{n} (n-k-1) c_k, \tag{4}$$

where c_k is given by (2).

Of course, the more zeros A has on its main diagonal, the faster the algorithm works. At worst, it is comparable to the evaluation of the characteristic polynomial.

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