An alternative approach for Quasi-Truth

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Abstract

In 1986, I. Mikenberg, N. da Costa and R. Chuaqui introduced the semantic notion of quasi-truth defined by means of partial structures. In such structures, the predicates are seen as triples of pairwise disjoint sets: the set of tuples which satisfies, does not satisfy and can satisfy or not the predicate, respectively. The syntactical counterpart of the logic of partial truth is a rather complicated first-order modal logic. In the present paper, the notion of predicates-as-triples is recursively extended, in a natural way, to any complex formula of the first-order object language. From this, a new definition of quasi-truth is obtained. The proof-theoretic counterpart of the new semantics is a first-order paraconsistent logic whose propositional base is a 3-valued logic belonging to hierarchy of paraconsistent logics known as Logics of Formal Inconsistency, which was proposed by W. Carnielli and J. Marcos in 2001.

1 Introduction

In 1986 Mikenberg, da Costa and Chuaqui (cf. [13]) introduced the semantic notion of *quasi-truth*, defined by means of *partial structures*, where the relations interpreted in this structures are partial. Thus, the membership (or not) of a given tuple of the domain in such a relation is not always defined, and so any partial relation R is a triple of sets $\langle R_+, R_-, R_u\rangle$ such that R_+ is the set of tuples which effectively belong to R; R_- is the set of tuples which effectively do not belong to R, and R_u is the set of tuples whose membership to R is (still) undetermined.

In this way, by introducing the notion of partial structure into the model-theoretical approach, the conceptual framework of quasi-truth provides a way of accommodating the conceptual incompleteness inherent in scientific theories. The philosophical importance of quasi truth was analyzed, for instance, in [2], [8] and [9].

Looking for a logical system that can serve as the underlying logic to theories that have the quasi-truth as their truth conception, the paraconsistent modal logic PT is introduced by da Costa (cf. [7], p.136). According to da Costa, in general, this logic can be used as a deductive logic for empirical sciences. Posteriorly, in [11], da Costa's system PT is analyzed as a paraconsistent modal system associated to $S5Q^=$, a kind of Jaśkowski's discussive logic, and new results are obtained. However, the logic PT is established through two modal systems associated with each other, and they use normal structures.

In order to provide a complementary formulation of quasi-truth, Bueno and de Souza (cf. [2]) introduced in 1996 a different definition of quasi-truth with the purpose of presenting a distinct philosophical outlook, in the sense of establishing a framework for the notion of truth according to empiricism and the dynamics of scientific knowledge. Bueno and de Souza's strategy avoids constructing normal structures from partial ones, and introduces the concept of quasi-truth by means of the notion of *quasi-satisfaction*. However, Bueno and de Souza's strategy presents formal limitations, because their underlying logic coincides with classical logic (cf. Theorem 2.8 below).

In this paper, we propose a new approach to the concept of quasi-truth in order to establish a methodology for generating first-order non-modal logics, and on the other hand, avoiding the use of normal structures and so become closer to the original notion of quasi-truth for predicates presented in [13]. Among other things, the obtained logic will be a paraconsistent one.

The strategy is as follows: the notion of predicates as triples is extended recursively to any complex (i.e. non-atomic) formula of the object first-order language. Thus, the interpretation of any formula φ, in a partial structure $\mathfrak{A}$ inductively originates a triple $\langle \varphi_+^{\mathfrak{A}}, \varphi_-^{\mathfrak{A}}, \varphi_u^{\mathfrak{A}}\rangle$, generalizing the approach introduced in [13].

Moreover, this proposal naturally generalizes the usual tarskian perspective of considering a given first-order formula φ (with at most n free variables) within a structure $\mathfrak{A}$ as being a relation $R = \{\vec{a} \in D^n : \mathfrak{A} \models \varphi[\vec{a}]\}$, which is inductively defined. From this, a new definition of quasi-truth via the notion of pragmatic satisfaction is obtained. After that, a first-order axiomatic system for the logic of quasi-truth is presented, which is proved to be sound and complete with respect to the semantics of triples. It is also shown that the underlying propositional logic is 3-valued, and it belongs to the hierarchy of paraconsistent logics known as *Logics of Formal Inconsistency*, introduced by W. Carnielli and J. Marcos (see [5] and [3]).

We conclude by pointing out the similarities and differences among the presented approach to quasi-truth and two related frameworks: the first-order logic of paradox LP of G. Priest (cf. [15]) and the logic LFI1* of evolutionary databases of W. Carnielli, J. Marcos and S. de Amo (cf. [4]).

2 The concept of quasi-truth

In this section, we briefly recall the main definitions of the model theory of quasi-truth as introduced in [13] (see also [9]).

Definition 2.1. Let D be a nonempty set. An n-ary *partial relation* R defined on D is an ordered triple $\langle R_+, R_-, R_u\rangle$, where R_+, R_-, and R_u are mutually disjoint sets such that $R_+ \cup R_- \cup R_u = D^n$. ∎

Remark 2.2. The intended meaning of $\langle R_+, R_-, R_u\rangle$ is as follows:

(i) R_+ is the set of n-tuples that we know that belong to R;

(ii) R_- is the set of n-tuples that we know that do not belong to R;

(iii) R_u is the set of n-tuples for which it is undetermined whether or not they are in the relation R.

If $R_u = \emptyset$ then R is an usual n-ary relation, which can be identified with R_+. In this case R is called a *total relation*. ∎

The following definition will play a central role in the theory of quasi-truth.

Definition 2.3. A *partial structure* for a first-order language $\mathbb{L}$, or a *partial model* for $\mathbb{L}$, is an ordered pair $\mathfrak{A} = \langle D, (\cdot)^{\mathfrak{A}}\rangle$, where D is a nonempty set and $(\cdot)^{\mathfrak{A}}$ is a function defined on $\mathbb{L}$ such that, for every symbol of n-ary relation R, $R^{\mathfrak{A}} = \langle R^{\mathfrak{A}}_+, R^{\mathfrak{A}}_-, R^{\mathfrak{A}}_u\rangle$ is an n-ary partial relation. For symbols of constants and functions the mapping $(\cdot)^{\mathfrak{A}}$ is defined as in classical first-order structures.

∎

In order to establish a relationship with the tarskian notion of truth, Mikenberg *et alia* introduced the concept of $\mathfrak{A}$-normal structure. A classical first-order structure $\mathfrak{B}$ is a $\mathfrak{A}$-*normal* structure if it is defined over the same domain than $\mathfrak{A}$, the interpretation of constants and symbol of functions coincides with that of $\mathfrak{A}$, and the total relation $R^{\mathfrak{B}}$ extends the corresponding partial relation $R^{\mathfrak{A}}$, for every symbol of n-ary relation R. That is, $R^{\mathfrak{A}}_{+} \subseteq R^{\mathfrak{B}}$ and $R^{\mathfrak{A}}_{-} \subseteq D^n - R^{\mathfrak{B}}$.

Given a partial structure $\mathfrak{A}$ and a sentence α:

(i) α is *quasi-true* in $\mathfrak{A}$ with respect to a $\mathfrak{A}$-normal structure $\mathfrak{B}$, denoted by $\mathfrak{A} \Vdash_{\mathfrak{B}} \alpha$, if $\mathfrak{B} \models \alpha$, i.e. α is true in $\mathfrak{B}$ in the tarskian sense.

(ii) α is *quasi-true* in $\mathfrak{A}$, denoted by $\mathfrak{A} \Vdash \alpha$, if $\mathfrak{A} \Vdash_{\mathfrak{B}} \alpha$, for some $\mathfrak{A}$-normal structure $\mathfrak{B}$. Otherwise α is *quasi-false*.

(iii) α is *true* in $\mathfrak{A}$, denoted by $\mathfrak{A} \mid\approx \alpha$, if $\mathfrak{A} \Vdash_{\mathfrak{B}} \alpha$ for every $\mathfrak{A}$-normal structure $\mathfrak{B}$.

The main properties of the relation $\Vdash$ can be stated. As we shall see, these properties will be satisfied by our proposed notion of quasi-truth:

Theorem 2.4. Let R be an n-ary predicate symbol of a language $\mathbb{L}$, $\tau_1, \ldots, \tau_n$ closed terms and let $\mathfrak{A}$ be a partial structure for $\mathbb{L}$.

(#1) $\mathfrak{A} \Vdash R(\tau_1, \ldots, \tau_n)$ iff $(\tau_1^{\mathfrak{A}}, \ldots, \tau_n^{\mathfrak{A}}) \in R^{\mathfrak{A}}_{+} \cup R^{\mathfrak{A}}_{u}$;

(#2) $\mathfrak{A} \Vdash \neg R(\tau_1, \ldots, \tau_n)$ iff $(\tau_1^{\mathfrak{A}}, \ldots, \tau_n^{\mathfrak{A}}) \in R^{\mathfrak{A}}_{-} \cup R^{\mathfrak{A}}_{u}$;

(#3) $\mathfrak{A} \Vdash \alpha \wedge \beta$ implies $\mathfrak{A} \Vdash \alpha$ and $\mathfrak{A} \Vdash \beta$;

(#4) $\mathfrak{A} \nVdash \alpha$ implies $\mathfrak{A} \Vdash \neg\alpha$;

(#5) $\mathfrak{A} \Vdash \alpha \vee \beta$ iff $\mathfrak{A} \Vdash \alpha$ or $\mathfrak{A} \Vdash \beta$;

(#6) $\mathfrak{A} \Vdash \forall x \alpha$ implies $\mathfrak{A} \Vdash \alpha[x/\bar{a}]$, for all $a \in D$.[1]

Bueno and de Souza (cf. [2]) introduced a different definition of quasi-truth with the purpose of establishing a framework for the notion of truth according to the empiricism and the dynamic of the scientific knowledge.

Bueno and de Sousa's strategy (cf. [2], p. 192) avoids constructing the normal structures and introduces the concept of quasi-truth by means of the notion of *quasi-satisfaction*.

[1]By $\varphi[x/\tau]$ we mean the formula obtained from φ by replacing every free occurrence of x in φ by the term τ. As usual, the constant of the diagram language of $\mathfrak{A}$ representing $a \in D$ will be denoted by $\bar{a}$, and $\mathfrak{A}$ will be identified with its canonical expansion to the diagram language.

Definition 2.5. Let $\varphi(x_1, \ldots, x_n)$ be a formula, $\langle D, (\cdot)^{\mathfrak{A}} \rangle$ a partial structure, and $\vec{a}$ a sequence in D. We say that $\vec{a}$ *quasi-satisfies* φ in $\langle D, (\cdot)^{\mathfrak{A}} \rangle$, which is denoted by $\mathfrak{A} \Vdash\!\!= \varphi[\vec{a}]$, if:

(1) Suppose that φ is the atomic formula $R(\tau_1, \ldots, \tau_k)$ and that R is a k-ary relation symbol, then

$$\mathfrak{A} \Vdash\!\!= R(\tau_1, \ldots, \tau_n)[\vec{a}] \quad \text{iff} \quad (\tau_1^{\mathfrak{A}}[\vec{a}], \ldots, \tau_k^{\mathfrak{A}}[\vec{a}]) \in R_+^{\mathfrak{A}} \cup R_u^{\mathfrak{A}};$$

(2) $\vec{a}$ quasi-satisfies $\neg\psi$ in $\langle D, (\cdot)^{\mathfrak{A}} \rangle$ iff $\vec{a}$ does not quasi-satisfy ψ in $\langle D, (\cdot)^{\mathfrak{A}} \rangle$.

This definition continues recursively, *mutatis mutandis*, as in the classical notion of satisfaction. ∎

Definition 2.6. A formula φ is *quasi-true* in a partial structure $\mathfrak{A}$ iff φ is quasi-satisfied in $\mathfrak{A}$ by all sequences in the domain $|\mathfrak{A}|$ of $\mathfrak{A}$. We denote this by $\mathfrak{A} \Vdash\!\!= \varphi$. Finally, φ is *quasi-valid* if it is quasi-true in every partial structure $\mathfrak{A}$. We denote this by $\Vdash\!\!= \varphi$. ∎

Remark 2.7. If R is an n-ary predicate symbol and $\mathfrak{A}$ is a partial structure. Then, as a straightforward consequence from Definition 2.5 (2), we have: $\mathfrak{A} \Vdash\!\!= \neg R(\tau_1, \ldots, \tau_n)[\vec{a}] \quad \text{iff} \quad (\tau_1^{\mathfrak{A}}[\vec{a}], \ldots, \tau_n^{\mathfrak{A}}[\vec{a}]) \in R_-^{\mathfrak{A}}$. ∎

From the previous results, it is clear that there is a difference between the definition of quasi-truth introduced by da Costa and collaborators and the one presented by Bueno and de Souza. In fact, it is enough to contrast Theorem 2.4 (#2) with Remark 2.7. Moreover, as mentioned above, it can be proved that the logic underlying Bueno and de Souza's notion of quasi-truth is exactly classical logic:

Theorem 2.8. Let φ be a formula. Then φ is quasi-valid iff it is valid in classical first-order logic.

Proof. For every partial structure $\mathfrak{A} = \langle D, (\cdot)^{\mathfrak{A}} \rangle$, we consider a classical structure $\mathfrak{A}' = \langle D, (\cdot)^{\mathfrak{A}'} \rangle$ such that, for every predicate symbol R, $R^{\mathfrak{A}'} = R_+^{\mathfrak{A}} \cup R_u^{\mathfrak{A}}$; $f^{\mathfrak{A}'} = f^{\mathfrak{A}}$ for every symbol for function, and $c^{\mathfrak{A}'} = c^{\mathfrak{A}}$ for every constant c. Then, it is easy to check that $\mathfrak{A} \Vdash\!\!= \varphi[\vec{a}]$ iff $\mathfrak{A}' \models \varphi[\vec{a}]$. Since every classical structure is also a partial structure, the result follows. □

3 A new theory of quasi-truth

Despite the negative result shown in Theorem 2.8, the ideas contained in [2] concerning the possibility of defining a pure notion of quasi-truth – without using normal structures – are challenging.

Thus, inspired by [2], a new definition of quasi-truth will be proposed in this section, which does not need normal structures. The basic idea is to propagate the indeterminacy of partial predicates (given by the extra component R_u) to complex formulas, in a recursive way. In this manner, a key feature of the original proposal from da Costa and collaborators is preserved and generalized. On the other hand, the new semantics does not depend on normal structures. Finally, a sound and complete axiomatization will be obtained, in terms of a very natural system of first-order logic without modalities, in contrast to the first-order modal logic proposed in [7] for quasi-truth.

Previous to this, the basic notions of Logics of Formal Inconsistency will be briefly described.

3.1 The logics of formal inconsistency

The logics of formal inconsistency (LFIs), introduced by W. Carnielli and J. Marcos in [5], are logics that allow to internalize the concepts of consistency and inconsistency by means of formulas of their language. Contradictoriness, on the other hand, it can always be expressed in any logic, provided its language includes a symbol for negation. Besides being able to represent the distinction between contradiction and inconsistency, LFIs are non-explosive logics, in the sense that a contradiction does not entail arbitrary statements, i.e. does not hold the Principle of Explosion: (PE) $\alpha \to (\neg\alpha \to \beta)$. However, LFIs are gently explosive, in the sense that, adjoining the additional requirement of consistency, then contradictoriness does cause explosion.

Several logics can be seen as LFIs (cf. [3]), among them the great majority of paraconsistent logics developed under the Brazilian tradition, as well as the systems developed under the Polish tradition.

One of the main contributions of LFIs with respect to the original proposal of paraconsistent logic from da Costa is the use of primitive operators representing consistency (and/or inconsistency), which are not definable, in general, in terms of other connectives of the language.

Definition 3.1. Let **L** be a logic defined over a set For of formulas in a signature containing a negation $\neg$. Then **L** is an LFI (with respect to $\neg$) if the following holds:

(a) $\alpha, \neg\alpha \nvdash \beta$ for some α and β, i.e. the logic is not explosive; and

(b) there exists a set of formulas $\bigcirc(p)$ depending exactly on the propositional variable p, which satisfies: (i) $\bigcirc(\alpha), \alpha \nvdash \beta$ for some α and β; (ii) $\bigcirc(\alpha), \neg\alpha \nvdash \beta$ for some α and β; and(iii) $\bigcirc(\alpha), \alpha, \neg\alpha \vdash \beta$ for every α and β, i.e. the *gentle principle of explosion* holds. ∎

We will show in Section 4 that the propositional logic underlying quasi-truth is an LFI.

3.2 Pragmatic satisfaction

As observed in the previous sections, the approach to quasi-truth in [13] in which the predicates are partial, applies only to atomic formulas, taking into account that the complex formulas are treated in a classic way. A question that arises naturally is whether the approach of considering predicates as ordered triples could be extended to all the formulas of a language. This would originate a generalization of the tarskian notion of the truth, on one hand, and an alternative notion of quasi-truth, on the other hand.

From these motivations, we introduce in this section an original proposal to extend, in a recursive way, the notion of predicates as triples. Thereby, the interpretation of each formula $\varphi(x_1, \ldots, x_n)$ in a partial structure $\mathfrak{A}$, originates inductively a triple $\langle \varphi_+^{\mathfrak{A}}, \varphi_-^{\mathfrak{A}}, \varphi_u^{\mathfrak{A}} \rangle$ of pairwise disjoint set of tuples such that $\varphi_+^{\mathfrak{A}} \cup \varphi_-^{\mathfrak{A}} \cup \varphi_u^{\mathfrak{A}} = D^n$.

This proposal generalizes the usual tarskian perspective of a given formula of first-order φ (with at most n free variables) in a structure $\mathfrak{A}$ seen as a relation $R = \{\vec{a} \in D^n : \mathfrak{A} \models \varphi[\vec{a}]\}$ defined inductively. From this, a new definition of quasi-truth is obtained.

From now on, given a first-order signature Σ (formed by symbols for predicates, for functions and for constants, with their arity) the language to be considered is the usual first-order language $\mathbb{L}(\Sigma)$ based on the connectives $\wedge, \rightarrow, \neg$ and the quantifier $\forall$. The disjunction $\varphi \vee \lambda$ is defined as $\neg(\neg\varphi \wedge \neg\lambda)$, and the existencial quantifier es defined as $\exists x\varphi =_{def} \neg\forall x \neg\varphi$. Given a term τ of $\mathbb{L}(\Sigma)$ depending at most on the variables $x_1, \ldots, x_n$, a partial structure $\mathfrak{A}$ with domain D and $\vec{a} = (a_1, \ldots, a_n)$ in D^n such that a_i interprets x_i (for $i = 1, \ldots, n$) then $\tau^{\mathfrak{A}}[\vec{a}]$ is the element of D obtained by interpreting τ in $\mathfrak{A}$ using $\vec{a}$.

Definition 3.2. Let $\varphi(x_1, \ldots, x_n)$ be a formula whose free variables occur in the list $x_1, \ldots, x_n$, and let $\mathfrak{A}$ be a partial structure with domain D. Then, the triple $\varphi^{\mathfrak{A}} = \langle \varphi_+^{\mathfrak{A}}, \varphi_-^{\mathfrak{A}}, \varphi_u^{\mathfrak{A}} \rangle$ is defined recursively as follows:

(i) if $\varphi = P(\tau_1, \ldots, \tau_n)$ is atomic then $\varphi^{\mathfrak{A}} = \langle \varphi_+^{\mathfrak{A}}, \varphi_-^{\mathfrak{A}}, \varphi_u^{\mathfrak{A}} \rangle$ is such that, for $* \in \{+, -, u\}$, $\varphi_*^{\mathfrak{A}} = \{\vec{a} \in D^n : (\tau_1^{\mathfrak{A}}[\vec{a}], \ldots, \tau_n^{\mathfrak{A}}[\vec{a}]) \in P_*^{\mathfrak{A}}\}$;

(ii) $(\neg\varphi)^{\mathfrak{A}} \stackrel{def}{=} \langle \varphi_-^{\mathfrak{A}}, \varphi_+^{\mathfrak{A}}, \varphi_u^{\mathfrak{A}} \rangle$;

(iii) $(\varphi \wedge \lambda)^{\mathfrak{A}} \stackrel{def}{=} \langle \varphi_+^{\mathfrak{A}} \cap \lambda_+^{\mathfrak{A}}, \varphi_-^{\mathfrak{A}} \cup \lambda_-^{\mathfrak{A}}, D^n - \big[\big(\varphi_+^{\mathfrak{A}} \cap \lambda_+^{\mathfrak{A}}\big) \cup \big(\varphi_-^{\mathfrak{A}} \cup \lambda_-^{\mathfrak{A}}\big)\big] \rangle$;

(iv) $(\varphi \to \lambda)^{\mathfrak{A}} \stackrel{def}{=} \langle \varphi_{-}^{\mathfrak{A}} \cup \left(\lambda_{+}^{\mathfrak{A}} \cup \lambda_{u}^{\mathfrak{A}}\right), \left(\varphi_{+}^{\mathfrak{A}} \cup \varphi_{u}^{\mathfrak{A}}\right) \cap \lambda_{-}^{\mathfrak{A}}, \emptyset \rangle$. ■

From the last definition, it follows that:

$$(\varphi \vee \lambda)^{\mathfrak{A}} = \langle \varphi_{+}^{\mathfrak{A}} \cup \lambda_{+}^{\mathfrak{A}}, \varphi_{-}^{\mathfrak{A}} \cap \lambda_{-}^{\mathfrak{A}}, D^{n} - \left[\left(\varphi_{+}^{\mathfrak{A}} \cup \lambda_{+}^{\mathfrak{A}}\right) \cup \left(\varphi_{-}^{\mathfrak{A}} \cap \lambda_{-}^{\mathfrak{A}}\right)\right] \rangle.$$

Proposition 3.3. The triples obtained in clauses (i)-(iv) of Definition 3.2 are formed by pairwise disjoint sets whose union is D^n.

Proof. It is trivial using basic notions from set theory. □

From the Definition 3.2 we note that:

$$(\neg\lambda \to \neg\varphi)^{\mathfrak{A}} = \langle \lambda_{+}^{\mathfrak{A}} \cup \left(\varphi_{-}^{\mathfrak{A}} \cup \varphi_{u}^{\mathfrak{A}}\right), \left(\lambda_{-}^{\mathfrak{A}} \cup \lambda_{u}^{\mathfrak{A}}\right) \cap \varphi_{+}^{\mathfrak{A}}, \emptyset \rangle$$

i.e. the contrapositive does not coincide, in general, with the given conditional. Moreover, from the previous definition it is easy to show that, if $\varphi^{\mathfrak{A}} \equiv \psi^{\mathfrak{A}}$ denotes that the formulas φ and ψ originate the same triple, we have: (i) $\varphi^{\mathfrak{A}} \equiv (\neg\neg\varphi)^{\mathfrak{A}}$; (ii) $[\neg(\varphi \wedge \lambda)]^{\mathfrak{A}} \equiv [\neg\varphi \vee \neg\lambda]^{\mathfrak{A}}$; (iii) $[\neg(\varphi \vee \lambda)]^{\mathfrak{A}} \equiv [\neg\varphi \wedge \neg\lambda]^{\mathfrak{A}}$.

Since by definition $\varphi_{+}^{\mathfrak{A}}$ and $\varphi_{-}^{\mathfrak{A}}$ are disjoint sets, notice that:

$$\begin{aligned}(\varphi \wedge \neg\varphi)^{\mathfrak{A}} &= \langle \varphi_{+}^{\mathfrak{A}} \cap \varphi_{-}^{\mathfrak{A}}, \varphi_{-}^{\mathfrak{A}} \cup \varphi_{+}^{\mathfrak{A}}, D^{n} - \left[\left(\varphi_{+}^{\mathfrak{A}} \cap \varphi_{-}^{\mathfrak{A}}\right) \cup \left(\varphi_{-}^{\mathfrak{A}} \cup \varphi_{+}^{\mathfrak{A}}\right)\right] \rangle \\ &= \langle \emptyset, \varphi_{+}^{\mathfrak{A}} \cup \varphi_{-}^{\mathfrak{A}}, \varphi_{u}^{\mathfrak{A}} \rangle\end{aligned}$$

This can be interpreted in the following way: giving a partial relation R, the partial relation $R \wedge \neg R$ is not supported by any positive evidence; the negative evidences are given by the reliable knowledge of the relation R, that is, by the negative and positive evidences of R, while the lack of knowledge of the new relation coincides with the lack of knowledge of the original relation.

Dually, the following holds: $(\varphi \vee \neg\varphi)^{\mathfrak{A}} = \langle \varphi_{+}^{\mathfrak{A}} \cup \varphi_{-}^{\mathfrak{A}}, \emptyset, \varphi_{u}^{\mathfrak{A}} \rangle$.

We now analyze the quantificational case.[2] For this purpose, we shall consider $A \subseteq D^{n+1}$ and the sets $\forall(A) \subseteq D^n$ and $\exists(A) \subseteq D^n$ defined, as usual, in the following way:

$$\begin{aligned}\forall(A) &= \{\vec{a} \in D^n : (b, \vec{a}) \in A, \text{for all b} \in D\} \\ \exists(A) &= \{\vec{a} \in D^n : (b, \vec{a}) \in A, \text{for some b} \in D\}.\end{aligned}$$

Moreover, if $A \subseteq D$ (i.e. $n = 0$), then:

$$\forall(A) = \begin{cases} 1, & \text{if } A = D \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \exists(A) = \begin{cases} 1, & \text{if } A \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

[2]The definition of quantifiers presented here differs from the related approach [18].

Definition 3.4. Assume that $\varphi^{\mathfrak{A}} = \langle \varphi^{\mathfrak{A}}_+, \varphi^{\mathfrak{A}}_-, \varphi^{\mathfrak{A}}_u \rangle$ is defined on D^{n+1} for $\varphi(x_0, \ldots, x_n)$, $n \geq 1$. Then:[3]

$$(\forall x_0 \varphi)^{\mathfrak{A}} \stackrel{def}{=} \left\langle \forall \left(\varphi^{\mathfrak{A}}_+\right), \exists \left(\varphi^{\mathfrak{A}}_-\right), D^n - \left[\forall \left(\varphi^{\mathfrak{A}}_+\right) \cup \exists \left(\varphi^{\mathfrak{A}}_-\right)\right] \right\rangle. \qquad \blacksquare$$

Since $\exists x_0 \varphi$ denotes $\neg \forall x_0 \neg \varphi$, it follows that:

$$(\exists x_0 \varphi)^{\mathfrak{A}} = \left\langle \exists \left(\varphi^{\mathfrak{A}}_+\right), \forall \left(\varphi^{\mathfrak{A}}_-\right), D^n - \left[\exists \left(\varphi^{\mathfrak{A}}_+\right) \cup \forall \left(\varphi^{\mathfrak{A}}_-\right)\right] \right\rangle.$$

Proposition 3.5. If a triple $\left\langle \varphi^{\mathfrak{A}}_+, \varphi^{\mathfrak{A}}_-, \varphi^{\mathfrak{A}}_u \right\rangle$ is composed by mutually disjoint sets whose union is D^{n+1}, then the triple $(\forall x_0 \varphi)^{\mathfrak{A}}$ obtained as in Definition 3.4 is formed by mutually disjoint sets whose union is D^n.

Proof. Straightforward. □

From the Definition 3.2, we obtain a new concept of quasi-truth, following the original proposal from da Costa and collaborators, via the *pragmatic satisfaction* notion. This strategy avoids the construction of total structures, and it is given *mutatis mutandis* by the tarskian notion of satisfaction, but based now in formulas interpreted as triples of sets (that can be considered as positive, negative evidences and lack of reliable information), instead of considering the formulas as representing only sets (having only the positive information). In order to achieve our goal, we shall consider the following cases for a formula $\varphi^{\mathfrak{A}} = \langle \varphi^{\mathfrak{A}}_+, \varphi^{\mathfrak{A}}_-, \varphi^{\mathfrak{A}}_u \rangle$:

(i) For φ with n free variables: $\varphi^{\mathfrak{A}}_+, \varphi^{\mathfrak{A}}_-, \varphi^{\mathfrak{A}}_u \subseteq D^n$, and $n > 0$. Besides, $\varphi^{\mathfrak{A}}_+$, $\varphi^{\mathfrak{A}}_-$, and $\varphi^{\mathfrak{A}}_u$ are mutually disjoint sets and also $\varphi^{\mathfrak{A}}_+ \cup \varphi^{\mathfrak{A}}_- \cup \varphi^{\mathfrak{A}}_u = D^n$.

(ii) For φ a sentence: $\varphi^{\mathfrak{A}}_+, \varphi^{\mathfrak{A}}_-, \varphi^{\mathfrak{A}}_u \in \{0, 1\}$ in which only one of $\varphi^{\mathfrak{A}}_*$ is 1, for $* \in \{+, -, u\}$. So, $\langle 1, 0, 0 \rangle$, $\langle 0, 0, 1 \rangle$ and $\langle 0, 1, 0 \rangle$ represent φ true, true by lack of evidence to the contrary and false, respectively. From the definition of the universal quantifier (see Definition 3.4), considering $A = \varphi^{\mathfrak{A}}_+, B = \varphi^{\mathfrak{A}}_-$ and $C = \varphi^{\mathfrak{A}}_u$, it follows that $\forall \langle A, B, C \rangle = \langle \forall(A), \exists(B), D^n - [\forall(A) \cup \exists(B)] \rangle$. In the case where $n = 0$:

$$\forall \langle A, B, C \rangle = \begin{cases} \langle 1, 0, 0 \rangle & \text{if} \quad A = D; \\ \langle 0, 1, 0 \rangle & \text{if} \quad A \neq D, \ B \neq \emptyset; \\ \langle 0, 0, 1 \rangle & \text{if} \quad A \neq D, \ B = \emptyset. \end{cases}$$

We now introduce the definition of pragmatic satisfaction and the definition of quasi-truth.

[3] Clearly, it is always possible to rearrange the variables in order to quantify with respect to the first variable of the list. This fact will be tacitly assumed from now on.

Definition 3.6. Let $\varphi(x_1, \ldots, x_n)$ be a formula, $\mathfrak{A} = \langle D, (\cdot)^{\mathfrak{A}} \rangle$ a partial structure, and $\vec{a}$ in D^n. The sequence $\vec{a}$ *pragmatically satisfies* φ in $\mathfrak{A}$, denoted by $\mathfrak{A} \Vdash \varphi[\vec{a}]$, in the following cases:

(1) for φ an atomic formula and R a k-ary relation symbol, we have:
$\mathfrak{A} \Vdash\ R(\tau_1, \ldots, \tau_k)[\vec{a}]$ iff $(\tau_1^{\mathfrak{A}}[\vec{a}], \ldots, \tau_k^{\mathfrak{A}}[\vec{a}]) \in R_+^{\mathfrak{A}} \cup R_u^{\mathfrak{A}}$;

(2) $\mathfrak{A} \Vdash\ \neg\psi[\vec{a}]$ iff $\vec{a} \in \psi_-^{\mathfrak{A}} \cup \psi_u^{\mathfrak{A}}$;

(3) $\mathfrak{A} \Vdash (\varphi \wedge \psi)[\vec{a}]$ iff $\mathfrak{A} \Vdash \varphi[\vec{a}]$ and $\mathfrak{A} \Vdash \psi[\vec{a}]$;

(4) $\mathfrak{A} \Vdash (\varphi \to \psi)[\vec{a}]$ iff $\mathfrak{A} \nVdash \varphi[\vec{a}]$ or $\mathfrak{A} \Vdash \psi[\vec{a}]$;

(5) $\mathfrak{A} \Vdash \forall x\varphi[\vec{a}]$ iff $\mathfrak{A} \Vdash \varphi[b, \vec{a}]$, for all $b \in D$.[4] ∎

Observe that $\mathfrak{A} \Vdash (\varphi \vee \psi)[\vec{a}]$ iff $\mathfrak{A} \Vdash \varphi[\vec{a}]$ or $\mathfrak{A} \Vdash \psi[\vec{a}]$.

Definition 3.7. A formula $\varphi(x_1, \ldots, x_n)$ is *quasi-true* in a partial structure $\mathfrak{A}$ if for every sequence $\vec{a}$, $\mathfrak{A} \Vdash \varphi[\vec{a}]$. We denote that the formula φ is quasi-true in $\mathfrak{A}$ by $\mathfrak{A} \Vdash \varphi$ and we say that $\mathfrak{A}$ satisfies pragmatically φ, or φ is pragmatically satisfied by $\mathfrak{A}$. ∎

From this formalization, we generalize a well-known feature of the classical tarskian semantics, namely that, given a structure $\mathfrak{A}$, each first-order formula φ (with at most n free variables, for $n \geq 1$) defines inductively a set formed by n-tuples $\vec{a}$ in D^n for which the structure $\mathfrak{A}$ satisfies φ with parameters $\vec{a}$. More precisely, we can establish the following relations.

Proposition 3.8. Let $\mathfrak{A} = \langle D, (\cdot)^{\mathfrak{A}} \rangle$ be a partial structure, and $\varphi(x_1, \ldots, x_n)$ (for $n \geq 1$) a formula. Then:

(i) $\varphi_+^{\mathfrak{A}} \cup \varphi_u^{\mathfrak{A}} = \{\vec{a} \in D^n\ :\ \mathfrak{A} \Vdash \varphi[\vec{a}]\}$;

(ii) $\varphi_-^{\mathfrak{A}} \cup \varphi_u^{\mathfrak{A}} = \{\vec{a} \in D^n\ :\ \mathfrak{A} \Vdash (\neg\varphi)\,[\vec{a}]\}$;

(iii) $\varphi_+^{\mathfrak{A}} = \{\vec{a} \in D^n\ :\ \mathfrak{A} \Vdash \varphi[\vec{a}]$ and $\mathfrak{A} \nVdash (\neg\varphi)\,[\vec{a}]\}$;

(iv) $\varphi_-^{\mathfrak{A}} = \{\vec{a} \in D^n\ :\ \mathfrak{A} \nVdash \varphi[\vec{a}]$ and $\mathfrak{A} \Vdash (\neg\varphi)\,[\vec{a}]\}$;

(v) $\varphi_u^{\mathfrak{A}} = \{\vec{a} \in D^n\ :\ \mathfrak{A} \Vdash \varphi[\vec{a}]$ and $\mathfrak{A} \Vdash (\neg\varphi)\,[\vec{a}]\}$.

Proof. These items follow from Definitions 3.2 and 3.6. □

[4] As mentioned in Definition 3.4, it will be assumed, without loss of generality, that the quantified variable is the first variable of the resulting list of variables encompassing the free variables of φ.

Remark 3.9. From the last result and by definition of $\exists$ it is clear that: $\mathfrak{A} \Vdash \exists x\varphi[\vec{a}]$ iff $\mathfrak{A} \Vdash \varphi[b, \vec{a}]$, for at least one element $b \in D$. ∎

Observe that, when $n = 0$ (that is, in the case of sentences), the sequence $\vec{a}$ of Definition 3.6 is no longer necessary, since D^n is a singleton. As in the classical case, it is easy to see that, if φ is a sentence and $n \geq 1$ then, in a given partial structure $\mathfrak{A}$, $\mathfrak{A} \Vdash \varphi[\vec{a}]$ for every $\vec{a} \in D^n$ or $\mathfrak{A} \nVdash \varphi[\vec{a}]$ for every $\vec{a} \in D^n$. In particular, the following holds from Definitions 3.6 and 3.7:

Proposition 3.10. Let φ be a sentence and $\mathfrak{A}$ a partial structure. Then $\mathfrak{A} \Vdash \varphi$ iff $\varphi^{\mathfrak{A}}_{-} = 0$.[5] □

Now, from Definition 3.7 a notion of logical consequence between closed sentences is naturally induced:[6]

Definition 3.11. Let $\Gamma \cup \{\varphi\}$ be a set of sentences. We say that φ *is a pragmatic consequence of* Γ, denoted by $\Gamma \Vdash \varphi$, if $\mathfrak{A} \Vdash \varphi$ for every partial structure $\mathfrak{A}$ such that $\mathfrak{A} \Vdash \psi$ for all $\psi \in \Gamma$. ∎

From Proposition 3.8 (iii) it follows that the classical semantics is a particular case of the new one:

Proposition 3.12. If $\mathfrak{A}$ is a total structure, that is, $R^{\mathfrak{A}}_{u} = \emptyset$ for every relation symbol R, then $\varphi^{\mathfrak{A}}_{u} = \emptyset$ for every formula φ. □

From the previous propositions, we can see that the notion of quasi-truth via pragmatic satisfaction (cf. definitions 3.6 and 3.7) generalizes the tarskian notion of truth, the latter being, where all the relations are total, a particular case of the former. Clearly, the notion of truth and of quasi-truth coincide for the total structures, as well as their consequence relations. Furthermore, this semantics notion generalizes, to the whole language, a basic feature of the original concept of quasi-truth of [13], namely the conception of predicates as triples.

4 The 3-valued logic of quasi-truth

The preceding semantic framework motivates the development of a first-order logic underlying the notion of quasi-truth via pragmatic satisfaction. As a first step, a propositional axiomatic system called LPT will be introduced,

[5] We recall that, for sentences (that is, when $n = 0$), the triples associated to the formulas by a partial structure $\mathfrak{A}$ are of the form $\langle 1, 0, 0\rangle$, $\langle 0, 1, 0\rangle$ or $\langle 0, 0, 1\rangle$.

[6] Of course this notion can be naturally extended to arbitrary sets of formulas, provided that the set of variables occurring free is finite.

in which the semantics will be given by 3-valued matrices. This matrix logic, denoted by MPT, is the logic associated to sentences by means of the notion of pragmatic satisfaction introduced in the previous section. In other words, LPT is the propositional logic underlying the logic of quasi-truth introduced in Section 3.2. The corresponding theorem of soundness and completeness of LPT with respect to MPT will be established in Section 5 with the help of a bivaluation semantics. The second step, which will be done in Section 6, consists of extending LPT to a first-order system, denoted by LPT1, proving that the new system is sound and complete with respect to the notion of pragmatic consequence defined at the end of the previous section.

The 3-valued matrices of MPT will be defined now following the guidelines of the definition of pragmatic satisfaction. We recall that, for the case of the sentences (that is, when $n = 0$), the triples associated to any formula within a given partial structure are of the form $\langle 1, 0, 0\rangle$, $\langle 0, 0, 1\rangle$ and $\langle 0, 1, 0\rangle$. By simplicity, we will denote these triples as follows: $1 = \langle 1, 0, 0\rangle$, $\frac{1}{2} = \langle 0, 0, 1\rangle$ and $0 = \langle 0, 1, 0\rangle$. From Definition 3.2, we obtain, for example:

$$\begin{aligned} \neg \langle 1, 0, 0\rangle &= \langle 0, 1, 0\rangle \\ \langle 0, 0, 1\rangle \to \langle 0, 1, 0\rangle &= \langle 0, 1, 0\rangle \\ \langle 1, 0, 0\rangle \wedge \langle 0, 0, 1\rangle &= \langle 0, 0, 1\rangle \end{aligned}$$

Using the abbreviations introduced above, this can be simply written as follows: $\neg 1 = 0$, $\frac{1}{2} \to 0 = 0$ and $1 \wedge \frac{1}{2} = \frac{1}{2}$. Thereby, the truth-tables of MPT can be constructed straightforwardly:

$\to$	1	1/2	0
1	1	1	0
1/2	1	1	0
0	1	1	1

$\wedge$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

	$\neg$
1	0
1/2	1/2
0	1

Using Definition 3.6, it follows that the designated values of MPT are 1 and $\frac{1}{2}$. The language of MPT is the classical propositional one based on the connectives $\wedge$, $\to$, $\neg$, namely, conjunction, implication and negation, respectively. We denote by For the set of formulas over this signature generated by a denumerable set of propositional variables. The defined connectives are as follow:

$$\begin{aligned} \alpha \vee \beta &\overset{def}{=} \neg(\neg\alpha \wedge \neg\beta) && \text{disjunction} \\ \top_\alpha &\overset{def}{=} \alpha \to \alpha && \text{top} \\ \bot_\alpha &\overset{def}{=} \neg(\alpha \to \alpha) && \text{bottom} \end{aligned}$$

$$\sim\alpha \stackrel{def}{=} \alpha \to \bot_\alpha \qquad \text{classical negation}$$
$$\circ\alpha \stackrel{def}{=} \sim(\alpha \wedge \neg\alpha) \qquad \text{consistency}$$
$$\alpha \leftrightarrow \beta \stackrel{def}{=} (\alpha \to \beta) \wedge (\beta \to \alpha) \qquad \text{biconditional}$$

The truth-tables of the defined connectives are given below:

$\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

	$\sim$
1	0
1/2	0
0	1

	$\circ$
1	1
1/2	0
0	1

$\leftrightarrow$	1	1/2	0
1	1	1	0
1/2	1	1	0
0	0	0	1

	$\top$
1	1
1/2	1
0	1

	$\bot$
1	0
1/2	0
0	0

Remark 4.1. From Proposition 3.10 it is easy to see that, if φ is a sentence and $\mathfrak{A}$ is a partial structure:

(i) $\mathfrak{A} \Vdash \varphi$ iff $\varphi^{\mathfrak{A}} \in \{1, \frac{1}{2}\}$;

(ii) $\mathfrak{A} \Vdash \neg\varphi$ iff $\varphi^{\mathfrak{A}} \in \{0, \frac{1}{2}\}$. ■

Concerning the truth-tables presented above, it can be observed that this logic is one of the 8Kb 3-valued LFIs introduced in [12] (see also [3]). We will return to this point in Section 6.2.

Remark 4.2. Through the Definition 3.2, we can obtain the interpretation of the derived connectives by a partial structure $\mathfrak{A}$ in the context of formulas $\varphi(x_1, \ldots, x_n)$ having at most n free variables. Hence:

(i) $(\top_\varphi)^{\mathfrak{A}} = (\varphi \to \varphi)^{\mathfrak{A}} = \langle D^n, \emptyset, \emptyset\rangle$

(ii) $(\bot_\varphi)^{\mathfrak{A}} = (\neg\top_\varphi)^{\mathfrak{A}} = \langle \emptyset, D^n, \emptyset\rangle$

(iii) $(\sim\varphi)^{\mathfrak{A}} = (\varphi \to \bot_\varphi)^{\mathfrak{A}} = \left\langle \varphi_-^{\mathfrak{A}}, \varphi_+^{\mathfrak{A}} \cup \varphi_u^{\mathfrak{A}}, \emptyset\right\rangle$

(iv) $(\circ\varphi)^{\mathfrak{A}} = [\sim(\varphi \wedge \neg\varphi)]^{\mathfrak{A}} = \left[\sim\left\langle \emptyset, \varphi_+^{\mathfrak{A}} \cup \varphi_-^{\mathfrak{A}}, \varphi_u^{\mathfrak{A}}\right\rangle\right]^{\mathfrak{A}} = \left\langle \varphi_+^{\mathfrak{A}} \cup \varphi_-^{\mathfrak{A}}, \varphi_u^{\mathfrak{A}}, \emptyset\right\rangle$. ■

The matrix logic MPT is defined as usual.

Definition 4.3. A function $w : For \longrightarrow \left\{1, \frac{1}{2}, 0\right\}$ is a *valuation* for MPT if it is an homomorphism. That is, $w(\alpha \# \beta) = w(\alpha) \# w(\beta)$, for $\# \in \{\wedge, \rightarrow\}$, and $w(\neg\alpha) = \neg w(\alpha)$.

The consequence relation of MPT is defined as follows: for every set $\Gamma \cup \{\alpha\} \subseteq For$, $\Gamma \vDash_{TLP} \alpha$ iff there is a finite set $\Gamma_0 \subseteq \Gamma$ such that, for every valuation w for MPT, if $w(\Gamma_0) \subseteq \{1, \frac{1}{2}\}$ then $w(\alpha) \in \{1, \frac{1}{2}\}$. ■

We can easily verify that the logic MPT is a LFI where $\circ\alpha$ denotes the consistency of the formula α.

Inspired by [3], the logic MPT may be axiomatized by the following schemas of a Hilbert calculus defined over the language For. The resulting axiomatic system will be called LPT.

System LPT: Axiom schemas

(A1) $\alpha \rightarrow (\beta \rightarrow \alpha)$

(A2) $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow \gamma))$

(A3) $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$

(A4) $(\alpha \wedge \beta) \rightarrow \alpha$

(A5) $(\alpha \wedge \beta) \rightarrow \beta$

(A6) $\alpha \rightarrow (\alpha \vee \beta)$

(A7) $\beta \rightarrow (\alpha \vee \beta)$

(A8) $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma))$

(A9) $\alpha \vee (\alpha \rightarrow \beta)$

(A10) $\alpha \vee \neg\alpha$

(A11) $\neg\neg\alpha \leftrightarrow \alpha$

(A12) $\circ\alpha \rightarrow (\alpha \rightarrow (\neg\alpha \rightarrow \beta))$

(A13) $\neg\circ\alpha \rightarrow (\alpha \wedge \neg\alpha)$

(A14) $\circ(\alpha \rightarrow \beta)$

(A15) $(\circ\alpha \wedge \circ\beta) \rightarrow \circ(\alpha \wedge \beta)$

(A16) $(\alpha \wedge \neg\alpha \wedge \beta) \rightarrow \neg(\alpha \wedge \beta) \wedge \neg(\beta \wedge \alpha)$

Rule of inference: (MP) infer β from α and $\alpha \to \beta$. ■

We denote that α is a syntactic consequence of LPT from a set of formulas Γ by $\Gamma \vdash_{LPT} \alpha$.

In the following sections, it will be proved that the consequence relations of LPT and MPT coincide.

5 Bivaluation semantics for LPT

In this section, and adapting techniques from [3], we show that LPT is sound and complete with respect to a paraconsistent *bivaluation semantics*, i.e. truth functions (not truth-functional) that assign, for each sentence of the language, a truth-value 1 or 0.

Definition 5.1. Let $\mathbf{2} \overset{def}{=} \{0, 1\}$ be the set of truth-values, where 1 denotes 'true' and 0 denotes 'false'. A $\mathbf{2}_{LPT}$-valuation is any mapping $v : For \longrightarrow \mathbf{2}$ satisfying the following conditions (recall that $\circ$ is a defined connective):

($v1$) $v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = v(\beta) = 1$;

($v2$) $v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$;

($v3$) $v(\neg\alpha) = 0$ implies $v(\alpha) = 1$;

($v4$) $v(\alpha) = v(\neg\neg\alpha)$;

($v5$) $v(\circ\alpha) = 1$ implies $v(\alpha) \neq v(\neg\alpha)$;

($v6$) $v(\neg\circ\alpha) = 1$ implies $v(\alpha) = v(\neg\alpha) = 1$;

($v7$) $v(\circ(\alpha \to \beta)) = 1$;

($v8$) $v(\circ\alpha) = v(\circ\beta) = 1$ implies $v(\circ(\alpha \wedge \beta)) = 1$;

($v9$) $v(\alpha) = v(\neg\alpha) = v(\beta) = 1$ implies $v(\neg(\alpha \wedge \beta)) = v(\neg(\beta \wedge \alpha)) = 1$. ■

It is easy to prove that $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$; $v(\top) = 1$; $v(\bot) = 0$; and $v({\sim}\alpha) = 1$ iff $v(\alpha) = 0$.

For a set $\Gamma \cup \{\alpha\}$ of formulas of LPT, $\Gamma \vDash_{\mathbf{2}} \alpha$ denotes that there exists a finite set $\Gamma_0 \subseteq \Gamma$ such that $v(\alpha) = 1$ for every $\mathbf{2}_{LPT}$-valuation v such that $v(\Gamma_0) \subseteq 1$.

The soundness proof for LPT, with respect to bivaluation semantics introduced above is straightforward.

Theorem 5.2. [Soundness] Let $\Gamma \cup \{\alpha\}$ be a set of formulas in For. Then: $\Gamma \vdash_{LPT} \alpha$ implies $\Gamma \vDash_{\mathbf{2}} \alpha$.

Proof. It is sufficient check that all the axioms of LPT assume only the designated value 1 in any $\mathbf{2}_{LPT}$-valuation, and that (MP) preserves satisfaction: if $v(\alpha) = v(\alpha \to \beta) = 1$ then $v(\beta) = 1$, for for every $\mathbf{2}_{LPT}$-valuation v. □

In order to obtain the completeness of LPT, it is necessary to prove some auxiliary results.

Recall (see, for instance, [19]) that a logic $\mathbf{L}$ defined over a set of formulas $\mathbb{L}$ and with a consequence relation $\vdash_{\mathbf{L}}$ is *tarskian* if it satisfies the following properties: (i) if $\alpha \in \Gamma$ then $\Gamma \vdash_{\mathbf{L}} \alpha$; (ii) if $\Gamma \vdash_{\mathbf{L}} \alpha$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_{\mathbf{L}} \alpha$; and (iii) if $\Gamma, \beta \vdash_{\mathbf{L}} \alpha$ and $\Gamma \vdash_{\mathbf{L}} \beta$ then $\Gamma \vdash_{\mathbf{L}} \alpha$. A tarskian logic $\mathbf{L}$ is *finitary* if it satisfies: (iv) if $\Gamma \vdash_{\mathbf{L}} \alpha$ then there exists a finite subset Γ_0 of Γ such that $\Gamma_0 \vdash_{\mathbf{L}} \alpha$.

Definition 5.3. Let $\Delta \cup \{\alpha\}$ be a set of formulas in $\mathbb{L}$. We say that Δ is *maximal relatively to* α in $\mathbf{L}$, or α*-saturated* in $\mathbf{L}$, if $\Delta \nvdash_{\mathbf{L}} \alpha$ but $\Delta, \beta \vdash_{\mathbf{L}} \alpha$, for any formula β in $\mathbb{L}$ such that $\beta \notin \Delta$. ■

We say that a set of formulas Γ is *closed* in $\mathbf{L}$ if it contains all of its consequences in $\mathbf{L}$, that is: $\Gamma \vdash_{\mathbf{L}} \alpha$ iff $\alpha \in \Gamma$. It is easy to prove the following:

Lemma 5.4. Any set of formulas relatively maximal to α in $\mathbf{L}$ is closed, provided that $\mathbf{L}$ is tarskian. □

In Theorem 22.2 of [19] there is a proof of the following classical result:

Theorem 5.5. [Lindenbaum-Łos] Let $\mathbf{L}$ be a tarskian and finitary logic over the set of formulas $\mathbb{L}$. Let $\Gamma \cup \{\alpha\} \subseteq \mathbb{L}$ such that $\Gamma \nvdash_{\mathbf{L}} \alpha$. Then there exists an α-saturated set Δ in $\mathbf{L}$ such that $\Gamma \subseteq \Delta$. □

Since every logic $\mathbf{L}$ defined by a Hilbert calculus where the inference rules are finitary is tarskian and finitary, it is immediate that Theorem 5.5 holds in $\mathbf{L}$. In particular, Theorem 5.5 applies to LPT.

Now some properties specific of LPT will be proved.

Lemma 5.6. Let Δ be an α-saturated set in LPT. Then:

(i) $(\beta \wedge \lambda) \in \Delta$ iff $\beta \in \Delta$ and $\lambda \in \Delta$;

(ii) $(\beta \vee \lambda) \in \Delta$ iff $\beta \in \Delta$ or $\lambda \in \Delta$;

(iii) $(\beta \to \lambda) \in \Delta$ iff $\beta \notin \Delta$ or $\lambda \in \Delta$;

(iv) $\beta \notin \Delta$ implies $\neg\beta \in \Delta$;

(v) $\beta \in \Delta$ iff $\neg\neg\beta \in \Delta$;

(vi) $\bot \notin \Delta$;

(vii) $\circ\beta \in \Delta$ iff $\beta \notin \Delta$ or $\neg\beta \notin \Delta$;

(viii) $\neg\circ\beta \in \Delta$ iff $\beta \in \Delta$ and $\neg\beta \in \Delta$;

(ix) $\circ(\beta \to \lambda) \in \Delta$;

(x) $\circ\beta, \circ\lambda \in \Delta$ implies $\circ(\beta \wedge \lambda) \in \Delta$;

(xi) $\beta, \neg\beta, \lambda \in \Delta$ implies $\neg(\beta \wedge \lambda) \in \Delta$ and $\neg(\lambda \wedge \beta) \in \Delta$.

Proof: The proof of items (i) to (viii) can be found in [3], where similar systems were treated. We now prove:
(ix) By axiom (A14), $\Delta \vdash_{LPT} \circ(\beta \to \lambda)$. Hence, by Lemma 5.4, it follows that $\circ(\beta \to \lambda) \in \Delta$.
(x) Assume that $\circ\beta \in \Delta$ and $\circ\lambda \in \Delta$. Hence, by item (i), $(\circ\beta \wedge \circ\lambda) \in \Delta$. So, $\Delta \vdash_{LPT} \circ\beta \wedge \circ\lambda$. By the axiom (A15), $\Delta \vdash_{LPT} (\circ\beta \wedge \circ\lambda) \to \circ(\beta \wedge \lambda)$. Hence, $\Delta \vdash_{LPT} \circ(\beta \wedge \lambda)$. Thus, by the Lemma 5.4, $\circ(\beta \wedge \lambda) \in \Delta$.
(xi) It is proved by item (i) and axiom (A16). □

Recall that, given a subset B of a set A, its *characteristic function* is the function $\chi_B : A \longrightarrow \{0, 1\}$ such that, for every $x \in A$, $\chi_B(x) = 1$ iff $x \in B$.

Corollary 5.7. The characteristic function of an α-saturated set of formulas in LPT defines a $\mathbf{2}_{LPT}$-valuation.

Proof. Let Δ be a set of formulas maximal relatively to α in LPT and define a function $\upsilon : For \longrightarrow \{0, 1\}$ such that, for any formula λ in For, $\upsilon(\lambda) = 1$ iff $\lambda \in \Delta$. Hence, by Lemma 5.6, it is easy to prove that υ satisfies clauses $(\upsilon 1)$ to $(\upsilon 9)$ of Definition 5.1. □

The next step towards the proof of completeness of LPT is to establish a link between the bivaluation semantics and the valuations over the matrices of MPT. Thus, the completeness of LPT will be a consequence of its completeness with respect to bivaluations.

Lemma 5.8. For any $\mathbf{2}_{LPT}$-valuation υ, there exists a valuation w for MPT such that $\upsilon(\alpha) = 1$ iff $w(\alpha) \in \left\{1, \frac{1}{2}\right\}$, for every formula α.

Proof: Let υ be a $\mathbf{2}_{LPT}$-valuation. Consider a valuation w for LPT such that, for every atomic formula p, $w(p)$ is defined by:

$$w(p) = \begin{cases} 1 & \text{iff} \quad \upsilon(p) = 1, \text{and } \upsilon(\neg p) = 0 \\ \frac{1}{2} & \text{iff} \quad \upsilon(p) = 1, \text{and } \upsilon(\neg p) = 1 \\ 0 & \text{iff} \quad \upsilon(p) = 0. \end{cases}$$

By induction on the complexity of a formula α, it will be proved the following:

$$w(\alpha) = \begin{cases} 1 & \text{iff} \quad \upsilon(\alpha) = 1, \text{and } \upsilon(\neg\alpha) = 0 & \text{(a)} \\ \frac{1}{2} & \text{iff} \quad \upsilon(\alpha) = 1, \text{and } \upsilon(\neg\alpha) = 1 & \text{(b)} \\ 0 & \text{iff} \quad \upsilon(\alpha) = 0 & \text{(c)} \end{cases}$$

[If α is atomic:] the result follows by definition of w.
(IH) Assume that the result holds for every formula with complexity $k < n$.
[If $\alpha = \neg\beta$:]

(a)($\Rightarrow$) If $w(\alpha) = w(\neg\beta) = 1$, then (by the truth-table of $\neg$) $w(\beta) = 0$. Hence, by (IH), $\upsilon(\beta) = 0$. So, by ($\upsilon3$), $\upsilon(\neg\beta) = \upsilon(\alpha) = 1$. And, by ($\upsilon4$) $\upsilon(\neg\alpha) = \upsilon(\neg\neg\beta) = \upsilon(\beta) = 0$.

(a)($\Leftarrow$) If $\upsilon(\alpha) = 1$ and $\upsilon(\neg\alpha) = 0$, then by ($\upsilon4$), $\upsilon(\neg\neg\beta) = \upsilon(\beta) = 0$. Hence, by (IH) $w(\beta) = 0$. Thus, $w(\alpha) = w(\neg\beta) = \neg w(\beta) = 1$.

(b)($\Rightarrow$) If $w(\alpha) = w(\neg\beta) = \frac{1}{2}$, then $w(\beta) = \frac{1}{2}$. So, by (IH), $\upsilon(\beta) = 1$ and $\upsilon(\neg\beta) = 1$. Hence, $\upsilon(\alpha) = \upsilon(\neg\beta) = 1$. Thus, by ($\upsilon4$), $\upsilon(\neg\alpha) = \upsilon(\neg\neg\beta) = \upsilon(\beta) = 1$.

(b)($\Leftarrow$) If $\upsilon(\alpha) = \upsilon(\neg\alpha) = 1$, then by ($\upsilon4$), $\upsilon(\neg\beta) = \upsilon(\beta) = 1$. Hence, by (IH), $w(\beta) = \frac{1}{2}$. So $w(\alpha) = w(\neg\beta) = \neg w(\beta) = \frac{1}{2}$.

(c)($\Rightarrow$) If $w(\alpha) = w(\neg\beta) = 0$, then $w(\beta) = 1$. Hence, by (IH), $\upsilon(\beta) = 1$ and $\upsilon(\alpha) = \upsilon(\neg\beta) = 0$.

(c)($\Leftarrow$) If $\upsilon(\alpha) = \upsilon(\neg\beta) = 0$, then by ($\upsilon3$), $\upsilon(\beta) = 1$. So, by (IH), $w(\beta) = 1$. Hence, $w(\alpha) = w(\neg\beta) = \neg w(\beta) = 0$.

If α is $\beta \wedge \lambda$ or $\beta \to \lambda$, the result is similarly proved. □

From this result, we have:

Lemma 5.9. For every valuation w for MPT, there exists a $\mathbf{2}_{LPT}$-valuation υ, such that $\upsilon(\alpha) = 1$ iff $w(\alpha) \in \left\{1, \frac{1}{2}\right\}$ for every formula α.

Proof. Let w be a valuation for TLP. Define a mapping $\upsilon : For \longrightarrow \mathbf{2}$ such that $\upsilon(\alpha) = 1$ iff $w(\alpha) \in \mathcal{D} = \left\{1, \frac{1}{2}\right\}$. It will be proved that υ satisfies the clauses ($\upsilon1$) to ($\upsilon9$) of Definition 5.1.
[$\upsilon1$]: $\upsilon(\beta \wedge \lambda) = 1$ iff $w(\beta \wedge \lambda) \in \mathcal{D}$ iff $w(\beta) \in \mathcal{D}$ and $w(\lambda) \in \mathcal{D}$ iff $\upsilon(\beta) = \upsilon(\lambda) = 1$.
[$\upsilon2$]: $\upsilon(\beta \to \lambda) = 1$ iff $w(\beta \to \lambda) \in \mathcal{D}$ iff either $w(\beta) = 0$ or $w(\lambda) \in \mathcal{D}$ iff either $\upsilon(\beta) = 0$ or $\upsilon(\lambda) = 1$.

[υ3]: If $\upsilon(\neg\beta) = 0$, then $w(\neg\beta) = 0$. Hence, $w(\beta) = 1$. Therefore, $\upsilon(\beta) = 1$.
[υ4]: Suppose that $\upsilon(\beta) = 1$; then $w(\beta) \in \mathcal{D}$. Since $w(\beta) = w(\neg\neg\beta)$, it follows that $\upsilon(\neg\neg\beta) = 1$. On the other hand, if $\upsilon(\beta) = 0$ then $w(\beta) = 0 = w(\neg\neg\beta)$, so $\upsilon(\neg\neg\beta) = 0$.
[υ5]: $\upsilon(\circ\beta) = 1$ implies $w(\circ\beta) \in \mathcal{D}$ implies $w(\circ\beta) = 1$, so $w(\beta) \neq w(\neg\beta)$, where $w(\beta), w(\neg\beta) \in \{0, 1\}$. Hence, $\upsilon(\beta) \neq \upsilon(\neg\beta)$.
[υ6]: If $\upsilon(\neg\circ\beta) = 1$, then $w(\neg\circ\beta) \in \mathcal{D}$. By the truth-table of $\circ$, $w(\neg\circ\beta) = 1$. Hence, $w(\circ\beta) = 0$ and so $w(\beta) = \frac{1}{2}$. Thus, $w(\neg\beta) = \frac{1}{2}$. From this $\upsilon(\beta) = \upsilon(\neg\beta) = 1$.

[υ7] to [υ9] can be proved in a similar way. □

Corollary 5.10. $\Gamma \vDash_{MPT} \alpha$ iff $\Gamma \vDash_{\mathbf{2}} \alpha$.

Proof. ($\Rightarrow$) Assume that $\Gamma \vDash_{MPT} \alpha$ and let υ be a $\mathbf{2}_{LPT}$-valuation such that, for every $\beta \in \Gamma$, $\upsilon(\beta) = 1$. Let w be a valuation for MPT defined from υ as in Lemma 5.8. Then, for every $\beta \in \Gamma$, $w(\beta) \in \mathcal{D}$. So, by hypothesis, $w(\alpha) \in \mathcal{D}$. Hence, by definition of w, $\upsilon(\alpha) = 1$. Therefore, $\Gamma \vDash_{\mathbf{2}} \alpha$.
($\Leftarrow$) Suppose that $\Gamma \vDash_{\mathbf{2}} \alpha$ and let w be a valuation of MPT such that, for every $\beta \in \Gamma$, $w(\beta) \in \mathcal{D}$. Let υ be a $\mathbf{2}_{LPT}$-valuation defined from w as in Lemma 5.9. Then, for every $\beta \in \Gamma$, $\upsilon(\beta) = 1$. By hypothesis, $\upsilon(\alpha) = 1$ and so, by definition of υ, $w(\alpha) \in \mathcal{D}$. Therefore, $\Gamma \vDash_{MPT} \alpha$. □

From the results above, the completeness of LPT with respect to the bivaluation semantics can be stated.

Theorem 5.11. [Completeness of LPT w.r.t. bivaluations]
Let $\Gamma \cup \{\alpha\}$ be a set of formulas in For. Then: $\Gamma \vDash_{\mathbf{2}} \alpha$ implies $\Gamma \vdash_{LPT} \alpha$.

Proof. Let $\Gamma \cup \{\alpha\}$ be a set of formulas in For such that $\Gamma \nvdash_{LPT} \alpha$. By Theorem 5.5 there exists an α-saturated set Δ in LPT such that $\Gamma \subseteq \Delta$. Since $\Delta \nvdash_{LPT} \alpha$ then $\alpha \notin \Delta$. By Corollary 5.7, the characteristic function υ of Δ is a $\mathbf{2}_{LPT}$-valuation such that, for every $\beta \in \Delta$, $\upsilon(\beta) = 1$, and $\upsilon(\alpha) = 0$. Therefore, $\Delta \nvDash_{\mathbf{2}} \alpha$. Since $\Gamma \subseteq \Delta$, it follows that $\Gamma \nvDash_{\mathbf{2}} \alpha$. □

Theorem 5.12. [Soundness and Completeness of LPT w.r.t. MPT]
Let $\Gamma \cup \{\alpha\}$ be a set of formulas in For. Then: $\Gamma \vDash_{MPT} \alpha$ iff $\Gamma \vdash_{LPT} \alpha$.

Proof. Assume that $\Gamma \vDash_{MPT} \alpha$. By Corollary 5.10, it follows that $\Gamma \vDash_{\mathbf{2}} \alpha$. Hence, by Theorem 5.11, we get that $\Gamma \vdash_{LPT} \alpha$. Conversely, if $\Gamma \vdash_{LPT} \alpha$ then $\Gamma \vDash_{\mathbf{2}} \alpha$, by Theorem 5.2, and so $\Gamma \vDash_{MPT} \alpha$, by Corollary 5.10. □

As an application of the last result, the following can be easily proved by using the truth-tables of MPT:

Proposition 5.13. The following formulas are theorems in LPT:
(i) $\neg(\alpha \wedge \beta) \leftrightarrow \neg\alpha \vee \neg\beta$; (ii) $\neg(\alpha \vee \beta) \leftrightarrow \neg\alpha \wedge \neg\beta$;
(iii) $(\alpha \to \beta) \leftrightarrow ({\sim}\alpha \vee \beta)$; (iv) $\neg(\alpha \to \beta) \leftrightarrow ({\sim}(\alpha \to \beta))$;
(v) $\neg(\alpha \to \beta) \leftrightarrow (\alpha \wedge {\sim}\beta)$; (vi) ${\sim}{\sim}\alpha \leftrightarrow \alpha$. □

6 The first-order logic for quasi-truth

In this section the logic LPT1, the first-order version of LPT, will be defined. Besides, the completeness of LPT1 with respect to the pragmatic semantics of partial structures introduced at the end of Section 2 will be established.[7]

The language of LPT1 is the usual first-order language $\mathbb{L}(\Sigma)$ based on the connectives $\wedge$, $\to$, $\neg$, the quantifier $\forall$ and a first-order signature Σ. The logic LPT1 is given by means of the Hilbertian axiomatic method.

System LPT1: Axiom schemas
All the axioms of LPT in the language $\mathbb{L}(\Sigma)$ plus the following:

(A17) $\forall x\varphi \to \varphi[x/\tau]$, where τ is a term free for x in φ.

Inference rules:
In addition to (MP), we have:

(I$\forall$) Infer $\alpha \to \forall x\beta$ from $\alpha \to \beta$, whenever x does not occur free in α.

The next step is to obtain the adequacy (soundness and completeness) of the calculus LPT1 with respect to the semantics of pragmatic satisfaction by partial structures introduced in Definition 3.6. As usual, the proof of soundness is an easy task:

Theorem 6.1. [Soundness] Let $\Gamma \cup \{\varphi\}$ be a set of closed first-order sentences. Then: $\Gamma \vdash_{LPT1} \varphi$ implies $\Gamma \Vdash \varphi$.

Proof. It is easy to prove that the axioms of LPT1 are valid in every partial structure, using Definition 3.6. Finally, by the same definition, the rules of inference preserve satisfaction. □

The proof of completeness of LPT1 with respect to the semantics of partial structures will follow, with minor modifications, the steps of the Henkin's proof of completeness of the calculus of classical first-order logic with respect to tarskian structures.

[7]Since the quantifiers are defined here in a different way that in the related approach [18], the first-order logic LPT1 presented here differs from the logic LPT1 proposed in [18].

We begin by observing that Lemma 5.6 holds in LPT1 for sets Δ of sentences of the language $\mathbb{L}(\Sigma)$ which are α-saturated. However, being α-saturated in LPT1 is not enough to satisfy a crucial property for the Henkin proof of completeness. In fact, as in the classical case, it will be needed to extend conservatively a given α-saturated set to a Henkin set. Recall that $\exists x\varphi$ is an abbreviation for $\neg\forall x\neg\varphi$. The suitable notion of Henkin set for LPT1 is the following:

Definition 6.2. A set of formulas Δ in $\mathbb{L}(\Sigma)$ is a *Henkin set* for LPT1 if:

(H1) for every sentence of the form $\exists x\varphi$ there is some constant c in the signature Σ such that $\Delta \vdash_{LPT1} \exists x\varphi \to \varphi[x/c]$;

(H2) for every sentence of the form $\forall x\varphi$, if $\Delta \vdash_{LPT1} \varphi[x/c]$ for every constant c in the signature Σ, then $\Delta \vdash_{LPT1} \forall x\varphi$. ■

Theorem 6.3. Let Δ be a set of formulas in $\mathbb{L}(\Sigma)$. Then, there exists a signature Σ' extending Σ and a Henkin set Δ' for LPT1 in $\mathbb{L}(\Sigma')$ such that $\Delta \subseteq \Delta'$. Moreover, Δ' is a conservative extension of Δ, that is: $\Delta' \vdash_{LPT1} \varphi$ iff $\Delta \vdash_{LPT1} \varphi$, for every sentence φ in $\mathbb{L}(\Sigma)$. In addition, if Δ'' is a set of formulas in $\mathbb{L}(\Sigma')$ such that $\Delta' \subseteq \Delta''$ then Δ'' is a Henkin set for LPT1.

Proof. The proof is similar to the corresponding proof for classical first-order logic. This is a consequence of the fact that LPT1 is an extension of classical logic.[8] □

Theorem 6.4. Let Δ be a set of formulas in $\mathbb{L}(\Sigma)$ which is α-saturated in LPT1 for some sentence α in $\mathbb{L}(\Sigma)$. Then, there exists a signature Σ' extending Σ and a Henkin set Δ'' for LPT1 in $\mathbb{L}(\Sigma')$ such that $\Delta \subseteq \Delta''$ and Δ'' is α-saturated in LPT1 over $\mathbb{L}(\Sigma')$. Moreover, Δ'' is a conservative extension of Δ, that is: $\Delta'' \vdash_{LPT1} \varphi$ iff $\Delta \vdash_{LPT1} \varphi$, for every sentence φ in $\mathbb{L}(\Sigma)$.

Proof. Given an α-saturated set Δ, there exists a Henkin set Δ' over a signature Σ' conservatively extending Δ, by Theorem 6.3. Since $\Delta \nvdash_{LPT1} \alpha$ then $\Delta' \nvdash_{LPT1} \alpha$. Given that LPT1 is tarskian and finitary, there exists a set Δ'' of formulas in $\mathbb{L}(\Sigma')$ which is α-saturated in LPT1 such that $\Delta' \subseteq \Delta''$, by Theorem 5.5. Moreover, Δ'' is a Henkin set over $\mathbb{L}(\Sigma')$, by Theorem 6.3. Suppose now that φ is a sentence of $\mathbb{L}(\Sigma)$ such that $\Delta'' \vdash_{LPT1} \varphi$, and suppose that $\Delta \nvdash_{LPT1} \varphi$. Then $\Delta, \varphi \vdash_{LPT1} \alpha$, since Δ is α-saturated. But then $\Delta'', \varphi \vdash_{LPT1} \alpha$ and so $\Delta'' \vdash_{LPT1} \alpha$, a contradiction. Therefore $\Delta \vdash_{LPT1} \varphi$. This shows that Δ'' is a conservative extension of Δ. □

[8]For a proof of this result for first-order LFIs similar to LPT1, see [16].

Lemma 6.5. Let Δ be an α-saturated set of sentences over a given first-order signature Σ. If φ is a sentence in $\mathbb{L}(\Sigma)$ such that $\Delta \nvdash_{LPT1} \varphi$ then $\Delta \vdash_{LPT1} {\sim}\varphi$.

Proof. Suppose that $\Delta \nvdash_{LPT1} {\sim}\alpha$. Then, $\Delta, {\sim}\alpha \vdash_{LPT1} \alpha$ and so $\Delta \vdash_{LPT1} \alpha$, a contradiction. Hence $\Delta \vdash_{LPT1} {\sim}\alpha$. Now, assume that φ is a sentence in $\mathbb{L}(\Sigma)$ such that $\Delta \nvdash_{LPT1} \varphi$. Therefore $\Delta, \varphi \vdash_{LPT1} \alpha$ and so $\Delta, \varphi \vdash_{LPT1} \bot$, since $\Delta, \varphi \vdash_{LPT1} {\sim}\alpha$. From this, $\Delta \vdash_{LPT1} {\sim}\varphi$. □

Since the propositional basis of LPT1 is LPT, it follows:

Lemma 6.6. Let Δ be an α-saturated set of sentences in $\mathbb{L}(\Sigma)$, and let Δ'' be an α-saturated Henkin set in $\mathbb{L}(\Sigma')$ conservatively extending Δ as in Theorem 6.4. Then Δ'' is a closed theory which satisfies the properties (i)-(xi) of Lemma 5.6 for sentences in $\mathbb{L}(\Sigma')$. Additionally, Δ'' satisfies the following properties:
(xii) $\quad \forall x\varphi \in \Delta''$ iff $\varphi[x/\tau] \in \Delta''$, for all $\tau \in D$
(xiii) $\quad \exists x\varphi \in \Delta''$ iff $\varphi[x/\tau] \in \Delta''$, for some $\tau \in D$. □

Definition 6.7. Let Δ and Δ'' as in Lemma 6.6. The partial structure $\mathfrak{A}(\Delta'')$ is defined over the domain D of closed terms of $\mathbb{L}(\Sigma')$ (i.e. terms without variables) such that the interpretation of the symbols of Σ' is as follows (here, P, f and c denote symbols of predicates, functions and constants, respectively):

$P_{+}^{\mathfrak{A}(\Delta'')} = \{(\tau_1, \ldots, \tau_n) \in D^n : P(\tau_1, \ldots, \tau_n) \in \Delta'' \text{ and } \neg P(\tau_1, \ldots, \tau_n) \notin \Delta''\}$;
$P_{-}^{\mathfrak{A}(\Delta'')} = \{(\tau_1, \ldots, \tau_n) \in D^n : P(\tau_1, \ldots, \tau_n) \notin \Delta'' \text{ and } \neg P(\tau_1, \ldots, \tau_n) \in \Delta''\}$;
$P_{u}^{\mathfrak{A}(\Delta'')} = \{(\tau_1, \ldots, \tau_n) \in D^n : P(\tau_1, \ldots, \tau_n) \in \Delta'' \text{ and } \neg P(\tau_1, \ldots, \tau_n) \in \Delta''\}$;
$f^{\mathfrak{A}(\Delta'')} : D^n \longrightarrow D$ is such that $f^{\mathfrak{A}(\Delta'')}(\tau_1, \ldots, \tau_n) = f(\tau_1, \ldots, \tau_n)$ for all $(\tau_1, \ldots, \tau_n) \in D^n$;
$c^{\mathfrak{A}(\Delta'')} = c$. ■

Lemma 6.8. Let Δ and Δ'' as in Lemma 6.6. Let $\varphi(x)$ be a formula in $\mathbb{L}(\Sigma')$ such that x is the unique variable occurring free. Let τ be a closed term in $\mathbb{L}(\Sigma')$ and $\bar{\tau}$ be the corresponding constant of the diagram language of $\mathfrak{A}(\Delta'')$.[9] Then $\mathfrak{A}(\Delta'') \Vdash \varphi[x/\bar{\tau}]$ iff $\mathfrak{A}(\Delta'') \Vdash \varphi[x/\tau]$.

[9] Notice that τ can be seen simultaneously as a term (and so it can replace free occurrences of variables inside a formula) and as an element of the domain D of the structure $\mathfrak{A}(\Delta'')$ (and so it is represented by the constant $\bar{\tau}$ of the diagram language of $\mathfrak{A}(\Delta'')$.

Proof. Observe that $\bar{\tau}^{\mathfrak{A}(\Delta'')} = \tau = \tau^{\mathfrak{A}(\Delta'')}$. From this, it can be proved by induction on the complexity of φ that $\mathfrak{A}(\Delta'') \Vdash \varphi[x/\bar{\tau}]$ iff $\mathfrak{A}(\Delta'') \Vdash \varphi[\tau]$ iff $\mathfrak{A}(\Delta'') \Vdash \varphi[x/\tau]$.[10] $\square$

Lemma 6.9. Let Δ and Δ'' as in Lemma 6.6. Let $\mathfrak{A}(\Delta'')$ be the partial structure constructed as in Definition 6.7. Then, for every sentence φ in $\mathbb{L}(\Sigma')$:

$$\varphi^{\mathfrak{A}(\Delta'')} = \begin{cases} 1 & \text{iff} \quad \varphi \in \Delta'' \text{ and } \neg\varphi \notin \Delta'' \\ \frac{1}{2} & \text{iff} \quad \varphi \in \Delta'' \text{ and } \neg\varphi \in \Delta'' \\ 0 & \text{iff} \quad \varphi \notin \Delta'' \text{ and } \neg\varphi \in \Delta''. \end{cases}$$

Proof. By induction on the complexity of the sentence φ.
[**Base:** $\varphi = P(\tau_1, \ldots, \tau_k)$ is atomic]
Then: $\varphi^{\mathfrak{A}(\Delta'')} = 1$ iff $\mathfrak{A}(\Delta'') \Vdash \varphi$ and $\mathfrak{A}(\Delta'') \nVdash \neg\varphi$ iff $(\tau_1, \ldots, \tau_k) \in P_+^{\mathfrak{A}(\Delta'')}$ iff $\varphi \in \Delta''$ and $\neg\varphi \notin \Delta''$, by Definition 6.7. The cases when $\varphi^{\mathfrak{A}(\Delta'')} = \frac{1}{2}$ and $\varphi^{\mathfrak{A}(\Delta'')} = 0$ are proved similarly.
(IH) Assume that the proposition holds for every formula with complexity $k < n$.

[**Case a**: $\varphi = \neg\psi$]
Then: $\varphi^{\mathfrak{A}(\Delta'')} = 1$ iff $(\neg\psi)^{\mathfrak{A}(\Delta'')} = 1$ iff $\psi^{\mathfrak{A}(\Delta'')} = 0$ iff, by (IH), $\psi \notin \Delta''$ and $\neg\psi \in \Delta''$ iff $\neg\psi \in \Delta''$ and $\neg\neg\psi \notin \Delta''$ iff $\varphi \in \Delta''$ and $\neg\varphi \notin \Delta''$. The cases when $\varphi^{\mathfrak{A}(\Delta'')} = \frac{1}{2}$ and $\varphi^{\mathfrak{A}(\Delta'')} = 0$ are proved similarly.

[**Case b**: $\varphi = \beta \to \psi$]
(b.1) Suppose that $\varphi^{\mathfrak{A}(\Delta'')} = (\beta \to \psi)^{\mathfrak{A}(\Delta'')} = 1$. Then, either $\beta^{\mathfrak{A}(\Delta'')} = 0$ or $\psi^{\mathfrak{A}(\Delta'')} \in \{1, \frac{1}{2}\}$. By (IH), either $\beta \notin \Delta''$ and $\neg\beta \in \Delta''$, or $\psi \in \Delta''$. In the first case $\sim\beta \in \Delta''$, by Lemma 6.5. Then $(\sim\beta \vee \psi) \in \Delta''$ and so $(\beta \to \psi) \in \Delta''$, that is, $\varphi \in \Delta''$. In the second case it follows, by monotonicity, that $\Delta'', \beta \vdash_{LPT1} \psi$ and so, by the deduction theorem (which is valid for sentences) $\Delta'' \vdash_{LPT1} \beta \to \psi$. That is, $\varphi \in \Delta''$. This shows that, in any case, $\varphi \in \Delta''$. Suppose that $\neg\varphi \in \Delta''$. Then, by (A12) and (A14) it follows that $\Delta'' \vdash_{LPT1} \alpha$, a contradiction. Therefore $\neg\varphi \notin \Delta''$.

Conversely, assume that $\varphi \in \Delta''$ and $\neg\varphi \notin \Delta''$. Suppose that $\beta^{\mathfrak{A}(\Delta'')} \in \{1, \frac{1}{2}\}$ and $\psi^{\mathfrak{A}(\Delta'')} = 0$. By (IH), $\beta \in \Delta''$ and $\psi \notin \Delta''$. Then $\Delta'', \psi \vdash_{LPT1} \alpha$, since Δ'' is α-saturated. But $\Delta'', \beta \vdash_{LPT1} \psi$, since $\varphi \in \Delta''$. Then $\Delta'' \vdash_{LPT1} \psi$ (since $\beta \in \Delta''$) and so $\Delta'' \vdash_{LPT1} \alpha$, a contradiction. Therefore, either $\beta^{\mathfrak{A}(\Delta'')} = 0$ or $\psi^{\mathfrak{A}(\Delta'')} \in \{1, \frac{1}{2}\}$. Thus $\varphi^{\mathfrak{A}(\Delta'')} = (\beta \to \psi)^{\mathfrak{A}(\Delta'')} = 1$.

(b.2) Suppose that $\varphi^{\mathfrak{A}(\Delta'')} = (\beta \to \psi)^{\mathfrak{A}(\Delta'')} = 0$. Then, $\beta^{\mathfrak{A}(\Delta'')} \in \{1, \frac{1}{2}\}$ and

[10] Recall that $\beta[x/t]$ denotes the formula obtained by β by substituting every free occurrence of x by the term t; on the other hand, $\mathfrak{A} \Vdash \beta[a]$ means that $a \in D$ pragmatically satisfies β in the partial structure $\mathfrak{A}$, cf. Definition 3.6.

$\psi^{\mathfrak{A}(\Delta'')} = 0$. By (IH), $\beta \in \Delta''$, $\psi \notin \Delta''$ and $\neg\psi \in \Delta''$. If $\beta \to \psi \in \Delta''$ then, as proved in (b.1), a contradiction arises. Hence $\beta \to \psi \notin \Delta''$, that is, $\varphi \notin \Delta''$. Then, by Lemma 5.6(iv), $\neg\varphi \in \Delta''$.

Conversely, assume that $\varphi \notin \Delta''$ and $\neg\varphi \in \Delta''$. Since $\neg\varphi$ implies $\beta \wedge {\sim}\psi$ in LPT1 [11] then $\beta \in \Delta''$ and $\psi \notin \Delta''$. By (IH), $\beta^{\mathfrak{A}(\Delta'')} \in \{1, \frac{1}{2}\}$ and $\psi^{\mathfrak{A}(\Delta'')} = 0$, and so $\varphi^{\mathfrak{A}(\Delta'')} = (\beta \to \psi)^{\mathfrak{A}(\Delta'')} = 0$.

(b.3) Observe that $\varphi^{\mathfrak{A}(\Delta'')} = (\beta \to \psi)^{\mathfrak{A}(\Delta'')} \neq \frac{1}{2}$.

On the other hand, if $\beta \to \psi \in \Delta''$ and $\neg(\beta \to \psi) \in \Delta''$ then, by (A12) and (A14) $\alpha \in \Delta''$, a contradiction.

[**Case c**: $\varphi = \forall x\psi$]

(c.1) $(\forall x\psi)^{\mathfrak{A}(\Delta'')} = 1$ iff, by Remark 4.1, $\mathfrak{A}(\Delta'') \Vdash \forall x\psi$ and $\mathfrak{A}(\Delta'') \nVdash \neg\forall x\psi$, iff $\mathfrak{A}(\Delta'') \Vdash \forall x\psi$ and $\mathfrak{A}(\Delta'') \nVdash \exists x\neg\psi$, by definition of $\exists$, iff $(\psi[x/\bar{\tau}])^{\mathfrak{A}(\Delta'')} = 1$ for every $\tau \in D$ iff, by Lemma 6.8, $(\psi[x/\tau])^{\mathfrak{A}(\Delta'')} = 1$ for every $\tau \in D$ iff, by (IH), $\psi[x/\tau] \in \Delta''$ and $\neg\psi[x/\tau] \notin \Delta''$ for every $\tau \in D$ iff, by Lemma 6.6, $\forall x\psi \in \Delta''$ and $\exists x\neg\psi \notin \Delta''$ iff, by definition of $\exists$, $\forall x\psi \in \Delta''$ and $\neg\forall x\psi \notin \Delta''$.

(c.2) $(\forall x\psi)^{\mathfrak{A}(\Delta'')} = 0$ iff $(\psi[x/\bar{\tau}])^{\mathfrak{A}(\Delta'')} = 0$ for some $\tau \in D$ iff, by Lemma 6.8, $(\psi[x/\tau])^{\mathfrak{A}(\Delta'')} = 0$ for some $\tau \in D$ iff, by (IH), $\psi[x/\tau] \notin \Delta''$ and $\neg\psi[x/\tau] \in \Delta''$ for some $\tau \in D$ iff, by Lemma 6.6, $\forall x\psi \notin \Delta''$ and $\neg\forall x\psi \in \Delta''$ (in the last two steps we also use Lemma 5.6(iv) and definition of $\exists$).

(c.3) $(\forall x\psi)^{\mathfrak{A}(\Delta'')} = \frac{1}{2}$ iff, by Remark 4.1, $\mathfrak{A}(\Delta'') \Vdash \forall x\psi$ and $\mathfrak{A}(\Delta'') \Vdash \neg\forall x\psi$, iff $\mathfrak{A}(\Delta'') \Vdash \forall x\psi$ and $\mathfrak{A}(\Delta'') \Vdash \exists x\neg\psi$, by definition of $\exists$, iff $(\psi[x/\tau])^{\mathfrak{A}(\Delta'')} \in \{1, \frac{1}{2}\}$ for every $\tau \in D$, and $(\psi[x/\tau])^{\mathfrak{A}(\Delta'')} \in \{0, \frac{1}{2}\}$ for some $\tau \in D$, by Remark 3.9, iff $\psi[x/\tau] \in \Delta''$ for every $\tau \in D$ and $\neg\psi[x/\tau] \in \Delta''$ for some $\tau \in D$, by (IH), iff $\forall x\psi \in \Delta''$ and $\exists x\neg\psi \in \Delta''$, by Lemma 6.6, iff $\forall x\psi \in \Delta''$ and $\neg\forall x\psi \in \Delta''$, by definition of $\exists$.

[**Case d**: $\varphi = \beta \wedge \psi$]. It is proved analogously. □

From the last lemma it follows the fundamental result:

Theorem 6.10. [Canonical Model] Let Δ and Δ'' as in Lemma 6.6. Let $\mathfrak{A}(\Delta'')$ be the partial structure constructed as in Definition 6.7. Then, for every sentence φ in $\mathbb{L}(\Sigma')$: $\mathfrak{A}(\Delta'') \Vdash \varphi$ iff $\varphi \in \Delta''$. In particular, for every sentence φ in $\mathbb{L}(\Sigma)$: $\mathfrak{A}(\Delta'') \Vdash \varphi$ iff $\varphi \in \Delta''$.

Proof. It is a direct consequence of Lemma 6.9. □

Using the previous results, we shall prove that LPT1 is complete with respect to the semantics of pragmatic satisfaction by partial structures introduced in Section 2.

[11] The easy proof of this fact is left to the reader.

Theorem 6.11. [Completeness] Let $\Gamma \cup \{\varphi\}$ be a set of closed sentences in $\mathbb{L}(\Sigma)$. Then: $\Gamma \Vdash \varphi$ implies $\Gamma \vdash_{LPT1} \varphi$.

Proof. Assume that $\Gamma \nvdash_{LPT1} \varphi$. Hence, by Theorem 5.5, there exists Δ, relatively maximal with respect to φ, which extends Γ. Let Δ'' be a φ-saturated Henkin set in $\mathbb{L}(\Sigma')$ conservatively extending Δ as in Theorem 6.4, and let $\mathfrak{A}(\Delta'')$ be the partial structure constructed as in Definition 6.7. By Theorem 6.10, $\mathfrak{A}(\Delta'') \Vdash \psi$ iff $\psi \in \Delta''$, for every sentence ψ in $\mathbb{L}(\Sigma)$. Since $\Gamma \subseteq \Delta''$, then $\mathfrak{A}(\Delta'') \Vdash \psi$, for every $\psi \in \Gamma$. Since $\varphi \notin \Delta''$, then $\mathfrak{A}(\Delta'') \nVdash \varphi$. Therefore $\Gamma \nVdash \varphi$, because there is a partial structure that satisfies pragmatically Γ, but it does not satisfy pragmatically φ. □

6.1 Relationship between LPT1 and LP

In this section a relationship between the first-order version of logic LP and LPT1 will be established.

The logic *LP* is one of the first 3-valued paraconsistent logics introduced in the literature. It consists of the same matrices for negation and conjunction as in Łukasiewicz and Kleene's 3-valued logics (which are the same as in LPT) together with Kleene's matrix for implication:

$\supset$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	1/2
0	1	1	1

Clearly, $\alpha \supset \beta$ is $\neg(\alpha \wedge \neg\beta)$. Different to Kleene's logic, the designated truth-values are 1 and $\frac{1}{2}$. This logic was introduced by F. Asenjo in [1] as a formal framework for studying antinomies. In [14], G. Priest studied this logic in detail, from the perspective of matrix logics, baptizing it as LP (the logic of paradox).

The semantics for the first-order version of LP, proposed by Priest, (see, for instance, [15] and [6]) is given by means of *LP-models* $\mathcal{A}$, which are usual tarskian structures, except that any n-ary relation symbol P is interpreted as an ordered pair $\langle P_+^{\mathcal{A}}, P_-^{\mathcal{A}}\rangle$, such that $P_+^{\mathcal{A}} \cup P_-^{\mathcal{A}} = D^n$, where D (a nonempty set) is the domain of the structure $\mathcal{A}$.

The truth and falsehood of sentences in a structure $\mathcal{A}$ are defined inductively in the following way:

$$\mathcal{A} \models^+ P(\tau_1 \dots \tau_n) \text{ iff } \big((\tau_1)^{\mathcal{A}}, \dots, (\tau_n)^{\mathcal{A}}\big) \in P_+^{\mathcal{A}};$$
$$\mathcal{A} \models^- P(\tau_1 \dots \tau_n) \text{ iff } \big((\tau_1)^{\mathcal{A}}, \dots, (\tau_n)^{\mathcal{A}}\big) \in P_-^{\mathcal{A}};$$

$\mathcal{A} \models^+ \neg\varphi$ iff $\mathcal{A} \models^- \varphi$;

$\mathcal{A} \models^- \neg\varphi$ iff $\mathcal{A} \models^+ \varphi$;

$\mathcal{A} \models^+ (\varphi \wedge \psi)$ iff $\mathcal{A} \models^+ \varphi$ and $\mathcal{A} \models^+ \psi$;

$\mathcal{A} \models^- (\varphi \wedge \psi)$ iff $\mathcal{A} \models^- \varphi$ or $\mathcal{A} \models^- \psi$;

$\mathcal{A} \models^+ \exists x\varphi$ iff $\mathcal{A} \models^+ \varphi[x/\bar{a}]$ for at least one $a \in D$;

$\mathcal{A} \models^- \exists x\varphi$ iff $\mathcal{A} \models^- \varphi[x/\bar{a}]$ for all $a \in D$.

Finally, the (semantical) consequence relation in LP is defined as follows: $\Gamma \models_{LP} \varphi$ iff, for every LP-model $\mathcal{A}$, if $\mathcal{A} \models^+ \psi$ for every $\psi \in \Gamma$ then $\mathcal{A} \models^+ \varphi$.

It will be shown that the first-order semantics of LP coincides with the pragmatic satisfaction of LPT1, and so the former is a fragment of the latter.

Since $P_+^{\mathcal{A}}$ and $P_-^{\mathcal{A}}$ are not necessarily mutually disjoint, the set $P_+^{\mathcal{A}} \cap P_-^{\mathcal{A}}$ interprets the conjunction $P \wedge \neg P$. Thus, given a partial structure $\mathfrak{A}$, an LP-model $\mathcal{A}_{\mathfrak{A}}$ can be constructed as follows: the set $P_+^{\mathcal{A}_{\mathfrak{A}}}$ is given by $P_+^{\mathfrak{A}} \cup P_u^{\mathfrak{A}}$, and the set $P_-^{\mathcal{A}_{\mathfrak{A}}}$ is given by $P_-^{\mathfrak{A}} \cup P_u^{\mathfrak{A}}$. The interpretation of the symbols for function and the constants remains the same. Notice that $P_+^{\mathcal{A}_{\mathfrak{A}}} \cap P_-^{\mathcal{A}_{\mathfrak{A}}} = P_u^{\mathfrak{A}}$. Conversely, given an LP-model $\mathcal{A}$, it is defined a partial structure $\mathfrak{A}_{\mathcal{A}}$ as follows: $P_+^{\mathfrak{A}_{\mathcal{A}}} = P_+^{\mathcal{A}} - (P_+^{\mathcal{A}} \cap P_-^{\mathcal{A}})$; $P_-^{\mathfrak{A}_{\mathcal{A}}} = P_-^{\mathcal{A}} - (P_+^{\mathcal{A}} \cap P_-^{\mathcal{A}})$; and $P_u^{\mathfrak{A}_{\mathcal{A}}} = P_+^{\mathcal{A}} \cap P_-^{\mathcal{A}}$. The symbols for function and the constants are interpreted in the same way.

It is easy to see that, given $\mathcal{A}$, then $\mathcal{A} = \mathcal{A}_{(\mathfrak{A}_{\mathcal{A}})}$. On the other hand, $\mathfrak{A} = \mathfrak{A}_{(\mathcal{A}_{\mathfrak{A}})}$ for every $\mathfrak{A}$. This shows that the class of models of LP and LPT1 are essentially the same.

From this, a comparison between both semantics, LP and LPT1, can now be established (assume, without loss of generality, that $\exists$ is primitive in LPT1, while $\forall$ is defined from $\exists$ by using the negation $\neg$):

Proposition 6.12. Let $\mathfrak{A}$ be a partial structure for a signature Σ and let $\mathbb{L}(\Sigma)$ be the language of first-order LP over a signature Σ:

$$\mathfrak{A} \Vdash \varphi \quad \text{iff} \quad \mathcal{A}_{\mathfrak{A}} \models^+ \varphi;$$

$$\mathfrak{A} \Vdash \neg\varphi \quad \text{iff} \quad \mathcal{A}_{\mathfrak{A}} \models^- \varphi.$$

Proof. Straightforward, by induction on the complexity of the sentence φ. □

Proposition 6.13. let $\mathbb{L}(\Sigma)$ be the language of first-order LP over a signature Σ. Then:

$$\Gamma \models_{LP} \varphi \quad \text{iff} \quad \Gamma \Vdash \varphi \quad \text{iff} \quad \Gamma \vdash_{LPT1} \alpha.$$

Proof. Immediate, by Proposition 6.12 and the definition of semantic entailment in both logics. □

Corollary 6.14. First-order LP is the $\{\neg, \wedge, \exists\}$-fragment of LPT1. □

The difference between logic LPT1 and first-order LP is that the former includes a classical implication $\to$ and a bottom $\bot$, which produce together a strong negation $\sim$. This feature is not present in the logic LP.

6.2 LPT1 and evolutionary databases

In [4], W. Carnielli, J. Marcos and S. de Amo introduced the so-called *evolutionary databases*, which are databases based on the paraconsistent first-order logic LFI1*. The idea was to explore the possibility of using the first-order extension of a 3-valued paraconsistent logic in order to analyze inconsistent databases.

The 3-valued logic underlying evolutionary databases is LFI1, an LFI (re)introduced in [4]. This logic was independently studied by several authors at different times and with different names, signature and motivations. It is equivalent to the well-known logic J_3, introduced in 1970 by I. D'Ottaviano and N. da Costa in [10]. More surprisingly, it was also introduced by K. Schütte in 1960, in the context of Proof Theory (see [17]).[12]

The logic LFI1 can be defined over the signature formed by the connectives $\wedge$, $\vee$, $\Rightarrow$, $\neg$ and $\bullet$. The truth-tables for $\wedge$, $\vee$ and $\neg$ are as in LPT; the *inconsistency* operator $\bullet$ coincides with the operator $\neg\circ$ of LPT (see truth-table below), while the (deductive) implication $\Rightarrow$ is defined by the truth-table below. The distinguished truth-values are 1 and $\frac{1}{2}$.

$\Rightarrow$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	0
0	1	1	1

	$\bullet$
1	0
1/2	1
0	0

All the truth-tables in the class of 8Kb maximal (w.r.t. classical logic) 3-valued logics can be defined inside of LFI1 (cf. [3]). In particular, the matrices of LPT are definable in LFI1: $\alpha \to \beta = \neg{\sim}(\alpha \Rightarrow \beta)$, where ${\sim}\alpha = \neg\alpha \wedge \neg\bullet\alpha$. Thus, LPT is a fragment of LFI1. The interesting fact to be noted is that the implication $\Rightarrow$ of LFI1 can be defined in LPT as $\alpha \Rightarrow \beta = {\sim}\alpha \vee \beta$. This means that LPT coincides with LFI1 (and so with J_3 and with Schütte's logic), being therefore one more reincarnation of this extremely interesting

[12] For more details about the logic LFI1 and its history see [3].

3-valued logic. It is worth noting that $\alpha \to \beta$ and $\alpha \Rightarrow \beta = {\sim}\alpha \vee \beta$ are semantically equivalent, but they are not intersubstitutable, since both logics does not satisfy replacement. For instance, $\circ(\alpha \to \beta)$ and $\circ(\alpha \Rightarrow \beta)$ are inequivalent.

The logic LFI1 was extended in [4] to first-order languages, producing the logic LFI1* for evolutionary databases. The semantics for LFI1* is briefly described in the sequel.

An *LFI1*-structure* is a classical tarskian structure I such that its domain $|I|$ contains two distinguished elements denoted by $\checkmark$ and $\ltimes$. Terms and predicate symbols are interpreted in I as usual. Given a symbol for n-ary relation P, and its classical interpretation $P^I \subseteq |I|^n$, its *extended* interpretation is a new relation $P^{I^+} \subseteq P^I \times \{\checkmark, \ltimes\}$, such that elements of the form $(\vec{a}, \checkmark)$ and $(\vec{a}, \ltimes)$ do not occur simultaneously in P^{I^+}. The notion of satisfaction of sentences by a structure I is defined inductively as follows:

(1) $I \vDash P(\tau_1, \ldots, \tau_n)$ iff $(\tau_1^I, \ldots, \tau_n^I, \checkmark) \in P^{I^+}$ or $(\tau_1^I, \ldots, \tau_n^I, \ltimes) \in P^{I^+}$

(2) $I \vDash \neg P(\tau_1, \ldots, \tau_n)$ iff $(\tau_1^I, \ldots, \tau_n^I, \ltimes) \in P^{I^+}$,
or both $(\tau_1^I, \ldots, \tau_n^I, \checkmark) \notin P^{I^+}$ and $(\tau_1^I, \ldots, \tau_n^I, \ltimes) \notin P^{I^+}$

(3) $I \vDash \bullet P(\tau_1, \ldots, \tau_n)$ iff $(\tau_1^I, \ldots, \tau_n^I, \ltimes) \in P^{I^+}$

(4) $I \vDash \alpha \wedge \beta$ iff $I \vDash \alpha$ and $I \vDash \beta$

(5) $I \vDash \alpha \vee \beta$ iff $I \vDash \alpha$ or $I \vDash \beta$

(6) $I \vDash \alpha \Rightarrow \beta$ iff $I \nvDash \alpha$ or $I \vDash \beta$

(7) $I \nvDash \neg\alpha$ implies $I \vDash \alpha$

(8) $I \vDash \neg\neg\alpha$ iff $I \vDash \alpha$

(9) $I \vDash \neg(\alpha \wedge \beta)$ iff $I \vDash \neg\alpha$ or $I \vDash \neg\beta$

(10) $I \vDash \neg(\alpha \vee \beta)$ iff $I \vDash \neg\alpha$ and $I \vDash \neg\beta$

(11) $I \vDash \neg(\alpha \Rightarrow \beta)$ iff $I \vDash \alpha$ and $I \vDash \neg\beta$

(12) $I \vDash \bullet\alpha$ iff $I \vDash \alpha$ and $I \vDash \neg\alpha$

(13) $I \vDash \neg\bullet\alpha$ iff $I \nvDash \bullet\alpha$

(14) $I \vDash \forall x\alpha$ iff $I \vDash \alpha[x/\bar{a}]$ for every $a \in |I|$

(15) $I \vDash \exists x\alpha$ iff $I \vDash \alpha[x/\bar{a}]$ for some $a \in |I|$

(16) $I \vDash \neg\forall x\alpha$ iff $I \vDash \exists x\neg\alpha$

(17) $I \vDash \neg\exists x\alpha$ iff $I \vDash \forall x\neg\alpha$

(18) $I \vDash \bullet\forall x\alpha$ iff $I \vDash \forall x\alpha$ and $I \vDash \exists x\bullet\alpha$

(19) $I \vDash \bullet\exists x\alpha$ iff $I \vDash \forall x\neg\alpha$ and $I \vDash \exists x\bullet\alpha$.

Now, the relationship between LFI1* and LPT1 will be analyzed, showing that they coincide.

Given an LFI1*-structure I, define a partial structure $\mathfrak{A}_I$ over the same domain $D = |I|$ as follows: if P is a symbol for a n-ary relation then:

(a) $P_{+}^{\mathfrak{A}_I} = \{\vec{a} \in D^n \ : \ (\vec{a}, \checkmark) \in P^{I^+}\}$;

(b) $P_{u}^{\mathfrak{A}_I} = \{\vec{a} \in D^n \ : \ (\vec{a}, \ltimes) \in P^{I^+}\}$;

(c) $P_{-}^{\mathfrak{A}_I} = \{\vec{a} \in D^n \ : \ (\vec{a}, \checkmark) \notin P^{I^+} \text{ and } (\vec{a}, \ltimes) \notin P^{I^+}\}$.

The symbols for functions and constants are interpreted in $\mathfrak{A}_I$ as in I.

Conversely, given a partial structure $\mathfrak{A}$, let $I_{\mathfrak{A}}$ be the LFI1*-structure defined as follows: if P is a symbol for a n-ary relation then:

(d) $P^{I_{\mathfrak{A}}} = P_{+}^{\mathfrak{A}} \cup P_{u}^{\mathfrak{A}}$;

(e) $P^{I_{\mathfrak{A}}^+} = \{(\vec{a}, \checkmark) \ : \ \vec{a} \in P_{+}^{\mathfrak{A}}\} \cup \{(\vec{a}, \ltimes) \ : \ \vec{a} \in P_{u}^{\mathfrak{A}}\}$.

The symbols for functions and constants are interpreted in $I_{\mathfrak{A}}$ as in $\mathfrak{A}$.

It is worth noting that, given $\mathfrak{A}$ and I, it holds that $\mathfrak{A} = \mathfrak{A}_{(I_{\mathfrak{A}})}$ and $I = I_{(\mathfrak{A}_I)}$. This shows that the class of structures of both logics are essentially the same. From this, it is obtained the following result (cf. [18]):

Proposition 6.15. Let $\mathfrak{A}$ be a partial structure and consider the LFI1*-structure $I_{\mathfrak{A}}$. Then, for every symbol for n-ary relation P and for every $\vec{a} \in D^n$:

(i) $\vec{a} \in P_{+}^{\mathfrak{A}}$ iff $(\vec{a}, \checkmark) \in P^{I_{\mathfrak{A}}^+}$;

(ii) $\vec{a} \in P_{u}^{\mathfrak{A}}$ iff $(\vec{a}, \ltimes) \in P^{I_{\mathfrak{A}}^+}$;

(iii) $\vec{a} \in P_{-}^{\mathfrak{A}}$ iff $(\vec{a}, \checkmark) \notin P^{I_{\mathfrak{A}}^+}$ and $(\vec{a}, \ltimes) \notin P^{I_{\mathfrak{A}}^+}$.

(iv) $I_{\mathfrak{A}} \models P(\tau_1, \ldots, \tau_n)$ iff $\mathfrak{A} \Vdash P(\tau_1, \ldots, \tau_n)$;

(v) $I_{\mathfrak{A}} \models \neg P(\tau_1, \ldots, \tau_n)$ iff $\mathfrak{A} \Vdash \neg P(\tau_1, \ldots, \tau_n)$. □

Given a formula φ of LFI1*, let φ^* be the corresponding formula of LPT1 obtained from φ by replacing every occurrence of $\bullet\alpha$ by $\neg\circ\alpha$ and every ocurrence of $\alpha \Rightarrow \beta$ by $\sim\alpha \vee \beta$. From the last result, it follows:

Proposition 6.16. Let $\mathfrak{A}$ be a partial structure and consider the LFI1*-structure $I_{\mathfrak{A}}$. Then, for every sentence α of LFI1*: $I_{\mathfrak{A}} \models \alpha$ iff $\mathfrak{A} \Vdash \alpha^*$. □

Given that every I is of the form $I_{\mathfrak{A}}$, the last result shows that LFI1* coincides with LPT1:

Corollary 6.17. LPT1 coincides with LFI1* up to language. □

From corollaries 6.14 and 6.17 it follows:

Corollary 6.18. First-order LP is the $\{\neg, \wedge, \exists\}$-fragment of LFI1*. □

7 Final Remarks

This paper presents an alternative approach to the study of the quasi-truth theory introduced by Mikenberg, da Costa and Chuaqui. One of the main contributions is that the notion of predicates as ordered triples was extended recursively to every first-order complex formula. Thereby, the interpretation of each formula φ in a partial structure $\mathfrak{A}$ originates inductively a triple $\langle \varphi_+^{\mathfrak{A}}, \varphi_-^{\mathfrak{A}}, \varphi_u^{\mathfrak{A}} \rangle$.

From this, it follows that the new semantics generalizes the usual perspective that any first-order formula φ (with at most n free variables) in a structure $\mathfrak{A}$ can be seen as a relation $R = \{\vec{a} \in D^n : \mathfrak{A} \models \varphi[\vec{a}]\}$, defined inductively. In other words, the semantics proposed in this paper generalizes simultaneously the notion from Tarski's truth and the notion of quasi-truth from Mikenberg *et alia*.

Besides, our strategy avoids constructing the total structures, and the notion of pragmatic satisfaction is given, *mutatis mutandis*, by the tarskian notion of satisfaction.

Another contribution of this paper is the axiomatization of the new notion of quasi-truth, and the presentation of a 3-valued paraconsistent logic that represents its propositional base.

Finally, the proposed logic for quasi-truth LPT1 was compared with two related systems: the (first-order) paraconsistent logic LP and the logic LFI1* of evolutionary databases, showing that the former is a fragment of LPT1, while the latter is a version of LPT1 defined in a slightly different language. As a consequence, we obtain a characterization of the logic of evolutionary databases which is simpler than the original, and closer to the tarskian definition of truth. From this, it follows that first-order LP is a fragment of LFI1*.

It is worth noting that, through the semantic of ordered triples, that is, by means of the set-theoretic operations associated to the connectives and quantifiers, it is possible to originate different theories of quasi-truth. Besides, this approach would allow us to define different 3-valued logics according to the technique introduced in Section 2. For instance, it would be interesting to characterize other well-known 3-valued logics such as Łukasiewicz's $Ł_3$ and Kleene's logic K_3.

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