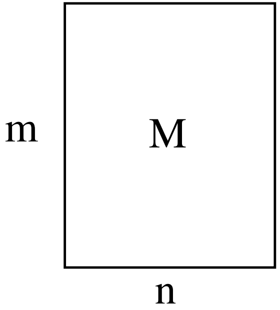
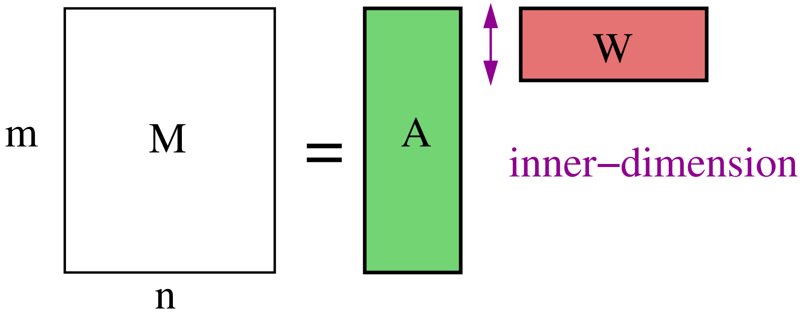
An Almost Optimal Algorithm for Computing Nonnegative Rank

Ankur Moitra

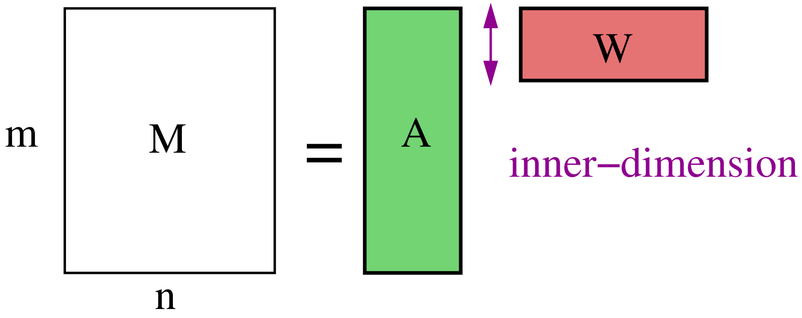
Institute for Advanced Study

January 8, 2013

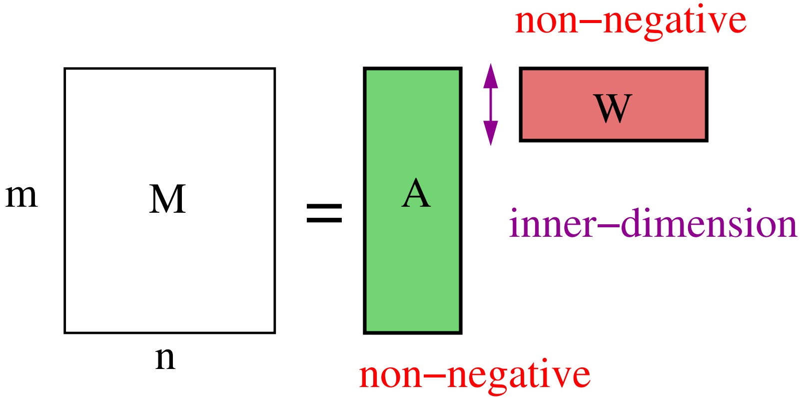

m
M
n

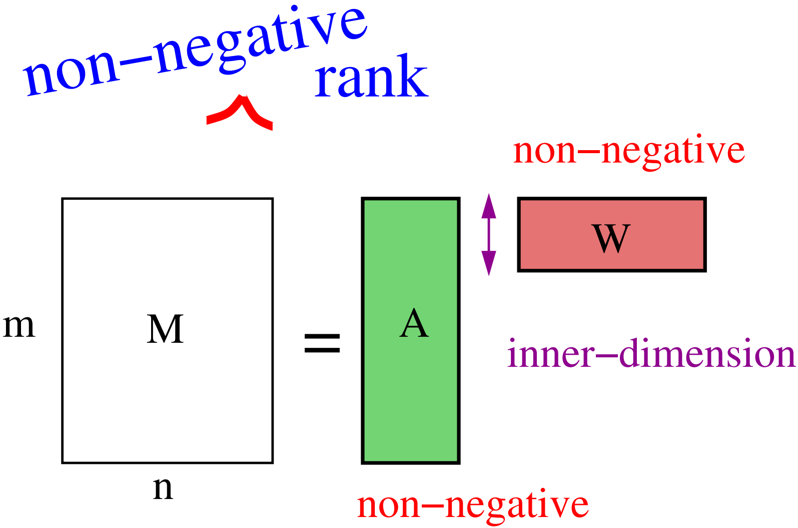

m
M
n
=
A
W
inner–dimension

rank



rank




non–negative
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Applications

- Statistics and Machine Learning:
  - extract latent relationships in data
  - image segmentation, text classification, information retrieval, collaborative filtering, ...

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- Physical Modeling:
  - interaction of components is additive
  - visual recognition, environmetrics

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...these algorithms are about an algebraic question, about how to best encode nonnegative rank as a systems of polynomial inequalities

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Semi-algebraic sets: s polynomials, k variables, Boolean function B

$$S = \{x_1, x_2 ... x_k | B(\text{sgn}(f_1), \text{sgn}(f_2), ... \text{sgn}(f_s)) = \text{"true"}\}$$

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In fact, best known algorithms (e.g. [Renegar]) for finding a point in S run in $(ds)^{O(k)}$ time

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What is the smallest formulation, measured in the number of variables? Can we use only $f(r)$ variables? $O(r^2)$ variables?

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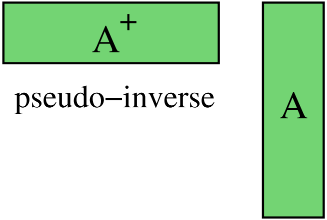

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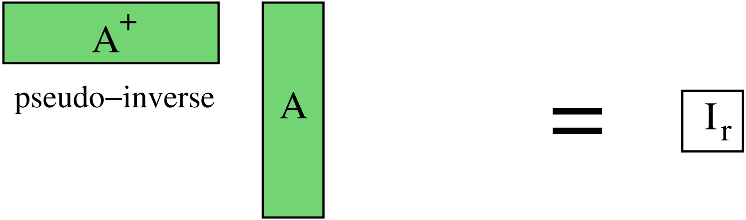

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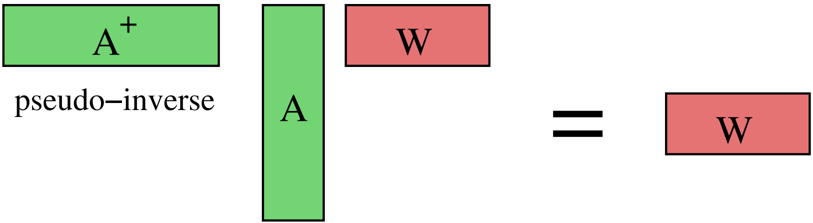

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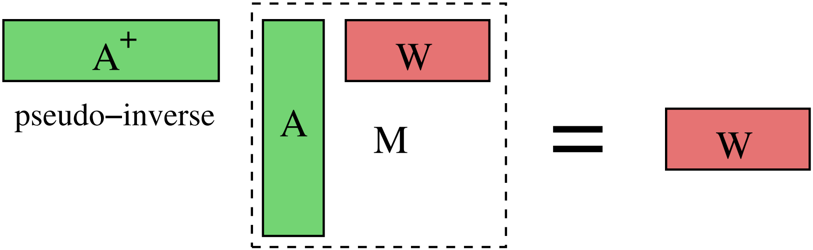

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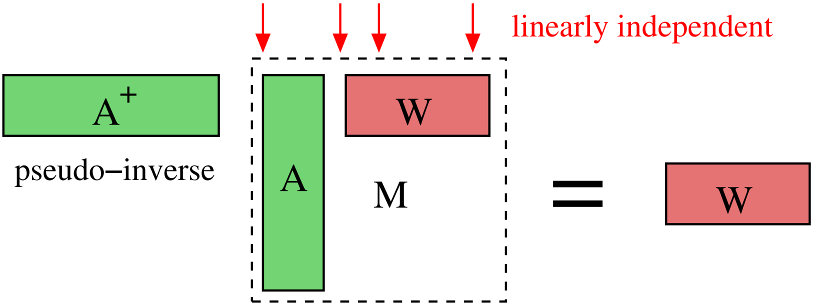

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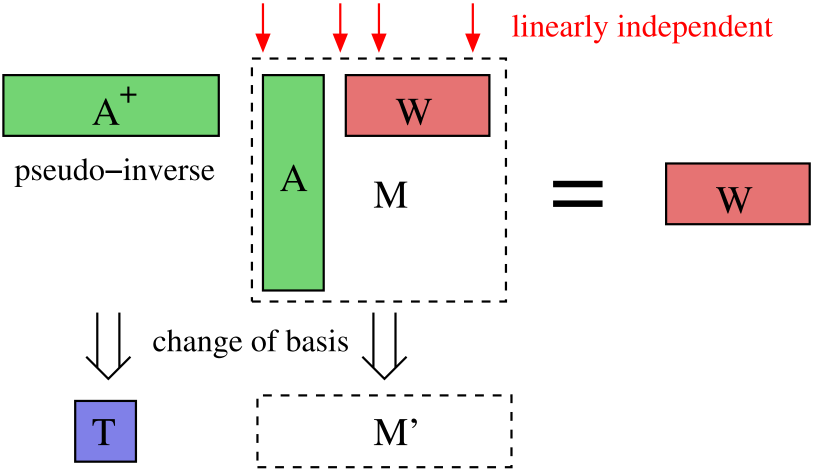
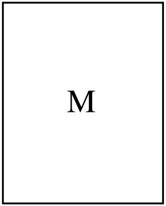


Putting it Together: $2r^2$ Variables



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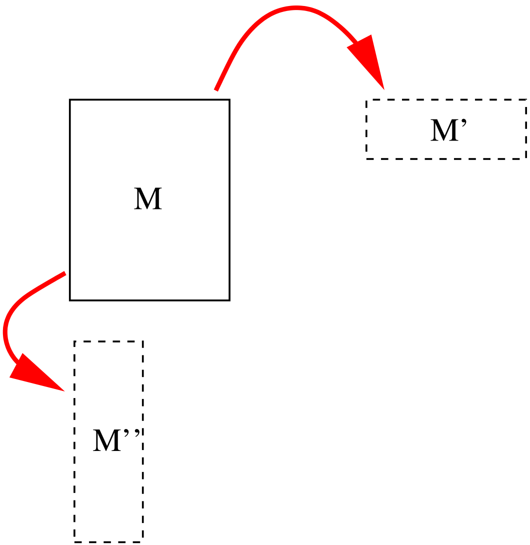

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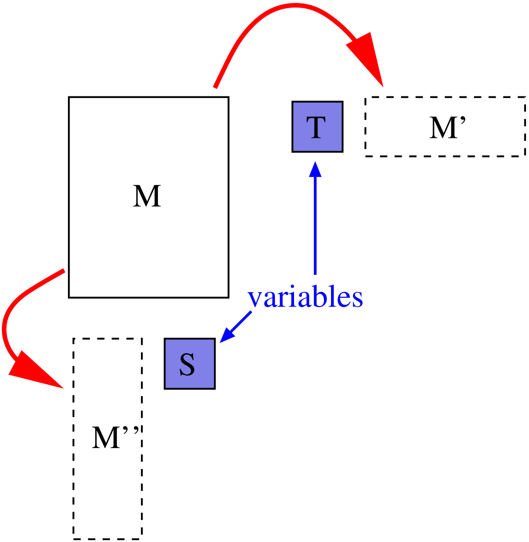

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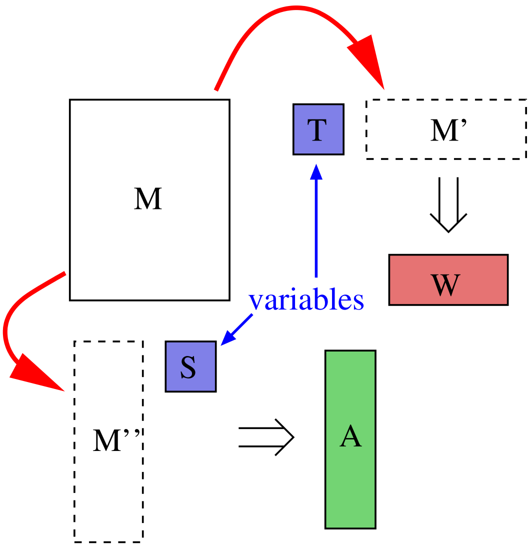

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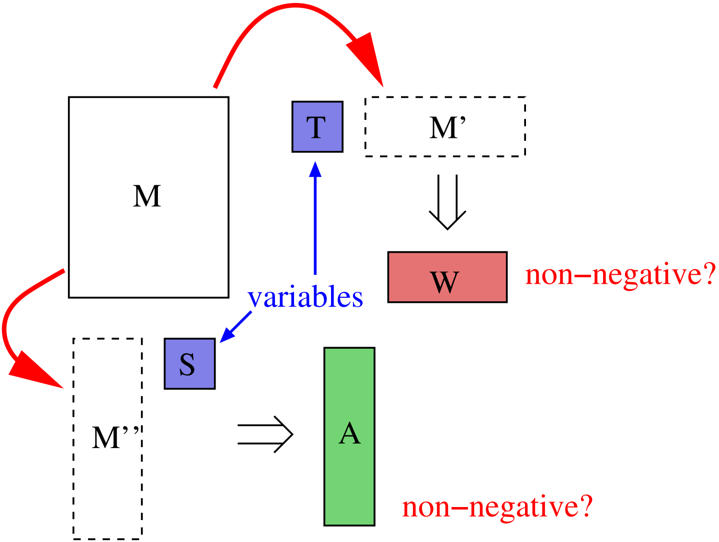

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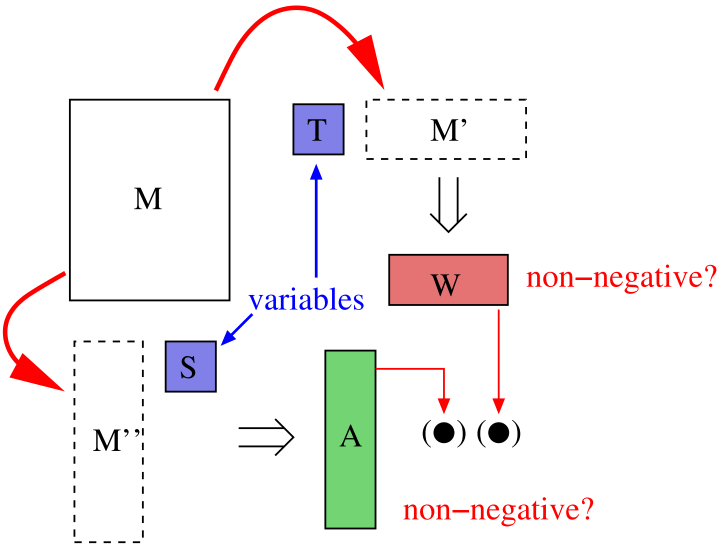

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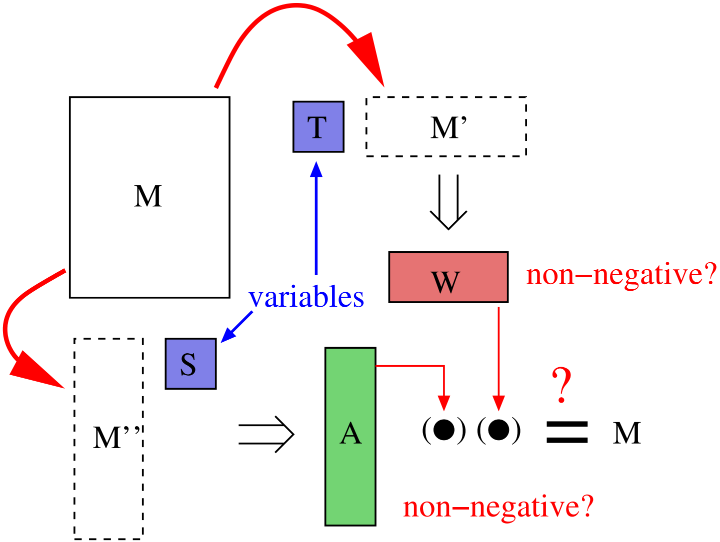

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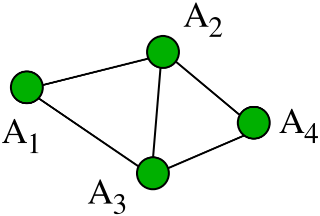
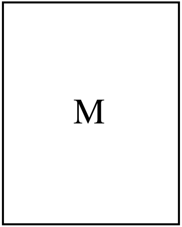

A2
A1
A4
A3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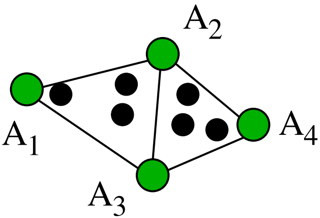
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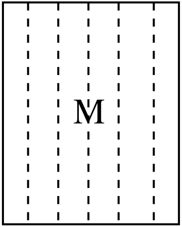
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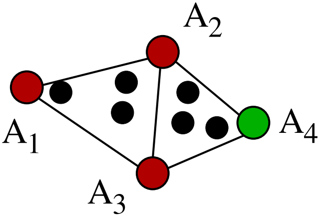

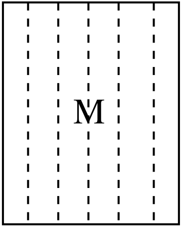

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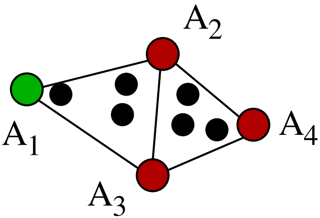

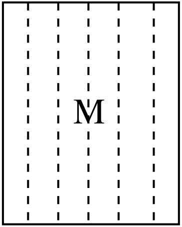

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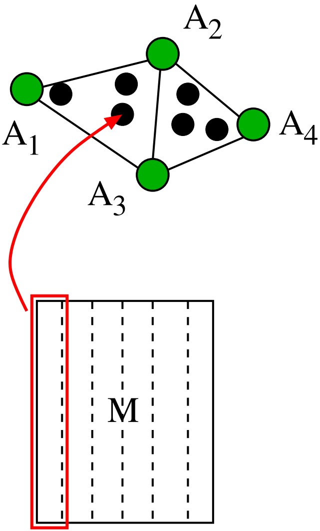

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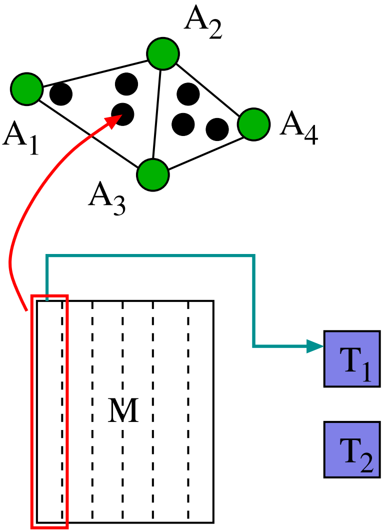

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Key

These linear transformations can be defined using a common set of r^2 variables!

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Is there an elementary proof of the Milnor-Warren bound?

Any Questions?

Thanks!