An algorithm for the solution of constrained generalised polynomial programming problems

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An algorithm is presented for the solution of a class of constrained, nonlinear programming problems. The problems considered may be formulated as generalised polynomials. This class of problems, which encompasses linear, quadratic and geometric programming problems, can be extended to include functions which are the ratios of generalised polynomials. Computational experience with some typical examples is also reviewed.

(Received November 1972)

1. Introduction

Geometric programming (GP) is a method for solving constrained minimisation problems in which the cost (objective) function and the constraint functions are in the form of generalised polynomials (a generalised polynomial is a finite sum of terms which are the product of a real coefficient and a finite set of nonnegative variables, each raised to a real power). These problems are frequently encountered in engineering design, although a wider field of application is steadily developing. The early work of Duffin, Peterson and Zener (1967) was restricted to 'posynomials' (generalised polynomials with positive coefficients) but was later extended by several authors to 'signomials' (unrestricted coefficients) as well. This work is well documented in the literature and the reader is referred to Wilde and Beightler (1967) and Avriel and Williams (1971) for a complete set of references. Less has been published on algorithms and implementations for the computer. The purpose of this paper is to describe an efficient procedure which has been successfully applied to GP problems and variations thereof. The algorithm is based on an iterative solution of the Kuhn-Tucker necessary conditions for an optimum (Kuhn and Tucker, 1951). A linear approximation is employed which leads to a sparse matrix which is easily decomposed. Accordingly the problem dimensionality is reduced, leading to an efficient computer implementation.

2. Problem statement

The problem is posed in the following form:
minimise

$$y_0(\mathbf{x}) \tag{1}$$

subject to

$$y_m(\mathbf{x}) \leqslant \sigma_m;\ m = 1, \ldots, M \tag{2}$$

$$x_n > 0;\ n = 1, \ldots, N \tag{3}$$

where y_0 and the y_m are generalised polynomials described by

$$y_m(\mathbf{x}) = \sum_{t=1}^{T_m} K_{mt} \prod_{n=1}^{N} x_n^{a_{mtn}} \tag{4}$$

with $\mathbf{x} = (x_1, \ldots, x_N)$ the vector of variables; $\mathbf{x}^*$ at optimum

N the number of variables

M the number of constraints

T_m the number of terms in the mth constraint

K_{mt} the coefficient of the tth in the mth constraint, a real number

a_{mtn} the exponent of the nth variable in the tth term of the mth constraint

σ_m the normalised limit of the mth constraint $= \pm 1$.

For convenience we shall introduce

$$\sigma_{mt} = \operatorname{sgn}[K_{mt}] \text{ and } C_{mt} = |K_{mt}| \tag{5}$$

so that

$$K_{mt} = \sigma_{mt} C_{mt};$$

and

$$\sigma_0 = \operatorname{sgn}[y_0(\mathbf{x}^*)] = \pm 1 \tag{6}$$

so that $\sigma_0 y_0^{\sigma_0}$ remains a positive minimand.

Furthermore we shall occasionally use

$$T = T_0 + T_1 + \ldots + T_M .$$

The notation is illustrated by means of a simple example:
Minimise

$$y_0(\mathbf{x}) = \frac{4}{x_1 x_2^{0\cdot 5}} \tag{7}$$

subject to

$$y_1(\mathbf{x}) = x_1 + 2x_2^2 \leqslant 1 \tag{8}$$

$$x_1, x_2 > 0$$

where

$$N = 2,\ M = 1,\ T_0 = 1,\ T_1 = 2,\ T = 3$$
$$C_{01} = 4,\ C_{11} = 1,\ C_{12} = 2$$
$$\sigma_{01} = 1,\ \sigma_{11} = 1,\ \sigma_{12} = 1$$
$$\sigma_0 = 1,\ \sigma_1 = 1$$

$$\begin{bmatrix} a_{011} & a_{012} \\ a_{111} & a_{112} \\ a_{121} & a_{122} \end{bmatrix} = \begin{bmatrix} -1 & -0\cdot 5 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}$$

3. Approach and necessary conditions

Our approach parallels that of Passy and Wilde (1967), although a modification will be introduced in the use of sigmas to obtain a symmetrical matrix in the final set of equations.

Define weights for the terms of the objective function as

$$w_{0t} \equiv C_{0t} \prod_{n=1}^{N} x_n^{a_{0tn}} > 0;\ t = 1, \ldots, T_0 \tag{9}$$

so that

$$\sum_{t=1}^{T_0} \sigma_{0t} w_{0t} = \sigma_0 \tag{10}$$

Note that w_{0t} is always positive and it is assumed that $y_0 \neq 0$. This can be assured by the addition of an auxiliary term to y_0. Generalised weights can be defined for the constraints as

$$w_{mt} \equiv C_{mt} \prod_{n=1}^{N} x_n^{a_{mtn}} > 0;\ m = 1, \ldots, M \quad t = 1, \ldots, T_m \tag{11}$$

so that

$$\sum_{t=1}^{T_m} \sigma_{mt} w_{mt} \leqslant \sigma_m;\ m = 1, \ldots, M \tag{12}$$

By defining

$$u_n \equiv \ln x_n;\ n = 1, \ldots, N \tag{13}$$

$$u_0 \equiv \ln \sigma_0 y_0^{\sigma_0} = \sigma_0 \ln \sigma_0 y_0 \tag{14}$$

and taking logarithms of (10) and (11), a set of equations is

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obtained which is linear in the u's.

$$-\sigma_0 u_0 + \sum_{n=1}^{N} a_{0tn} u_n = \ln (w_{0t}/C_{0t});\ t = 1, \ldots, T_0 \quad (15)$$

$$\sum_{n=1}^{N} a_{mtn} u_n = \ln (w_{mt}/C_{mt});\ m = 1, \ldots, M;\ t = 1, \ldots, T_m \quad (16)$$

Combining u, the transformed cost function, with (10), (12), (15) and (16) to form a $\mathscr{L}$ Lagrangian with multipliers Ω and λ we have:

$$\begin{aligned}\mathscr{L}(w, u, \lambda, \Omega) \equiv u_0 &+ \lambda_0 \Big(\sigma_0 - \sum_{t=1}^{T_0} \sigma_{0t} w_{0t}\Big) \\ &- \sum_{m=1}^{M} \lambda_m \Big(\sigma_m - \sum_{t=1}^{T_m} \sigma_{mt} w_{mt}\Big) \\ &- \sum_{t=1}^{T_0} \Omega_{0t} \Big\{\ln (w_{0t}/C_{0t}) - \sum_{n=1}^{N} a_{0tn} u_n + \sigma_0 u_0\Big\} \\ &- \sum_{m=1}^{M} \sum_{t=1}^{T_m} \Omega_{mt} \Big\{\ln (w_{mt}/C_{mt}) - \sum_{n=1}^{N} a_{mtn} u_n\Big\} \quad (17)\end{aligned}$$

Next, the Kuhn-Tucker necessary conditions are used to obtain expressions for the stationary points of (17).

Observe that the monotonicity of the logarithmic transformation has not compromised any properties of convexity of the original problem. The function is to be minimised on the w's and u's but, alternatively, it can be maximised over the λ and Ω: the necessary condition equations are the same. Note, however, that the w_{mt} are restricted to the positive orthant and that the inequality (12) dictates non-negative λ_m. Accordingly the complementary slackness conditions to be met at an optimum are $w_{mt}\ \partial\mathscr{L}/\partial w_{mt} = 0$ $(m = 0, 1, \ldots, M;\ t = 1, \ldots, T_m)$ and $\lambda_m\ \partial\mathscr{L}/\partial\lambda_m = 0$ $(m = 1, \ldots, M)$. The remaining necessary conditions are derived by equating to zero the partial derivatives of $\mathscr{L}$ with respect to the variables u_n $(n = 0, 1, \ldots, N)$, λ_0, and Ω_{mt} $(m = 0, 1, \ldots, M;\ t = 1, \ldots, T_m)$, all of which are unrestricted in sign.

Thus it is found that

$$\Omega_{mt} = \lambda_m \sigma_{mt} w_{mt},\ m = 0, 1, \ldots, M;\ t = 1, \ldots, T_m \quad (18)$$

and

$$\lambda_0 = 1\,. \quad (19)$$

Rearranged and referred to Ω_{mt} the relevant conditions appear as

$$-\ln (\Omega_{mt}\sigma_{mt}/\sigma_{mt}\lambda_m) + \sum_{n=1}^{N} a_{0tn} u_n - \sigma_m u_m \delta_{m0} = 0 \quad (20)$$

$$\sum_{t=1}^{T_m} \sigma_m \Omega_{mt} = \lambda_m \quad (21)$$

$$\sum_{m=1}^{M} \sum_{t=1}^{T_m} a_{mtn} \Omega_{mt} = 0 \quad (22)$$

for

$$m = 0, 1, \ldots, M;\ t = 1, \ldots, T_m;\ n = 1, \ldots, N$$

with

$$\sigma_{mt}\Omega_{mt} \geqslant 0,\ \lambda_0 = 1,\ \delta_{00} = 1,\ \delta_{m0} = 0 \text{ when } m \neq 0\,.$$

Further manipulation of the preceding equations and use of the Kuhn-Tucker saddlepoint theorem would allow derivation of the familiar dual problem:

$$\text{Maximise } d(\Omega, \lambda) = \sigma_0 \prod_{m=0}^{M} \prod_{t=1}^{T_m} \left[\frac{C_{mt}\lambda_m}{\Omega_{mt}\sigma_{mt}}\right]^{\Omega_{mt}\sigma_0} \quad (23)$$

subject to equation (21), the 'normality condition' when $m = 0$, and equation (22), the 'orthogonality conditions'.

When all sigma's are $+1$ we have the geometric programming dual which may also be derived from the arithmetic-geometric mean inequality (Duffin, 1962) from which geometric programming derives its name. Duality implies $y_0(x) \leqslant d(\Omega, \lambda)$ and $y_0(x^*) = d(\Omega^*, \lambda^*)$, where the asterisk denotes the optimum.

In the general case $(\sigma = \pm 1)$ some of the concepts of duality are lost, for instance the ability to bound the problems as indicated in Wilde *et al.* (1967). Also the optimum may not be unique, and one refers to the 'pseudomaxima' of $d(\Omega, \lambda)$.

4. Methods and solution

The dual problem given by (23) is attractive in that only linear constraints appear in its formulation. A fortuitous case arises when $T = N + 1$ because then the linear constraints uniquely determine the optimal dual variables. Our earlier numerical example (7), (8) falls into this 'zero degrees of difficulty' category:

$$\begin{aligned} \Omega_{01} \qquad\qquad\qquad &= 1 \\ -\Omega_{01} + \Omega_{11} \qquad\quad &= 0 \\ -0{\cdot}5\Omega_{01} \qquad + 2\Omega_{12} &= 0 \\ \Omega^*_{01} = 1,\ \Omega^*_{11} = 1,\ \Omega^*_{12} &= 1/4 \end{aligned}$$

From Ω^* the optimal $d^* = y_0^*$ can be computed:

$$y_0^* = d^* = \left(\frac{4}{1}\right)^1 \left(\frac{1{\cdot}5/4}{1}\right)^1 \left(\frac{2{\cdot}5/4}{1/4}\right)^{1/4} = 8{\cdot}9\,.$$

When $T < N + 1$ the problem can be transformed into one with trivial solutions. The case $T > N + 1$, with $D = T - (N + 1) =$ 'degrees of difficulty', requires that the dual objective function be maximised subject to its linear constraints. A number of procedures have been proposed. For instance the linear side conditions may be used to reduce the total number of variables, then a search procedure can be applied to (23) in a reduced search space (Frank, 1965; Duffin *et al.*, 1967; or Wilde *et al.*, 1967). Alternately some authors view the problem as a convex programming problem with linear constraints (Dinkel and Kochenberger, 1972).

Another approach employs a successive approximation method, based on the logarithm of the dual objective function (Duffin, 1962; Schinzinger, 1965; Beck and Ecker, 1972). The procedure presented is not unlike the latter method, but instead of starting from the dual function the method begins with the Lagrangian of the primal problem. The main advantages are (*a*) a more obvious symmetry in the matrix formulation of the linear approximation to the necessary conditions, (*b*) a straight forward transformation from dual to primal variables. The extraction of the primal variables has, in other methods, proceeded along more cumbersome lines. (See for example Schinzinger, 1965). For instance one could retrace the relationships between Ω, λ and x at the optimum and solve a set of equations which are linear in terms of the logarithm of the primal variables, but the set may be overdetermined. In contrast the procedure described in the next section solves for $u_n = \ln x_n$ directly along with the dual variables Ω and λ.

5. The algorithm

Equation (20) is non-linear in Ω and λ. Linearising the non-linear terms of these equations about some initial guess, indicated by $\bar{\lambda}_m$, $\bar{\Omega}_{0t}$, and $\bar{\Omega}_{mt}$ yields:

$$\ln\left[\frac{\Omega_{mt}\sigma_{mt}}{C_{mt}\lambda_m}\right] \approx \ln\left[\frac{\bar{\Omega}_{mt}\sigma_{mt}}{C_{mt}\bar{\lambda}_m}\right] + \frac{\Omega_{mt}}{\bar{\Omega}_{mt}} - \frac{\lambda_m}{\bar{\lambda}_m} \quad (24)$$

Substituting (24) into (20) and making use of (2) we have:

$$-\frac{\Omega_{0t}}{\bar{\Omega}_{0t}} + \sum_{n=1}^{N} a_{0tn} u_n + \mu_0 = \ln\left[\frac{\bar{\Omega}_{0t}\sigma_{0t}}{C_{0t}}\right] \quad (25a)$$

and

$$-\frac{\Omega_{mt}}{\bar{\Omega}_{mt}} + \sum_{n=1}^{N} a_{mtn} u_n + \frac{\lambda_m}{\bar{\lambda}_m} = \ln\left[\frac{\bar{\Omega}_{mt}\sigma_{mt}}{C_{mt}\bar{\lambda}_m}\right] \quad (25b)$$

$$m = 1, \ldots, M;\ t = 1, \ldots, T_m$$

where

$$\mu_0 \overset{\triangle}{=} (1 - \sigma_0 u_0) \tag{26}$$

with

$$\sigma_0 u_0 = \ln(\sigma_0 y_0) = (1 - \mu_0) \tag{27}$$

or

$$y_0^* = \sigma_0 \exp(1 - \mu_0^*)\,. \tag{28}$$

Normalising (28) to preserve matrix symmetry yields

$$\sum_{t=1}^{T_m} \frac{\Omega_{mt}}{\bar{\lambda}_m} - \frac{\lambda_m \sigma_m}{\bar{\lambda}_m} = 0 \tag{29}$$

Equations (25), (22) and (29), when collected in the form of a partitioned matrix as shown in **Fig. 1**, have the following interesting symmetry

$$\left[\begin{array}{c|c|c} D & A & K \\ \hline A^T & \varnothing & \varnothing \\ \hline K^T & \varnothing & L \end{array}\right] \left[\begin{array}{c} \Omega \\ \hline u \\ \hline \lambda \end{array}\right] = \left[\begin{array}{c} b \\ \hline e \\ \hline \varnothing \end{array}\right] \tag{30}$$

where $\varnothing$ is a zero matrix of appropriate dimensions. From the above we have

$$D(\bar{\Omega})\Omega + Au + K(\bar{\lambda})\lambda = b(\bar{\Omega}, \bar{\lambda}) \tag{31}$$

$$A^T \Omega = e \tag{32}$$

$$K^T(\bar{\lambda})\Omega + L(\bar{\lambda})\lambda = \varnothing \tag{33}$$

Matrices D and L are diagonal and invertible under the assumption that $\Omega < \infty$ and $\lambda < \infty$, which is reasonable for most well posed problems. Thus we can rewrite (31) as

$$\Omega = D^{-1}(\bar{\Omega})\,\{b(\bar{\Omega}, \bar{\lambda}) - K(\bar{\lambda})\lambda - Au\} \tag{34}$$

Multiplying by A^T and using (32) we can solve for u from

$$Ru = A^T D^{-1}(\bar{\Omega})\,\hat{b}(\bar{\Omega}, \bar{\lambda}) - e \tag{35}$$

where

$$\hat{b} = (b - K\lambda) \text{ and } R = (A^T D^{-1} A)\,. \tag{36}$$

We solve sequentially; equation (33) for λ, equation (35) for u, equation (34) for Ω. Since D and L are diagonal matrices, the only computationally significant effort is in solving (35). This, however, is a greatly reduced matrix; while the tableau of Fig. 1 shows a total of $T + N + M + 1$ variables the rank of R is merely $N + 1$.

Once the Ω^*, u^*, λ^* are known the original variables are obtained from equations (28) and (13).

Sensitivity analysis, which is perhaps just as important as numerical answers, allows us to examine changes in the objective function for a small change in the coefficient, K_{mt}:

$$\frac{\partial y_0(x^*)}{\partial K_{mt}} = \frac{\Omega^*_{mt}}{K_{mt}\lambda_m^*}\, y_0(x^*) \tag{40}$$

This may be obtained directly from equation (23) if duality holds.

The rate of change of the objective function with respect to an active constraint value is given by Zener (1971):

$$\frac{\partial y_0(x^*)}{\partial y_m(x^*)} = -\lambda_m^* \left(\frac{y_0(x^*)}{y_m(x^*)}\right) \tag{41}$$

Also for changes in the exponent a_{0tk} we have

$$\frac{\partial y_0^*}{\partial a_{0tk}} = K_{0t} \prod_{n=1}^{N} x_n^{a_{0tk}} (\ln x_k) = \Omega_{0t}^* \sigma_{0t} u_k^* \tag{42}$$

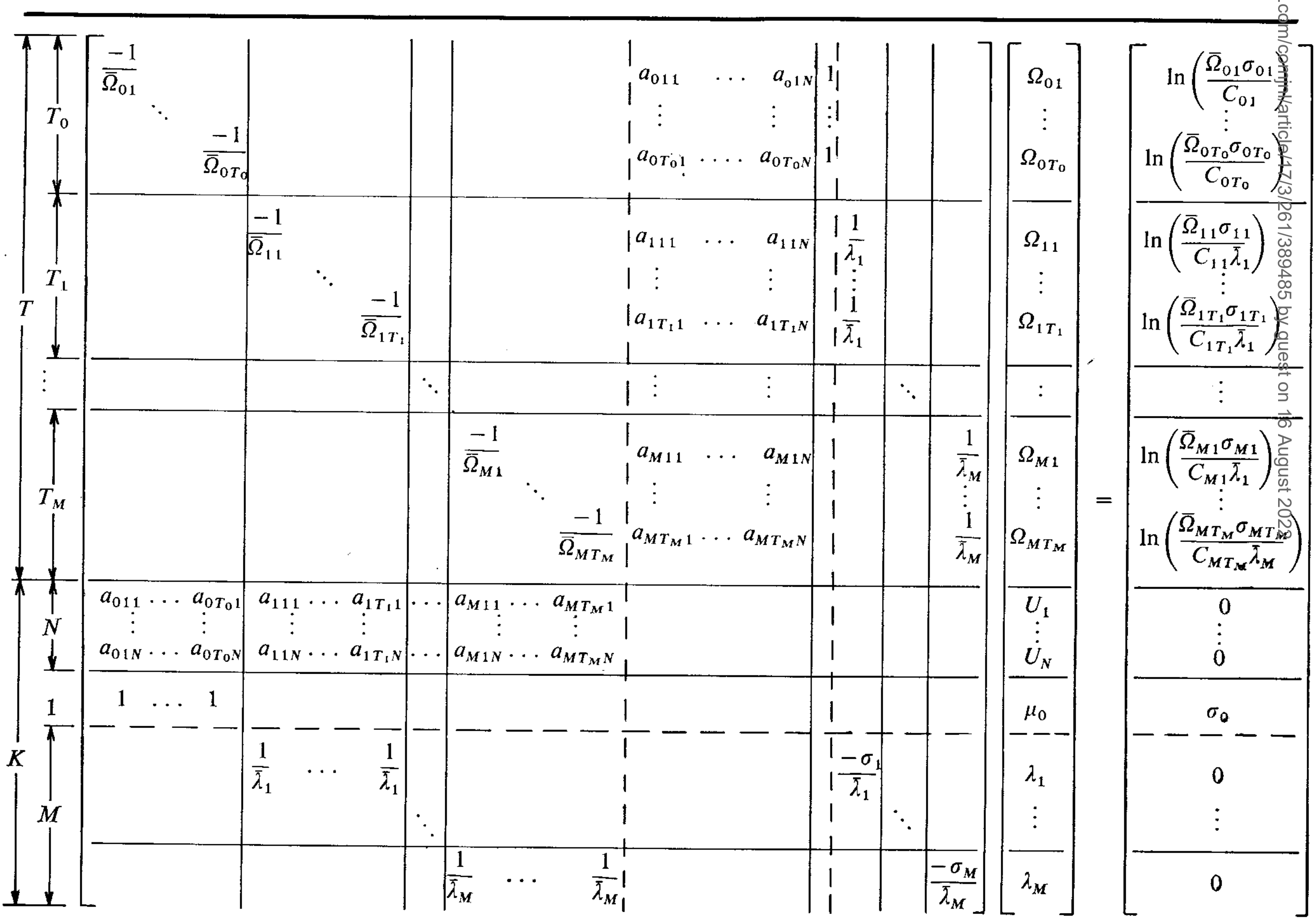

Fig. 1 Linearised set of necessary conditions Dots indicate missing terms

6. Conditions for solutions

At this point some conditions which must be met for the algorithm to yield the optimum will be enumerated.

(*a*) In the case of posynomials, positive X_n will assure a positive y_0. If the original problem allows negative X_n, then simple transformations may be resorted to (Duffin *et al.*, 1967; Duffin, 1970; Zener, 1971; Avriel and Williams, 1970). If the sign of y_0 is not known a priori both $\sigma_0 = +1$ and -1 may be tried, or the problem may be transformed by the addition of a large artificial variable, constrained from below.

(*b*) A posynomial is convex if each variable X_n which appears with a positive exponent also occurs at least once with a negative exponent, or vice versa (Erlicki and Applebaum, 1964). This condition is also reflected by the requirement that the $\sigma_{mt}\Omega_{mt}$ which satisfy Equation (22) be non-negative. In the more general case of signomials the algorithm may converge onto a local optimum or a saddlepoint. Wherever possible problems should be formulated as posynomials.

(*c*) Use of $\sigma_0 = -1$ when $y_0 < 0$, in order to render $\sigma_0 y_0$ positive, can lead to difficulties as demonstrated graphically by **Fig. 2.** Appropriate starting values for X_n (from which $\bar{\Omega}$ may be determined) become important.

(*d*) The rows of matrix A in (30) must be independent or $R = A^T D^{-1} A$ will be singular. The introduction of artificial variables as suggested by Beck and Ecker (1972) can overcome this difficulty. Singularity may also occur when certain variables appear always in the same grouping, in which case a single variable, say $x_4 = x_1^\alpha x_2^\beta x_3^\alpha$, can be substituted (Zener, 1971, p. 12).

(*e*) Convergence usually proceeds quite rapidly and could be further accelerated by employing second order methods. As a stopping rule either relative or absolute changes in the magnitudes of all variables can be used, with particular precaution for variables which approach zero or change sign. The requirement $\sigma_{mt}\Omega_{mt} \geqslant 0$ is imposed where necessary. Convergence tests are applied to the Ω_{mt}, u_n, λ_m and μ_0 as well as all x_n.

Even with the above restrictions a large class of optimisation problems can be solved. The next section is intended to indicate the scope of the areas of application and problem formulation.

7. Application

The examples given in the appendix were all solved using the algorithm and are intended to give the reader a sampling of the many areas of application. Due to the limitation of space only the problem statement, the solution and, where appropriate, remarks concerning formulation are presented. However most of the problems have been gleaned from the open literature on optimisation theory, which may be consulted for a more complete description.

Numerical results are summarised in **Table 1.**

The number of iterations reported indicates the total number of solutions of the $(N + 1)$-order set of linear equations before the problem converged.

The computer (a XDS Sigma 7) was operated in a time-share mode, and therefore run times of only two problems with artificially high iteration counts caused by a restrictive convergence criteria are given.

8. Conclusions

We have presented an algorithm related to geometric programming. It solves optimisation problems which can be formulated as generalised polynomials. The ready inclusion of constraints is particularly attractive. The algorithm handles GP type problems rapidly with little computer storage requirement. The examples presented indicate its broad area of application. Convergence has occurred in all problems attempted thus far, provided feasible solutions existed. In a number of signomial cases care had to be exercised in selecting starting points, and

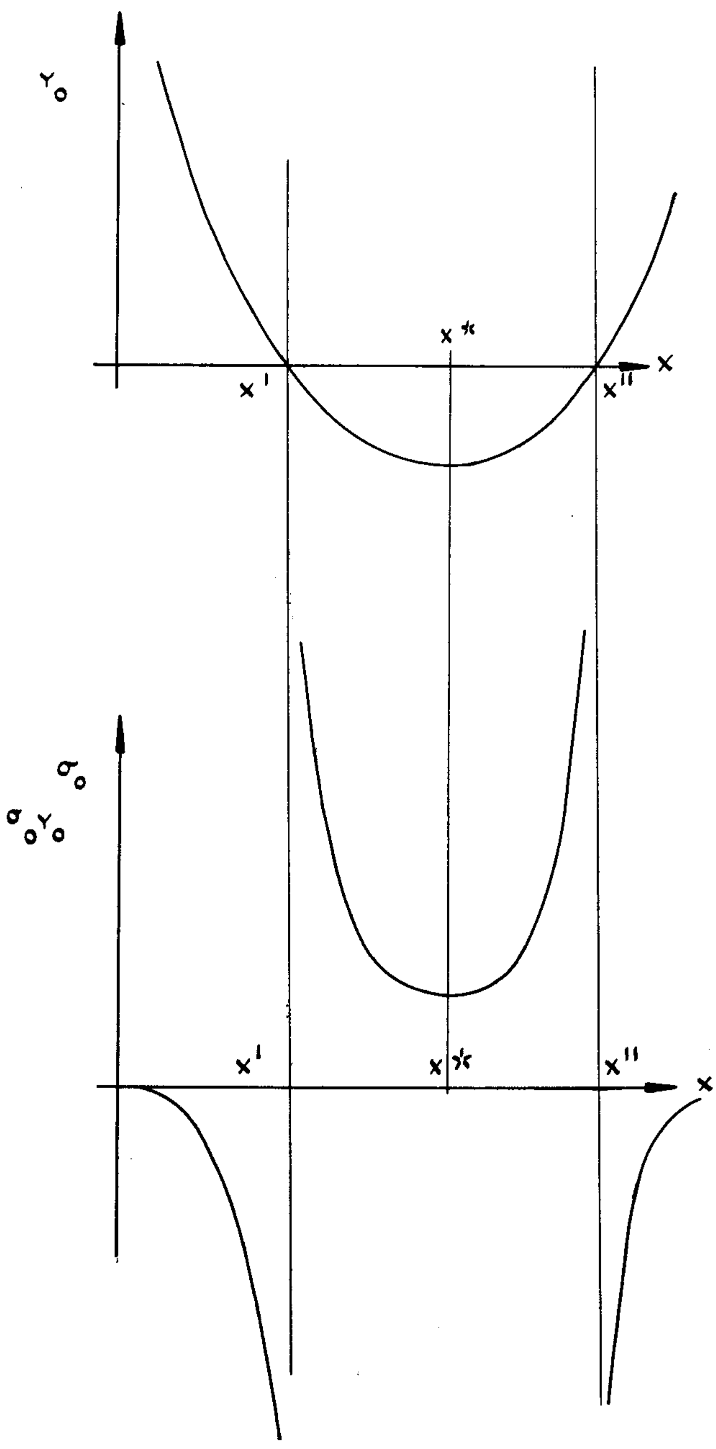

Fig. 2 Transformation of the objective function with negative minimum

solution times of problems with constraints were at times sensitive to the manner in which the constraints were formulated. The authors were pleased with the results and hope that the numerical data provided here will invite comparison by others.

Acknowledgement

C. O. Jelinek's work and the computational effort were supported in part by USAF Office of Scientific Research under Grant No. AFOSR-71-2116A.

Appendix Numerical examples

Problem 1:

Waste Treatment Plant Design (Scherfig *et al.*, 1969). Minimise the variable annual cost of waste treatment plant:

Table 1 Numerical results

PROBLEM NO.	1	2	3	4	5
Optimal x_n	0·6169	$5{\cdot}628 \times 10^{-5}$	5·3336	1,000·0	1·054
$(n = 1, \ldots, N)$	$5{\cdot}814 \times 10^5$	$2{\cdot}450 \times 10^{-7}$	4·6585	99·962	0·122
	$2{\cdot}999 \times 10^5$	$2{\cdot}332 \times 10^{-6}$	10·4365	4·607	
			12·0840	52·336	
			0·7525	276·299	
			0·8781	21·453	
				202·248	
Obj. Fct., y_0	$71{\cdot}765 \times 10^3$	$5{\cdot}525 \times 10^{20}$	135·1023	2,420·284	0·500
Variables Ω_{0t}	0·1464	1·000	0·0519 0·0874	0·517	0·991
$(t = 1, \ldots, T_0)$	0·0800		0·0453 0·0764	0·051	0·009
	0·4298		0·1016 0·0647	0·432	
	0·3438		0·0359 0·0888		
			0·0491 0·1465		
			0·0813 0·1711		
1st Constraint	0·9996	1·000	0·9999	0·999	1·000
Sens. Coeff. λ_1	0·2223	1·638	1·0954	0·485	1·009
Variables Ω_{1t}	0·2223	0·041 0·001	1·0954	0·243	0·009
$(t = 1, \ldots, T_1)$		0·148 0·027		0·027	1·00
		0·783 0·018		0·216	
		0·001 0·037			
		0·485 0·097			
2nd Constraint			0·9997	1·000	
Sens. Coeff. λ_2			0·4606	0·510	
Variables Ω_{2t}			0·0556 0·0534	0·510	
$(t = 1, \ldots, T_2)$			0·0485 0·0732		
			0·1088 0·1211		
3rd Constraint				0·998	
Sens. Coeff. λ_3				0·225	
Variables Ω_{3t}				0·124	
$(t = 1, \ldots, T_3)$				0·010	
				0·091	
Convergence $\lvert\varepsilon\rvert$	0·0001	0·001	0·0001	0·001	0·001
Iterations	32	7	50	8	8
Time (Sec)	3·2		6·2		

$$y_0 = 2{\cdot}1.10^{-11}\, x_2^{2\cdot 55} + 6{\cdot}29.10^7\, x_2^5/x_3^6 + 8{\cdot}5.10^{10}/(x_2 x_3^{0\cdot 2}\, x_1^2)$$
$$+ 1{\cdot}6.10^5\, x_1^{2\cdot 5}\, x_3/x_2$$
$$y_1 = (1/3)\, 10^{-5}\, x_3 \leqslant 1$$

x_1: fraction of feed chemical oxygen demand not met (dimensionless) x_2, x_3: influent, and effluent, volatile solids concentration (lb/million) gal.

Problem 2:
Chemical Equilibrium Problem (Duffin, *et al.*, 1967). Consider the combustion of a stochiometric mixture of hydrazine and oxygen at 3,500°K, 750 psi:

$$y_0 = 1/(x_1^2\, x_2\, x_3)$$
$$y_1 = 440{\cdot}98.x_1 + 2{\cdot}846.10^7\, x_1^2 + 6{\cdot}1584.10^{14}\, x_1^2\, x_2$$
$$+ 370{\cdot}18\, x_3\; 5{\cdot}4474.10^{10}\, x_3^2 + 3{\cdot}2236.10^6\, x_1\, x_3$$
$$+ 2{\cdot}920.10^{10}\, x_2\, x_3 + 4{\cdot}4712.\; 10^4\, x_2$$
$$+ 3{\cdot}7964.10^{11}\, x_2^2 + 4{\cdot}2876.10^9\, x_1\, x_2 \leqslant 1$$

x: Composition variables, y_1: equilibrium mole fraction balance.

Problem 3:
Transformer Design (Hamaker and Hehenkamp, 1950; Schinzinger, 1965). Minimise the present worth of a transformer, including operating costs over 20 years.

$$y_0 = 0{\cdot}0204\,(x_1^2\, x_4 + x_1\, x_2\, x_4 + x_1\, x_3\, x_4) + 0{\cdot}0187$$
$$(x_1\, x_2\, x_3 + 1{\cdot}57\, x_2^2\, x_3 + x_2\, x_3\, x_4) + 0{\cdot}0607$$
$$(x_1^2\, x_4\, x_5^2 + x_1\, x_2\, x_4\, x_5^2) + 0{\cdot}0437$$
$$(x_1\, x_2\, x_3\, x_6^2 + 1{\cdot}57\, x_2^2\, x_3\, x_6^2 + x_2\, x_3\, x_4\, x_6^2)$$
$$+ 0{\cdot}0607\, x_1\, x_3\, x_4\, x_5^2$$
$$y_1 = 2070/(x_1\, x_2\, x_3\, x_4\, x_5\, x_6) \leqslant 1$$
$$y_2 = 0{\cdot}00062\,(x_1^2\, x_4\, x_5^2 + x_1\, x_2\, x_4\, x_5^2 + x_1\, x_3\, x_4\, x_5^2)$$
$$+ 0{\cdot}00058\,(x_1\, x_2\, x_3\, x_6^2 + x_2\, x_3\, x_4\, x_6^2$$
$$+ 1{\cdot}57\, x_2^2\, x_3\, x_6^2) \leqslant 1$$

x_1 through x_4: physical dimensions of winding and core; x_5: magn. flux density; x_6: current density; y_0: in arbitrary monetary units; y_1: rating; y_2: loss constraint.

Problem 4:
Catalogue Planning (Sakolich and Harting, 1969; Dinkel and Kochenberger, 1972). Allocation of resources to cataloguing. Maximise demand for products, minimise negative thereof.

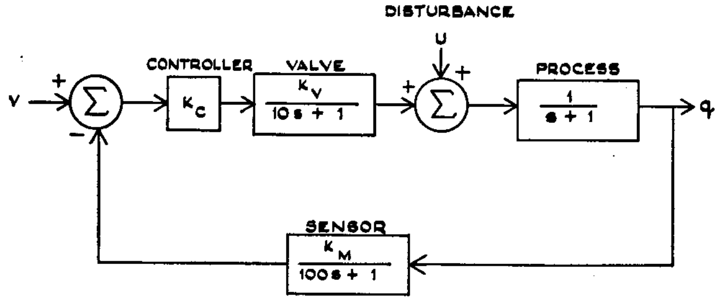

Fig. 3

$$y_0 = -1{\cdot}1\, x_1^{0{\cdot}51}\, x_2^{0{\cdot}47}\, x_3^{0{\cdot}24} - 0{\cdot}9\, x_1^{0{\cdot}51}\, x_4^{0{\cdot}53}\, x_5^{0{\cdot}19} - 1{\cdot}4\, x_1^{0{\cdot}51}\, x_6^{0{\cdot}5}\, x_7^{0{\cdot}21}$$
$$y_1 = (50\, x_2 + 120\, x_4 + 85\, x_6)/10{,}000 \leqslant 1$$
$$y_2 = x_1/1{,}000 \leqslant 1$$
$$y_3 = (x_3 + x_5 + x_7)/500 \leqslant 1$$

x_1: distribution quantity; x_{2i}: no. of pages devoted to line $i = 1, 2, 3$; x_{2i+1}: no. of items in line i; constraints: y_1 on development and printing costs, y_2 on distribution, y_3 on total number of items.

Problem 5:

Chemical Process Control Problem (Gould, 1971; pp. 83-86). This example is a chemical process control system. The block diagram is shown in **Fig. 3.**

The input v is nominally zero, u is a white noise disturbance having a power spectral density $\Phi_{uv}(s) = 1/\pi$ and q is the process output flow rate.

To minimise the effect of a pressure disturbance, it is desired to determine the system parameter $K - (1 + K_c K_v K_m) \times 10^{-3}$, which yields the least mean-square value of q where

$$q^2 = \frac{0{\cdot}1211K + 1{\cdot}11 \times 10^{-6}}{2K(0{\cdot}12321 - K)}$$

Notice that this is not in the form of a geometric programming problem. Stability of the system requires $0 \leqslant K \leqslant 0{\cdot}12321$; this is also necessary to obtain a positive value of q^2. For the formulation it is convenient to set $x_1 = K$ and $x_2 \leqslant 0{\cdot}12321 - x_1$ so the problem becomes

$$\min_{x_1 x_2} q^2 = \frac{1}{2}\left[\frac{0{\cdot}1211}{x_2} + \frac{1{\cdot}11 \times 10^{-6}}{x_1 x_2}\right]$$

subject to $y_1 = 8{\cdot}1162243[x_1 + x_2] \leqslant 1$

$$x_1, x_2 \geqslant 0$$

References

APPLEBAUM, J., and ERLICKI, M. S. (1964). A Problem of Economic Optimization of Electric Equipment Design, *IEEE Transactions, Communications and Electronics*, Vol. 88, pp. 773-776.

AVRIEL, M., and WILLIAMS, A. C. (1970). Complementary Geometric Programming, *SIAM J. Appl. Math*, Vol. 19, No. 1, pp. 125-141.

AVRIEL, M., and WILLIAMS, A. C. (1971). An Extension of Geometric Programming with Applications in Engineering Optimization, *J. of Engineering Mathematics*, Vol. 5, No. 3, pp. 187-194.

BECK, P. A., and ECKER, J. G. (1972). *Some Computational Experience with a Modified Convex Simplex Algorithm for Geometric Programming*, Armament Development and Test Center, Elgin A.F.B., Florida, ADTC-72-20.

DINKEL, J., and KOCHENBERGER, G. (1972). *Applications of Geometric Programming to Marketing Decisions*, a paper presented to the 41st National meeting of Operations Research Society of America at New Orleans, La.

DUFFIN, R. J. (1967). Cost Minimization Problems Treated by Geometric Means, *J. ORSA*, Vol. 10. No. 5, pp. 668-675.

DUFFIN, R. J., PETERSON, E. L., and ZENER, C. (1967). *Geometric Programming Theory and Application*, New York, John Wiley and Sons.

DUFFIN, R. J. (1972). *Harmonized Geometric Programming*, a paper presented at the meeting of the Institute of Management Science, Houston, Texas, April 6, 1972.

FRANK, C. J. (1965). An Algorithm for Geometric Programming, in *Recent Advances in Optimization Techniques*, edited by A. Lavi and T. P. Vogl, John Wiley and Sons, pp. 145-162.

GOULD, L. A. (1969). *Chemical Process Control: Theory and Applications*, Addison-Wesley Publishing Co., pp. 85-86.

HAMAKER, H. C., and HEHENKAMP, G. (1950-51). Minimum Cost Transformers and Chokes, Part 1, *Philips Research Report* 5, pp. 357-394, R150; Part II, *Report* 6, pp. 105-134, R163.

KUHN, H. W., and TUCKER, A. W. (1951). Nonlinear Programming in *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, J. Neyman, editor. Berkeley and Los Angeles, California, University of California Press, 1951, pp. 481-492.

PASSY, U., and WILDE, D. J. (1967). Generalized Polynomial Optimization, *SIAM J. Appl. Math*, September, Vol. 15, No. 5, pp. 1344-1356.

SAKOLICH, W. D., and HARTING, P. H. (1969). Catalog Advertising Allocation, *Management Science*, June 1969.

SCHERFIG, J., SCHINZINGER, R., and MORGAN, T. W. (1969). Waste Treatment Optimization by Geometric Programming, *Proceedings of the Fifth International Conference on Water Pollution Control*, San Francisco, California.

SCHINZINGER, R. (1965). Optimization in Electromagnetic System Design, in *Recent Advances in Optimization Techniques*, edited by A. Lavi and T. P. Vogl, John Wiley and Sons, pp. 163-213.

WILDE, D. J., and BEIGHTLER, C. S. (1967). *Foundations of Optimization*, Prentice-Hall.

ZANGWILL, W. I. (1969). *Nonlinear Programming, A Unified Approach*, Prentice-Hall.

ZENER, C. (1971). *Engineering Design by Geometric Programming*, John Wiley and Sons.