

## AN ALGORITHMIC DESINGULARIZATION OF 3-DIMENSIONAL TORIC VARIETIES

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**Abstract.** We present an algorithmic procedure to desingularize every 3-dimensional toric variety, while keeping under control the Euler characteristic of the varieties computed during the process. We prove that our upper bounds for the Euler characteristic of the desingularized toric varieties are the best possible.

**Introduction.** We refer to [4] and [5] for resolutions of singularities of toric varieties in general. In [3, p. 48] and in [2, 8.2] it is explained how to get a simplicial subdivision of a general nonsimplicial fan.

We present a method to desingularize every 3-dimensional toric variety  $X = X_{\Delta}$  (with  $\Delta$  a simplicial fan): a subdivision  $\nabla(\Delta)$  of  $\Delta$  is constructed by successively starring each 3-dimensional cone  $\sigma \in \Delta$  of multiplicity  $\text{mult}(\sigma) > 1$  at a nonzero primitive vector  $p_{\sigma} \in \sigma \cap \mathbf{Z}^3$  such that the sum of the multiplicities of the cones obtained by the starring is the smallest possible. Here, as usual,  $\text{mult}(\sigma)$  is the index of the subgroup of  $\mathbf{Z}^3$  generated by the primitive (integral) generators of  $\sigma$ .

When applied to the 2-dimensional case, our method yields the familiar construction of the coarsest nonsingular subdivision of  $\Delta$ : this is uniquely determined by the set  $S$  of integral points of the compact faces of the boundary of the convex hull of the set of integral nonzero points in each cone of  $\Delta$ . As is well known, for every dimension  $> 2$ ,  $S$  generally contains too few points, and these are not very useful to construct desingularizations.

In the 3-dimensional case we rather focus attention on the set  $S'$  of  $\leq_{\sigma}$ -minimal integral vectors of  $\sigma$  with respect to the order induced by each cone  $\sigma \in \Delta$ . As shown in our paper,  $S'$  is the set of primitive generators of those rays which belong to every nonsingular subdivision of  $\Delta$ . While in the 2-dimensional case  $S = S'$ , in the 3-dimensional case  $S'$  is strictly larger than  $S$ , and is a key tool in our desingularization algorithm.

Our method also keeps under control the number of cones obtained during the desingularization process and yields, for every 3-dimensional compact toric variety  $X = X_{\Delta}$  (where each 3-dimensional cone  $\sigma \in \Delta$  is simplicial) a desingularization  $X'$  of  $X$  such that  $E(X') \leq -E(X) + 2 \sum_{\sigma \in \Delta^{(3)}} \text{mult}(\sigma)$ . Here,  $E(X)$  is the Euler characteristic, which coincides with the number  $\#\Delta^{(3)}$  of 3-dimensional cones in  $\Delta$ . We show that our upper bound is the best possible.

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After the writing of the second revised version of this paper we were informed by Professor Oda, Managing Editor of the Tohoku Mathematical Journal, that there exist considerable overlaps between our results and the results of the paper by Catherine Bouvier and Gérard Gonzalez-Sprinberg: *Système Générateur minimal, Diviseurs essentiels et G-Désingularisations de Variétés toriques*. Following Professor Oda's suggestion, during May 1994 we contacted these authors, sent them a copy of our paper, and received from them a copy of their paper. We leave it to the reader to single out all overlaps: for instance, the set  $G_\sigma$  of Définition 1.1 in their paper coincides with the set of nonzero minimal integral points with respect to the order  $\leq_\sigma$  introduced in Section 3 of our paper. It follows that Théorème 1.10 in their paper is similar to Proposition 3.3 in our paper. (The editor's note: The paper by C. Bouvier and G. Gonzalez-Sprinberg will appear in Tôhoku Math. J. 47, No. 1 (1995).)

**NOTATION.**

$\sigma, \tau, \rho$	simplicial rational cones in $\mathbf{R}^3$
$\sigma + (-\sigma)$	the $\mathbf{R}$ -subspace spanned by $\sigma$
$\langle v_1, \dots, v_k \rangle$	the $k$ -dimensional cone generated by the vectors $v_1, \dots, v_k \in \mathbf{Z}^3$ , $k = 1, 2, 3$
$X_\sigma$	the affine toric variety associated with $\sigma$
$\Delta$	a fan in $\mathbf{R}^3$
$\Delta^{(k)}$	the set of $k$ -dimensional cones in a fan $\Delta$ ; 1-dimensional cones are called <i>rays</i>
$X_\Delta$	the toric variety associated with a fan $\Delta$
$E(X_\Delta)$	$= \#\Delta^{(3)}$ the Euler characteristic of a fan $\Delta$ (the symbol $\#$ denotes cardinality)
$\text{int} S$	the interior of $S$

**1. Cones and nonsingular subdivisions.** We refer to [5] for background on cones, fans, and their subdivisions. All cones considered in this paper will be rational, simplicial, and will be contained in  $\mathbf{R}^3$ . When writing  $\sigma = \langle v_1, \dots, v_k \rangle$ , we mean that  $\sigma$  is the  $k$ -dimensional cone determined by its generators  $v_i \in \mathbf{Z}^3$ . A nonzero vector  $w \in \mathbf{Z}^3$  is said to be *primitive* if it is minimal along its ray.

Let  $\sigma = \langle v_1, v_2, v_3 \rangle$ . The cones  $\langle v_1, v_2 \rangle$ ,  $\langle v_1, v_3 \rangle$ ,  $\langle v_2, v_3 \rangle$  are called the (2-dimensional) *faces* of  $\sigma$ . For each nonzero  $p \in \sigma \cap \mathbf{Z}^3$  we let  $(\sigma * p)^{(3)}$  be the set of 3-dimensional cones of the fan  $\sigma * p$  obtained by *starring*  $\sigma$  at  $p$ . Thus in particular, if  $p \in \text{int } \sigma$  we have  $(\sigma * p)^{(3)} = \{\langle p, v_2, v_3 \rangle, \langle v_1, p, v_3 \rangle, \langle v_1, v_2, p \rangle\}$ ; in case  $p$  lies in the relative interior, say, of the face  $\langle v_1, v_2 \rangle$ , then  $(\sigma * p)^{(3)} = \{\langle p, v_2, v_3 \rangle, \langle v_1, p, v_3 \rangle\}$ ; finally, if  $p \in \langle v_i \rangle$  for some  $i \in \{1, 2, 3\}$ , then  $(\sigma * p)^{(3)} = \{\sigma\}$ . Compare [6, p. 15].

Let  $\sigma = \langle v_1, \dots, v_k \rangle$ , where each  $v_i$  is primitive and  $k=2, 3$ . Note that the set  $\{v_1, \dots, v_k\}$  is uniquely determined by  $\sigma$ . Then the *half-open parallelepiped (half-open parallelogram, in case  $k=2$ )*  $P_\sigma$  is defined by

$$P_\sigma = \{ \sum \lambda_i v_i \mid 0 \leq \lambda_i < 1 \} \subseteq \mathbf{R}^3 .$$

By definition, the *multiplicity* of  $\sigma$ ,  $\text{mult}(\sigma)$ , is the index in the lattice  $\mathbf{Z}^3 \cap (\sigma + (-\sigma))$  of the subgroup generated by  $v_1, \dots, v_k$ . Equivalently,

$$\text{mult}(\sigma) = \#(P_\sigma \cap \mathbf{Z}^3) = \text{the number of integral points in } P_\sigma .$$

In case  $k=3$  we also have

$$\text{mult}(\sigma) = | \det(v_1, v_2, v_3) | = \text{the volume of } P_\sigma .$$

By definition, a simplicial fan  $\Delta$  in  $\mathbf{R}^3$  is *nonsingular* if the multiplicity of each cone in  $\Delta^{(2)} \cup \Delta^{(3)}$  is equal to 1. We say that a point  $p$  is *indispensable for  $\sigma$*  if for every nonsingular subdivision  $\Delta$  of  $\sigma$ ,  $p$  is a primitive generator of some ray in  $\Delta^{(1)}$ . In particular, every primitive generator of  $\sigma$  is indispensable for  $\sigma$ .

Let  $\sigma = \langle v, w \rangle$  be a (2-dimensional, rational, simplicial) cone in  $\mathbf{R}^3$ , with  $v$  and  $w$  primitive vectors in  $\mathbf{Z}^3$ . Let  $\Theta$  be the convex hull in  $\mathbf{R}^3$  of the set  $(\sigma \cap \mathbf{Z}^3) \setminus \{0\}$ . With reference to [5, 1.6], let us display, in their natural order, the points

$$v = l_0, l_1, \dots, l_t, l_{t+1} = w$$

in  $\mathbf{Z}^3$  lying on the compact edges of the boundary polygon  $\partial\Theta$  of  $\Theta$ . Let  $\Delta'$  be the subdivision of  $\sigma$  determined by the cones  $\langle l_j, l_{j+1} \rangle$  for each  $j=0, \dots, t$ . Then  $\Delta'$  is the *coarsest* nonsingular subdivision of  $\sigma$ , in the sense that whenever  $\Delta$  is a nonsingular subdivision of  $\sigma$ , then for each  $i=0, \dots, t+1$  the vector  $l_i$  is a primitive generator of some ray in  $\Delta^{(1)}$ . Stated otherwise,  $\{l_0, l_1, \dots, l_t, l_{t+1}\}$  is the set of indispensable points for  $\sigma$ . Let us write

$$\text{indisp}(\sigma) = \{l_0, l_1, \dots, l_t, l_{t+1}\} .$$

There is no ambiguity in this terminology: as shown by the following proposition, for every 3-dimensional cone  $\rho = \langle u, v, w \rangle$  having  $\sigma = \langle v, w \rangle$  among its faces,  $\text{indisp}(\sigma)$  coincides with the set of indispensable points for  $\rho$  lying on  $\sigma$ .

1.1. PROPOSITION. *Let  $\sigma = \langle v, w \rangle$  be a 2-dimensional cone in  $\mathbf{Z}^3$ , with  $v$  and  $w$  primitive. Let  $q \in \mathbf{Z}^3 \cap (\sigma + (-\sigma))$  be a primitive vector. Then  $q \in \text{indisp}(\sigma)$  if and only if for some (equivalently, for all) primitive  $u \in \mathbf{Z}^3$  with  $u \notin \sigma + (-\sigma)$ ,  $q$  is indispensable for the cone  $\langle u, v, w \rangle$ .*

PROOF. Suppose there exists a primitive vector  $u \in \mathbf{Z}^3 \setminus (\sigma + (-\sigma))$  together with a nonsingular subdivision  $\Delta$  of  $\langle u, v, w \rangle$  such that  $q$  is not a primitive generator of any ray in  $\Delta^{(1)}$ . Letting  $\Delta'' = \{ \rho \in \Delta \mid \rho \subseteq \langle v, w \rangle \}$ , we have that  $\Delta''$  is a nonsingular subdivision of  $\langle v, w \rangle$ ; since  $\Delta''$  contains the coarsest subdivision of  $\langle v, w \rangle$ , then  $q \notin \text{indisp}(\sigma)$ .

Conversely, assume  $q \notin \text{indisp}(\sigma)$ . Let  $\Delta'$  be the coarsest nonsingular subdivision of  $\sigma$ , and let  $v=l_0, l_1, \dots, l_t, l_{t+1}=w$  be the primitive generators of the rays in  $\Delta'^{(1)}$ , listed in their natural order. Pick an arbitrary primitive vector  $u \in \mathbb{Z}^3 \setminus (\sigma + (-\sigma))$ . Let  $\Delta_1$  be the simplicial fan whose 3-dimensional cones are  $\langle u, l_0, l_1 \rangle, \langle u, l_1, l_2 \rangle, \dots, \langle u, l_t, l_{t+1} \rangle$ . By construction, for each  $j=0, 1, \dots, t$ ,  $\langle l_j, l_{j+1} \rangle$  is a cone in  $(\Delta_1)^{(2)}$ .

Claim. There exists a nonsingular subdivision  $\Delta_2$  of  $\Delta_1$  such that each  $\langle l_j, l_{j+1} \rangle$  is a cone in  $(\Delta_2)^{(2)}$ .

Although the proof is a routine exercise [3, p. 48], we shall give it here in detail to make this paper self-contained. Let  $d$  be the maximum multiplicity of the 3-dimensional cones in  $\Delta_1$ . If  $d=1$  then also every cone in  $(\Delta_1)^{(2)}$  has multiplicity=1, and we have nothing to prove. Otherwise, let  $\tau = \langle w_1, w_2, w_3 \rangle$  be a 3-dimensional cone in  $\Delta_1$  such that  $\text{mult}(\tau) = d$ , where  $w_1, w_2, w_3$  are primitive. Since  $\text{mult}(\tau) = \#(P_\tau \cap \mathbb{Z}^3)$ , the half-open parallelepiped  $P_\tau$  contains a nonzero integral point  $a$ , which we can safely assume to be primitive. Let  $\Delta_{1,a}$  be the smallest subdivision of  $\Delta_1$  containing  $\langle a \rangle$  among its rays. In detail,  $\Delta_{1,a}$  is constructed as follows (compare [6, p. 15]):

Case 1:  $a$  lies in the interior of  $\tau$ , or  $a$  lies on the boundary of  $\Delta_1$ .

Then replace  $\tau$  and its faces by the cones in  $(\tau * a)^{(3)}$ , together with their faces and the ray  $\langle a \rangle$ .

Case 2:  $a$  lies on the boundary of  $\tau$  but not on the boundary of  $\Delta_1$ .

Then let  $\tau' \neq \tau$  be the unique 3-dimensional cone in  $\Delta_1$  such that  $a$  lies on the common 2-dimensional face  $\tau \cap \tau'$ . To obtain  $\Delta_{1,a}$ , replace  $\tau$  and  $\tau'$  (and their faces) by the cones in  $(\tau * a)^{(3)}$  and  $(\tau' * a)^{(3)}$ , together with their faces and the ray  $\langle a \rangle$ .

In either case,  $a$  does not lie in any cone  $\langle l_j, l_{j+1} \rangle$ , because  $a$  is a primitive integral point in  $P_\tau$  and, by construction, the half-open parallelogram  $P_{\langle l_j, l_{j+1} \rangle}$  does not contain any nonzero integral points. Thus, each  $\langle l_j, l_{j+1} \rangle$  is a 2-dimensional cone also in  $\Delta_{1,a}$ .

Write  $a = \lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3$  ( $0 \leq \lambda_k < 1$ ) and assume that  $\rho$  is a 3-dimensional cone in  $\Delta_{1,a}$  having  $a$  among its primitive generators. Then for some  $i=1, 2, 3$  we have  $\text{mult}(\rho) = \lambda_i |\det \langle w_1, w_2, w_3 \rangle| < d$ . Then the number of 3-dimensional cones in  $\Delta_{1,a}$  having multiplicity  $d$  is strictly smaller than the number of 3-dimensional cones in  $\Delta_1$  having multiplicity  $d$ . Proceeding in this way, after a finite number of steps we obtain a subdivision  $\Delta^*$  of  $\Delta_1$  such that the maximum multiplicity  $d^*$  of the 3-dimensional cones in  $\Delta^*$  is strictly smaller than  $d$ , while each  $\langle l_j, l_{j+1} \rangle$  is still a cone in  $(\Delta^*)^{(2)}$ . By induction, after a finite number of steps, we obtain a nonsingular subdivision  $\Delta_2$  of  $\Delta_1$  having the required properties.

Having settled our claim, we note that  $\Delta_2$  is also a nonsingular subdivision of  $\langle u, v, w \rangle$ , and  $q$  is not the primitive generator of any ray in  $(\Delta_2)^{(1)}$ . Thus,  $q$  is not indispensable for  $\langle u, v, w \rangle$ . q.e.d.

The following *additivity* property for indispensable points in 2-dimensional cones is implicit in [5, 1.6]:

1.2. PROPOSITION. Let  $\sigma = \langle v, w \rangle$ , with both  $v$  and  $w$  primitive; assume

$\text{indisp}(\sigma) = \{l_0, l_1, \dots, l_t, l_{t+1}\}$ . Then for each  $j=1, \dots, t$  we have  $\text{indisp}\langle v, w \rangle = \text{indisp}\langle v, l_j \rangle \cup \text{indisp}\langle l_j, w \rangle$ .

**2. A desingularization algorithm.**

2.1. DEFINITION. Let  $\sigma = \langle v_1, v_2, v_3 \rangle$ , with primitive vectors  $v_1, v_2, v_3$ . Let  $p$  be a (possibly, not primitive) vector of  $\sigma \cap \mathbf{Z}^3$ , and write  $p = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$  for uniquely determined rational numbers  $\lambda_i \geq 0$ . Then the *norm*  $\|p\|_\sigma$  of  $p$  in  $\sigma$  is defined by

$$\|p\|_\sigma = |\det(p, v_2, v_3)| + |\det(v_1, p, v_3)| + |\det(v_1, v_2, p)| = (\lambda_1 + \lambda_2 + \lambda_3) \text{mult}(\sigma).$$

Note that if  $p$  is primitive,  $\|p\|_\sigma$  is the sum of the multiplicities of the cones in  $(\sigma * p)^{(3)}$ .

The following are immediate consequences of the definition:

- (a)  $\|0\|_\sigma = 0$ . For each  $i = 1, 2, 3$ ,  $\|v_i\|_\sigma = \text{mult}(\sigma) = |\det(v_1, v_2, v_3)|$ .
- (b) The function  $\| \cdot \|_\sigma : \sigma \cap \mathbf{Z}^3 \rightarrow \mathbf{Q}$  takes integral values  $> 0$  for all nonzero points  $p \in \sigma \cap \mathbf{Z}^3$  and has the following linearity property:

$$\|p + q\|_\sigma = \|p\|_\sigma + \|q\|_\sigma \quad \text{for all } p \text{ and } q.$$

- (c) Let  $A_\sigma$  denote the lattice in  $\mathbf{Z}^3$  generated by  $v_1, v_2, v_3$ . Given  $r$  and  $s$  in  $\sigma \cap \mathbf{Z}^3$ , suppose that  $r - s \in A_\sigma$ . Then by (a) and (b), the integers  $\|r\|_\sigma$  and  $\|s\|_\sigma$  differ by an integral multiple of  $\text{mult}(\sigma)$ , in symbols:

$$\|r\|_\sigma \equiv \|s\|_\sigma \pmod{\text{mult}(\sigma)}.$$

We are interested in those nonzero vectors  $p \in \sigma \cap \mathbf{Z}^3$ , other than the primitive generators of  $\sigma$ , having the smallest possible norm in  $\sigma$ . Since there may be many such points, throughout this paper we fix, once and for all, a total order  $\leq_{\text{lex}}$  over  $\mathbf{Z}^3$ , by the following stipulation: For any two distinct points  $q = (q_1, q_2, q_3)$  and  $r = (r_1, r_2, r_3)$  we write  $q <_{\text{lex}} r$  if and only if

- either  $\text{dist}(0, q) < \text{dist}(0, r)$ ,
- or  $q$  and  $r$  are equidistant from the origin and, letting  $i \in \{1, 2, 3\}$  be the first index such that  $q_i \neq r_i$ , we have  $q_i < r_i$ .

Here,  $\text{dist}(p, q)$  is the Euclidean distance between  $p$  and  $q$ .

2.2. DEFINITION. Given  $\sigma = \langle v_1, v_2, v_3 \rangle$ , with primitive vectors  $v_1, v_2, v_3$ , we denote by  $p_\sigma$  the *first* point (with respect to the total order  $\leq_{\text{lex}}$ ) having the *smallest* norm in  $\sigma$  among all points in the set  $(\sigma \cap \mathbf{Z}^3) \setminus \{0, v_1, v_2, v_3\}$ . We call  $p_\sigma$  the point of minimal norm in  $\sigma$ .

The following are immediate consequences of the definition:

- (a)  $p_\sigma$  is necessarily primitive.
- (b) If  $\text{mult}(\sigma) > 1$  then  $p_\sigma \in P_\sigma$ .
- (c) Assume  $\text{mult}(\sigma) > 1$ , and let  $\rho$  be a cone in  $(\sigma * p_\sigma)^{(3)}$ . Writing  $p_\sigma = \sum_i \lambda_i v_i$ , by (a) the multiplicity of  $\rho$  is given by  $\lambda_j \text{mult}(\sigma)$  for some  $j \in \{1, 2, 3\}$ . Therefore, by (b),

$\text{mult}(\rho) < \text{mult}(\sigma)$ .

We shall never need to consider  $p_\sigma$  in case  $\text{mult}(\sigma) = 1$ .

2.3. DEFINITION. Let  $\sigma = \langle v_1, v_2, v_3 \rangle$ , with primitive vectors  $v_1, v_2, v_3$ . For each  $n = 0, 1, 2, \dots$ , the set  $\nabla_n(\sigma)$  is defined inductively as follows:

$$\nabla_0(\sigma) = \{\sigma\},$$

$$\nabla_{n+1}(\sigma) = \{\tau \in \nabla_n(\sigma) \mid \text{mult}(\tau) = 1\} \cup \{(\tau * p_\tau)^{(3)} \mid \tau \in \nabla_n(\sigma), \text{mult}(\tau) > 1\}.$$

2.4. PROPOSITION. *Adopt the above notation. Then we have:*

(i) *There is an integer  $0 \leq m < \text{mult}(\sigma)$  such that  $\nabla_m(\sigma) = \nabla_{m+1}(\sigma) = \dots$ , while  $\nabla_l(\sigma) \neq \nabla_{l+1}(\sigma)$  for all  $l < m$ .*

(ii)  *$\nabla_m(\sigma)$  is the set of 3-dimensional cones of a finite fan  $\nabla(\sigma)$  which is a nonsingular subdivision of  $\sigma$ . Moreover, the primitive generators of those rays in  $\nabla(\sigma)^{(1)}$  lying on the boundary of  $\sigma$  are exactly the indispensable points of the faces of  $\sigma$ .*

PROOF. By induction on  $d = \text{mult}(\sigma)$ . If  $d = 1$  we have nothing to prove. Assume  $d > 1$ . Assume  $\nabla_t(\sigma)$  has already been obtained, and let  $d_t > 1$  be the maximum multiplicity of the cones in  $\nabla_t(\sigma)$ . Let  $\tau$  be a cone in  $\nabla_t(\sigma)$  of multiplicity  $d_t$ . Let  $p_\tau$  be the point of minimal norm in  $\tau$ . By Remark (c) following Definition 2.2, the multiplicity of each cone in  $(\tau * p_\tau)^{(3)}$  is strictly smaller than  $d_t$ . Thus, whenever  $t$  is such that  $d_t > 1$ , the step leading from  $\nabla_t(\sigma)$  to  $\nabla_{t+1}(\sigma)$  guarantees that the maximum multiplicity  $d_{t+1}$  of the cones in  $\nabla_{t+1}(\sigma)$  is strictly smaller than  $d_t$ . This shows that the process must terminate: more precisely, there is a smallest integer  $m$ , with  $0 \leq m < d$ , such that  $\nabla_m(\sigma) = \nabla_{m+1}(\sigma)$ ;  $m$  is the smallest integer such that the maximum multiplicity  $d_m$  equals 1. This completes the proof of (i).

In order to prove (ii), focusing attention on the initial step from  $\nabla_0(\sigma)$  to  $\nabla_1(\sigma)$ , we argue by cases:

Case 1:  $p_\sigma \in \text{int } P_\sigma$ .

Writing, as above,  $(\sigma * p_\sigma)^{(3)} = \{\sigma_1, \sigma_2, \sigma_3\}$ , we again have  $\text{mult}(\sigma_j) < d$  for each  $j = 1, 2, 3$ . By the induction hypothesis, we have a nonsingular subdivision  $\nabla(\sigma_j)$  of  $\sigma_j$ , such that the primitive generators of the rays in  $\nabla(\sigma_j)^{(1)}$  lying on the boundary of  $\sigma_j$  are exactly the indispensable points of the faces of  $\sigma_j$ . It follows that  $\nabla(\sigma) = \nabla(\sigma_1) \cup \nabla(\sigma_2) \cup \nabla(\sigma_3)$  is a nonsingular fan. Further, the primitive generators of the rays in  $\nabla(\sigma)^{(1)}$  lying on the boundary of  $\sigma$  coincide with the indispensable points of those faces of  $\sigma_j$  which are also faces of  $\sigma$ , i.e., those faces of  $\sigma_j$  not containing  $p_\sigma$ . The proof of Case 1 is complete.

Case 2:  $p_\sigma \notin \text{int } P_\sigma$ .

Let us write  $\sigma = \langle v_1, v_2, v_3 \rangle$  for primitive vectors  $v_1, v_2, v_3$ . Arguing as in (i) above, and writing without loss of generality  $p_\sigma \in \langle v_1, v_2 \rangle$  with  $p_\sigma \neq v_1$  and  $p_\sigma \neq v_2$ , it follows that the primitive vector  $p_\sigma$  lies in the half-open parallelogram  $P_{\langle v_1, v_2 \rangle} = \{\lambda_1 v_1 + \lambda_2 v_2 \mid 0 \leq \lambda_i < 1, i = 1, 2\}$ . Thus, by the definition of multiplicity,  $\text{mult}\langle v_1, v_2 \rangle > 1$ . Let  $\sigma_1 = \langle p_\sigma, v_1, v_3 \rangle$  and  $\sigma_2 = \langle p_\sigma, v_2, v_3 \rangle$ . Again recalling Remark (c) after Definition 2.2, we

can write  $\text{mult}(\sigma_1) < d$  and  $\text{mult}(\sigma_2) < d$ . By the induction hypothesis, for each  $i = 1, 2$ , we have that  $\nabla(\sigma_i)$  is a nonsingular fan, and the primitive generators of the rays in  $\nabla(\sigma_i)^{(1)}$  lying on the boundary of  $\sigma_i$  coincide with the indispensable points of the faces of  $\sigma_i$ . Therefore,  $\nabla(\sigma) = \nabla(\sigma_1) \cup \nabla(\sigma_2)$  is a nonsingular fan such that the primitive generators of the rays in  $\nabla(\sigma)^{(1)}$  lying on either face  $\langle v_1, v_3 \rangle$  or  $\langle v_2, v_3 \rangle$  coincide with the indispensable points of these faces. There remains to be proved that the primitive generators of the rays in  $\nabla(\sigma)^{(1)}$  lying on the face  $\langle v_1, v_2 \rangle$  coincide with the indispensable points of  $\langle v_1, v_2 \rangle$ . In the light of Proposition 1.2, it suffices to settle the following:

Claim. The point  $p_\sigma$  is indispensable for  $\langle v_1, v_2 \rangle$ .

For otherwise (absurdum hypothesis),  $p_\sigma$  would not be among the primitive vectors  $v_1 = l_0, l_1, \dots, l_t, l_{t+1} = v_2$  of the coarsest nonsingular subdivision of  $\langle v_1, v_2 \rangle$ . Trivially,  $t \geq 1$  because, as we have seen,  $p_\sigma$  lies in the half-open parallelogram  $P_{\langle v_1, v_2 \rangle}$ . There is a uniquely determined  $k \in \{0, \dots, t\}$  such that  $p_\sigma$  lies in the cone  $\langle l_k, l_{k+1} \rangle$ . By construction,  $\text{mult}\langle l_k, l_{k+1} \rangle = 1$ , or equivalently,  $\{l_k, l_{k+1}\}$  is a part of a basis of  $\mathbf{Z}^3$ . Since 0 is the only integral point of the half-open parallelogram  $P_{\langle l_k, l_{k+1} \rangle}$ , it follows that  $p_\sigma = \lambda_1 l_k + \lambda_2 l_{k+1}$  for suitable integers  $\lambda_1, \lambda_2 \geq 1$ . The linearity property of the function  $\|\cdot\|_\sigma$  (Remark (b) after Definition 2.1) implies that  $\|p_\sigma\|_\sigma = \lambda_1 \|l_k\|_\sigma + \lambda_2 \|l_{k+1}\|_\sigma$ . Since the 2-dimensional cone  $\langle l_k, l_{k+1} \rangle$  is strictly contained in  $\langle v_1, v_2 \rangle$ , either  $l_k$  or  $l_{k+1}$  belongs to the set  $(\sigma \cap \mathbf{Z}^3) \setminus \{0, v_1, v_2, v_3\}$ , and its norm in  $\sigma$  is strictly smaller than  $\|p_\sigma\|_\sigma$ . This contradicts our assumptions about  $p_\sigma$ .

Having proved our claim, we have also completed the proof of the proposition.

q.e.d.

2.5. **REMARK** (to be continued in 4.3). In case  $\tau$  is a 2-dimensional cone, one can similarly define a nonsingular subdivision  $\nabla(\tau)$  of  $\tau$  by choosing points of minimal norm. In this way we get that the set of generating vectors of the rays in  $\nabla(\tau)^{(1)}$  coincides with  $\text{indisp}(\tau)$ : as a matter of fact, arguing as in the above claim, we see that each point of minimal norm is indispensable; now apply the additivity property of Proposition 1.2. Thus, our algorithm yields a 3-dimensional generalization of the traditional construction [5, 1.6] of the coarsest nonsingular subdivision of a 2-dimensional cone.

Suppose now we are given a finite fan  $\Delta$  of rational simplicial cones in  $\mathbf{R}^3$  whose support is the whole space  $\mathbf{R}^3$ . By Proposition 2.4, for any two 3-dimensional cones  $\sigma$  and  $\sigma'$  of  $\Delta$  having a common 2-dimensional face  $\sigma \cap \sigma'$ , the set  $G$  of primitive generators of the cones in  $\nabla(\sigma)$  lying on  $\sigma \cap \sigma'$  only depends on  $\sigma \cap \sigma'$ : specifically,  $G$  coincides with the set of indispensable points of  $\sigma \cap \sigma'$ ; therefore,  $G$  is also equal to the set of primitive generators of the cones in  $\nabla(\sigma')$  lying on  $\sigma \cap \sigma'$ . This key observation, together with Proposition 2.4, shows that by patching together the  $\nabla(\sigma)$ 's we obtain a nonsingular subdivision of the fan  $\Delta$ . This establishes the following result:

2.6. **THEOREM.** *Let  $\Delta$  be a finite fan of rational simplicial cones in  $\mathbf{R}^3$  as above. Let*

$\nabla(\Delta) = \bigcup \{ \nabla(\sigma) \mid \sigma \in \Delta^{(3)} \}$ . Then  $\nabla(\Delta)$  is a fan yielding a nonsingular subdivision of  $\Delta$ .

**3. Upper bounds for nonsingular subdivisions.** Throughout this section,  $\sigma = \langle v_1, v_2, v_3 \rangle$  will denote a 3-dimensional cone generated by primitive vectors  $v_1, v_2, v_3$ .

3.1. LEMMA. *Let  $\sigma = \langle v_1, v_2, v_3 \rangle$  and assume that  $d = \text{mult}(\sigma) > 1$ . Then there is a point  $p \in (P_\sigma \cap \mathbf{Z}^3) \setminus \{0\}$  such that  $\|p\|_\sigma \leq d + 1$ . If in addition  $(P_\sigma \cap \mathbf{Z}^3) \setminus \{0\}$  has non-empty intersection with the boundary of  $\sigma$ , then  $p$  can be so chosen that  $\|p\|_\sigma \leq d$ .*

PROOF. Case 1:  $(P_\sigma \cap \mathbf{Z}^3) \setminus \{0\} \subseteq \text{int } P_\sigma$ .

Then let  $\pi = \langle v_1, v_2 \rangle + (-\langle v_1, v_2 \rangle)$  be the plane in  $\mathbf{R}^3$  spanned by  $v_1$  and  $v_2$ . Our standing assumption implies that  $\pi$  contains exactly one integral point of  $P_\sigma$ , namely 0. Therefore, all integral points of  $\pi$  are obtainable as linear combinations of  $v_1$  and  $v_2$  with integral coefficients; stated otherwise,  $\text{mult}\langle v_1, v_2 \rangle = 1$ . By hypothesis, the half-open parallelepiped  $P_\sigma$  contains exactly  $d$  integral points. Let  $r$  be a nonzero integral point of  $P_\sigma$ . Let  $\pi(r)$  be the plane through  $r$  and parallel to  $\pi$ . Then  $r$  must be the only integral point of  $P_\sigma \cap \pi(r)$ ; for otherwise, if  $r' \neq r$  were another such point, then  $r' - r$  would be an integral point of the plane  $\pi = \pi(0)$  not belonging to the lattice generated by  $v_1$  and  $v_2$ , thus contradicting  $\text{mult}\langle v_1, v_2 \rangle = 1$ .

Following now [1, VII.2.4], and denoting by  $A_\sigma$  be the lattice in  $\mathbf{Z}^3$  generated by  $v_1, v_2, v_3$ , we observe that every point  $x \in \mathbf{R}^3$  can be written uniquely in the form  $x = \xi_1 v_1 + \xi_2 v_2 + \xi_3 v_3$  for some real numbers  $\xi_1, \xi_2, \xi_3$ ; and  $x \in A_\sigma$  if and only if  $\xi_1, \xi_2, \xi_3$  are integers. Hence, for every  $x \in \mathbf{R}^3$  there is a unique  $t \in A_\sigma$  such that  $x^\diamond = x - t$  is a point of  $P_\sigma$ ; the function  $\diamond : x \rightarrow x^\diamond$  maps  $\mathbf{Z}^3$  onto  $P_\sigma \cap \mathbf{Z}^3$ . Two points have the same image if and only if they differ by a lattice point of  $A_\sigma$ . Let  $g$  be the only nonzero integral point of  $P_\sigma$  lying at the smallest possible (Euclidean) distance  $\delta$  from  $\pi = \pi(0)$ . Uniqueness of  $g$ , as well as the fact that  $\delta > 0$ , are guaranteed by our initial discussion.

Claim.  $\{0^\diamond, g^\diamond, (2g)^\diamond, \dots, ((d-1)g)^\diamond\} = P_\sigma \cap \mathbf{Z}^3$ .

It is sufficient to show that for each  $j = 1, \dots, d-1$ , the plane  $\pi((jg)^\diamond)$  is different from  $\pi(0)$ . We already know that  $\pi(g^\diamond) = \pi(g) \neq \pi(0)$ . By way of contradiction, let  $j$  be the smallest integer such that  $\pi((jg)^\diamond) = \pi(0)$  and  $j \in \{2, \dots, d-1\}$ . Since for each  $r \in P_\sigma \cap \mathbf{Z}^3$  the plane  $\pi(r)$  contains precisely one point of  $P_\sigma \cap \mathbf{Z}^3$ , it follows that not all  $d$  points of  $P_\sigma \cap \mathbf{Z}^3$  belong to the set  $\pi(0) \cup \pi(g^\diamond) \cup \dots \cup \pi(((j-1)g)^\diamond)$ . So, for a suitable  $i = 0, \dots, j-2$ , there exists  $s \in P_\sigma \cap \mathbf{Z}^3$  lying in the open region in  $\mathbf{R}^3$  between the two planes  $\pi((ig)^\diamond)$  and  $\pi(((i+1)g)^\diamond)$ . Let  $\delta^*$  be the distance between  $\pi(0)$  and  $(s-ig)^\diamond \in P_\sigma \cap \mathbf{Z}^3$ . Then  $0 < \delta^* < \delta$ , which contradicts the assumed minimality of  $\delta$ . The claim is settled.

Let  $n = 0, 1, \dots, d-1$ . By the linearity property of the function  $\| \cdot \|_\sigma$  (Remark (b) after Definition 2.1) we have  $n\|g\|_\sigma = \|ng\|_\sigma$  whence, by Remark (c).

$$(*) \quad \|(ng)^\diamond\|_\sigma \equiv n\|g\|_\sigma \pmod{d}.$$

Let us now denote  $a = \|g\|_\sigma$ .

Subcase 1.1:  $\gcd(a, d) = 1$ .

Then pick an integer  $n' \in \{1, \dots, d-1\}$  satisfying the congruence  $n'a \equiv 1 \pmod{d}$ . By  $(*)$ , we must either have  $\|(n'g)^\diamond\|_\sigma = d+1$ , or  $\|(n'g)^\diamond\|_\sigma = 2d+1$  (the other cases being manifestly impossible). Note that the integral point  $v_1 + v_2 + v_3 - (n'g)^\diamond$  lies in  $\text{int } P_\sigma$ . Again by linearity,  $\|v_1 + v_2 + v_3 - (n'g)^\diamond\|_\sigma = \|v_1 + v_2 + v_3\|_\sigma - \|(n'g)^\diamond\|_\sigma = 3d - \|(n'g)^\diamond\|_\sigma$ , whence either  $(n'g)^\diamond$  or  $(v_1 + v_2 + v_3 - (n'g)^\diamond)$  has the desired properties.

Subcase 1.2:  $\gcd(a, d) = m > 1$ .

Then for suitable integers  $h, k \geq 1$  we have  $a = hm$  and  $d = km$ . By  $(*)$ , we obtain  $\|(kg)^\diamond\|_\sigma \equiv 0 \pmod{km}$ . Since, by our claim,  $(kg)^\diamond \neq 0$ , it follows that either  $\|(kg)^\diamond\|_\sigma = d$ , or  $\|(kg)^\diamond\|_\sigma = 2d$ . Noting that the integral point  $v_1 + v_2 + v_3 - (kg)^\diamond$  lies in  $\text{int } P_\sigma$ , we conclude that either  $(kg)^\diamond$  or  $(v_1 + v_2 + v_3 - (kg)^\diamond)$  has the desired properties.

Case 2: Some point  $q \in (P_\sigma \cap \mathbb{Z}^3) \setminus \{0\}$  lies on the boundary of  $P_\sigma$ .

Without loss of generality, we may assume that  $q$  is primitive and belongs to the half-open parallelogram  $P_{\langle v_1, v_2 \rangle}$ . Taking, if necessary, symmetric points with respect to the center  $(v_1 + v_2)/2$ , and recalling that  $v_1$  and  $v_2$  are primitive, we may safely write  $q = \lambda v_1 + \mu v_2$  for suitable  $\lambda, \mu > 0$  with  $\lambda + \mu \leq 1$ . Letting now  $\sigma_1$  and  $\sigma_2$  denote the 3-dimensional cones obtained by starring  $\sigma$  at  $q$ , we conclude that  $\|q\|_\sigma = \text{mult}(\sigma_1) + \text{mult}(\sigma_2) = (\lambda + \mu)d \leq d$ . q.e.d.

The reader will recall that  $\nabla(\sigma)^{(k)}$  denotes the set of  $k$ -dimensional cones in  $\nabla(\sigma)$ , and that for any set  $S$  we write  $\#S$  to denote the cardinality of  $S$ . We also write

$$\text{gen } \nabla(\sigma)^{(1)}$$

to denote the set of primitive generators of the rays in  $\nabla(\sigma)^{(1)}$ .

**3.2. PROPOSITION.** *Let  $\sigma = \langle v_1, v_2, v_3 \rangle$  be a cone with  $\text{mult}(\sigma) = d$ . Then we have the inequalities  $\#\nabla(\sigma)^{(3)} \leq 2d - 1$  and  $\#(\text{gen } \nabla(\sigma)^{(1)} \setminus \{v_1, v_2, v_3\}) \leq d - 1$ .*

**PROOF.** By induction on  $d$ . The case  $d = 1$  is trivial. Assume  $d > 1$ . Let  $p_\sigma$  be as given by Definition 2.2. Note that  $p_\sigma$  is primitive (Remark (a) following Definition 2.2). We argue by cases:

Case 1:  $p_\sigma \in \text{int } P_\sigma$ .

Then letting  $(\sigma * p_\sigma)^{(3)} = \{\sigma_1, \sigma_2, \sigma_3\}$ , by Definition 2.1 together with Lemma 3.1 we have  $\text{mult}(\sigma_1) + \text{mult}(\sigma_2) + \text{mult}(\sigma_3) = \|p_\sigma\|_\sigma \leq d + 1$ ; by Remark (c) following 2.2, we have the inequality  $\text{mult}(\sigma_i) < d$ . By induction  $\#\nabla(\sigma)^{(3)} = \sum_i \#\nabla(\sigma_i)^{(3)} \leq -3 + 2 \sum_i \text{mult}(\sigma_i) = -3 + 2\|p_\sigma\|_\sigma \leq 2d - 1$ .

Similarly, from the identity  $\text{gen } \nabla(\sigma)^{(1)} \setminus \{v_1, v_2, v_3\} = (\text{gen } \nabla(\sigma_1)^{(1)} \setminus \{v_1, p_\sigma, v_3\}) \cup (\text{gen } \nabla(\sigma_2)^{(1)} \setminus \{p_\sigma, v_2, v_3\}) \cup (\text{gen } \nabla(\sigma_3)^{(1)} \setminus \{v_1, v_2, p_\sigma\}) \cup \{p_\sigma\}$ , by induction we get  $\#(\text{gen } \nabla(\sigma)^{(1)} \setminus \{v_1, v_2, v_3\}) \leq \text{mult}(\sigma_1) - 1 + \text{mult}(\sigma_2) - 1 + \text{mult}(\sigma_3) - 1 + 1 = \|p_\sigma\|_\sigma - 2 \leq d - 1$ .

Case 2:  $p_\sigma$  lies on the boundary of  $P_\sigma$ .

Then assume without loss of generality  $p_\sigma \in \{\lambda_1 v_1 + \lambda_2 v_2 \mid 0 \leq \lambda_i < 1\}$ . Let  $(\sigma * p_\sigma)^{(3)} = \{\sigma_1, \sigma_2\}$ . From the proof of Case 2 in Lemma 3.1 it follows that  $\text{mult}(\sigma_1) + \text{mult}(\sigma_2) = \|p_\sigma\|_\sigma \leq d$ . Since  $\text{mult}(\sigma_i) < d$ , by induction we get  $\#\nabla(\sigma)^{(3)} = \#\nabla(\sigma_1)^{(3)} + \#\nabla(\sigma_2)^{(3)} \leq 2 \text{mult}(\sigma_1) - 1 + 2 \text{mult}(\sigma_2) - 1 = 2\|p_\sigma\|_\sigma - 2 < 2d - 1$ .

Similarly, using the identity  $\text{gen } \nabla(\sigma)^{(1)} \setminus \{v_1, v_2, v_3\} = (\text{gen } \nabla(\sigma_1)^{(1)} \setminus \{v_1, p_\sigma, v_3\}) \cup (\text{gen } \nabla(\sigma_2)^{(1)} \setminus \{p_\sigma, v_2, v_3\}) \cup \{p_\sigma\}$  by induction we get  $\#(\text{gen } \nabla(\sigma)^{(1)} \setminus \{v_1, v_2, v_3\}) \leq \|p_\sigma\|_\sigma - 1 \leq d - 1$ . q.e.d.

For later use, we shall now give a characterization of indispensable points. For this purpose, we first observe that any 3-dimensional cone  $\sigma$  induces a partial order relation  $\leq_\sigma$  in  $\mathbb{R}^3$  by the usual stipulation:

$$x \leq_\sigma y \quad \text{if and only if} \quad y - x \in \sigma.$$

For each  $z \in \mathbb{R}^3$  we let  $[0, z]_\sigma = \{x \in \mathbb{R}^3 \mid 0 \leq_\sigma x \leq_\sigma z\}$ .

**3.3. PROPOSITION.** *Let  $p \in (\sigma \cap \mathbb{Z}^3) \setminus \{0\}$ . Then  $p$  is indispensable for  $\sigma$  if and only if  $p$  is  $\leq_\sigma$ -minimal among all nonzero integral points of  $\sigma$ , in symbols,  $[0, p]_\sigma \cap \mathbb{Z}^3 = \{0, p\}$ .*

**PROOF.** To avoid trivialities, throughout this proof we assume that  $p$  is primitive.

First assume that  $p$  is not indispensable for  $\sigma$ . Let  $\Delta$  be a nonsingular subdivision of  $\sigma$  such that  $p$  is not a primitive generator of any ray in  $\Delta^{(1)}$ . Let  $\tau$  be a 3-dimensional cone of  $\Delta$  such that  $p \in \tau$ . The nonsingularity of  $\tau$  implies that  $p \notin P_\tau$ , and we must have  $v \leq_\tau p$  for some primitive generator  $v$  of  $\tau$ ; a fortiori, we have  $v \leq_\sigma p$ . By our hypotheses about  $\Delta$ , we have that  $v \neq p$  and  $v \neq 0$ , which shows that  $p$  is not a  $\leq_\sigma$ -minimal point in  $\sigma$ .

Conversely, assume that  $p$  is not  $\leq_\sigma$ -minimal in  $\sigma$ , and let  $q' \in ([0, p]_\sigma \cap \mathbb{Z}^3) \setminus \{0, p\}$ . Let  $r' = p - q'$ , and note that  $r' \in ([0, p]_\sigma \cap \mathbb{Z}^3) \setminus \{0, p\}$ . Let  $q$  and  $r$  be the primitive vectors in  $\sigma \cap \mathbb{Z}^3$  such that  $q' = nq$  and  $r' = mr$  for suitable integers  $m, n > 0$ . Note that  $q \neq r$ , for otherwise  $p$  would not be primitive. Let  $\Delta$  be the coarsest nonsingular subdivision of the 2-dimensional cone  $\rho = \langle q, r \rangle$  as defined in [5, p. 24]. Let us display the primitive generators of the rays of  $\Delta^{(1)}$  in their natural order as follows:  $q = l_0, l_1, \dots, l_t, l_{t+1} = r$ . For each  $j = 0, \dots, t + 1$ , the point  $p$  is different from  $l_j$ : as a matter of fact,  $p$  does not belong to the half-open parallelogram  $P_\rho = \{\lambda q + \mu r \mid 0 \leq \lambda < 1, 0 \leq \mu < 1\}$  while, by construction, each  $l_i$  ( $i = 1, \dots, t$ ) does belong to  $P_\rho$ ; moreover, by assumption,  $p$  is different from both  $q$  and  $r$ . Let  $\sigma = \langle v_1, v_2, v_3 \rangle$ , with primitive vectors  $v_1, v_2, v_3$ . Since  $p$  is not  $\leq_\sigma$ -minimal in  $\sigma$ ,  $p$  is necessarily distinct from  $v_1, v_2, v_3$ .

**Claim.** There exists a simplicial fan  $\Delta_1$  which is a subdivision of  $\sigma$  and such that for each  $i = 0, \dots, t$ , the cone  $\langle l_i, l_{i+1} \rangle$  is in  $(\Delta_1)^{(2)}$ .

The proof is a routine exercise [6, Chapter 2]. However, to make this paper self-contained we shall give an explicit construction of such  $\Delta_1$ . We argue by cases:

Case 1: Both  $q = l_0$  and  $r = l_{t+1}$  lie in the interior of  $\sigma$ .

Let  $\pi = \langle q, r \rangle + (-\langle q, r \rangle)$  be the plane in  $\mathbf{R}^3$  spanned by  $q$  and  $r$ . Letting  $\pi^+$  and  $\pi^-$  be the two open half-spaces in  $\mathbf{R}^3$  determined by  $\pi$ , it is no loss of generality to assume that  $v_1 \in \pi^+$  and  $v_2 \in \pi^-$ . The intersection of the cone  $\langle v_1, v_2 \rangle$  with  $\pi$  is a ray  $\langle w \rangle$ , with  $w \in \mathbf{Z}^3$  the primitive generating vector of  $\langle w \rangle$ . It is no loss of generality to assume that  $l_{t+1}$  belongs to the cone  $\langle w, l_0 \rangle$  (otherwise,  $l_0$  belongs to  $\langle w, l_{t+1} \rangle$  and the proof is similar). Then  $\{\langle v_3, l_0, v_1 \rangle, \langle v_3, l_0, v_2 \rangle, \langle v_1, l_0, l_1 \rangle, \langle v_2, l_0, l_1 \rangle, \dots, \langle v_1, l_t, l_{t+1} \rangle, \langle v_2, l_t, l_{t+1} \rangle, \langle v_1, v_2, l_{t+1} \rangle\}$  is the set of 3-dimensional cones of a fan  $\Delta_1$  with the required properties.

Case 2:  $l_0$  lies in the interior of  $\sigma$ , and  $l_{t+1}$  lies on the boundary.

Subcase 2.1:  $l_{t+1}$  lies in the relative interior of a face of  $\sigma$ .

We can assume that  $l_{t+1}$  lies in the relative interior of  $\langle v_1, v_2 \rangle$ . Then  $\{\langle v_3, l_0, v_1 \rangle, \langle v_3, l_0, v_2 \rangle, \langle v_1, l_0, l_1 \rangle, \langle v_2, l_0, l_1 \rangle, \dots, \langle v_1, l_t, l_{t+1} \rangle, \langle v_2, l_t, l_{t+1} \rangle\}$  is the set of 3-dimensional cones of a fan  $\Delta_1$  with the required properties.

Subcase 2.2:  $l_{t+1}$  coincides with some vertex of  $\sigma$ .

We can assume  $l_{t+1} = v_1$ . Then  $\{\langle v_3, l_0, v_2 \rangle, \langle v_3, l_0, l_1 \rangle, \langle v_2, l_0, l_1 \rangle, \dots, \langle v_3, l_t, l_{t+1} \rangle, \langle v_2, l_t, l_{t+1} \rangle\}$  is the set of 3-dimensional cones of a fan  $\Delta_1$  with the required properties.

Case 3: Both  $l_{t+1}$  and  $l_0$  lie on the boundary of  $\sigma$ .

This case is handled similarly to the previous cases.

Having settled our claim, the same construction as the one given in the proof of the claim in Proposition 1.1 yields a *nonsingular* subdivision  $\Delta_2$  of  $\Delta_1$  such that each  $\langle l_i, l_{i+1} \rangle$  is still a 2-dimensional cone of  $\Delta_2$ . Since the primitive generators of the rays of  $(\Delta_2)^{(1)}$  lying on  $\langle l_i, l_{i+1} \rangle$  coincide with the two indispensable points  $l_i$  and  $l_{i+1}$ , by the above discussion it follows that  $p$  is not a primitive generator of any ray in  $(\Delta_2)^{(1)}$ . Since  $\Delta_2$  is a nonsingular subdivision of  $\sigma$ , we conclude that  $p$  is not indispensable for  $\sigma$ . q.e.d.

**4. Tightness of the upper bounds.** Given any rational, simplicial, 3-dimensional cone  $\sigma$ , recall from 2.1 and 2.2 the definition of  $\| \cdot \|_\sigma$  and of  $p_\sigma$ .

4.1. LEMMA. *Let  $\sigma = \langle v_1, v_2, v_3 \rangle$  be a cone with  $d = \text{mult}(\sigma) > 1$ , and with primitive vectors  $v_1, v_2, v_3$ . Let  $I_\sigma$  denote the set of indispensable points for  $\sigma$ , other than  $v_1, v_2, v_3$ . Assume that  $\|p_\sigma\|_\sigma = d + 1$ . Then  $I_\sigma = (P_\sigma \cap \mathbf{Z}^3) \setminus \{0\} = \text{gen } \nabla(\sigma)^{(1)} \setminus \{v_1, v_2, v_3\} \subseteq \text{int } P_\sigma$ .*

PROOF. If, by absurdum hypothesis, there exists a (possibly, not primitive) point in  $(P_\sigma \cap \mathbf{Z}^3) \setminus \{0\}$  lying on the boundary of  $P_\sigma$ , then by Lemma 3.1 it would follow that  $\|p_\sigma\|_\sigma \leq d$ , which contradicts our hypothesis. Thus,  $(P_\sigma \cap \mathbf{Z}^3) \setminus \{0\} \subseteq \text{int } P_\sigma$ .

Claim.  $(P_\sigma \cap \mathbf{Z}^3) \setminus \{0\} \subseteq I_\sigma$ .

By way of contradiction, assume  $p \in (P_\sigma \cap \mathbf{Z}^3) \setminus \{0\}$ , but  $p$  is not indispensable for  $\sigma$ . Then by Proposition 3.3, there exists a point  $q' \in ([0, p]_\sigma \cap \mathbf{Z}^3) \setminus \{0, p\}$ . Letting  $q'' = p - q'$ , we also have  $q'' \in ([0, p]_\sigma \cap \mathbf{Z}^3) \setminus \{0, p\}$ . By hypothesis, for each  $q \in (P_\sigma \cap \mathbf{Z}^3) \setminus \{0\}$  we have  $d + 1 \leq \|q\|_\sigma$ . Since by our initial discussion  $v_1 + v_2 + v_3 - q \in \mathbf{Z}^3 \cap \text{int } P_\sigma$ , by Remarks (a) and (b) after 2.1 we also have  $d + 1 \leq \|v_1 + v_2 + v_3 - q\|_\sigma =$

$\|v_1 + v_2 + v_3\|_\sigma - \|q\|_\sigma = 3d - \|q\|_\sigma$ , whence  $\|q\|_\sigma \leq 2d - 1$ . It follows that  $\|p\|_\sigma \leq 2d - 1$  and  $\|q'\|_\sigma \geq d + 1$ , and hence  $\|q''\|_\sigma = \|p\|_\sigma - \|q'\|_\sigma \leq 2d - 1 - (d + 1) = d - 2$ , which is impossible. Our claim is settled.

Since by the definition of multiplicity, the half-open parallelepiped  $P_\sigma$  has precisely  $d - 1$  nonzero integral points, it follows that  $I_\sigma$  has at least  $d - 1$  elements. On the other hand, since by Proposition 2.4  $\nabla(\sigma)$  is a nonsingular subdivision of  $\sigma$ , we obtain  $I_\sigma \subseteq \text{gen } \nabla(\sigma)^{(1)} \setminus \{v_1, v_2, v_3\}$ . By Proposition 3.2, the latter set has at most  $d - 1$  elements, whence it has exactly  $d - 1$  elements. In conclusion, the equinumerous sets  $(P_\sigma \cap \mathbf{Z}^3) \setminus \{0\} \subseteq I_\sigma \subseteq \text{gen } \nabla(\sigma)^{(1)} \setminus \{v_1, v_2, v_3\}$  coincide, as required to complete the proof. q.e.d.

4.2. PROPOSITION. *For each integer  $d \geq 1$ , the cone  $\sigma = \langle v_1, v_2, v_3 \rangle$  with  $v_1 = (0, 0, 1)$ ,  $v_2 = (0, 1, 1)$  and  $v_3 = (d, d - 1, d)$ , has multiplicity equal to  $d$ , and every nonsingular subdivision  $\Delta$  of  $\sigma$  satisfies the inequalities*

- (i)  $\#\Delta(\sigma)^{(3)} \geq 2d - 1$  and
- (ii)  $\#(\text{gen } \Delta(\sigma)^{(1)} \setminus \{v_1, v_2, v_3\}) \geq d - 1$ .

PROOF. To avoid trivialities assume  $d > 1$ . Direct inspection shows that  $(P_\sigma \cap \mathbf{Z}^3) \setminus \{0\}$  coincides with the set  $\{(1, 1, 2), (2, 2, 3), \dots, (d - 1, d - 1, d)\}$ , and for each  $q \in (P_\sigma \cap \mathbf{Z}^3) \setminus \{0\}$ , a straightforward computation yields  $\|q\|_\sigma \geq d + 1$ . By Lemma 4.1, all  $d - 1$  points in  $(P_\sigma \cap \mathbf{Z}^3) \setminus \{0\}$  are indispensable for  $\sigma$ , and  $\#(\text{gen } \nabla(\sigma)^{(1)} \setminus \{v_1, v_2, v_3\}) = d - 1$ . Further, for every nonsingular subdivision  $\Delta$  of  $\sigma$  we must have  $\Delta^{(1)} \supseteq \nabla(\sigma)^{(1)}$ , whence (ii) immediately follows.

With reference to the proof of Proposition 2.4 (i), for each  $t = 0, 1, \dots, d - 2$ , the step from  $\nabla_t(\sigma)$  to  $\nabla_{t+1}(\sigma)$  amounts to starring the cone  $\tau_t = \langle (t, t, t + 1), (0, 1, 1), (d, d - 1, d) \rangle$  at the point  $(t + 1, t + 1, t + 2)$ . Thus from  $\tau_t$  we obtain three 3-dimensional cones, two of which have multiplicity = 1. Since the number of 3-dimensional cones is increased by two, we conclude that  $\#\nabla(\sigma)^{(3)} = 2d - 1$ . A simple counting argument now shows that for every nonsingular subdivision  $\Delta$  of  $\sigma$ ,  $\#\Delta^{(3)} \geq 2d - 1$ , which settles (i).

4.3. REMARK (continuation of 2.5). Let  $\sigma = \langle v_1, v_2, v_3 \rangle$  be the same as in Proposition 4.2, with  $d > 1$ . Let  $\Theta$  be the convex hull in  $\mathbf{R}^3$  of the set  $(\sigma \cap \mathbf{Z}^3) \setminus \{0\}$ . Let  $X$  be the set of integral points lying on the compact faces of the boundary polyhedron  $\partial\Theta$  of  $\Theta$ . Then each  $q \in X$  belongs to the closed tetrahedron with vertices  $0, v_1, v_2$  and  $v_3$ . It follows that  $\|q\|_\sigma \leq d = \|v_i\|_\sigma$  ( $i = 1, 2, 3$ ). Actually,  $X$  coincides with the set  $\{v_1, v_2, v_3\}$ , since each  $p \in (P_\sigma \cap \mathbf{Z}^3) \setminus \{0\}$  satisfies the inequality  $\|p\|_\sigma \geq d + 1$ . Thus, in contrast to the 2-dimensional case,  $X$  gives no information on how to construct (coarse) nonsingular subdivisions of  $\sigma$ ; compare with the analogous remarks in [5, p. 34]. On the other hand, for every nonsingular subdivision  $\Delta$  of  $\sigma$ , each  $r \in \{(1, 1, 2), (2, 2, 3), \dots, (d - 1, d - 1, d)\} = \text{gen } \nabla(\sigma)^{(1)} \setminus \{v_1, v_2, v_3\}$  must be a primitive generator of some ray in  $\Delta^{(1)}$ , because  $r$  is indispensable for  $\sigma$ . This shows that the nonsingular subdivision  $\nabla(\sigma)$  given by Proposition 2.4 is a *nontrivial* generalization of the construction of the coarsest

nonsingular subdivision of a 2-dimensional cone.

The rest of this section is devoted to the other extreme case, where all integral points of  $P_\sigma$  lie on the two-dimensional faces of  $\sigma$ . Given  $\sigma = \langle v_1, v_2, v_3 \rangle$ , with primitive  $v_1, v_2, v_3$ , for any two distinct indices,  $i, j \in \{1, 2, 3\}$  we denote by  $P_{ij}$  the half-open parallelogram

$$P_{ij} = P_{\langle v_i, v_j \rangle} = \{ \lambda v_i + \mu v_j \mid 0 \leq \lambda < 1, 0 \leq \mu < 1 \} .$$

We also let

$$d_{ij} = \text{mult} \langle v_i, v_j \rangle = \#(P_{ij} \cap \mathbf{Z}^3) .$$

It is not hard to see [4, p. 35] that each  $d_{ij}$  is a divisor of  $\text{mult}(\sigma)$ . Actually, the following stronger result holds:

4.4. PROPOSITION. *Let  $\sigma = \langle v_1, v_2, v_3 \rangle$  with primitive vectors  $v_1, v_2, v_3$ . Let  $i, j, k \in \{1, 2, 3\}$  be pairwise distinct. Then the product  $d_{ij}d_{ik}$  is a divisor of  $\text{mult}(\sigma)$ .*

PROOF. Let  $d = \text{mult}(\sigma)$ . To avoid trivialities, assume  $d > 1$ . Without loss of generality we may assume  $d_{ij} > 1$ . It is easy to see that the  $d_{ij}$  integral points of  $P_{ij}$  lie on  $d_{ij}$  equidistant (with respect to Euclidean distance) parallel lines  $\Lambda(0), \dots, \Lambda(d_{ij} - 1)$ , where  $\Lambda(0)$  is the line spanned by the vector  $v_i$ , and for each  $t = 0, \dots, d_{ij} - 1$ ,  $\Lambda(t)$  is the line parallel to  $\Lambda(0)$  and passing through point  $(t/d_{ij})v_j$ . Let  $p$  be the unique integral point of  $P_{ij}$  lying on  $\Lambda(1)$ . We can write

$$(1) \quad p = \lambda v_i + (1/d_{ij})v_j \quad \text{for some } 0 < \lambda < 1 .$$

Now consider the half-open parallelogram  $P_{ik}$ . Arguing as in the proof of Case 1 in Lemma 3.1, we see that the  $d$  integral points of  $P_\sigma$  lie on  $d/d_{ik}$  equidistant parallel planes  $\pi(0), \dots, \pi(d/d_{ik} - 1)$ , where  $\pi(0)$  is the plane spanned by  $v_i$  and  $v_k$ , and, for each  $t = 1, 2, \dots, d/d_{ik} - 1$ ,  $\pi(t)$  is the plane parallel to  $\pi(0)$  and passing through the point  $(td_{ik}/d)v_j$ . Each plane  $\pi(t)$  contains precisely  $d/d_{ik}$  many integral points of  $P_\sigma$ . For some integer  $r$  with  $0 < r < d/d_{ik}$ , the point  $p$  lies on the plane  $\pi(r)$ , and we can write

$$(2) \quad p = \lambda v_i + (rd_{ik}/d)v_j .$$

From (1) and (2) we get  $d = rd_{ik}d_{ij}$ , which yields the desired conclusion. q.e.d.

4.5. PROPOSITION. *Let  $\sigma = \langle v_1, v_2, v_3 \rangle$  with primitive vectors  $v_1, v_2, v_3$ , and with  $d = \text{mult}(\sigma) > 1$ . Assume without loss of generality  $d_{23} \geq d_{13} \geq d_{12}$ . If  $\text{int } P_\sigma \cap \mathbf{Z}^3 = \emptyset$  then one of the following conditions holds:*

- (a) either  $d = d_{23}$  and  $d_{12} = d_{13} = 1$ ,
- (b) or  $d = 4$  and  $d_{23} = d_{13} = d_{12} = 2$ .

PROOF. By hypothesis we have  $d_{23} + d_{13} + d_{12} = d + 2$ . For suitable integers,  $h_1, h_2, h_3$  we can write  $d = h_1d_{23} = h_2d_{13} = h_3d_{12}$ . It follows that

$$(3) \quad d(1/h_1 + 1/h_2 + 1/h_3) = d + 2, \quad 1 \leq h_1 \leq h_2 \leq h_3,$$

whence  $1/h_1 + 1/h_2 + 1/h_3 > 1$  and  $h_1 \in \{1, 2\}$ .

Case 1:  $h_1 = 1$ .

Then  $d = d_{23}$ . By (3),  $d_{13} + d_{12} = 2$ , whence  $d_{13} = d_{12} = 1$ . Thus, condition (a) holds.

Case 2:  $h_1 = 2$ .

Then  $d$  is an even number, say  $d = 2n$  with  $d_{23} = n$ . From (3) we get  $1/h_2 + 1/h_3 > 1/2$ , whence  $h_2 \in \{2, 3\}$ .

Subcase 2.1:  $h_2 = 2$ .

Then  $d_{13} = n$ . By (3) we get  $d_{12} = 2$ . By Proposition 4.4, the only possibility is  $n = 2$ , whence  $d = 4$ , and each  $d_{ij} = 2$ ; we are in condition (b).

Subcase 2.2:  $h_2 = 3$ .

Then by (3),  $1/h_3 > 1/6$ , whence  $h_3 \in \{3, 4, 5\}$ . Suppose  $h_3 = 3$  (absurdum hypothesis). Then  $d = 12$ ,  $d_{23} = 6$ , and  $d_{12} = d_{13} = 4$ , which contradicts Proposition 4.4. Similarly, if  $h_3 = 4$  we get  $d = 24$ , whence  $d_{23} = 12$ ,  $d_{13} = 8$ , and  $d_{12} = 6$ , another contradiction. Finally, if  $h_3 = 5$ , then  $d = 60$ ,  $d_{23} = 30$ ,  $d_{13} = 20$ ,  $d_{12} = 12$ , which is also impossible. In conclusion, when  $h_1 = 2$ , the only possibility left open by Proposition 4.4 is  $d = 4$  and  $d_{ij} = 2$  for each  $i \neq j$ . q.e.d.

When  $\text{int } P_\sigma$  contains no integral points,  $\nabla(\sigma)$  satisfies the following minimality conditions:

4.6. PROPOSITION. *Given  $\sigma = \langle v_1, v_2, v_3 \rangle$  with primitive vectors  $v_1, v_2, v_3$ , assume  $\text{int } P_\sigma \cap \mathbf{Z}^3 = \emptyset$ . Then for any arbitrary nonsingular subdivision  $\Delta$  of  $\sigma$  we have:*

(i)  $\nabla(\sigma)^{(1)} \subseteq \Delta^{(1)}$ ; indeed,  $\text{gen } \nabla(\sigma)^{(1)}$  coincides with the set of indispensable points of (the faces of)  $\sigma$ .

(ii)  $\# \nabla(\sigma)^{(3)} \leq \# \Delta^{(3)}$ .

PROOF. To avoid trivialities, assume  $d = \text{mult}(\sigma) > 1$ . Assume without loss of generality  $d_{23} \geq d_{13} \geq d_{12}$ . In the light of Proposition 4.5, we have only to consider the following two cases:

Case 1:  $d_{23} = d$  and  $d_{13} = d_{12} = 1$ .

Let  $p_\sigma$  be the point of minimal norm in  $\sigma$ . By hypothesis,  $p_\sigma \in P_{23}$ . Arguing as in the proof of the Claim in Proposition 2.4, we see that  $p_\sigma$  is indispensable for the 2-dimensional cone  $\langle v_3, v_2 \rangle$ . By Proposition 1.1,  $p_\sigma$  is also indispensable for  $\sigma$ . In fact, by [5, 1.6],  $p_\sigma$  is contained in the triangle with vertices  $0, v_2$  and  $v_3$ . Let  $(\sigma * P_\sigma)^{(3)} = \{\sigma_2, \sigma_3\}$ , where  $\sigma_2 = \langle v_1, v_2, p_\sigma \rangle$  and  $\sigma_3 = \langle v_1, v_3, p_\sigma \rangle$ .

Claim.  $(P_{\sigma_2} \cap \mathbf{Z}^3) \setminus \{0\} \subseteq P_{\langle v_2, p_\sigma \rangle}$  and  $(P_{\sigma_3} \cap \mathbf{Z}^3) \setminus \{0\} \subseteq P_{\langle v_3, p_\sigma \rangle}$ .

As a matter of fact, pick an arbitrary point  $q \in (P_{\sigma_2} \cap \mathbf{Z}^3) \setminus \{0\}$ . Since by hypothesis  $P_{12} \cap \mathbf{Z}^3 = \{0\}$ , and since the half-open parallelogram  $P_{\langle v_1, p_\sigma \rangle} = \{\lambda v_1 + \mu p_\sigma \mid 0 \leq \lambda < 1, 0 \leq \mu < 1\}$  is contained in  $P_\sigma$ , it follows that  $P_{\langle v_1, p_\sigma \rangle} \cap \mathbf{Z}^3 = \{0\}$ , whence  $q$  cannot lie in  $P_{\langle v_1, p_\sigma \rangle}$ . Further,  $q$  cannot lie in  $\text{int } P_{\sigma_2} \cap \mathbf{Z}^3$ , for otherwise, either point  $q$  or

$v_1 + v_2 + p_\sigma - q$  would lie in  $\text{int } P_\sigma \cap \mathbf{Z}^3$ , which is impossible. The only remaining possibility is that  $q$  lies in  $P_{\langle v_2, p_\sigma \rangle}$ . We have proved that  $(P_{\sigma_2} \cap \mathbf{Z}^3) \setminus \{0\} \subseteq P_{\langle v_2, p_\sigma \rangle}$ . Similarly,  $(P_{\sigma_3} \cap \mathbf{Z}^3) \setminus \{0\} \subseteq P_{\langle v_3, p_\sigma \rangle}$ , and the claim is settled.

Proceeding inductively as in the proof of Proposition 2.4, since both  $\text{mult}(\sigma_2)$  and  $\text{mult}(\sigma_3)$  are strictly smaller than  $d$ , we obtain the nonsingular subdivision  $\nabla(\sigma)$  after exactly  $-2 + \# \text{indisp}\langle v_2, v_3 \rangle$  many starring operations, involving precisely the indispensable points of  $\langle v_2, v_3 \rangle$  other than  $v_2$  and  $v_3$ . It is now easy to see that conditions (i) and (ii) are satisfied in the present case.

Case 2:  $d=4$  and  $d_{23}=d_{13}=d_{12}=2$ .

Then, without loss of generality, we can write  $p_\sigma = (v_2 + v_3)/2$ . Let  $(\sigma * p_\sigma)^{(3)} = \{\sigma_2, \sigma_3\}$ , where  $\sigma_2 = \langle v_1, v_2, p_\sigma \rangle$  and  $\sigma_3 = \langle v_1, v_3, p_\sigma \rangle$ . Then for each  $i \in \{2, 3\}$ ,  $\text{mult}(\sigma_i) = 2$ , and the half-open parallelogram  $P_{\sigma_i}$  contains precisely one nonzero integral point, namely the point  $q_i = (v_i + v_1)/2$ . Therefore, by Definition 2.2,  $p_{\sigma_i}$  must coincide with  $q_i$ . Proceeding in this way, after two steps the construction of  $\nabla(\sigma)$  is completed. Since  $p_\sigma, q_1$  and  $q_2$  are all indispensable for  $\sigma$ , condition (i) holds. A trivial counting shows that also (ii) holds. q.e.d.

**5. Epilogue.** In this section we assume familiarity with the first chapters of [3] and [5]. Recall [2, 12.8] that for every (finite simplicial) complete fan  $\Delta$  in  $\mathbf{R}^3$  the Euler characteristic  $E(X_\Delta)$  of the associated toric variety coincides with the number  $\#\Delta^{(3)}$  of 3-dimensional cones in  $\Delta$ .

5.1. THEOREM. (i) *Let  $X = X_\Delta$  be a compact 3-dimensional toric variety, where each  $\sigma \in \Delta^{(3)}$  is simplicial. Then  $X' = X_{\nabla(\Delta)}$  is a desingularization of  $X$  satisfying the inequality*

$$E(X') \leq -E(X) + 2 \sum_{\sigma \in \Delta^{(3)}} \text{mult}(\sigma).$$

(ii) *This upper bound is the best possible. Indeed, for any two integers  $e$  and  $m$  with  $4 \leq e \leq m$ , there exists a compact 3-dimensional toric variety  $X = X_\Delta$  such that each cone  $\sigma \in \Delta^{(3)}$  is simplicial,  $E(X) = e$ , and  $\sum_{\sigma \in \Delta^{(3)}} \text{mult}(\sigma) = m$ , having the additional property that every desingularization  $X''$  of  $X$  satisfies the inequality*

$$E(X'') \geq -E(X) + 2 \sum_{\sigma \in \Delta^{(3)}} \text{mult}(\sigma).$$

PROOF. (i) By [5, Theorem 1.11],  $\Delta$  is a finite fan whose support coincides with  $\mathbf{R}^3$ . By Theorem 2.6 and Proposition 3.2,  $\nabla(\Delta)$  is a nonsingular subdivision of  $\Delta$  satisfying the inequality  $\#\nabla(\Delta)^{(3)} \leq -\#\Delta^{(3)} + 2 \sum_{\sigma \in \Delta^{(3)}} \text{mult}(\sigma)$ . By [5, Theorem 1.18], the toric variety  $X' = X_{\nabla(\Delta)}$  arises as a desingularization of  $X$ . By [2, Corollary 12.8],  $E(X') \leq -E(X) + 2 \sum_{\sigma \in \Delta^{(3)}} \text{mult}(\sigma)$ . For the proof of (ii), one uses the same results from [2] and [5], together with Proposition 4.2. q.e.d.

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