AN ALORITHMIC CONSTRUCTION OF ENTROPIES IN HIGHER-ORDER NONLINEAR PDES

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Abstract. A new approach to the construction of entropies and entropy productions for a large class of nonlinear evolutionary PDEs of even order in one space dimension is presented. The task of proving entropy dissipation is reformulated as a decision problem for polynomial systems. The method is successfully applied to the porous medium equation, the thin film equation, and the quantum drift-diffusion model. In all cases, an infinite number of entropy functionals together with the associated entropy productions is derived. Our technique can be extended to higher-order entropies, containing derivatives of the solution, and to several space dimensions. Furthermore, logarithmic Sobolev inequalities can be obtained.

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1. Introduction

The analysis of nonlinear evolution equations arising from applications relies on appropriate a priori estimates of the solutions. Often, the physical energy or entropy of the underlying physical system proves to be a conserved or at least a non-increasing quantity with respect to time. However, additional estimates are usually necessary in order to prove mathematical properties of the solutions of the differential equation. It is a difficult task to derive new estimates. In this paper we present a novel approach to construct such non-increasing functionals, which we call *entropies*, and the corresponding integral bounds, called *entropy productions*. Our approach is based on a reformulation of the problem as a decision problem known in real algebraic geometry.

More specifically, we consider nonlinear partial differential equations of even order K of the form

$$\partial_t n = \partial_x \left(n^{\beta+1} P\left(\frac{\partial_x n}{n}, \frac{\partial_x^2 n}{n}, \ldots, \frac{\partial_x^{K-1} n}{n} \right) \right), \quad t > 0, \qquad n(\cdot, 0) = n_I, \tag{1}$$

in a bounded interval $(0, L)$ supplemented with periodic boundary conditions, for instance. Here, $P(\xi_1, \ldots, \xi_{K-1})$ is a polynomial in the variables $\xi_1, \ldots, \xi_{K-1} \in \mathbb{R}$ and $\beta \in \mathbb{R}$. In

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the main part of this paper, we restrict ourselves to the one-dimensional situation; the multi-dimensional case is studied in section 5.4.

A large class of equations from applications can be written in the form (1). In the following we give some examples.

- The *porous medium equation*
$$n_t = (n^\beta n_x)_x, \quad \beta > 0,$$
is of the form (1) with $P(\xi_1) = \xi_1$. It describes the flow of an isentropic gas through a porous medium with density $n(x,t)$ but it also appears in the modeling of heat radiation in plasmas, water infiltration etc. (see, e.g., the survey [33]).
- The *thin film equation*
$$n_t = -(n^\beta n_{xxx})_x, \quad \beta > 0, \tag{2}$$
is also of the form (1) with $P(\xi_1,\xi_2,\xi_3) = -\xi_3$. This equation models the flow of a thin liquid along a solid surface with film height $n(x,t)$ ($\beta = 2$ or $\beta = 3$) or the thin neck of a Hele-Shaw flow in the lubrication approximation ($\beta = 1$). For details, we refer to the reviews [6, 8, 29, 30].
- The *Derrida-Lebowitz-Speer-Spohn (DLSS) equation*
$$n_t = -(n(\log n)_{xx})_{xx} = \left(n\left(-\frac{n_{xxx}}{n} + \frac{2n_x n_{xx}}{n^2} - \frac{n_x^3}{n^3}\right)\right)_x \tag{3}$$
can be written as in (1) with $\beta = 0$ and $P(\xi_1,\xi_2,\xi_3) = -\xi_3 + 2\xi_1\xi_2 - \xi_3$. It arises as a scaling limit in the study of interface fluctuations in a certain spin system [22] and in quantum semiconductor modeling as the zero-temperature zero-field quantum drift-diffusion equation [1, 25]. Here, the function $n(x,t)$ describes the particle density.
- The sixth-order equation
$$n_t = \left(n\left(\frac{1}{n}(n(\log n)_{xx})_{xx} + \frac{1}{2}(\log n)_{xx}^2\right)_x\right)_x \tag{4}$$
can also be written as in (1) with $\beta = 0$ and
$$P(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5) = 6\xi_1^5 - 18\xi_1^3\xi_2 + 11\xi_1\xi_2^2 + 8\xi_1^3\xi_3 - 3\xi_1\xi_4 - 5\xi_2\xi_3 + \xi_5.$$
This equation is derived from the generalized quantum drift-diffusion model for semiconductors of Degond et al. [19] in the $O(\hbar^6)$ approximation, with $\hbar$ denoting the reduced Planck constant (see the appendix for an outline of the derivation). Again, $n(x,t)$ represents the particle density.

In general, a priori estimates (for smooth positive solutions) are obtained by multiplying (1) by a nonlinear function $\sigma(n)$ and integrating by parts,
$$\frac{d}{dt}\int s(n)\,dx = \int \sigma(n)n_t\,dx = -\int n^{\beta+2}\sigma'(n)P\left(\frac{\partial_x n}{n},\ldots,\frac{\partial_x^{K-1}n}{n}\right)\frac{\partial_x n}{n}\,dx,$$
where $s(n)$ is a primitive of $\sigma(n)$ and here and in the following, the integral has to be understood as an integral from 0 and L. Notice that we use periodic boundary conditions

only in order to avoid boundary integrals. Clearly, any other boundary conditions with the same property can be chosen instead. We refer, for instance, to [24] for the treatment of the DLSS equation with more complicated boundary conditions.

We assume that $s(n) > 0$ for all $n > 0$. Denoting by

$$\mathcal{S}(t) = \int s(n)\,dx \quad \text{and} \quad \mathcal{P}(t) = \int n^{\beta+2}\sigma'(n)P\left(\frac{\partial_x n}{n}, \ldots, \frac{\partial_x^{K-1} n}{n}\right)\frac{\partial_x n}{n}\,dx,$$

we can write the above equation as

$$\frac{d\mathcal{S}}{dt} + \mathcal{P} = 0, \quad t > 0. \tag{5}$$

If $\mathcal{P}(t)$ is nonnegative, $\mathcal{S}(t)$ is non-increasing and is referred to as an *entropy* (see section 2.1 for a precise definition). In some sense, $\mathcal{S}(t)$ can be interpreted as a Lyapunov functional. Additional estimates may be obtained from the time-integrated *production term* $\mathcal{P}(t)$.

The key point is to prove the nonnegativity of the production term which is usually done by appropriate integrations by parts and other estimates. However, the proof can be quite involved. We are able to present an *algorithmic approach* to prove this property. In this framework, the claim $\mathcal{P}(t) \geq 0$ is reformulated as a so-called quantifier elimination problem for polynomial systems which is always solvable in an algorithmic way.

1.1. **Idea of the method.** We illustrate the idea of the reformulation by means of the thin film equation (2) as an example. We multiply (2) by $\sigma(n) = n^{\alpha-1}/(\alpha-1)$ with $\alpha \neq 0, 1$ and integrate by parts once:

$$\frac{d}{dt}\int \frac{n^\alpha}{\alpha(\alpha-1)}\,dx = \int n^{\alpha+\beta-2} n_{xxx} n_x\,dx, \tag{6}$$

which is of the form (5) with

$$\mathcal{S}(t) = \int \frac{n^\alpha}{\alpha(\alpha-1)}\,dx, \quad \mathcal{P}(t) = -\int n^{\alpha+\beta-2} n_{xxx} n_x\,dx\,ds.$$

Some ingenious integrations by parts allow to show that the right-hand side of (6) is nonpositive for $\frac{3}{2} - \beta < \alpha < 3 - \beta$ [5, 10, 28] (also see [17, (4)]).

In a systematic way, this result can be obtained as follows. First, identify possible integration-by-parts formulas:

$$\begin{aligned}
\left(n^{\alpha+\beta}\left(\frac{n_x}{n}\right)^3\right)_x &= n^{\alpha+\beta}\left[(\alpha+\beta-3)\left(\frac{n_x}{n}\right)^4 + 3\left(\frac{n_x}{n}\right)^2\frac{n_{xx}}{n}\right], \\
\left(n^{\alpha+\beta}\frac{n_x}{n}\frac{n_{xx}}{n}\right)_x &= n^{\alpha+\beta}\left[(\alpha+\beta-2)\left(\frac{n_x}{n}\right)^2\frac{n_{xx}}{n} + \left(\frac{n_{xx}}{n}\right)^2 + \frac{n_x}{n}\frac{n_{xxx}}{n}\right], \\
\left(n^{\alpha+\beta}\frac{n_{xxx}}{n}\right)_x &= n^{\alpha+\beta}\left[(\alpha+\beta-1)\frac{n_{xxx}}{n}\frac{n_x}{n} + \frac{n_{xxxx}}{n}\right].
\end{aligned}$$

Integrating these expressions over the interval $(0, L)$ and taking into account the periodic boundary conditions, we obtain

$$\begin{aligned}
\mathcal{I}_1 &= \int n^{\alpha+\beta}\left[(\alpha+\beta-3)\left(\frac{n_x}{n}\right)^4 + 3\left(\frac{n_x}{n}\right)^2\frac{n_{xx}}{n}\right]dx = 0,\\
\mathcal{I}_2 &= \int n^{\alpha+\beta}\left[(\alpha+\beta-2)\left(\frac{n_x}{n}\right)^2\frac{n_{xx}}{n} + \left(\frac{n_{xx}}{n}\right)^2 + \frac{n_x}{n}\frac{n_{xxx}}{n}\right]dx = 0,\\
\mathcal{I}_3 &= \int n^{\alpha+\beta}\left[(\alpha+\beta-1)\frac{n_{xxx}}{n}\frac{n_x}{n} + \frac{n_{xxxx}}{n}\right]dx = 0.
\end{aligned}$$

Therefore, the production term can be written as

$$\mathcal{P} = \mathcal{P} + c_1\mathcal{I}_1 + c_2\mathcal{I}_2 + c_3\mathcal{I}_3, \tag{7}$$

with arbitrary real constants c_1, c_2, and c_3. The goal is to find c_1, c_2, and c_3 such that the integrand of $\mathcal{P}$ proves to be a nonnegative function. In fact, we will show in section 4.2 that there exists $\varepsilon > 0$ such that

$$\mathcal{P} \geq \varepsilon \int \left(n^{\alpha+\beta-2}n_{xx}^2 + (n^{(\alpha+\beta)/2})_{xx}^2 + (n^{(\alpha+\beta)/4})_x^4\right)dx. \tag{8}$$

The terms on the right-hand side are called *entropy productions* (see section 2.1).

The above integration-by-parts formulas can be translated into polynomials by identifying $(\partial_x^k n/n)^m$ with ξ_k^m. Then

$$\begin{aligned}
\mathcal{P} &\quad \text{corresponds to} \quad S_0(\xi) &=& -\xi_1\xi_3,\\
\mathcal{I}_1 &\quad \text{corresponds to} \quad T_1(\xi) &=& (\alpha+\beta-3)\xi_1^4 + 3\xi_1^2\xi_2,\\
\mathcal{I}_2 &\quad \text{corresponds to} \quad T_2(\xi) &=& (\alpha+\beta-2)\xi_1^2\xi_2 + \xi_1\xi_3 + \xi_2^2,\\
\mathcal{I}_3 &\quad \text{corresponds to} \quad T_3(\xi) &=& (\alpha+\beta-1)\xi_1\xi_3 + \xi_4,
\end{aligned}$$

where $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$. Thus, translating (7) shows that it is sufficient to prove that

$$\exists c_1, c_2, c_3 \in \mathbb{R}: \ \forall \xi \in \mathbb{R}^4: \ (S_0 + c_1T_1 + c_2T_2 + c_3T_3)(\xi) \geq 0. \tag{9}$$

Problems of this kind are well known in real algebraic geometry. Quantified formulas as (9) are referred to as *Tarski sentences*. Determing the truth or falsity of such a sentence for a particular value of α is the associated *decision problem*. It was shown by Tarski [32] that such problems for polynomial systems are always solvable in an algorithmic way. We refer to section 2.2 for further comments.

Summarizing, our algorithm consists of the following four steps:

- ***Step 1:*** Calculate the functional $\mathcal{P}$ and "translate" it into a polynomial S_0.
- ***Step 2:*** Determine the polynomials $T_1, \ldots, T_d$ corresponding to integral expressions which can be obtained by integration by parts.
- ***Step 3:*** Decide for which functions the variety of all linear combinations of S_0 and $T_1, \ldots, T_d$ contains a polynomial which is nonnegative. This corresponds to a pointwise positivity estimate for the integrand of $\mathcal{P}$ (for a certain value of α).

Step 4: Check if the production term $\mathcal{P}$ can be estimated by an entropy production in the sense of $\mathcal{P} \geq \varepsilon\mathcal{E}$ for some $\varepsilon > 0$ and $\mathcal{E}$ is of a form similar to the right-hand side of (8) (see section 2.2 for a more precise description).

We notice that our method is formal in the sense that smooth positive solutions have to be assumed in order to justify the calculations. In the existence proofs, usually an appropriate approximation of the entropy functional has to be employed to overcome the lack of regularity and to ensure positivity of the approximations (see, e.g., [5, 7, 10, 31] for the thin film equation in one space dimension and [17, 18] for several space dimensions and [12, 24, 25] for the DLSS equation). However, the formal computations are a necessary first step to identify possible entropies and, even more importantly, they reveal a lot about the structure of the nonlinear equation.

Further, we mention that our method is exhaustive in the sense that the solution to the quantifier elimination problem in (9) reveals all $\alpha \in \mathbb{R}$, for which the respective production term $\mathcal{P}$ can be rewritten as an integral over a pointwise non-negative function by means of integration by parts of the type (7). The latter seems to be the most common technique in proofs of the dissipation property, so it is not surprising that previously know results are completely "rediscovered" by the algorithm. On the other hand, some of the remaining values α still might correspond to entropies. But the proof of the dissipation property in these cases neccessarily involves other techniques than suitable integrations by parts. For further comments on the absence of entropies, see section 5.5.

1.2. **Main results.** Our method allows to derive *all* known entropies for the thin film and DLSS equation. In the following we summarize some of our results. For this, we introduce the functions

$$\mathcal{S}_\alpha = \int \frac{n^\alpha}{\alpha(\alpha-1)}\,dx, \quad \mathcal{S}_0 = \int (n - \log n)\,dx, \quad \mathcal{S}_1 = \int (n(\log n - 1) + 1)\,dx, \tag{10}$$

$$\mathcal{S}_\alpha^1 = \int (n^{\alpha/2})_x^2\,dx. \tag{11}$$

We call $\mathcal{S}_\alpha$ an *entropy* if its time derivative is nonnegative for all $t > 0$. For all examples, we assume smooth positive solutions. Clearly, all functions in (10) are entropies for the porous medium equation.

- *Thin film equation:* The functions (10) with $\frac{3}{2} - \beta \leq \alpha \leq 3 - \beta$ are entropies for (2). (This holds also true in the multi-dimensional case; see section 5.4). For all $\frac{3}{2} - \beta < \alpha < 3 - \beta$, there exists a constant $\varepsilon > 0$ such that for all $t > 0$, the entropy production inequality (8) holds. Moreover, the functions (11) are entropies if (α, β) belong to the region shown in Figure 1. This region is characterized by a system of algebraic inequalities, which are at most quadratic in α and β.
- *DLSS equation:* The functions (10) with $0 \leq \alpha \leq \frac{3}{2}$ are entropies. In particular, there are entropy productions terms for all $0 < \alpha < \frac{3}{2}$. For instance, in the cases

$\alpha = 0, \frac{1}{2}, 1, \frac{3}{2}$, there exists $\varepsilon > 0$ such that for all $t > 0$,

$$\frac{d}{dt}\int (n - \log n)\,dx \;\leq\; -\varepsilon \int (\log n)_{xx}^2\,dx, \tag{12}$$

$$\frac{d}{dt}\int 4(\sqrt{n}-1)^2\,dx \;\leq\; -\varepsilon \int 4(\sqrt[4]{n})_{xx}^2\,dx, \tag{13}$$

$$\frac{d}{dt}\int (n(\log n - 1) + 1)\,dx \;\leq\; -\varepsilon \int (\sqrt{n})_{xx}^2\,dx, \tag{14}$$

$$\frac{d}{dt}\int n^{3/2}\,dx \;\leq\; -\varepsilon \int \sqrt{n}(\sqrt{n})_{xx}^2\,dx. \tag{15}$$

Furthermore, the functions (11) are entropies if $\frac{2}{53}(25-6\sqrt{10}) \leq \alpha \leq \frac{2}{53}(25+6\sqrt{10})$.

- *Sixth-order equation:* The functions (10) are entropies for all α which lie between the two real roots of the polynomial $1125\alpha^4 - 2700\alpha^3 + 2406\alpha^2 - 1020\alpha + 125$, namely $0.1927\ldots \leq \alpha \leq 1.1572\ldots$. For instance, $\alpha = 1$ satisfies this property, and there exists an $\varepsilon > 0$ such that for all $t > 0$,

$$\frac{d}{dt}\int (n(\log n - 1) + 1)\;dx \leq -\varepsilon \int (\sqrt{n})_{xxx}^2\,dx.$$

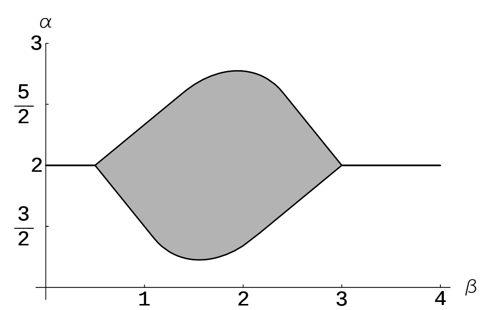

FIGURE 1. Values of α and β providing an entropy.

Most of the above results are well known: estimate (8) for the thin film equation has been shown in [5, 10] in an existence study. The dissipation property for (11) in the case $\alpha = 2$ has been shown in [18]; for more general α this property has been recently proved by Laugesen [27] using a different method. The entropies (10) and (11) (if $\alpha = 2$) for the DLSS equation have been reported in [12, 13, 25]. The entropies (11) for the DLSS equation in the case $\alpha \neq 2$ and the results on the sixth-order equation are new.

We stress the fact that, although most of the above results are known, the main focus of this paper is to present a *new systematic* method for deriving entropies and entropy productions. This method is not only able to reproduce the known results; it can be applied to *any* equation of the form (1) for *any* even order K. Moreover, our technique allows for several extensions which we sketch now.

1.3. **Extensions.** We already stated above that we are able to derive bounds on entropies containing derivatives of the solution. The most prominent example is the *Fisher information*,

$$\mathcal{S} = \int (\sqrt{n})_x^2 dx.$$

In section 5.1, first-order entropies (11) are determined for the example of the thin film and DLSS equation. Clearly, even entropies containing more than one derivative can be theoretically treated.

Secondly, our technique can be employed to prove functional inequalities which resemble logarithmic Sobolev inequalities. For instance, we are able to show that, for $\alpha > 0$,

$$\begin{aligned} \int n^\alpha (\log n)_x^4 dx &\leq \left(\frac{3}{\alpha}\right)^2 \int n^\alpha (\log n)_{xx}^2 dx, \\ \int n^\alpha (\log n)_{xx}^3 dx &\leq \frac{5}{12\alpha} \int n^\alpha (\log n)_{xxx}^2 dx \end{aligned}$$

for smooth positive functions n. We refer to section 5.2 for details of the computations.

Thirdly, we can consider *compound* equations of the form

$$n_t = \partial_x \left[n^{\beta+1} \left(P\left(\frac{\partial_x n}{n}, \frac{\partial_x^2 n}{n}, \ldots, \frac{\partial_x^{K-1} n}{n} \right) + Q\left(\frac{\partial_x n}{n}, \frac{\partial_x^2 n}{n}, \ldots, \frac{\partial_x^{K-1} n}{n} \right) \right) \right],$$

where P and Q are polynomials of different orders. A simple example is the thin film equation with a perturbation of porous medium type with a “bad” sign:

$$n_t = -(n^\beta (n_{xxx}))_x - q(n^\beta n_x)_x, \quad q > 0.$$

We show in section 5.3 that for $0 < q \leq 2\pi/L$ (recall that L is the interval length), the function (10) with $\alpha \in [\gamma_1(\beta, q), \gamma_2(\beta, q)]$ is non-increasing. The interval $[\gamma_1(\beta, q), \gamma_2(\beta, q)]$ is non-empty if and only if $\frac{3}{2} - \beta \leq \alpha \leq 3 - \beta$ and it always contains the value $2 - \beta$.

Finally, we are able to treat *multi-dimensional equations*. For instance, entropies of the form (10) are obtained for the thin film equation

$$\partial_t n = -\mathrm{div}(n^\beta \nabla \Delta n)$$

in any space dimension when $\frac{3}{2} - \beta \leq \alpha \leq 3 - \beta$ (see section 5.4). This result has previously been found in [11, 18] in the case of two or three space dimensions.

These examples show that our algorithmic construction of entropies is quite powerful and can be applied to a variety of important mathematical questions concerning the structure of nonlinear equations.

A variety of further possible applications of the method is obvious. For instance, it is natural to extend our technique to differential equations of type (1) with *odd* order K or to nonlinear conservation laws, obtaining first integrals rather than Lyapunov functionals. Furthermore, more general (e.g. convex) entropies could be studied. The method could be applied to other higher-order equations like the doubly nonlinear thin film equation [2],

$$n_t = -(|n|^\beta |n_{xxx}|^{p-2} n_{xxx})_x, \quad \text{where } p \geq 2,\ 0 < \beta \leq p + 1.$$

Finally, the multi-dimensional case should be studied systematically; we examine only one example here. All these topics are currently under investigation.

The paper is organized as follows. In the next section we give the precise definitions of entropy and entropy production and present in detail the general scheme for their determination. In section 3 some nonnegativity results for polynomials are proved. Section 4 is devoted to a detailed study of the four examples presented above and to the proofs of our main results. Finally, details about the extension of our method to more general situations are given in section 5.

2. The general scheme

2.1. **Definitions.** First we make precise which differential operators are admissible in (1).

Definition 1. *The real polynomial*

$$P(\xi_1,\ldots,\xi_k) = \sum_{p_1,\ldots,p_k} c_{p_1,\ldots,p_k}\xi_1^{p_1}\cdots\xi_k^{p_k}$$

*is called a k-*order symbol *if at most those coefficients $c_{p_1,\ldots,p_k}$ with $1\cdot p_1+2\cdot p_2+\cdots+k\cdot p_k = k$ are non-zero. We denote by Σ_k the set of all k-order symbols. We associate to $P\in\Sigma_k$ the following nonlinear ordinary differential operator of order k:*

$$D_P(n) = P\left(\frac{\partial_x n}{n}, \frac{\partial_x^2 n}{n}, \ldots, \frac{\partial_x^k n}{n}\right).$$

In this notation, we are concerned with equations of the type

$$n_t = (n^{\beta+1}D_P(n))_x, \quad x\in(0,L),\ t>0, \quad n(\cdot,0)=n_I,$$

where $P\in\Sigma_K$ and $\beta\in\mathbb{R}$; recall that K is an even positive integer. For simplicity, periodic boundary conditions are imposed on n,

$$\partial_x^\ell n(0,t) = \partial_x^\ell n(L,t), \quad \ell = 0,1,\ldots,k-1,\ t>0.$$

The notions of entropy and entropy production are formalized in the following definition.

Definition 2. *For a real number α, we define:*

- *An α-*functional *$\mathcal{S}(t)$ is an integral of the form*

$$\mathcal{S}(t) = \int s(n(x,t))\,dx,$$

where the function s is positive for positive arguments, and is such that $s''(n) = n^{\alpha-2}$ for $n>0$.

- *The α-*production term *$\mathcal{P}_\alpha$ is the negative time derivative of an α-functional, i.e. $\mathcal{P}_\alpha(t) = -(d/dt)\mathcal{S}(t)$.*
- *An* entropy *is an α-functional $\mathcal{S}(t)$ which is non-increasing along any sufficiently smooth solution $n(x,t)$ of (1), i.e., the α-production term is nonnegative.*

- *An* entropy production *for the entropy $\mathcal{S}(t)$ is an integral expression*

$$\mathcal{E}(n) = \int n^{\alpha+\beta} D_E(n)\, dx \tag{16}$$

with a K-order symbol E such that

$$\mathcal{P}_\alpha(t) \geq \varepsilon \mathcal{E}(n(t)) \quad \text{for all } t > 0, \quad \text{and for some } \varepsilon > 0. \tag{17}$$

- *An A-entropy $\mathcal{S}$ is called* generic *if and only if (16) yields an entropy production for $\mathcal{S}$ for* any *K-order symbol E.*

The production term is – as will be seen below – completely determined by α. The function $s(n)$ inside an α-functional is *almost* defined by the exponent α. Indeed, the definition implies

$$\begin{aligned}
\alpha \neq 0, 1 \quad &: \quad s(n) = \frac{n^\alpha}{\alpha(\alpha - 1)} + An + B, \\
\alpha = 1 \quad &: \quad s(n) = n(A + \log n) + B, \\
\alpha = 0 \quad &: \quad s(n) = -\log n + An + B,
\end{aligned}$$

and the constants A and B are chosen such that $s(n) > 0$ for $n > 0$.

On the other hand, the variety of entropy productions for one α-functional may be large. It is reasonable to restrict oneself to positive functionals. Typical (and for the analysis of the equations, useful) examples are

$$\mathcal{E}(n) = \int n^{\alpha+\beta-2} (\partial_x^{K/2} n)^2\, dx \quad \text{and} \tag{18}$$

$$\mathcal{E}(n) = \int (\partial_x^j n^{(\alpha+\beta)/\ell})^\ell\, dx \quad \text{or} \quad \mathcal{E}(n) = \int (\partial_x^j \log n)^\ell\, dx, \tag{19}$$

with positive integers j and ℓ such that $j\ell = K$. The first integral in (19) is defined for $\alpha + \beta \neq 0$, the second one for $\alpha + \beta = 0$. Some examples of entropies and entropy productions are given in (12)-(15).

2.2. **Determining entropies and entropy productions.** In the following, we propose an algorithm to decide for which values of α the α-functional $\mathcal{S}$ is indeed an entropy for (1) and what are possible choices for the entropy productions $\mathcal{E}$. According to the introduction, one has to perform four steps, which we repeat below using the terminology introduced in section 2.1:

Step 1: Calculate the polynomial S_0 corresponding to the α-production term $\mathcal{P}_\alpha$.
Step 2: Determine the polynomials $T_1, \ldots, T_d$ corresponding to integral expressions which can be obtained by integration by parts.
Step 3: Decide for which values of α there are $c_1, \ldots, c_d \in \mathbb{R}$ such that the polynomial $S = S_0 + c_1 T_1 + \cdots + c_d T_d$ is nonnegative for all arguments.
Step 4: Check the stability of the positive polynomials under perturbations, implying genericity of the entropy.

These four steps are now analyzed in detail and accompanied by the example of the thin film equation.

Step 1: Characteristic symbols. First we establish a canonical link between the integral expression of an α-production term and symbols of order K.

Definition 3. *A* characteristic symbol *for the α-production term $\mathcal{P}_\alpha$ is a K-order symbol $S \in \Sigma_K$ such that*

$$\mathcal{P}_\alpha = -\int n^{\alpha+\beta} D_S(n)\, dx.$$

There is at least one characteristic symbol for each α, namely $S_0(\xi_1, \ldots, \xi_K) = \xi_1 P(\xi_1, \ldots, \xi_{K-1})$, with P being the $(K-1)$-order symbol in (1). This fact follows from

$$\begin{aligned} \frac{d}{dt}\mathcal{S}(t) &= \int s'(n) n_t\, dx = \int s'(n)(n^{\beta+1} D_P(n))_x\, dx \\ &= -\int s''(n) n_x (n^{\beta+1} D_P(n))\, dx \\ &= -\int n^{\alpha+\beta} \left(\frac{n_x}{n}\right) P\left(\frac{\partial_x n}{n}, \ldots, \frac{\partial_x^{K-1} n}{n}\right) dx. \end{aligned}$$

The characteristic symbol S_0 is called the *canonical symbol.* The canonical symbol is independent of α and characterizes the equation (1).

Example 4. We recall from the introduction that the thin film equation can be written in the form (1) with $P(\xi) = -\xi_3$. Therefore, $S_0(\xi) = \xi_1 P(\xi) = -\xi_1\xi_3$. This simply expresses the fact that for any $\alpha \neq 0, 1$,

$$\frac{d}{dt}\int \frac{n^\alpha}{\alpha(\alpha-1)}\, dx = -\int n^{\alpha+\beta}\left(-\frac{n_x}{n}\frac{n_{xxx}}{n}\right) dx \tag{20}$$

(see (6)). □

Step 2: Shift polynomials. There exist infinitely many characteristic symbols S for the α-production term $\mathcal{P}_\alpha$. As this function can be rewritten in various ways using integration by parts, the coefficients of S vary. We give a systematic description of how the polynomial S changes.

Definition 5. *Let $P \in \Sigma_k$ be a k-order symbol and $\gamma \in \mathbb{R}$. Define the $(k+1)$-order symbol $\delta_\gamma P$ by*

$$(n^\gamma D_P(n))_x = n^\gamma D_{\delta_\gamma P}(n).$$

The operator δ_γ is a linear map from the space of k-order symbols Σ_k to the space of $(k+1)$-order symbols Σ_{k+1}. An explicit calculation shows that the image of the monomial $P(\xi) = \xi_1^{p_1}\xi_2^{p_2}\cdots\xi_k^{p_k}$ can be represented as

$$\delta_\gamma P(\xi) = \big(\gamma - (p_1 + \cdots + p_k)\big)\xi_1 P(\xi) + p_1\frac{\xi_2}{\xi_1}P(\xi) + \cdots + p_k\frac{\xi_{k+1}}{\xi_k}P(\xi). \tag{21}$$

The next simple lemma is essential for our theory.

Lemma 6. *Let α, $\beta \in \mathbb{R}$. If $S \in \Sigma_K$ is a characteristic symbol for the α-production term $\mathcal{P}_\alpha$ and $P \in \Sigma_{K-1}$ then $S' = S + \delta_{\alpha+\beta} P$ is another characteristic symbol for $\mathcal{P}_\alpha$.*

Proof. For any P, we obtain, abbreviating $\delta P = \delta_{\alpha+\beta} P$,

$$-\int n^{\alpha+\beta} D_{S+\delta P}(n)\,dx = -\int n^{\alpha+\beta}\big(D_S(n)+D_{\delta P}(n)\big)\,dx = \mathcal{P}_\alpha - \int \big(n^{\alpha+\beta} D_P(n)\big)_x\,dx = \mathcal{P}_\alpha,$$

since we have assumed periodic boundary conditions. □

By Lemma 6, all symbols S belonging to the affine subspace $S_0 + (\delta_{\alpha+\beta}\Sigma_{K-1}) \subset \Sigma_K$, where S_0 is a canonical symbol, are characteristic for the α-production term. A basis $T_1, \ldots, T_d$ of the linear space $\delta_{\alpha+\beta}\Sigma_{K-1}$ is simply obtained by evaluating $\delta_{\alpha+\beta}$ on any basis $R_1, \ldots, R_d$ of Σ_{K-1}. From the representation given in (21) it is clear that if the R_j are linear independent, so are the T_j. In the following, we choose monomials for the R_j. The corresponding symbols $T_j = \delta_{\alpha+\beta} R_j$ are called *shift polynomials*. Notice that, whereas the canonical symbol S_0 is independent of α and β, the shift polynomials are not.

Example 7. It is not difficult to check that the three monomials

$$R_1(\xi) = \xi_1^3, \quad R_2(\xi) = \xi_1\xi_2, \quad R_3(\xi) = \xi_3$$

form a basis of the space Σ_3. Formula (21) yields the shift polynomials in Σ_4:

$$\begin{aligned} T_1(\xi) &= \delta_{\alpha+\beta} R_1(\xi) = (\alpha+\beta-3)\xi_1^4 + 3\xi_1^2\xi_2, \\ T_2(\xi) &= \delta_{\alpha+\beta} R_2(\xi) = (\alpha+\beta-2)\xi_1^2\xi_2 + \xi_2^2 + \xi_1\xi_3, \\ T_3(\xi) &= \delta_{\alpha+\beta} R_3(\xi) = (\alpha+\beta-1)\xi_1\xi_3 + \xi_4. \end{aligned} \tag{22}$$

Adding some linear combination of the T_i to S_0 gives the characteristic symbol for an equivalent integral representation of the α-production term. The variety of integral representations which are connected to the original one by integration by parts is hence described by polynomials $S = S_0 + c_1T_1 + c_2T_2 + c_3T_3 \in \Sigma_4$ with arbitrary real parameters c_1, c_2 and c_3. For instance, rewriting the right-hand side of (20)

$$\int n^{\alpha+\beta-2} n_x n_{xxx}\,dx = -\int n^{\alpha+\beta}\left[(\alpha+\beta-2)\left(\frac{n_x}{n}\right)^2\left(\frac{n_{xx}}{n}\right) + \left(\frac{n_{xx}}{n}\right)^2\right]dx$$

corresponds to the passage from $S_0 = -\xi_1\xi_3$ to $S = S_0 + T_2 = (\alpha+\beta-2)\xi_1^2\xi_2 + \xi_2^2$. □

Step 3: Decision problem. If one can show that there exists a characteristic symbol S for $\mathcal{P}_\alpha$ which is nonnegative for all real arguments $\xi_1,\ldots,\xi_K$, then the corresponding α-functional $\mathcal{S}$ is an entropy. Indeed,

$$\frac{d}{dt}\mathcal{S}(t) = -\mathcal{P}_\alpha(t) = -\int n^{\alpha+\beta} D_S(n)\,dx = -\int n^{\alpha+\beta} S\left(\frac{\partial_x n}{n}, \ldots, \frac{\partial_x^K n}{n}\right)dx,$$

and if S is a nonnegative polynomial, then the expression under the last integral is nonnegative, for all functions n. This implies that the α-functionals are entropies. In other words, the statement

"The α-functional $\mathcal{S}$ is an entropy."

follows if one can show that

$$\exists c_1, \ldots, c_d \in \mathbb{R} : \ \forall \xi \in \mathbb{R}^K : \ (S_0 + c_1 T_1 + \cdots + c_d T_d)(\xi) \geq 0. \tag{23}$$

More precisely, one would like to find all values of $\alpha \in \mathbb{R}$ such that (23) is true; recall that the shift polynomials T_i depend on α, and so does the validity of (23).

We already noticed in the introduction that the determination of all parameters α for which (23) holds true is called a *quantifier elimination problem.* Such problems are always solvable in an algorithmic way [32]. Solution algorithms have been implemented, for instance, in the computer algebra system `Mathematica`. Moreover, there exists software which is specialized on quantifier elimination, like the tool `QEPCAD` [16]. Seemingly all available programs perform cylindrical algebraic decomposition [15], an algorithm whose complexity is doubly exponential in the number of variables ξ_i and c_i. Consequently, the solution of a decision problem turns out to be extremely time- and memory-consuming in practice. The apparently simple problems stated in section 4 are already at the edge of the ability of a standard computer today. In more complicated situations, like for general entropies in several space dimensions, more efficient algorithms are needed. The development of such algorithms is an active field of current research (see, e.g., the FRISCO project). A conceptually new algorithm has been recently proposed in [3]; it is yet not completely implemented.

Fortunately, some of the simpler quantifier elimination problems can be solved by hand, and others can be sufficiently simplified such that the available software produces a result in reasonable time (see section 4). Indeed, several decisive properties of the polynomials occuring in (23) are visible without going into algebraic geometry. For this, the following notion is useful.

Definition 8. *A characteristic symbol S for $\mathcal{P}_\alpha$ is in* normal form *if for each k, the highest exponent with which ξ_k occurs in S is even.*

In particular, a K-order symbol S in normal form is independent of $\xi_{K/2+1}$, $\xi_{K/2+2}$, $\ldots$, ξ_K. We claim that if a characteristic symbol is *not* in normal form, then the corresponding polynomial cannot have a definite sign. Indeed, assume that the highest power p_ℓ of ξ_ℓ is odd. Fix the variables ξ_k with $k \neq \ell$ at some values, thus considering S as a polynomial in ξ_ℓ only. Assume without loss of generality that the coefficient of $\xi_\ell^{p_\ell}$ is positive. Then for $\xi_\ell \to +\infty$ and $\xi_\ell \to -\infty$, $S(\xi)$ tends to $+\infty$ and $-\infty$, respectively, which shows the claim.

The requirement that S is in normal form helps to reduce the number of parameters c_i, as we will see in section 4. We do not investigate the question of whether a general symbol can be brought into normal form by means of integration by parts. We just notice that in the examples analyzed here, it is always possible.

Example 9. We consider again the thin film equation. The first step is to identify those symbols $S = S_0 + c_1T_1 + c_2T_2 + c_3T_3$ which are in normal form. Obviously, c_3 must vanish as T_3 contains ξ_4 in first power. Similarly, c_2 must be chosen to eliminate the first power of ξ_3 which stems from $S_0 = -\xi_1\xi_3$, i.e. $c_2 = 1$. There are no restrictions on c_1. Thus, the variety of equivalent normal forms for $\mathcal{P}_\alpha$ is given by the symbols

$$S = S_0 + c_1 \cdot T_1 + 1 \cdot T_2 + 0 \cdot T_3 = (\alpha + \beta - 3)c_1\xi_1^4 + (\alpha + \beta - 2 + 3c_1)\xi_1^2\xi_2 + \xi_2^2. \tag{24}$$

Recall that the number β is fixed by the model and α characterizes the entropy under consideration. In section 4.2 it is shown in detail how the corresponding quantifier elimination problem is explicitly solved. The final result is that there exists a suitable choice of c_1 turning S into a nonnegative polynomial if and only if $\frac{3}{2} \leq \alpha + \beta \leq 3$. $\square$

Step 4: Entropy production. Finally, we turn to an algebraic formulation of the entropy production. Obviously, (17) is a consequence of the following statement: There exists a characteristic symbol S for $\mathcal{P}_\alpha$ and there is an $\varepsilon > 0$ such that

$$S(\xi) - \varepsilon E(\xi) \geq 0 \quad \text{for all } \xi \in \mathbb{R}^K. \tag{25}$$

To simplify calculations, it is advisable to bring both S and E into a normal form (assuming that this is possible by adding shift polynomials) before property (25) is checked.

Recall that an entropy $\mathcal{S}$ is generic if (25) is true for all choices of K-order symbols E, with ε depending on E. This terminology reflects the following idea: S is a nonnegative polynomial, and $S - \varepsilon E$ is a polynomial with an ε-small perturbation in the coefficients. If the nonnegativity property (25) is retained for arbitrary but sufficiently small perturbations E, then one can say that the coefficients of S are in generic position in Σ_K.

For instance, the polynomial $P(\xi) = \xi^2 - 2\xi + 1$ is nonnegative, but not generic, as $P(\xi) - \varepsilon\xi$ becomes negative at some point, no matter how small $\varepsilon > 0$ is chosen. In contrast, $Q(\xi) = 2\xi^2 + 2\xi + 1$ *is* generic. In general, it is far from being trivial to decide whether a nonnegative polynomial is generic in this sense or not. It is possible in our examples.

Example 10. Exactly those α-functionals with $\frac{3}{2} < \alpha + \beta < 3$ are generic entropies for the thin film equation (see section 4.2). How do possible entropy productions for the thin film equation look like? As $K = 4$, the standard ansatz (18) reads as

$$\mathcal{E} = \int n^{\alpha+\beta-2} n_{xx}^2 \, dx,$$

while the ansatz (19) with $\ell = 2$ and $\ell = 4$, respectively, yields

$$\mathcal{E} = \int (n^{(\alpha+\beta)/2})_{xx}^2 \, dx \quad \text{and} \quad \mathcal{E} = \int (n^{(\alpha+\beta)/4})_x^4 \, dx.$$

For a generic entropy, one thus finds positive constants ε_1, ε_2, ε_3 such that

$$\frac{d}{dt} \int \frac{n^\alpha}{\alpha(\alpha-1)} \, dx \leq -\varepsilon_1 \int n^{\alpha+\beta-2}(n_{xx})^2 \, dx - \varepsilon_2 \int (n^{(\alpha+\beta)/2})_{xx}^2 \, dx - \varepsilon_3 \int (n^{(\alpha+\beta)/4})_x^4 \, dx.$$

$\square$

3. Some auxiliary results

In this section we present technical lemmas that help to solve some easy quantifier elimination problems "by hand", or at least to help to simplify them noticeably. The basic idea is to consider special polynomials and derive relations between the – unknown – coefficients guaranteeing nonnegativity.

Lemma 11. *Let the real polynomial* $P(\xi_1, \xi_2) = a_1\xi_1^4 + a_2\xi_1^2\xi_2 + a_3\xi_2^2$ *be given. Then the quantified expression*

$$\forall \xi_1, \xi_2 \in \mathbb{R}: \ P(\xi_1, \xi_2) \geq 0 \tag{26}$$

is equivalent to the quantifier-free statement that

$$\textit{either} \qquad a_3 > 0 \quad \textit{and} \quad 4a_1a_3 - a_2^2 \geq 0 \tag{27}$$

$$\textit{or} \qquad a_3 = a_2 = 0 \quad \textit{and} \quad a_1 \geq 0. \tag{28}$$

Proof. The sufficiency of (28) for (26) is obvious, while formula (27) implies

$$P(\xi) = \Big(a_1 - \frac{a_2^2}{4a_3}\Big)\xi_1^4 + a_3\Big(\xi_2 + \frac{a_2}{2a_3}\xi_1^2\Big)^2 \geq 0.$$

Conversely, (26) implies $0 \leq P(1,0) = a_1$ and $0 \leq P(0,1) = a_3$. If $a_3 > 0$,

$$0 \leq P(\sqrt{a_3}, -\frac{a_2}{2}) = \frac{a_3}{4}(4a_1a_3 - a_2^2)$$

yields $4a_1a_3 - a_2^2 \geq 0$, whereas $a_3 = 0$ implies $0 \leq P(a_2, -a_2 - a_1a_2) = -a_2^4$ and hence $a_2 = 0$. $\square$

If $a_3 = 1$ the statement of Lemma 11 simplifies:

$$\forall \xi_1, \xi_2 \in \mathbb{R}: \ P(\xi_1, \xi_2) \geq 0 \quad \text{if and only if} \quad 4a_1 - a_2^2 \geq 0.$$

Lemma 12. *Let the real polynomial*

$$P(\xi_1, \xi_2, \xi_3) = a_1\xi_1^6 + a_2\xi_1^4\xi_2 + a_3\xi_1^3\xi_3 + a_4\xi_1^2\xi_2^2 + a_5\xi_1\xi_2\xi_3 + \xi_3^2 \tag{29}$$

be given. Then the quantified formula

$$\forall \xi_1, \xi_2, \xi_3 \in \mathbb{R}: \ P(\xi_1, \xi_2, \xi_3) \geq 0 \tag{30}$$

is equivalent to the quantifier free formula

$$\textit{either} \qquad 4a_4 - a_5^2 > 0 \quad \textit{and} \quad 4a_1a_4 - a_1a_5^2 - a_2^2 - a_3^2a_4 + a_2a_3a_5 \geq 0 \tag{31}$$

$$\textit{or} \qquad 4a_4 - a_5^2 = 2a_2 - a_3a_5 = 0 \quad \textit{and} \quad 4a_1 - a_3^2 \geq 0. \tag{32}$$

Proof. The polynomial P is obviously nonnegative on the plane $\xi_1 = 0$. Thus, the formula (30) is equivalent to the statement that the quadratic polynomial

$$p(y,z) = a_1 + a_2y + a_3z + a_4y^2 + a_5yz + z^2$$

is nonnegative for all real values of y and z where $y = \xi_2/\xi_1^2$ and $z = \xi_3/\xi_1^3$.

For a fixed value of $y_0 \in \mathbb{R}$, the univariate polynomial

$$p(y_0, z) = (a_1 + a_2y_0 + a_4y_0^2) + (a_3 + a_5y_0)z + z^2$$

is non-negative (cf. Lemma 11 above) if and only if the expression

$$q(y_0) = 4(a_1 + a_2y_0 + a_4y_0^2) - (a_3 + a_5y_0)^2$$

is non-negative. It is obvious that p is non-negative if and only if $p(y_0, z)$ is non-negative for all $y_0 \in \mathbb{R}$, which means that

$$q(y) = (4a_4 - a_5^2)y^2 + (4a_2 - 2a_3a_5)y + (4a_1 - a_3^2)$$

is a non-negative polynomial. The latter is – again by Lemma 11 – equivalent to (31)-(32). □

Lemma 13. *Let a univariate polynomial* $P(x) = a_0 + a_1x + a_2x^2$ *with* $a_2 > 0$ *and a real number* $\hat{x}$ *be given. Then the quantified formula*

$$\exists x > \hat{x} : P(x) \leq 0 \tag{33}$$

is equivalent to the quantifier-free expression

$$\text{either} \qquad a_0 + a_1\hat{x} + a_2\hat{x}^2 < 0 \tag{34}$$

$$\text{or} \qquad 4a_0a_2 - a_1^2 \leq 0 \quad \text{and} \quad 2a_2\hat{x} + a_1 < 0. \tag{35}$$

Proof. If P possesses real roots, i.e. if $4a_0a_2 - a_1^2 \leq 0$, then the larger one is given by $x_+ = (\sqrt{a_1^2 - 4a_0a_2} - a_1)/2a_2$. Now observe that $\hat{x} < x_+$ if and only if either $\big(\hat{x} + a_1/2a_2)^2 < \big(\sqrt{a_1^2 - 4a_0a_2}\big)^2$ or $\hat{x} + a_1/2a_2 < 0$, which can be rephrased as (34) and (35). □

4. Examples

4.1. **The porous medium equation.** We associate to the porous medium equation

$$n_t = (n^\beta n_x)_x$$

the polynomial $P(\xi_1) = \xi_1$. Thus the canonical symbol is $S_0(\xi_1) = \xi_1^2$. Notice that S_0 is already a nonnegative polynomial. This immediately implies that all α-functionals are entropies for the porous medium equation.

For the sake of completeness we show that all entropies are generic. Observe that a general 2-order symbol E is of the form $E(\xi) = b_1\xi_1^2 + b_2\xi_2$. The only shift polynomial is given by

$$T(\xi_1, \xi_2) = \delta_{\alpha+\beta}\xi_1 = (\alpha + \beta - 1)\xi_1^2 + \xi_2.$$

The normal form of E reads $E(\xi) = b'\xi_1^2$ with $b' = b_1 - (\alpha + \beta - 1)b_2$ and is unique in this exceptional case. Therefore, property (25) reduces to the question if there exists an $\varepsilon > 0$ such that

$$\xi_1^2 - \varepsilon b'\xi_1^2 \;\; \geq \;\; 0.$$

The answer is affirmative, independently of $\alpha \in \mathbb{R}$. Hence, the functionals (19) with $j = 1$ and $\ell = 2$ are entropy productions for all α.

Theorem 14. All *α-functionals* ($\alpha \in \mathbb{R}$) *are generic entropies for the porous medium equation. In particular, the following estimate holds for all* $\alpha \neq 0, 1$*:*

$$\frac{d}{dt}\int \frac{n^\alpha}{\alpha(\alpha-1)}\,dx \leq -\varepsilon \int (n^{(\alpha+\beta)/2})_x^2\,dx, \quad \varepsilon > 0 \textit{ sufficiently small.}$$

4.2. **The thin film equation.** The thin film equation

$$n_t = -(n^\beta n_{xxx})_x$$

has already been discussed as a guiding example in section 2.2. It remains to prove that the α-functionals with $\frac{3}{2} \leq \alpha+\beta \leq 3$ indeed correspond to entropies, and that those with $\frac{3}{2} < \alpha+\beta < 3$ are generic.

For this, we recall the general normal form of the characteristic symbols for the α-production term $\mathcal{P}_\alpha$ from equation (24),

$$S_c(\xi) = (\alpha+\beta-3)c\xi_1^4 + (\alpha+\beta-2+3c)\xi_1^2\xi_2 + \xi_2^2$$

with the free parameter $c \in \mathbb{R}$. We need to find all $\alpha \in \mathbb{R}$ such that

$$\exists c \in \mathbb{R}: \ \forall \xi \in \mathbb{R}: \ S_c(\xi) \geq 0.$$

By Lemma 11, the nonnegativity of S_c at a certain value of c is equivalent to

$$0 \geq 9c^2 + 2(\alpha+\beta)c + (\alpha+\beta-2)^2 = 9\left(c + \frac{1}{9}(\alpha+\beta)\right)^2 + \frac{8}{9}(\alpha+\beta)^2 - 4(\alpha+\beta) + 4. \tag{36}$$

Choosing the minimizing value $c = -(\alpha+\beta)/9$, the requirement (36) is satisfied if and only if

$$8(\alpha+\beta)^2 - 36(\alpha+\beta) + 36 \leq 0$$

which is fulfilled if and only if $\frac{3}{2} \leq \alpha+\beta \leq 3$.

We turn to the entropy production. Let E be a K-order symbol and assume that E is already in normal form. Writing

$$S(\xi) = a_1\xi_1^4 + a_2\xi_1^2\xi_2 + a_3\xi_2^2, \quad E(\xi) = b_1\xi_1^4 + b_2\xi_1^2\xi_2 + b_3\xi_2^2,$$

we arrive at

$$S(\xi) - \varepsilon E(\xi) \ = \ (a_1 - \varepsilon b_1)\xi_1^4 + (a_2 - \varepsilon b_2)\xi_1^2\xi_2 + (a_3 - \varepsilon b_2)\xi_2^2.$$

To decide positivity of this expression, we use Lemma 11. Set $a_i' = a_i - \varepsilon b_i$, $i = 1, 2, 3$. The condition $a_3' > 0$ is always satisfied for $\varepsilon > 0$ small enough. It remains to be checked for which a_1, a_2 and a_3

$$q' := 4a_1'a_3' - (a_2')^2 \geq 0.$$

Notice that this is an ε-small purturbation of $q := 4a_1a_3 - a_2^2$, so if $q > 0$, then also $q' > 0$ for small ε, regardless of the values of b_1, b_2, and b_3. It is easily seen that $q > 0$ corresponds to the strict inequality $\frac{3}{2} < \alpha < 3$; this inequality characterizes the generic entropies.

In the non-generic cases $\alpha+\beta = \frac{3}{2}$ and $\alpha+\beta = 3$, the selection of entropy productions is restricted. It is easily seen that, for instance, $E(\xi) = \xi_2^2$ corresponding to $D_E(n) = n_{xx}^2/n^2$ does *not* yield an entropy production for $\alpha = \frac{3}{2}$. However, there are still non-trivial choices for $\mathcal{E}$. As it is possible to calculate explicitly the constants c which makes S_c non-negative, it is most canonical to take $E = S_c$. With $c = -\frac{1}{6}$ and $c = -\frac{1}{3}$ for $\alpha = \frac{3}{2} - \beta$ and

$\alpha = 3-\beta$, respectively, one obtains $E = (\xi_2 - \xi_1^2/2)^2$ and $E = \xi_2^2$, corresponding to the entropy productions

$$\mathcal{E} = 4\int \sqrt{n}(\sqrt{n})_{xx}^2\,dx \quad \text{and} \quad \mathcal{E} = \int n^2 n_{xx}\,dx.$$

Recalling Example 10, we have proved the following result.

Theorem 15. *For the thin film equation with $\beta > 0$, all α-functionals with $\frac{3}{2} - \beta \le \alpha \le 3 - \beta$ are entropies. All α-functionals with $\frac{3}{2} - \beta < \alpha < 3 - \beta$ are generic, i.e.,* (8) *holds for some $\varepsilon > 0$.*

4.3. **The DLSS equation.** The associated polynomial to the DLSS equation

$$n_t = -(n(\log n)_{xx})_{xx}$$

reads as $P(\xi_1, \xi_2, \xi_3) = -\xi_1^3 + 2\xi_1\xi_2 - \xi_3$; thus the canonical symbol is $S_0(\xi) = -\xi_1^4 + 2\xi_1^2\xi_2 - \xi_1\xi_3$. The shift polynomials are the same as for the thin film equation but with $\beta = 0$, see (22). Similarly, the most general normal form of S is given by

$$S_c(\xi) = S_0 + c\cdot T_1 + 1\cdot T_2 + 0\cdot T_3 = (c(\alpha-3)-1)\xi_1^4 + (3c+\alpha)\xi_1^2\xi_2 + \xi_2^2,$$

only depending on one parameter $c \in \mathbb{R}$. Performing essentially the same steps as in section 4.2, we arrive at the analogue of equation (36):

$$0 \ge 9c^2 + 2(\alpha+6)c + (4+\alpha^2) = 9\left(c + \frac{1}{9}(\alpha+6)\right)^2 + \frac{8}{9}\alpha\left(\alpha - \frac{3}{2}\right).$$

Choosing $c = -(\alpha+6)/9$, this inequality is satisfied if and only if $\alpha(\alpha - \frac{3}{2}) \le 0$ or

$$0 \le \alpha \le \frac{3}{2}.$$

As before, generic entropies are those for which $q = 4a_1a_3 - a_2^2 > 0$, corresponding to $0 < \alpha < \frac{3}{2}$. The non-generic entropies $\alpha = 0$ and $\alpha = \frac{3}{2}$ are treated as before: For $\alpha = 0$, we obtain $c = -\frac{2}{3}$ which gives $E(\xi) = (\xi_1^2 - \xi_2)^2$; for $\alpha = \frac{3}{2}$, we have $c = -\frac{5}{6}$, so that $E(\xi) = (\xi_2/2 - \xi_1^2)^2$. Eventually, this leads to the entropy productions

$$\mathcal{E} = \int (\log n)_{xx}^2\,dx \quad \text{and} \quad \mathcal{E} = 4\int \sqrt{n}(\sqrt{n})_{xx}^2\,dx.$$

Theorem 16. *All α-functionals with $0 \le \alpha \le 3/2$ are entropies for the DLSS equation. Generic entropies are those with $0 < \alpha < 3/2$. Furthermore, the estimates* (12)-(15) *hold for some $\varepsilon > 0$.*

4.4. **A sixth-order equation.** The canonical symbol of

$$n_t = \left(n\left(\frac{1}{n}\left(n(\log n)_{xx}\right)_{xx} + \frac{1}{2}(\log n)_{xx}^2\right)_x\right)_x$$

equals

$$S_0(\xi) = 6\xi_1^6 - 18\xi_1^4\xi_2 + 11\xi_1^2\xi_2^2 + 8\xi_1^3\xi_3 - 5\xi_1\xi_2\xi_3 - 3\xi_1^2\xi_4 + \xi_1\xi_5.$$

It can be seen that there are seven shift polynomials,

$$\begin{aligned}
T_1(\xi) &= (\alpha-5)\xi_1^6 + 5\xi_1^4\xi_2, \\
T_2(\xi) &= (\alpha-4)\xi_1^4\xi_2 + 3\xi_1^2\xi_2^2 + \xi_1^3\xi_3, \\
T_3(\xi) &= (\alpha-3)\xi_1^2\xi_2^2 + \xi_2^3 + 2\xi_1\xi_2\xi_3, \\
T_4(\xi) &= (\alpha-3)\xi_1^3\xi_3 + 2\xi_1\xi_2\xi_3 + \xi_1^2\xi_4, \\
T_5(\xi) &= (\alpha-2)\xi_1\xi_2\xi_3 + \xi_3^2 + \xi_2\xi_4, \\
T_6(\xi) &= (\alpha-2)\xi_1^2\xi_4 + \xi_2\xi_4 + \xi_1\xi_5, \\
T_7(\xi) &= (\alpha-1)\xi_1\xi_5 + \xi_6.
\end{aligned}$$

Straightforward considerations lead to the following general normal form of the characteristic symbol:

$$\begin{aligned}
S_{c_1,c_2}(\xi) &= (S_0 + c_1\cdot T_1 + c_2\cdot T_2 + 0\cdot T_3 + (1+\alpha)\cdot T_4 + 1\cdot T_5 - 1\cdot T_6 + 0\cdot T_7)(\xi) \\
&= (6+(\alpha-5)c_1)\xi_1^6 + (-18+5c_1+(\alpha-4)c_2)\xi_1^4\xi_2 + (11+3c_2)\xi_1^2\xi_2^2 \\
&\quad + (5-2\alpha+\alpha^2+c_2)\xi_1^3\xi_3 + (3\alpha-5)\xi_1\xi_2\xi_3 + \xi_3^2. \qquad (37)
\end{aligned}$$

The corresponding quantifier elimination problem can now be solved using computer algebra. For this example, we prefer to perform the quantifier elimination explicitly by application of Lemmas 12 and 13. By Lemma 12, the polynomial S_{c_1,c_2} is nonnegative with respect to ξ if and only if either case (31) is true,

$$\begin{aligned}
0 < 4a_4 - a_5^2 &= 44 - (3\alpha-5)^2 + 12c_2, \qquad (38) \\
0 \le q(c_1,c_2) &:= 4a_1a_4 - a_1a_5^2 - a_2^2 - a_3^2a_4 + a_2a_3a_5, \qquad (39) \\
&= (q_0 + q_1c_2 + q_2c_2^2 - 3c_2^3) + (q_3 + q_4c_2)c_1 - 25c_1^2
\end{aligned}$$

(the coefficients q_i depend on α only) or if (32) holds,

$$\begin{aligned}
0 = 4a_4 - a_5^2 &= 44 - (3\alpha-5)^2 + 12c_2, \qquad (40) \\
0 = 2a_2 - a_3a_5 &= -11 - 25\alpha + 11\alpha^2 - 3\alpha^3 + 10c_1 - (3+\alpha)c_2, \qquad (41) \\
0 \le 4a_1 - a_3^2 &= 8\alpha(2-\alpha) - (\alpha-1)^4 + 4(\alpha-5)c_1 - (10-4\alpha+2\alpha^2)c_2 - c_2^2. \qquad (42)
\end{aligned}$$

The solution of the system (40)-(42) is easily computed: Inserting the values for c_1 and c_2 obtained from the linear equations (40) and (41) into (42) yields the following condition on α:

$$0 \le \frac{1}{720}(-125 + 1020\alpha - 2406\alpha^2 + 2700\alpha^3 - 1125\alpha^4). \qquad (43)$$

Values of α lying between the two real roots (namely, $0.1927\ldots$ and $1.1572\ldots$) of the polynomial give rise to entropies.

To resolve (38)-(39) as well, first observe that (38) is satisfied if and only if $c_2 > \hat{c}_2 := ((3\alpha-5)^2 - 44)/12$. Moreover, the polynomial $q(c_1,c_2)$ is quadratic in c_1 for any fixed c_2;

it possesses a nonnegative point if and only if its discriminant is nonpositive,

$$\begin{aligned} 0 &\geq 4(-25)(q_0 + q_1 c_2 + q_2 c_2^2 - 3c_2^3) - (q_3 + q_4 c_2)^2 \\ &= (c_2 - \hat{c}_2)\Big[4(25 + 20\alpha + 14\alpha^2 + 10\alpha^3 + \alpha^4) + 4(25 + 10\alpha + 7\alpha^2)c_2 + 25c_2^2\Big] \\ &=: (c_2 - \hat{c}_2)\Delta(c_2). \end{aligned}$$

(It is a fortunate fact that the discriminant factors this way. Observe that $q(c_1, c_2)$ itself does not permit such a factorization.) Now apply Lemma 13 to determine the conditions under which there exists a $c_2^* > \hat{c}_2$ such that $\Delta(c_2) \leq 0$. We omit the details but report that formula (34) reproduces condition (43) above, with a strict inequality sign, while (35) does not provide any additional information.

Finally, one can use the same perturbation argument as in the examples before. It is then easily seen that generic entropies are singled out by the property that both inequalities in (31) are *strictly* satisfied. The latter corresponds to strict inequality in (43).

Theorem 17. *All α-functionals with $0.1927\ldots \leq \alpha \leq 1.1572\ldots$ are entropies for the sixth-order equation* (4)*. Generic entropies are associated to those α which fulfil the* strict *inequality. In particular, for $\alpha = 1$, the following estimate holds:*

$$\frac{d}{dt}\int (n(\log n - 1) + 1)\ dx \leq -\varepsilon \int (\sqrt{n})_{xxx}^2\, dx$$

for some $\varepsilon > 0$.

5. Extensions

5.1. **Higher-order entropies.** The definition of α-functionals and entropies allows a straight-forward extension in which also x-derivatives of n may occur under the integral sign. We do not intend to investigate the most general situation here, but limit ourselves to functionals of the form

$$\mathcal{S}^m = \frac{1}{2}\int (\partial_x^m s(n))^2\, dx \quad \text{with } s'(n) = n^{\alpha/2-1}.$$

We call such an integral an *m-order α-functional*. Naturally, an m-order α-functional is called an *m-order entropy* if it is non-increasing in time,

$$\frac{d}{dt}\mathcal{S}^m(t) \leq 0 \quad \text{for } t > 0.$$

First-order entropies are of special interest. As mentioned in the introduction, the most prominent example is the *Fisher information*, obtained for $\alpha = 1$,

$$\mathcal{S}^1 = 2\int (\sqrt{n})_x^2\, dx,$$

which is known to be an entropy for the thin-film or DLSS equation [12, 27].

The connection of m-order α-functionals to our algebraic framework becomes obvious by calculating the time derivative

$$\begin{aligned}\frac{d\mathcal{S}^m}{dt} &= \int \partial_x^m s(n) \frac{d}{dt}\partial_x^m s(n)\,dx = (-1)^m \int \big(\partial_x^{2m} s(n)\big) s'(n) n_t\,dx \\ &= (-1)^{m+1}\int n^{\beta+1} D_P(n)\,\big((\partial_x^{2m}s(n))s'(n)\big)_x\,dx.\end{aligned}$$

For example, in order to obtain first-order entropies

$$\mathcal{S}^1 = \frac{2}{\alpha}\int (n^{\alpha/2})_x^2\,dx$$

for the thin film equation, we have to determine those values of α for which

$$\begin{aligned}\exists c_1, c_2 \in \mathbb{R}:\ \forall \xi: \quad & (\alpha+\beta-5)c_1\xi_1^6 + (5c_1 + (\alpha+\beta-4)c_2)\xi_1^4\xi_2 + 3c_2\xi_1^2\xi_2^2 \\ & + \left(\tfrac{1}{2}(\alpha^2-5\alpha+6)+c_2\right)\xi_1^3\xi_3 + (2\alpha-4)\xi_1\xi_2\xi_3 + \xi_3^2 \geq 0.\end{aligned}$$

The situation is very similar to that of the sixth order equation in section 4.4. The quantifier elimination can be performed either by computer algebra or explicitly with the help of Lemmas 12 and 13. The result is displayed in Figure 1. Thus, there is always a "trivial" first-order entropy, corresponding to $\alpha = 2$, which reads $\mathcal{S}^1 = \int n_x^2\,dx$. Further entropies are available for $1/2 < \beta < 3$; then α belongs to an interval that contains 2 in its interior.

The notions of entropy productions and generic entropies carry over literally to higher-order entropies. We report that the points (α,β) lying in the interior of the entropy region in Figure 1 correspond to generic entropies. Hence, there holds

$$\frac{d}{dt}\int (n^{\alpha/2})_x^2\,dx \leq -\varepsilon_1 \int n^{\alpha+\beta-2}(n_{xxx})^2\,dx - \varepsilon_2 \int (n^{(\alpha+\beta)/2})_{xxx}^2\,dx - \varepsilon_3 \int (n^{(\alpha+\beta)/6})_x^6\,dx.$$

Similarly, first-order entropies for the DLSS equation (3) correspond to values of α for which there are c_1 and c_2, making

$$\begin{aligned}S(\xi) &= \left(3 - 5\frac{\alpha}{2} + \frac{\alpha^2}{2} + (\alpha-5)c_1\right)\xi_1^6 + (-10 + 7\alpha - \alpha^2 + 5c_1 + (\alpha-4)c_2)\xi_1^4\xi_2 \\ &\quad + (8 - 4\alpha + 3c_2)\xi_1^2\xi_2^2 + \left(4 - \frac{5\alpha}{2} + \frac{\alpha^2}{2} + c_2\right)\xi_1^3\xi_3 + (2\alpha-6)\xi_1\xi_2\xi_3 + \xi_3^2\end{aligned}$$

a nonnegative polynomial in ξ. Along the same lines as before, one determines the condition that α lies in between the two reals roots of $20-100\alpha+53\alpha^2$, i.e. $\alpha \in (0.2274\ldots, 1.6593\ldots)$.

5.2. **Logarithmic Sobolev-type inequalities.** The same technique used to determine entropies can be employed to prove functional inequalities which resemble the logarithmic Sobolev inequality

$$\int n^2 \log\frac{n^2}{N^2}\,dx \leq C \int n^2 (\log n)_x^2\,dx,$$

where $N^2 = \int n^2 dx/L$, L being the interval length. As an example of our method we prove the relation

$$\int n^\alpha (\log n)_x^4 \, dx \le \left(\frac{3}{\alpha}\right)^2 \int n^\alpha (\log n)_{xx}^2 \, dx$$

for all smooth (rapidly decaying or periodic) functions $n > 0$. More precisely, we determine the range of ε such that

$$\int n^\alpha \left((\log n)_{xx}^2 - \varepsilon (\log n)_x^4 \right) \, dx \ge 0. \tag{44}$$

This estimate is clearly satisfied if $D_{Q_0}(n) \ge 0$, with

$$Q_0(\xi) = (1-\varepsilon)\xi_1^4 - 2\xi_1^2\xi_2 + \xi_2^2.$$

Integration by parts in (44) is translated into the addition of a linear combination of the shift polynomials (22) with $\beta = 0$ to Q_0. We thus consider the general normal form

$$Q_c(\xi) = Q_0(\xi) + cT_1(\xi) = (1 - \varepsilon - c(\alpha - 3))\xi_1^4 + (3c - 2)\xi_1^2\xi_2 + \xi_2^2$$

and seek all values of ε for which there exists a real constant c such that Q_c is nonnegative. In other words, we need to solve a quantifier elimination problem for Q_c, establishing a relation between α and ε. To do so, we apply Lemma 11 and obtain eventually the following relation for c,

$$9c^2 + 4\alpha c + 4\varepsilon \le 0,$$

which is true if and only if $9\varepsilon \le \alpha^2$. By Theorem 19 in section 5.5, the choice $\varepsilon = (\alpha/3)^2$ is optimal: Observe that $Q_0(\check{\xi}) = Q_0(1, 1 - \alpha/3) = \varepsilon - (\alpha/3)^2$.

In a similar way, but with more technical effort, we can prove inequalities involving more derivatives, like (assuming $\alpha > 0$)

$$\int n^\alpha (\log n)_{xx}^3 \, dx \le \frac{5}{12\alpha} \int n^\alpha (\log n)_{xxx}^2 \, dx.$$

For more information on logarithmic Sobolev inequalities and particularly its optimal constants, we refer to [14, 23, 34] and, more recently, to [4, 21].

5.3. **Combinations of operators of different order.** Often, equations arising from applications contain several differential terms modeling various physical phenomena. Examples are the (zero-field) quantum drift-diffusion model for semiconductors [1, 25],

$$n_t = -(n(\log n)_{xx})_{xx} + n_{xx}, \tag{45}$$

and the thin-film porous-medium equation [9],

$$n_t = -(n^\beta n_{xxx})_x + (n^\gamma)_{xx}. \tag{46}$$

More generally, we wish to deal with *compound* equations of the form

$$n_t = \left(n^{\beta+1} \left(D_P(n) + D_Q(n)\right)\right)_x, \tag{47}$$

where the symbols P and Q define operators of different order. It is clear that if an α-functional is an entropy both for $n_t = (n^{\beta+1} D_P(n))_x$ and $n_t = (n^{\beta+1} D_Q(n))_x$, then it is also an entropy for (47). Therefore, the range of entropies for the compound equations (47) contains the intersection of those for the individual equations.

More interesting is the situation in which the compound equation possesses additional entropies. Consider the example

$$n_t = -(n^\beta n_{xxx} + qn^\beta n_x))_x, \quad q > 0. \tag{48}$$

As $q > 0$, there is *no* entropy for the equation associated to Q. The corresponding polynomials are $P(\xi) = -\xi_3$ and $Q(\xi) = -q\xi_1$. Here we employ the Poincaré inequality

$$\ell \int (n^{(\alpha+\beta)/2})_x^2 \, dx \leq \int (n^{(\alpha+\beta)/2})_{xx}^2 \, dx, \tag{49}$$

where $\ell = (2\pi/L)^2$ is the (optimal) Poincaré constant. (Recall that L is the length of the interval.) This inequality is equivalent to

$$\int n^{\alpha+\beta} \left[\left(\frac{n_{xx}}{n} + \frac{\alpha+\beta-2}{2} \left(\frac{n_x}{n}\right)^2 \right)^2 - \ell \left(\frac{n_x}{n}\right)^2 \right] dx \geq 0.$$

Although it is impossible to prove positivity of any characteristic symbol for α-production terms of (48), the following estimate is clearly sufficient to show the positivity of the α-production term itself:

$$\xi_1 P(\xi) + \xi_1 Q(\xi) + cT_1(\xi) + T_2(\xi) \geq \rho \left(\left(\xi_2 + \frac{\alpha+\beta-2}{2}\xi_1^2 \right)^2 - \ell\xi_1^2 \right), \tag{50}$$

where $\rho > 0$ and T_1 and T_2 are the shift polynomials (22). The right-hand side is not necessarily a pointwise positive expression but it represents the nonnegative integral (49). If there is a choice of the shift parameter c such that this inequality holds for some $\rho > 0$, then the decay of the α-functional is proven. An equivalent version of (50) is

$$\xi_1 P(\xi) + cT_1(\xi) + T_2(\xi) - \rho \left(\xi_2 + \frac{\alpha+\beta-2}{2}\xi_1^2 \right)^2 \geq (q - \rho\ell)\xi_1^2. \tag{51}$$

Observe that the polynomial on the left-hand side is homogeneous of degree two in ξ_1^2 and ξ_2, whereas the right-hand side is homogeneous of degree one in ξ_1^2. Therefore (51) is true for all ξ if and only if the left-hand side is always nonnegative and the right-hand side is nonpositive. Among all values for ρ which make the right-hand side nonpositive, the choice $\rho = q/\ell$ obviously maximizes the left-hand side. It thus suffices to determine the values of $\gamma = \alpha + \beta$ for which there exists a constant c such that the left-hand side is a nonnegative polynomial. In explicit form,

$$\left(c(\gamma-3) - \frac{\rho}{4}(\gamma-2)^2 \right) \xi_1^4 + ((1-\rho)(\gamma-2) + 3c)\xi_1^2\xi_2 + (1-\rho)\xi_2^2 \geq 0.$$

By Lemma 11, the conditions $\rho < 1$ (which implies $q = \ell\rho < \ell$) and

$$(1-\rho)(4c(\gamma-3) - \rho(\gamma-2)) \geq ((\gamma-2)(1-\rho) + 3c)^2$$

are sufficient. (The case $\rho = 1$ corresponds to $\gamma = 2$.) Rewrite this inequality as a polynomial equation for c,

$$9c^2 + 2(1-\rho)\gamma c + (1-\rho)(\gamma-2)^2 \leq 0.$$

The minimal value of this quadratic expression in c is nonpositive if and only if

$$(8+\rho)\gamma^2 - 36\gamma + 36 \leq 0.$$

In the limit $q = 0$ of vanishing perturbations, we derive the condition $\frac{3}{2} \leq \gamma \leq 3$ again. If $0 < q < \ell$, then γ can be chosen from the interval $[\gamma_1, \gamma_2]$, where

$$\gamma_{1/2} = \frac{18}{8 + q(L/2\pi)^2}\left(1 \pm \sqrt{1 - \frac{1}{9(8+q(L/2\pi)^2)}}\right). \tag{52}$$

For $q = \ell$, we obtain $\gamma_1 = \gamma_2$ and hence, only $\gamma = 2$ is possible.

Theorem 18. *Every α-functional is an entropy for* (48) *provided that $0 \leq q \leq (2\pi/L)^2$ and $\gamma_1 - \beta \leq \alpha \leq \gamma_2 - \beta$, where $\gamma_{1/2}$ are given by* (52).

5.4. **Multi-dimensional equations.** In principle, the generalization of the one-dimensional concept to two (or more) space dimensions is straightforward: The basic building blocks are differential expressions of $u(x_1, x_2)$ of the form $(\partial_{x_1}^k \partial_{x_2}^\ell u)/u$. Consequently, in two dimensions, the variables ξ_k become double-indexed quantities, $\eta_{k,\ell}$. The rules of integration by parts are obtained by differentiating products of η with respect to x_1 or to x_2.

Although this naive strategy works in theory, it leads (even in the simplest situations) to large polynomial expressions in many variables $\eta_{k,\ell}$ and a huge variety of shift polynomials. Solving the corresponding quantifier elimination problem would be far beyond the ability of today's computer technology.

A better approach is seemingly not to incorporate *all* products of differential expressions $(\partial_x^k \partial_y^\ell u)/u$, but to restrict oneself to an appropriate subclass. We notice that the stronger such a restriction is, the greater are the chances that on the one hand, the solution of the quantifier elimination problem is actually computable, and that one the other hand, some entropies are "lost", i.e. not seen by the method. A natural restriction is to focus on those expressions with the same basic symmetry properties as the original evolution equations.

This section is intended to sketch how our method works for functions u depending on $d > 1$ variables. We do not rigorously develop the concept of multi-dimensional entropies and shift polynomials, but only present the main ideas in its application to the multi-dimensional thin film equation

$$u_t = -\mathrm{div}\big(u^\beta \nabla \Delta u\big), \quad u : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}_+. \tag{53}$$

In analogy to the one-dimensional case, we are looking for entropies in the form $\mathcal{S}(u) = \int u^\alpha dx/\alpha(\alpha-1)$. Taking the time derivative and integrating by parts (assuming multi-periodic boundary conditions), yields the production term

$$\frac{d\mathcal{S}}{dt} = \int u^{\alpha+\beta} \frac{\nabla u}{u} \cdot \frac{\nabla \Delta u}{u}\, dx.$$

The basis for determining the shift polynomials is the divergence theorem,

$$\int \mathrm{div} D_R(u)\, dx = 0,$$

i.e., we are looking for multi-dimensional differential expressions $D_T(u)$ of the form $D_T(u) = \mathrm{div} D_R(u)$.

We focus on differential expressions which can be written in terms of scalar multiplication and the operator ∇ alone, i.e. without any reference to the individual partial derivatives $\partial/\partial x_j$. The resulting quantifier elimination problem can be easily solved; however, one has to be aware that some entropies might get lost by this restriction. Here is a list of the relevant monomic differential expressions:

- first order: $\nabla u/u$ (gradient, 1-tensor);
- second order: $\Delta u/u$ (Laplacian, scalar), $\nabla\nabla u/u$ (Hessian, 2-tensor);
- third order: $\nabla\Delta u/u$ (1-tensor), $\nabla\nabla\nabla u/u$ (3-tensor);
- fourth order: $\Delta\Delta u/u$ (scalar), $\nabla\nabla\Delta u/u$ (2-tensor), $\nabla\nabla\nabla\nabla u/u$ (4-tensor).

Next, we list all homogeneous *scalar* expressions containing exactly four derivatives (which appear as products of the monomials above)

$$|\nabla u|^4/u^4, \quad (\Delta u)^2/u^2, \quad \mathrm{tr}(\nabla\nabla u)^2/u^2, \quad |\nabla u|^2\Delta u/u^3, \tag{54}$$

$$\nabla u^T(\nabla\nabla u)\nabla u/u^3, \quad \nabla\Delta u\nabla\Delta u/u, \quad \Delta\Delta u/u. \tag{55}$$

Introducing the symbols $\eta_G = \nabla u/u$, $\eta_L = \Delta u/u$, $\eta_H = \nabla\nabla u/u$, $\eta_T = \nabla\Delta u/u$, and $\eta_D = \Delta\Delta u/u$, the expressions in (54) and (55) are abbreviated as

$$\begin{gathered} \eta_G^4, \quad \eta_L^2, \quad \mathrm{tr}(\eta_H^2), \quad \eta_G^2\eta_L, \\ \eta_G^2\eta_H, \quad \eta_G\eta_T, \quad \eta_D. \end{gathered}$$

These expressions have to be read as formal symbols and not as actual products. Notice that in contrast to the one-dimensional situation, multiplication of differential monomials in higher dimensions is rather sophisticated. For instance, the formal expression $(\nabla u/u)^2$ could represent both the scalar function $|\nabla u|^2/u^2$ as well as the 2-tensor Q which is such that $Q(v,w) = (v\cdot\nabla u)(w\cdot\nabla u)/u^2$.

Fortunately, ambiguities like this do not appear in the situation at hand. Moreover, the symbols involving the tensor expressions η_G and η_H (i.e. the gradient vector and the Hessian matrix) share essential properties with actual products. In particular, from the Cauchy-Schwarz estimate

$$(v^TAw)^2 \le \mathrm{tr}(A^2)\,\|v\|^2\,\|w\|^2$$

for arbitrary symmetric matrices A and vectors v, w, it follows that pointwise,

$$|\eta_G^2\eta_H| \le \sqrt{\mathrm{tr}(\eta_H^2)}\,\|\eta_G\|^2. \tag{56}$$

The shift polynomials corresponding to the divergence of

$$|\nabla u|^2\nabla u/u^3, \quad \Delta u\nabla u/u^2, \quad \nabla u^T(\nabla\nabla u)/u^2, \quad \nabla\Delta u/u,$$

are, writing $\gamma = \alpha + \beta$ and $\eta = (\eta_G, \eta_H, \eta_L)$,

$$\begin{aligned}
T_1(\eta) &= (\gamma-3)\eta_G^4 + \eta_G^2\eta_L + 2\eta_G^2\eta_H, \\
T_2(\eta) &= (\gamma-2)\eta_G^2\eta_L + \eta_L^2 + \eta_G\eta_T, \\
T_3(\eta) &= (\gamma-2)\eta_G^2\eta_H + \mathrm{tr}(\eta_H^2) + \eta_G\eta_T, \\
T_4(\eta) &= (\gamma-1)\eta_G\eta_T + \eta_D.
\end{aligned}$$

The different coefficients of $\eta_G^2\eta_L$ and $\eta_G^2\eta_H$ in T_1 underline the fact that the second-order symbols η_L and η_H are not completely interchangable and therefore cannot be combined into one single variable.

The canonical symbol for equation (53) is $S_0(\eta) = -\eta_G\eta_T$. As in one dimension, it only makes sense to consider the normal forms. The general normal form for S reads:

$$\begin{aligned}
S(\eta) &= (S_0 + cT_1 + fT_2 + f'T_3)(\eta) \\
&= c(\gamma-3)\eta_G^4 + f\eta_L^2 + f'\mathrm{tr}(\eta_H^2) + ((\gamma-2)f + c)\eta_G^2\eta_L + ((\gamma-2)f' + 2c)\eta_G^2\eta_H,
\end{aligned}$$

where f and f' are constants satisfying $f + f' = 1$.

We now argue that it is sufficient to do quantifier elimination for a corresponding polynomial on three *scalar* variables. Namely, define $\xi = (\xi_G, \xi_H, \xi_L) \in \mathbb{R}^3$ for given η by

$$\xi_G = \|\eta_G\|, \quad \xi_H = \sqrt{\mathrm{tr}(\eta_H^2)}, \quad \xi_L = \eta_L.$$

Then $S(\eta)$ is identical to $S(\xi)$, up to the term involving $\eta_G^2\eta_H$. Thanks to the basic relation (56), it follows $S(\eta) \geq \min\big(S(\xi_G, \xi_H, \xi_L), S(\xi_G, -\xi_H, \xi_L)\big)$ pointwise. Thus, it indeed suffices to resolve

$$\exists c,\ f + f' = 1 \ :\ \forall \xi \in \mathbb{R}^3 \ :\ S(\xi) \geq 0.$$

From the latter, one obtains, using e.g. the algebra tool `QEPCAD`, the equivalent condition

$$\frac{3}{2} \leq \alpha + \beta \leq 3.$$

5.5. **Absence of Entropies.** A variety of calculations is necessary to perform our algorithm in actual applications, some of which (in particular the quantifier elimination) can be very time-consuming. Below, a method of less computational effort is presented, which yields restrictions on the regions where entropies will be found.

Recall that K is the order of equation (1). For $\alpha \in \mathbb{R}$, define $\check{\xi} \in \mathbb{R}^K$ with components

$$\check{\xi}_1 = 1 \quad \text{and inductively} \quad \check{\xi}_{k+1} = \big(1 - \tfrac{k}{K-1}(\alpha+\beta)\big)\xi_k \quad \text{for } 1 \leq k < K.$$

All respective shift polynomials $T_i(\xi)$ vanish at $\xi = \check{\xi}$. This follows immediately by inserting the definition of $\check{\xi}$ into formula (21). Therefore, the values of any two characteristic symbols S and S' coincide at $\check{\xi}$. Hence, if the canonical symbol S_0 is negative at $\check{\xi}$, then *any* characteristic symbol is. If this is the case, then the α-production term cannot be rewritten as an integral over a pointwise positive function by means of integration by parts. In consequence, the corresponding α-functional $\mathcal{S}$ will not be identified as an entropy by our algorithm. This statement can be even strengthened:

Theorem 19. *Assume that, for any C^∞ periodic initial data n_I, equation (1) posseses a classical solution $n(\cdot,t)$ for $t \in [0,T)$. For $\alpha \in \mathbb{R}$ with $\alpha+\beta > 0$, let $\mathcal{S}$ be the corresponding α-functional with canonical symbol S_0. If $S_0(\check{\xi}) < 0$, then $\mathcal{S}$ is not an entropy.*

Proof. The claim of this theorem is essentially a generalization of Theorem 1(C) in [27]. We give an outline of the adaption to our situation, omitting the technical details.

The proposal is to take $n_I(x) = |x|^\tau$ with exponent $\tau = (K-1)/(\alpha+\beta)$. For $x \neq 0$,

$$\frac{\partial_x^k n_I}{n_I} = \tau(\tau-1)\cdots(\tau-k+1)\Big(\frac{\sigma(x)}{|x|}\Big)^k$$

with $\sigma(x)$ being the sign of x. Hence, formally, the integrand of $\mathcal{P}_\alpha$ at $t=0$ reads

$$\begin{aligned} n_I^{\alpha+\beta} D_{S_0}(n_I) &= |x|^{(\alpha+\beta)\tau}\, S_0\Big(\tau(\tfrac{\sigma(x)}{|x|}), \tau(\tau-1)(\tfrac{\sigma(x)}{|x|})^2, \ldots, \tau\cdots(\tau-K+1)(\tfrac{\sigma(x)}{|x|})^K\Big) \\ &= |x|^{(\alpha+\beta)\tau-K}\,\tau^K\, S_0\Big(1, \tfrac{\tau-1}{\tau}, \ldots, (\tfrac{\tau-1}{\tau})\cdots(\tfrac{\tau-K+1}{\tau})\Big) \\ &= \tau^K\, S_0(\check{\xi})\,|x|^{-1}. \end{aligned}$$

Here we have used the homogeneity of $S_0 \in \Sigma_K$ to pull $\tau^k|x|^{-k}$ out of the kth argument. Also, since K is an even number, the expressions $\sigma(x)^k$ cancel.

To make n_I a valid initial datum, there are three obstacles to overcome:

(1) *The expression $|x|^{-1}$ is not integrable at $x=0$.* Define $n_\delta(x) = |x|^{\tau+\delta}$, with $\delta > 0$. Then, for any fixed $\bar{x} > 0$, a direct computation reveals

$$\int_{-\bar{x}}^{\bar{x}} n_\delta^{\alpha+\beta} D_{S_0}(n_\delta)\,dx = \frac{1}{\delta}\left[\frac{2\,\tau^K}{\alpha+\beta} S_0(\check{\xi}) + o(1)\right] \quad \text{as } \delta \to +0. \tag{57}$$

(2) *The function n_δ is not periodic.* Replace n_δ by $\hat{n}_\delta(x) = |\sin x|^{\tau+\delta}$ (assuming without loss of generality that $L = 2\pi$.). The asymptotic behaviour (57) is not affected, as the integral is dominated by the almost-nonintegrable contribution near $x=0$.

(3) *$\hat{n}_\delta$ is neither smooth nor positive at $x=0$.* Use the regularized function

$$\hat{n}_{\delta,\varepsilon}(x) = \Big(\sqrt{\varepsilon + \sin^2 x}\Big)^{\tau+\delta},$$

which is smooth und positive for $\varepsilon > 0$. The dominated convergence theorem allows to pass to the pointwise limit $\lim_{\varepsilon\to+0} \hat{n}_{\delta,\varepsilon}(x) = \hat{n}_\delta(x)$ under the integral in (57) for any $\delta > 0$. At this point, estimates analogous to those in [27] are necessary.

In summary, one obtains a family of smooth positive 2π-periodic functions $\hat{n}_{\delta,\varepsilon}$ satisfying

$$\lim_{\delta\to+0}\left[\delta \lim_{\varepsilon\to+0} \int_{-\pi}^{\pi} \hat{n}_{\delta,\varepsilon}^{\alpha+\beta} D_{S_0}(\hat{n}_{\delta,\varepsilon})\,dx\right] = \frac{2\,\tau^K}{\alpha+\beta} S_0(\check{\xi}) \;<\; 0. \tag{58}$$

Use $\hat{n}_{\delta,\varepsilon}$ as initial condition n_I for (1). By assumption, a classical solution $n(\cdot,t)$ with $n(\cdot,0) = n_I$ exists. From (58) it follows that $\mathcal{P}_\alpha(0) = \int n_I^{\alpha+\beta} D_{S_0}(n_I)\,dx < 0$ for $0 < \varepsilon \ll \delta$ sufficiently small. Thus, $\mathcal{S}(t)$ is increasing for small times $t \geq 0$ and cannot be an entropy. $\square$

The necessary criterion $S_0(\check{\xi}) \geq 0$ produces exact bounds on the (zeroth-order) entropy ranges for, e.g., the DLSS equation. More precisely, one has $S_0(\check{\xi}) < 0$ if and only if $\alpha < 0$ or $\alpha > \frac{3}{2}$. The same is true for the thin film equation.

The criterion seems less useful in situations with $K \geq 6$ or for higher-order entropies. For example, recall the equation of sixth order from section 4.4. One calculates

$$S_0(\check{\xi}) = \frac{2\alpha^2}{625}(-10+3\alpha)(-5+4\alpha).$$

This expression is negative for $\frac{5}{4} < \alpha < \frac{10}{3}$; hence the corresponding α-functionals are not entropies. This estimate gives little information on the range of entropies which we have determined as $0.1927\ldots < \alpha < 1.1572\ldots$.

Appendix

In this appendix we give a sketch of the derivation of the sixth-order equation (4). In [20] the following generalized quantum drift-diffusion model has been derived in $\mathbb{R}^d$ ($d \geq 1$):

$$n_t = \operatorname{div}(n\nabla A), \tag{59}$$

where the particle density $n(x,t)$ and the function $A(x,t)$ are related through

$$n(x,t) = \frac{1}{2\pi\delta}\int_{\mathbb{R}^d} \operatorname{Exp}(A(x,t) - |p|^2/2)dp, \quad x \in \mathbb{R}^d,\ t > 0,$$

and $\delta > 0$ is the scaled Planck constant. The so-called *quantum exponential* Exp is defined by $\operatorname{Exp}(f) = W(\exp(W^{-1}(f)))$, where $f = f(x,p,t)$ is a function, W is the Wigner transform, W^{-1} its inverse, and "exp" is the operator exponential. We refer to [19, 20] for the precise definitions of the Wigner transform and its inverse since we do not need it in the following. Equation (59) is obtained from a collisional Wigner equation by a moment method in the diffusion limit, employing the quantum exponential as a closure function.

The aim of this section is to expand (59) in one space dimension in powers of δ up to $O(\delta^6)$. The expansion up to $O(\delta^4)$ has been performed in [19, Thm. 5.1] and leads to the (zero-field) quantum drift-diffusion model

$$n_t = n_{xx} - \frac{\delta^2}{12}(n(\log n)_{xx})_{xx}.$$

The crucial step is to find an $O(\delta^6)$ approximation of $\operatorname{Exp}(f)$ with $f(x,p) = A(x) - |p|^2/2$. Following [19], we define $F(s) = \operatorname{Exp}(sf)$ and express $F(s)$ as a series in δ, i.e. $F(s) = \sum_{k=0}^\infty \delta^k F_k(s)$. The functions $F_k(s)$ can be computed by pseudo-differential calculus. Then, $F_k(s) = 0$ for all odd indices k and

$$\frac{d}{ds}F_k(s) = f \circ_0 F_k(s) + f \circ_2 F_{k-2}(s) + \cdots + f \circ_k F_0(s)$$

for all even indices k, where the multiplication "$\circ_m$" is defined by

$$u \circ_m v = \left(-\frac{1}{4}\right)^{m/2} \sum_{\alpha+\beta=m} \frac{(-1)^\beta}{\alpha!\beta!}\partial_x^\alpha \partial_p^\beta u \cdot \partial_x^\beta \partial_p^\alpha v.$$

The functions F_0 and F_2 have already been calculated in [19, Lemma 5.6]:

$$
\begin{aligned}
F_0(s)(x,p) &= e^{sf(x,p)}, \\
F_2(s)(x,p) &= \frac{1}{8}e^{sf(x,p)}\left(A_{xx}\left(s^2-\frac{s^3}{3}p^2\right)+\frac{s^3}{3}A_x^2\right).
\end{aligned}
$$

As $F_1(s)=F_3(s)=F_5(s)=0$, it remains to solve

$$
\begin{aligned}
\frac{d}{ds}F_4(s) &= f\circ_0 F_4(s)+f\circ_2 F_2(s)+f\circ_4 F_0(s) \\
&= f\cdot F_4(s)+\frac{e^{sf}}{192}\Big(s^5A_x^4+(9s^4-2s^5p^2)A_x^2A_{xx}+(10s^3-9s^4p^2+s^5p^4)A_{xx}^2 \\
&\quad +(8s^3-2s^4p^2)A_xA_{xxx}+(3s^2-s^3p^2)A_{xxxx}\Big) \\
&\quad +\frac{e^{sf}}{384}\left((3s^2-6s^3p^2+s^4p^4)A_{xxxx}\right).
\end{aligned}
$$

By the variation-of-constants formula, the result is

$$
\begin{aligned}
F_4(1) &= \frac{e^f}{384}\left[\frac{1}{3}A_x^4+\left(\frac{18}{5}-\frac{2}{3}p^2\right)A_x^2A_{xx}+\left(5-\frac{18}{5}p^2+\frac{1}{3}p^4\right)A_{xx}^2\right. \\
&\quad \left.+\left(4-\frac{4}{5}p^2\right)A_xA_{xxx}+\left(3-2p^2+\frac{1}{5}p^4\right)A_{xxxx}\right].
\end{aligned}
$$

This completes the $O(\delta^6)$ expansion of the quantum exponential.

In order to represent the density n as a function of A, we integrate F_0, F_2, and F_4 with respect to $p\in\mathbb{R}$ and employ the formulas

$$
\int_{\mathbb{R}} e^{A-p^2/2}p^m\,dp=\begin{cases} \dfrac{\sqrt{2\pi}m!}{(m/2)!}\,e^A & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd.}\end{cases}
$$

This gives

$$
\begin{aligned}
(60)\ n &= \frac{1}{2\pi\delta}\int_{\mathbb{R}}(F_0(1)+\delta^2F_2(1)+\delta^4F_4(1))dp+O(\delta^6)=\frac{e^A}{\sqrt{2\pi}\delta}\left(1+\frac{\delta^2}{24}(A_x^2+2A_{xx})\right. \\
&\quad \left.+\frac{\delta^4}{5760}(5A_x^4+44A_x^2A_{xx}+36A_{xx}^2+48A_xA_{xxx}+24A_{xxxx})+O(\delta^6)\right).
\end{aligned}
$$

To obtain also an δ-expansion of A in terms of n, we insert the ansatz $A=A_0+\delta^2A_2+\delta^4A_4$ in (60). Equating equal powers of δ yields the system

$$
\begin{aligned}
n &= \sqrt{2\pi}\delta\,e^{A_0}, \\
0 &= A_2+\frac{1}{24}(A_{0,x}^2+2A_{0,xx}),
\end{aligned}
$$

$$
\begin{aligned}
0 &= A_4 + \frac{1}{2}A_2^2 + \frac{1}{12}(A_{0,x}A_{2,x} + A_{2,xx}) + \frac{1}{24}A_2(A_{0,x}^2 + 2A_{0,xx}) \\
&\quad + \frac{1}{5760}(5A_{0,x}^4 + 44A_{0,x}^2A_{0,xx} + 36A_{0,xx}^2 + 48A_{0,x}A_{0,xxx} + 24A_{0,xxxx}).
\end{aligned}
$$

Therefore,

$$
A_0 = \log n - \log(\sqrt{2\pi}\delta), \quad A_2 = -\frac{1}{6}\frac{\sqrt{n}_{xx}}{\sqrt{n}},
$$

$$
A_4 = -\frac{1}{360}\left(\frac{3(\log n)_{xx}^2}{2} - \frac{n_{xx}(\log n)_{xx}}{n} - \left(\frac{n_{xx}}{n}\right)_{xx}\right).
$$

Finally, (59) becomes

$$
n_t = n_{xx} - \frac{\delta^2}{12}(n(\log n)_{xx})_{xx} + \frac{\delta^4}{360}\left(n\left(\frac{(n(\log n)_{xx})_{xx}}{n} + \frac{(\log n)_{xx}^2}{2}\right)_x\right)_x
$$

up to an error of order $O(\delta^6)$. The sixth-order equation (4) is obtained by considering only the last expression in the above equation and setting $\delta^4 = 360$. The whole equation can be treated by our entropy method using the techniques of section 5.3.

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