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An alternative Dunford–Pettis Property

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Abstract. An alternative to the Dunford–Pettis Property, called the DP1-property, is introduced. Its relationship to the Dunford–Pettis Property and other related properties is examined. It is shown that ℓ_p-direct sums of spaces with DP1 have DP1 if $1 \le p < \infty$. It is also shown that for preduals of von Neumann algebras, DP1 is strictly weaker than the Dunford–Pettis Property, while for von Neumann algebras, the two properties are equivalent.

Introduction. A Banach space X is said to have the *Dunford–Pettis Property* (DP) if for any Banach space Y, every weakly compact operator $T : X \to Y$ is completely continuous, i.e., T maps weakly compact subsets of X onto norm compact subsets of Y. Equivalently, X has DP if and only if for any weakly null sequences $(f_n) \subseteq X$ and $(a_n) \subseteq X^*$, one has $f_n(a_n) \to 0$. For this and other basic results, the reader is directed to [8]. It is well-known that for any Radon measure μ, and for any locally compact Hausdorff space Ω, both $L^1(\mu)$ and $C_0(\Omega)$ have DP. Thus commutative von Neumann algebras and their preduals have DP, but DP fails to hold generally in the noncommutative case. For example, as shown in [2], for any von Neumann algebra $\mathcal{M}$, $\mathcal{M}_*$ has DP if and only if $\mathcal{M}$ is type I and finite.

This paper introduces a closely related property, called the DP1-property, which can be thought of as a compromise between DP and the Kadec–Klee Property. This is shown to hold for a larger class of Banach spaces, in particular for a larger class of preduals of von Neumann algebras.

Notation. If X is any Banach space, the closed unit ball of X will be denoted by X_1 and the unit sphere will be denoted by ∂X_1. Throughout the paper, X and Y will denote Banach spaces over the field of complex

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numbers. The Banach space of all bounded linear operators from X to Y will be denoted by $\mathcal{B}(X,Y)$. By the word "operator", we will always mean a bounded linear operator. Given $f \in X$ and $a \in X^*$ we often write $f(a)$ or $\langle f, a\rangle$ for $a(f)$.

Recall by [4, p. 80] that if $\{X_\alpha : \alpha \in \Lambda\}$ is a family of Banach spaces, and $1 \le p < \infty$, one defines the Banach spaces

$$\Big(\bigoplus_{\alpha\in\Lambda} X_\alpha\Big)_p = \Big\{x = (x_\alpha) \in \prod_{\alpha\in\Lambda} X_\alpha : \sum_{\alpha\in\Lambda} \|x_\alpha\|^p < \infty\Big\},$$

with norm $\|x\| = (\sum_\alpha \|x_\alpha\|^p)^{1/p}$, and

$$\Big(\bigoplus_{\alpha\in\Lambda} X_\alpha\Big)_\infty = \Big\{x = (x_\alpha) \in \prod_{\alpha\in\Lambda} X_\alpha : \sup_\alpha \|x_\alpha\| < \infty\Big\},$$

with norm $\|x\| = \sup_\alpha \|x_\alpha\|$.

For a sequence (X_i) of Banach spaces, one defines the Banach space

$$\Big(\bigoplus_{i=1}^\infty X_i\Big)_0 = \Big\{x = (x_i) \in \prod_{i=1}^\infty X_i : \|x_n\| \to 0\Big\},$$

with norm $\|x\| = \sup_n \|x_n\|$.

Note that if $1 \le p < \infty$, and $p^{-1} + q^{-1} = 1$, then

$$\Big(\bigoplus_{\alpha\in\Lambda} X_\alpha\Big)_p^* = \Big(\bigoplus_{\alpha\in\Lambda} X_\alpha^*\Big)_q,$$

and for a sequence (X_i) of Banach spaces,

$$\Big(\bigoplus_{i=1}^\infty X_i\Big)_0^* = \Big(\bigoplus_{i=1}^\infty X_i^*\Big)_1.$$

Throughout the paper, H will denote an arbitrary infinite-dimensional Hilbert space. The space of trace-class operators on H will be denoted by $\mathcal{L}^1(H)$; the compact operators by $K(H)$; and the von Neumann algebra of all bounded operators by $\mathcal{B}(H)$. For any von Neumann algebra $\mathcal{M}$, $\mathcal{M}_*$ denotes the predual of $\mathcal{M}$, the unique (up to isometric isomorphism) Banach space such that $(\mathcal{M}_*)^* = \mathcal{M}$. The predual $\mathcal{M}_*$ can be identified with the space of all normal (bounded) linear functionals on $\mathcal{M}$. In particular, $\mathcal{L}^1(H)$ is identified with $\mathcal{B}(H)_*$.

By the weak topology on a C^*-algebra A we will always mean the weak Banach space topology, i.e., the topology induced by A^*. Given a C^*-algebra A with $f \in A^*$, and $a \in A$, one defines fa and $af \in A^*$ by $fa(x) = f(ax)$ and $af(x) = f(xa)$, respectively, for all $x \in A$.

1. The DP1-property

1.1. Definition. A Banach space X has the DP1-*property* if for any weakly convergent sequences $f_n \to f$ in X, and $a_n \to a$ in X^*, such that $\|f_n\| = \|f\| = 1$, we have $f_n(a_n) \to f(a)$.

Note that the condition $\|f_n\| = \|f\| = 1$ can be replaced by the equivalent condition that $\|f_n\| \to \|f\|$, and we may also assume without loss of generality that $a = 0$.

Recall that a Banach space X has the *Kadec–Klee Property* (KKP) if whenever $f_n \to f$ weakly, with $\|f_n\| = \|f\| = 1$, then $\|f_n - f\| \to 0$.

1.2. Remarks. 1. Obviously, if X has either DP or KKP, then it has DP1.

2. It is easy to see that if X^* has DP, then X has DP. This is in contrast to DP1, for as will be seen (2.3 and Example 3.3(i)), $K(H)^* = \mathcal{L}^1(H)$ has DP1, while $K(H)$ does not.

3. It is easy to check that as in the case of DP, if E is a complemented closed subspace of X, and X has DP1, then E has DP1, while for any closed subspace $E \subseteq X$, if X has KKP, then E has KKP.

4. It is important to note that DP1 and KKP, in contrast to DP, are not preserved by isomorphisms in general (see Example 1.6), though it is easy to check that they are preserved by *isometric* isomorphisms.

Recall that an operator $T \in \mathcal{B}(X,Y)$ is called *weakly compact* if $T(X_1)$ is relatively weakly compact in Y. Equivalently, by the Eberlein–Shmul'yan theorem [7, p. 18], for any bounded sequence (x_n) in X, the sequence (Tx_n) has a weakly convergent subsequence.

An operator $T \in \mathcal{B}(X,Y)$ is called *completely continuous* if it maps weakly compact subsets of X onto norm compact subsets of Y. An easy application of the Eberlein–Shmul'yan theorem shows that T is completely continuous if and only if T maps weakly convergent sequences onto norm convergent sequences.

1.3. Definition. An operator $T \in \mathcal{B}(X,Y)$ is said to be a DP1-*operator* if whenever $f_n \to f$ weakly in X, with $\|f_n\| = \|f\| = 1$, we have $Tf_n \to Tf$ in norm.

Note that again an easy application of the Eberlein–Shmul'yan theorem shows that an operator is a DP1-operator if and only if it maps weakly compact subsets of ∂X_1 onto norm compact subsets of Y.

As is the case with DP (cf. [8, p. 17]), there are several useful conditions which are equivalent to DP1, as shown next.

1.4. THEOREM. *Let X be a Banach space. The following are equivalent:*

(a) *X has* DP1.

(b) *For every Banach space Y, every weakly compact operator $T : X \to Y$ is a* DP1-*operator.*

(c) *For every reflexive space Y, every operator $T : X \to Y$ is a* DP1-*operator.*

(d) *Every weakly compact operator $T : X \to c_0$ is a* DP1-*operator.*

(e) *Assume $f_n \to f$ weakly in X with $\|f_n\| = \|f\| = 1$ for all n. If $(a_n) \subseteq X^*$ is a weakly Cauchy sequence, then $(f_n - f)(a_n) \to 0$.*

Proof. (a)⇒(b). Suppose X has DP1 and there exists a Banach space Y, a weakly compact operator $T : X \to Y$, and a weakly convergent sequence $f_n \to f$ with $\|f_n\| = 1 = \|f\|$ such that (Tf_n) is not norm convergent in Y. By passing to a subsequence if necessary, we may assume that there exists $\delta > 0$ such that $\|Tf_{2k} - Tf_{2k+1}\| > \delta$ for all $k = 1, 2, \ldots$, and choose $b_k \in Y_1^*$ such that for all k,

$$\delta < |b_k(Tf_{2k} - Tf_{2k+1})|.$$

Let $a_k = T^* b_k \in X^*$. Since T^* is weakly compact [14, II.C.6], the sequence (a_k) has a weakly convergent subsequence, so we may assume without loss of generality that there exists $a \in X^*$ such that $a_k \to a$ weakly. Now, $f_{2k} \to f$ weakly, and $f_{2k+1} \to f$ weakly, so by (a) we have

$$\delta < |b_k(Tf_{2k}) - b_k(Tf_{2k+1})| = |f_{2k}(a_k) - f_{2k+1}(a_k)| \to 0,$$

a contradiction.

(b)⇒(c). Suppose Y is reflexive, and let $T \in \mathcal{B}(X, Y)$. Let (x_n) be a bounded sequence in X. Since Y is reflexive, the bounded sequence (Tx_n) has a weakly convergent subsequence. Hence T is weakly compact, so by (b), T is a DP1-operator.

(c)⇒(d). If $T : X \to c_0$ is weakly compact, then by the Davis–Figiel–Johnson–Pełczyński factorization theorem [5], T factors through a reflexive space. Hence by (c), T is a DP1-operator.

(d)⇒(a). As in the proof of (b)⇒(a), let $f_n \to f$ weakly in X with $\|f_n\| = \|f\| = 1$, and $a_n \to 0$ weakly in X^*. The map $T : X \to c_0$ defined by $Tg = (g(a_n))$ is then weakly compact, as shown in (b)⇒(a), so that $\|Tf_n - Tf\| \to 0$. Hence $\sup_i |(f_n - f)(a_i)| \to 0$, and so $|(f_n - f)(a_n)| \to 0$. Thus as $a_n \to 0$ weakly, we have $f_n(a_n) \to 0$.

(a)⇒(e). Suppose X has DP1 and $f_n \to f$ weakly in X with $\|f_n\| = \|f\| = 1$ for all n. Let (a_n) be a weakly Cauchy sequence in X^*, and let $\varepsilon > 0$ be given. Since $f_n - f \to 0$ weakly in X, for each n there exists $k_n > n$ such that $|(f_{k_n} - f)(a_n)| < \varepsilon$. Hence

$$\begin{aligned} |(f_{k_n} - f)(a_{k_n})| &\le |(f_{k_n} - f)(a_{k_n} - a_n)| + |(f_{k_n} - f)(a_n)| \\ &< |(f_{k_n} - f)(a_{k_n} - a_n)| + \varepsilon. \end{aligned}$$

Since (a_n) is weakly Cauchy, $(a_{k_n} - a_n)$ is weakly null, so that $f(a_{k_n} - a_n) \to 0$, but since X has DP1, $f_{k_n}(a_{k_n} - a_n) \to 0$ as well. It follows that $(f_{k_n} - f)(a_{k_n}) \to 0$, whence $(f_n - f)(a_n) \to 0$.

(e)⇒(a). Suppose $f_n \to f$ weakly and $\|f_n\| = \|f\| = 1$. If $a_n \to 0$ weakly in X^*, then certainly (a_n) is weakly Cauchy, so by (e), we have $(f_n - f)(a_n) \to 0$. But since $f(a_n) \to 0$, we then have $f_n(a_n) \to 0$. ∎

1.5. COROLLARY. *If X is reflexive, then X has* DP1 *if and only if X has* KKP. *In particular, every Hilbert space has* KKP, *and hence has* DP1 *as well.*

Proof. One direction is obvious, so assume X has DP1 and let $x_n \to x$ weakly in X with $\|x_n\| = \|x\| = 1$. Since X is reflexive, the identity operator on X is a DP1-operator by Theorem 1.4. Hence $\|x_n - x\| \to 0$, so X has KKP.

If X is a Hilbert space, with x_n and x as above, then

$$\|x_n - x\|^2 = \langle x_n - x, x_n - x\rangle = 2(1 - \mathrm{Re}(\langle x, x_n\rangle)) \to 0,$$

so X has KKP. ∎

1.6. EXAMPLE: *A reflexive space which is isomorphic to a Hilbert space but which fails to have* DP1. Let $X = \ell_2 \oplus_\infty \ell_2$. Clearly, X is isomorphic to the Hilbert space $\ell_2 \oplus_2 \ell_2$. Let $y_n \to y$ weakly in ℓ_2, with $\|y_n\| = \|y\| = 1$ (so in fact $\|y_n - y\| \to 0$). Let (e_n) be the standard basis in ℓ_2, so $e_n \to 0$ weakly. Let $x_n = (y_n, e_n) \in X$, and let $x = (y, 0) \in X$. Clearly, $x_n \to x$ weakly, and $\|x_n\| = \|x\| = 1$ for all n. Yet we have $\|x_n - x\| = \|(y_n - y, e_n)\| = \max\{\|y_n - y\|, \|e_n\|\} \ge 1$ for all n, so X fails to have KKP, and hence by Corollary 1.5, X fails to have DP1. In particular, let $g_n = (0, e_n) \in X^*$. Then $g_n \to 0$ weakly, but $g_n(x_n) = 1$ for all n.

In [9], the following characterization of DP is given: A Banach space X has DP if and only if for every Banach space Y, whenever $K \subseteq X$ and $J \subseteq Y$ are weakly compact sets in X and Y, then the set $K \otimes J = \{x \otimes y \in X \widehat{\otimes} Y : x \in K,\ y \in J\}$ is weakly compact in $X \widehat{\otimes} Y$, the projective tensor product of X and Y. Using the same notation, similar characterizations of the DP1, KKP and Schur properties are given in the following theorem.

1.7. THEOREM. *Let X be a Banach space.*

(a) *X has* DP1 *if and only if for every Banach space Y, whenever $K \subseteq \partial X_1$ and $J \subseteq Y$ are weakly compact sets in X and Y, then the set $K \otimes J$ is weakly compact in $X \widehat{\otimes} Y$.*

(b) *For any Banach space* Y, *the spaces* X *and* Y *have* KKP *if and only if for any weakly compact sets* $K \subseteq \partial X_1$, $J \subseteq \partial Y_1$, *the set* $K \otimes J$ *is norm compact in* $X \widehat{\otimes} Y$.

(c) *For any Banach space* Y, *the spaces* X *and* Y *have the Schur property if and only if for any weakly compact sets* $K \subseteq X$, $J \subseteq Y$, *the set* $K \otimes J$ *is norm compact in* $X \widehat{\otimes} Y$.

Proof. Throughout the proof, full use is made of the Eberlein–Shmul'yan theorem [7, Ch. III]: A subset $S \subseteq X$ is weakly compact if and only if it is weakly sequentially compact.

(a) Suppose X has DP1, and let K and J be given as in the statement of the theorem. Let $(x_n) \subseteq K$ and $(y_n) \subseteq J$ be given. Since K and J are weakly compact, by passing to subsequences if necessary, we may assume without loss of generality that there exist $x \in K$ and $y \in J$ such that $x_n \to x$ and $y_n \to y$ weakly. Recall from [12, IV.2] that we have $(X \widehat{\otimes} Y)^* = \mathcal{B}(Y, X^*)$, where $S \in \mathcal{B}(Y, X^*)$ acts on $x \otimes y$ by $S(x \otimes y) = \langle x, Sy \rangle$. Now, if $\phi \in \mathcal{B}(Y, X^*)$, then $\phi(y_n) \to \phi(y)$ weakly in X^*. Since X has DP1, we obtain $\langle x_n, \phi(y_n) \rangle \to \langle x, \phi(y) \rangle$, so that $x_n \otimes y_n \to x \otimes y$ weakly in $X \widehat{\otimes} Y$.

Conversely, let $x_n \to x$ weakly in X with $\|x_n\| = \|x\| = 1$, and let $f_n \to 0$ weakly in X^*. Set $x_0 = x$ and $f_0 = 0 \in X^*$. By hypothesis, the set $\{x_i \otimes f_j : i, j \geq 0\}$ is weakly compact. Suppose that $(x_n \otimes f_n)$ is not weakly null in $X \widehat{\otimes} X^*$. It follows that there is a subsequence $(x_{n_k} \otimes f_{n_k})$ and $x_a \neq 0$ and $f_b \neq 0$ such that $x_{n_k} \otimes f_{n_k} \to x_a \otimes f_b$ weakly. Choose $g \in X^*$ and $h \in X^{**}$ such that $g(x_a) \neq 0$ and $h(f_b) \neq 0$. Then $g \otimes h \in (X \widehat{\otimes} X^*)^*$ and

$$(g \otimes h)(x_{n_k} \otimes f_{n_k}) \to (g \otimes h)(x_a \otimes f_b) = g(x_a)h(f_b) \neq 0,$$

but

$$(g \otimes h)(x_{n_k} \otimes f_{n_k}) = g(x_{n_k})h(f_{n_k}) \to 0,$$

since $f_{n_k} \to 0$ weakly and $|g(x_{n_k})| \leq \|g\|$, a contradiction. Hence $(x_n \otimes f_n)$ is weakly null. Since $(X \widehat{\otimes} X^*)^* = \mathcal{B}(X^*, X^*)$, we have $1_{X^*}(x_n \otimes f_n) = f_n(x_n) \to 0$, whence X has DP1.

(b) Suppose X and Y have KKP and K and J are as in the statement of the theorem. Let $(x_n \otimes y_n) \subseteq K \otimes J$. As in (a), since K and J are weakly compact, we may assume that there exist $x \in K$ and $y \in J$ such that $x_n \to x$ and $y_n \to y$ weakly. Since X and Y have KKP, we have

$$\begin{aligned} \|x_n \otimes y_n - x \otimes y\| &\leq \|x_n \otimes y_n - x \otimes y_n\| + \|x \otimes y_n - x \otimes y\| \\ &= \|(x_n - x) \otimes y_n\| + \|x \otimes (y_n - y)\| \\ &= \|x_n - x\| \cdot \|y_n\| + \|x\| \cdot \|y_n - y\| \\ &= \|x_n - x\| + \|y_n - y\| \to 0, \end{aligned}$$

so that $K \otimes J$ is norm compact.

Conversely, choose $(x_n) \subseteq \partial X_1$ such that $x_n \to x$ weakly and $\|x\| = 1$. Let $y \in \partial Y_1$, and set $x_0 = x$. By hypothesis, the set $\{x_i \otimes y : i \geq 0\}$ is norm compact in $X \widehat{\otimes} Y$, so by passing to a subsequence if necessary, we may assume without loss of generality that $(x_n \otimes y)$ is norm convergent. Clearly, $x_n \otimes y \to x \otimes y$ weakly, and hence converges in norm as well. But then

$$\|x_n - x\| = \|x_n - x\| \cdot \|y\| = \|x_n \otimes y - x \otimes y\| \to 0,$$

whence X has KKP. The same argument shows, upon switching the roles of X and Y, that Y has KKP.

(c) Suppose X and Y both have the Schur property, and let K and J be as given in the statement of the theorem. Let $(x_n \otimes y_n) \subseteq K \otimes J$. As in parts (a) and (b), we may assume without loss that there exist $x \in K$ and $y \in J$ such that $x_n \to x$ and $y_n \to y$ weakly. Since X and Y both have the Schur property, we have $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$, but as in the proof of (b),

$$\|x_n \otimes y_n - x \otimes y\| \leq \|x_n - x\| \cdot \|y_n\| + \|x\| \cdot \|y_n - y\|,$$

so that $\|x_n \otimes y_n - x \otimes y\| \to 0$. Hence $K \otimes J$ is norm compact.

Conversely, assume that $x_n \to x$ weakly in X, choose $y \in \partial Y_1$, and set $x_0 = x$. As in part (b), by hypothesis, the set $\{x_i \otimes y : i \geq 0\}$ is norm compact, so by passing to a subsequence if necessary, we may assume without loss of generality that $(x_n \otimes y)$ is norm convergent. Clearly, $x_n \otimes y \to x \otimes y$ weakly, and hence converges in norm as well. But then

$$\|x_n - x\| = \|x_n - x\| \cdot \|y\| = \|x_n \otimes y - x \otimes y\| \to 0,$$

whence X has the Schur property. The same argument shows, upon switching the roles of X and Y, that Y has the Schur property as well. ■

1.8. It is a well-known fact that if (f_n) is a sequence in a Banach space X and $f \in X$ is such that $f_n \to f$ weakly, then $\limsup \|f_n\| \geq \|f\|$. This fact will be used several times in the rest of the paper.

The next theorem plays a central role in this paper, establishing in particular that infinite direct sums of spaces with DP1 have DP1, provided that the right norm is used.

1.9. Theorem. *Let* $\{X_\alpha : \alpha \in \Lambda\}$ *be a family of Banach spaces, let* $1 \leq p < \infty$, *and set*

$$X = \Big(\bigoplus_\alpha X_\alpha\Big)_p.$$

Then X *has* DP1 *or* KKP *if and only if for each* α, X_α *has* DP1 *or* KKP, *respectively.*

Proof. For each $\alpha \in \Lambda$, X_α is a complemented closed subspace of X, so if X has KKP or DP1, so does X_α. (See Remark 1.2.3.)

For the converse, let $f_n, f \in X$ with $f = (f^\alpha)$ and $f_n = (f_n^\alpha)$ be such that $f_n \to f$ weakly, and such that for all n,

$$1 = \|f_n\|^p = \sum_\alpha \|f_n^\alpha\|^p = \|f\|^p = \sum_\alpha \|f^\alpha\|^p.$$

Clearly, for all $\alpha \in \Lambda$ we have $f_n^\alpha \to f^\alpha$ weakly in X_α. Let $\beta \in \Lambda$. By the remark in 1.8, $\limsup \|f_n^\beta\| \geq \|f^\beta\|$, so if $\|f_n^\beta\| \not\to \|f^\beta\|$, then there exists $r > 0$ and a subsequence $(f_{n_k}^\beta)$ such that $\|f_{n_k}^\beta\| \to r > \|f^\beta\|$. But then

$$1 = \sum_\alpha \|f^\alpha\|^p = \sum_\alpha \|f_{n_k}^\alpha\|^p = \|f_{n_k}^\beta\|^p + \sum_{\alpha \neq \beta} \|f_{n_k}^\alpha\|^p,$$

so that

$$\sum_{\alpha\neq\beta} \|f_{n_k}^\alpha\|^p = 1 - \|f_{n_k}^\beta\|^p = \sum_{\alpha\neq\beta} \|f^\alpha\|^p + \|f^\beta\|^p - \|f_{n_k}^\beta\|^p.$$

Hence

$$(*) \qquad \sum_{\alpha\neq\beta} \|f_{n_k}^\alpha\|^p \to \sum_{\alpha\neq\beta} \|f^\alpha\|^p + \|f^\beta\|^p - r^p < \sum_{\alpha\neq\beta} \|f^\alpha\|^p.$$

For each n, let $\widehat{f}_n = (\widehat{f}_n^\alpha) \in X$ be defined by setting $\widehat{f}_n^\alpha = f_n^\alpha$ for all $\alpha \neq \beta$ and $\widehat{f}_n^\beta = 0$, and similarly define $\widehat{f} = (\widehat{f}^\alpha)$. Then clearly $\widehat{f}_n \to \widehat{f}$ weakly in X since $f_n \to f$ weakly; but by $(*)$, we have $\lim_k \|\widehat{f}_{n_k}\| < \|\widehat{f}\|$, which contradicts the remark made in 1.8. Hence $\|f_n^\beta\| \to \|f^\beta\|$.

Suppose now that X_α has DP1 for each $\alpha \in \Lambda$, but X fails to have DP1. Let $(a_n) \subseteq X^* = (\bigoplus_\alpha X_\alpha^*)_q$, where $p^{-1} + q^{-1} = 1$, with $a_n = (a_n^\alpha)$, and such that $a_n \to 0$ weakly. Clearly, for all $\alpha \in \Lambda$ we have $a_n^\alpha \to 0$ weakly in X_α^*, and by the uniform boundedness principle, we may assume without loss of generality that $\|a_n\| \leq 1$.

Suppose that $|f_n(a_n)| = |\sum_\alpha f_n^\alpha(a_n^\alpha)| \not\to 0$. Then there exists a sequence $(n_k) \subseteq \mathbb{N}$ and $0 < s \leq 1$ such that

$$\Big|\sum_\alpha f_{n_k}^\alpha(a_{n_k}^\alpha)\Big| \to s,$$

since for all n,

$$\Big|\sum_\alpha f_n^\alpha(a_n^\alpha)\Big| \leq \|f_n\| \cdot \|a_n\| \leq 1.$$

Let $0 < \varepsilon < 2^{-(p+1)} s^p$, so that $2(2\varepsilon)^{1/p} < s$. Since $2\varepsilon < 1$, we have $2\varepsilon \leq (2\varepsilon)^{1/p}$ so that $2\varepsilon + (2\varepsilon)^{1/p} < s$.

Choose a finite set $\Lambda_0 \subseteq \Lambda$ such that

$$\text{(i)} \qquad \sum_{\alpha\notin\Lambda_0} \|f^\alpha\|^p < \varepsilon$$

and choose $N > 0$ such that for all $k \geq N$ the following hold:

$$\text{(ii)} \qquad \Big|\Big|\sum_\alpha f_{n_k}^\alpha(a_{n_k}^\alpha)\Big| - s\Big| < \varepsilon,$$

$$\text{(iii)} \qquad \sum_{\alpha\in\Lambda_0} |f_{n_k}^\alpha(a_{n_k}^\alpha)| < \varepsilon,$$

which is possible since $\|f_{n_k}^\alpha\| \to \|f^\alpha\|$, and X_α has DP1 for all α, and

$$\text{(iv)} \qquad \sum_{\alpha\in\Lambda_0} |\,\|f_{n_k}^\alpha\|^p - \|f^\alpha\|^p| < \varepsilon.$$

Since

$$\sum_\alpha \|f_n^\alpha\|^p = \sum_\alpha \|f^\alpha\|^p = 1,$$

we have

$$\begin{aligned}\sum_{\alpha\notin\Lambda_0} \|f_{n_k}^\alpha\|^p &= 1 - \sum_{\alpha\in\Lambda_0} \|f_{n_k}^\alpha\|^p \\ &= \sum_{\alpha\in\Lambda_0} \|f^\alpha\|^p + \sum_{\alpha\notin\Lambda_0} \|f^\alpha\|^p - \sum_{\alpha\in\Lambda_0} \|f_{n_k}^\alpha\|^p \\ &= \sum_{\alpha\in\Lambda_0} (\|f^\alpha\|^p - \|f_{n_k}^\alpha\|^p) + \sum_{\alpha\notin\Lambda_0} \|f^\alpha\|^p \\ &\leq \sum_{\alpha\in\Lambda_0} |\,\|f_{n_k}^\alpha\|^p - \|f^\alpha\|^p| + \sum_{\alpha\notin\Lambda_0} \|f^\alpha\|^p,\end{aligned}$$

so that by (i) and (iv) we have

$$\text{(v)} \qquad \sum_{\alpha\notin\Lambda_0} \|f_{n_k}^\alpha\|^p < 2\varepsilon.$$

By (ii), (iii), and (v) we have

$$\begin{aligned} s - \varepsilon &< \Big|\sum_\alpha f_{n_k}^\alpha(a_{n_k}^\alpha)\Big| < \varepsilon + \Big|\sum_{\alpha\notin\Lambda_0} f_{n_k}^\alpha(a_{n_k}^\alpha)\Big| \\ &\leq \varepsilon + \Big(\sum_{\alpha\notin\Lambda_0} \|f_{n_k}^\alpha\|^p\Big)^{1/p} \|a_{n_k}\| < \varepsilon + (2\varepsilon)^{1/p},\end{aligned}$$

so that $s < 2\varepsilon + (2\varepsilon)^{1/p}$, contradicting the choice of ε. It follows that $f_n(a_n) \to 0$, whence X has DP1.

Now suppose that X_α has KKP for all $\alpha \in \Lambda$, but X fails to have KKP. Let f and f_n be as above. By the first part of the proof above, we have that for all $\beta \in \Lambda$, $f_n^\beta \to f^\beta$ weakly and $\|f_n^\beta\| \to \|f^\beta\|$, so that $\|f_n^\beta - f^\beta\| \to 0$. Suppose that $\|f_n - f\| \not\to 0$. It follows that since $\|f_n - f\|^p \leq 2^p$, there exists a subsequence (f_{n_k}) and $0 < s \le 2^p$ such that

$$\|f_{n_k} - f\|^p = \sum_\alpha \|f^\alpha_{n_k} - f^\alpha\|^p \to s.$$

Let

$$0 < \varepsilon < \frac{s}{2 + 2^p + 2^{p+1}},$$

and let $\Lambda_0 \subseteq \Lambda$ be finite such that (i) holds. Choose $N > 0$ such that (iv) (hence (v)) and the following hold for all $k \ge N$:

(ii′) $$\Big|\sum_\alpha \|f^\alpha_{n_k} - f^\alpha\|^p - s\Big| < \varepsilon,$$

(iii′) $$\sum_{\alpha\in\Lambda_0} \|f^\alpha_{n_k} - f^\alpha\|^p < \varepsilon.$$

By (ii′), (iii′), (i), and (v), we then have

$$\begin{aligned} s - \varepsilon &\le \sum_\alpha \|f^\alpha_{n_k} - f^\alpha\|^p < \varepsilon + \sum_{\alpha\notin\Lambda_0} \|f^\alpha_{n_k} - f^\alpha\|^p \\ &\le \varepsilon + \sum_{\alpha\notin\Lambda_0} (\|f^\alpha_{n_k}\| + \|f^\alpha\|)^p \\ &\le \varepsilon + \sum_{\alpha\notin\Lambda_0} (2\max\{\|f^\alpha_{n_k}\|, \|f^\alpha\|\})^p \\ &\le \varepsilon + 2^p \sum_{\alpha\notin\Lambda_0} (\|f^\alpha_{n_k}\|^p + \|f^\alpha\|^p) < \varepsilon(1 + 2^p + 2^{p+1}), \end{aligned}$$

so that $s < \varepsilon(2 + 2^p + 2^{p+1})$, which contradicts the choice of ε. It follows that $\|f_n - f\| \to 0$, whence X has KKP. ∎

1.10. COROLLARY. *Let $\{\mathcal{M}^\alpha : \alpha \in \Lambda\}$ be a family of von Neumann algebras and let $\mathcal{M} = (\bigoplus_\alpha \mathcal{M}^\alpha)_\infty$. Then $\mathcal{M}_*$ has* DP1 *if and only if $\mathcal{M}^\alpha_*$ has* DP1 *for each $\alpha \in \Lambda$.*

Proof. We have $\mathcal{M}_* = (\bigoplus_\alpha \mathcal{M}^\alpha_*)_1$, so the result follows immediately from the theorem. ∎

1.11. Remark. Since ℓ_1 and ℓ_∞ have DP, it follows from the theorem that ℓ_p has DP1 for any $1 \le p \le \infty$. Hence by Corollary 1.5, if $1 < p < \infty$, then ℓ_p has KKP. This follows also from the fact that ℓ_p is uniformly convex (see [11]) and the standard result that any uniformly convex space has KKP.

We now consider briefly the case of quotient spaces.

1.12. THEOREM. *Let X be a Banach space which is weakly sequentially complete. Let $E \subseteq X$ be a closed reflexive subspace. If X has* DP1 *or* KKP, *then X/E has* DP1 *or* KKP *respectively.*

Proof. Suppose that X has DP1. Let $y_n \to y$ weakly in X/E, with $\|y_n\| = \|y\| = 1$. By [8, Lemma 8], there exists a subsequence (y_{n_k}), an element $x \in X$, and a weakly convergent sequence $(x_k) \subseteq X$ with $x_k \to x$ weakly such that $y_{n_k} = \pi x_k$ for all k, where $\pi : X \to X/E$ is the canonical map. It follows immediately that $\pi x = y$, and so $\|\pi x\| = 1$.

For all k we have

$$\|y_{n_k}\| = 1 = \inf_{e\in E} \|x_k - e\|,$$

so for each k, choose $e_k \in E$ such that

$$1 \le \|x_k - e_k\| < 1 + 2^{-k},$$

so that $\|x_k - e_k\| \to 1$. By passing to a subsequence if necessary, we may assume without loss of generality that (e_k) is weakly convergent to an element $e \in E$. Hence, $x_k - e_k \to x - e$ weakly and $\|x_k - e_k\| \to 1$, so that $\|x - e\| \le 1$. On the other hand, we have $1 = \|\pi x\| = \|\pi(x - e)\| \le \|x - e\|$, so that $\|x - e\| = 1$. Let $f_n \to 0$ weakly in $(X/E)^*$. Then $\pi^* f_n \to 0$ weakly in X^*, so since X has DP1,

$$f_{n_k}(y_{n_k}) = f_{n_k}(\pi x_k) = f_{n_k}(\pi(x_k - e_k)) = (\pi^* f_{n_k})(x_k - e_k) \to 0,$$

whence X/E has DP1.

Suppose now that X has KKP, and let $y_n \to y$ weakly in X/E, with $\|y_n\| = \|y\| = 1$. As in the first part, we have a subsequence (y_{n_k}), a sequence (x_k) with $x_k \to x$ weakly and $\|\pi x\| = 1$, and (e_k) and $e \in E$ with $\|x - e\| = 1$ and $\|x_k - x\| \to 1$. Since X has KKP, we have $\|x_k - x - (e_k - e)\| \to 0$, but

$$\|y_{n_k} - y\| = \|\pi(x_k - x)\| = \|\pi(x_k - x - (e_k - e))\| \le \|x_k - x - (e_k - e)\|,$$

whence X/E has KKP. ∎

2. DP1 and KKP for preduals of von Neumann algebras. As mentioned in the introduction, for preduals of von Neumann algebras, the DP property is a rare thing: as seen in [2], $\mathcal{M}_*$ has DP if and only if $\mathcal{M}$ is type I and finite. Since $\mathcal{L}^1(H)$ and $L^1[0,1]$ do have DP1, it follows that DP1 is a natural candidate for a property which holds for a larger class of preduals of von Neumann algebras and which may help us to better understand their nature. The first result shows that for either DP1 or KKP, it suffices to consider weakly convergent sequences of *states*, i.e., positive norm one functionals in $\mathcal{M}_*$.

2.1. PROPOSITION. *Let $\mathcal{M}$ be a von Neumann algebra. Then*

(a) *$\mathcal{M}_*$ has* KKP *if and only if every weakly convergent sequence of states in $\mathcal{M}_*$ converges in norm.*

(b) *$\mathcal{M}_*$ has* DP1 *if and only if for any weakly convergent sequence of states $(f_n) \subseteq \mathcal{M}_*$, with $f_n \to f$ weakly, whenever $a_n \to 0$ weakly in $\mathcal{M}$, we have $f_n(a_n) \to 0$.*

Proof. (a) Suppose $f_n \to f$ weakly in $\mathcal{M}_*$, with $\|f_n\| = \|f\| = 1$ for all n. It follows from [12, III.4.11] that $|f_n| \to |f|$ weakly, so

$$\| |f_n| - |f| \| \to 0.$$

By the polar decomposition theorem [12, III.4], there exist unique partial isometries u_n and u in $\mathcal{M}$ such that for all n,

$$f_n = |f_n|u_n, \quad |f_n| = f_n u_n^*, \quad f = |f|u, \quad |f| = fu^*,$$
$$|f_n^*| = u_n^*|f_n|u_n, \quad f_n^* = |f_n^*|u_n^*, \quad |f^*| = u^*|f|u, \quad f^* = |f^*|u^*,$$

with $u_n u_n^* = s(|f_n|)$ and $uu^* = s(|f|)$, the support projections [12, p. 140] for $|f_n|$ and $|f|$ respectively. It follows immediately that $f^* = u^*|f|$ and $|f| = uf^*$.

We have

$$\begin{aligned}\|f_n - f\| &= \| |f_n|u_n - |f|u\| \\ &\le \| |f_n|u_n - |f_n|u\| + \| |f_n|u - |f|u\| \\ &\le \| |f_n|u_n - |f_n|u\| + \| |f_n| - |f| \|,\end{aligned}$$

and by hypothesis, $\| |f_n| - |f| \| \to 0$, so it is sufficient to show that $\| |f_n|u_n - |f_n|u\| \to 0$.

Let $x \in \mathcal{M}_1$. By the Cauchy–Schwarz inequality for positive functionals [12, I.9], we have

$$\begin{aligned}| |f_n|((u_n - u)x)|^2 &\le |f_n|(x^*x)|f_n|((u_n - u)(u_n^* - u^*)) \\ &\le |f_n|((u_n - u)(u_n^* - u^*)) \\ &= |f_n|(u_n u_n^*) - |f_n|u_n(u^*) - u_n^*|f_n|(u) + |f_n|(uu^*) \\ &= 1 - f_n(u^*) - f_n^*(u) + |f_n|(uu^*),\end{aligned}$$

using in particular the fact that $u_n u_n^* = s(|f_n|)$ for all n. It follows that

$$\| |f_n|u_n - |f_n|u\| \le 1 - f_n(u^*) - f_n^*(u) + |f_n|(uu^*),$$

but

$$\begin{aligned}f_n(u^*) &\to f(u^*) = fu^*(1) = |f|(1) = 1, \\ f_n^*(u) &\to f^*(u) = uf^*(1) = |f|(1) = 1, \quad \text{and} \\ |f_n|(uu^*) &\to |f|(uu^*) = (u^*|f|u)(1) = |f^*|(1) = \|f\| = 1.\end{aligned}$$

Hence $\| |f_n|(u_n - u)\| \to 0$, and so $\|f_n - f\| \to 0$.

The converse follows immediately from the definition of KKP.

(b) Let f_n, f, u_n, and u be as in (a) and suppose that $a_n \to 0$ weakly in $\mathcal{M}$. We have

$$\begin{aligned}|(f_n - f)(a_n)| &\le |(|f_n|(u_n - u))(a_n)| + | |f_n|(ua_n) - |f|(ua_n)| \\ &\le \| |f_n|(u_n - u)\| \cdot \|a_n\| + | |f_n|(ua_n) - |f|(ua_n)|.\end{aligned}$$

Since $ua_n \to 0$ weakly, we have $| |f_n|(ua_n) - |f|(ua_n)| \to 0$ by assumption. But by the proof of (a), we have $\| |f_n|(u_n - u)\| \to 0$, so that $(f_n - f)(a_n) \to 0$, whence $f_n(a_n) \to 0$.

The converse follows immediately from the definition of the DP1-property. ∎

Remark. In [1, Corollary 5] it is shown that whenever a sequence (f_n) of states in $\mathcal{M}_*$ converges weakly to a *pure* state f, then we always have $\|f_n - f\| \to 0$.

2.2. EXAMPLE: *$L^1[0,1]$ fails to have* KKP. Let (r_n) denote the sequence of Rademacher functions on $[0,1]$, where for $0 \le t \le 1$, and for all $n = 1, 2, \ldots$,

$$r_n(t) = \operatorname{sgn}(\sin 2^n \pi t).$$

It is easy to see that $\{r_n : n \ge 1\}$ is orthonormal in $L^2[0,1]$. Since $L^\infty[0,1] \subseteq L^2[0,1]$, it follows that for any $g \in L^\infty[0,1]$, we have $\int_0^1 g(t) r_n(t)\, dt \to 0$, i.e., as elements of $L^1[0,1]$, $r_n \to 0$ weakly. Setting $f_n(t) = r_n(t) + 1$ for all $0 \le t \le 1$, we then have $f_n \to 1$ weakly in $L^1[0,1]$, and it is easy to see that $\|f_n\|_1 = 1$ for all n, but we also have $\|f_n - 1\|_1 = \|r_n\|_1 = 1$ for all n.

2.3. In [6], Dell'Antonio defines property U for a von Neumann algebra $\mathcal{M}$ saying that $\mathcal{M}$ has property U if the condition in 2.1(a) is met, i.e., weakly convergent sequences of states in $\mathcal{M}_*$ converge uniformly. Thus by Proposition 2.1, $\mathcal{M}$ has property U if and only if $\mathcal{M}_*$ has KKP. It is then shown in [6] that if $\mathcal{M}$ is type I, then $\mathcal{M}_*$ has KKP if and only if Z, the center of $\mathcal{M}$, is atomic. From this, it follows immediately that for any Hilbert space H, $\mathcal{L}^1(H)$ has KKP.

2.4. EXAMPLE: *A von Neumann algebra whose predual has* DP1, *but has neither* DP *nor* KKP. Let $\mathcal{M} = \mathcal{B}(H) \oplus_\infty L^\infty[0,1]$, so $\mathcal{M}_* = \mathcal{L}^1(H) \oplus_1 L^1[0,1]$. Since $\mathcal{L}^1(H)$ fails to have DP, $\mathcal{M}_*$ fails to have DP, and by Example 2.2, $L^1[0,1]$ fails to have KKP, so $\mathcal{M}_*$ also fails to have KKP. On the other hand, $\mathcal{L}^1(H)$ has KKP and $L^1[0,1]$ has DP, so that both spaces have DP1. Thus by Theorem 1.9, $\mathcal{M}_*$ has DP1 as well.

2.5. THEOREM. *Let A be a C^*-algebra such that A^* is separable. Then A^* has* KKP.

Proof. By [13, Lemma 3], there exists a countable family $\{H_i : i = 1, 2, \ldots\}$ of Hilbert spaces such that

$$A^{**} = \Big(\bigoplus_{i=1}^{\infty} \mathcal{B}(H_i)\Big)_\infty,$$

and hence

$$A^* = \Big(\bigoplus_{i=1}^{\infty} \mathcal{L}^1(H_i)\Big)_1.$$

As remarked in 2.3 above, $\mathcal{L}^1(H)$ has KKP, so by Theorem 1.9, A^* has KKP. Alternatively, the von Neumann algebra A^{**} is type I and clearly its center is isomorphic to ℓ_∞, which is atomic, so by [6, 3.1] its predual A^* has KKP. ■

In [3], it is shown that $\mathcal{M}_*$ has DP if and only if whenever $a_n \to 0$ weakly in $\mathcal{M}$, then $a_n \to 0$ σ-strong*, that is, $a_n^* a_n + a_n a_n^* \to 0$ weak*. That result inspired the following proposition.

2.6. PROPOSITION. *Let $\mathcal{M}$ be a von Neumann algebra such that $\mathcal{M}_*$ has* DP1. *Let $a_n \to a$ weakly in $\mathcal{M}$ with $\|a_n\| = \|a\| = 1$. For all $f \in \mathcal{M}_*$ such that $\|fa^*\| = \|f\|$, we have $f(a_n^* a_n) \to f(a^*a)$. For all $f \in \mathcal{M}_*$ such that $\|a^*f\| = \|f\|$, we have $f(a_n a_n^*) \to f(aa^*)$. Hence for all $f \in \mathcal{M}_*$ such that $\|a^*f\| = \|f\| = \|fa^*\|$, we have $f(a_n^* a_n + a_n a_n^*) \to f(a^*a + aa^*)$.*

Proof. Suppose $\|fa^*\| = \|f\|$. We have $fa_n^* \to fa^*$ weakly in $\mathcal{M}_*$ and $\|fa_n^*\| \le \|f\| \cdot \|a_n^*\| = \|f\| = \|fa^*\|$. By the remark in 1.8, we have $\|fa_n^*\| \to \|fa^*\|$. Since $\mathcal{M}_*$ has DP1, we then obtain $(fa_n^*)(a_n) \to (fa^*)(a)$, i.e., $f(a_n^* a_n) \to f(a^*a)$. Similarly, if $\|a^*f\| = \|f\|$, we obtain $f(a_n a_n^*) \to f(aa^*)$, and hence the last assertion follows as well. ■

2.7. COROLLARY. *Let $\mathcal{M}$ be a von Neumann algebra such that $\mathcal{M}_*$ has* DP1. *Let $(a_n) \subseteq \partial\mathcal{M}_1$. If $a_n \to 1$ weakly, then $a_n \to 1$ σ-strong*.*

Proof. This follows immediately from the proposition with $a = 1$. ■

3. DP1 and KKP for C^*-algebras and von Neumann algebras.

We will now consider when a von Neumann algebra $\mathcal{M}$ or C^*-algebra A has DP1 or KKP. As shown in [3], A has DP if and only if whenever $x_n \to 0$ weakly in A, we have $x_n^* x_n \to 0$ weakly. As will now be shown, a similar condition is equivalent to A having DP1.

3.1. THEOREM. *Let A be a C^*-algebra. Then A has* DP1 *if and only if whenever $x_n \to x$ weakly in A with $\|x_n\| = \|x\| = 1$, then $x_n^* x_n \to x^*x$ weakly. In that case, $x_n x_n^* \to xx^*$ weakly, and so $x_n^* x_n + x_n x_n^* \to x^*x + xx^*$ weakly.*

Proof. Suppose A has DP1, and $x_n \to x$ weakly in A with $\|x_n\| = \|x\| = 1$. Let $g \in A^*$, and define $f_n = gx_n^*$, and $f = gx^*$. For any $z \in A^{**}$, we have $f_n(z) = g(x_n^* z) = zg(x_n^*) \to zg(x^*) = f(z)$, so that $f_n \to f$ weakly in A^*. Hence $g(x_n^* x_n) = f_n(x_n) \to f(x) = g(x^*x)$, and thus $x_n^* x_n \to x^*x$ weakly. Similarly, $x_n x_n^* \to xx^*$ weakly, from which the final conclusion follows. Conversely, let $\|x_n\| = \|x\| = 1$ for all n, with $x_n \to x$ weakly in A. Let $f_n \to 0$ weakly in A^*. By hypothesis, $x_n^* x_n + x_n x_n^* \to x^*x + xx^*$ weakly, i.e., $x_n \to x$ in the σ-strong* topology of A^{**}, whence $f_n(x_n) \to 0$ by [12, III.5.5]. ■

In [3], it is also shown that if A is a C^*-algebra having DP, then every C^*-subalgebra of A has DP as well. A similar result holds for DP1 as seen in the next result.

3.2. COROLLARY. *Let A be a C^*-algebra, with $B \subseteq A$ a C^*-subalgebra. If A has* DP1, *then B has* DP1 *as well.*

Proof. Let $b_n \to b$ weakly in B, with $\|b_n\| = \|b\| = 1$. Clearly, $b_n \to b$ weakly in A, so by the theorem, $b_n^* b_n \to b^*b$ weakly in A, and hence in B as well. Thus by the theorem, B has DP1. ■

3.3. EXAMPLES. (i) *$K(H)$, and hence $\mathcal{B}(H)$, fails to have* DP1. Let (e_n) be an orthonormal sequence in H. For each n, let $x_n = e_n \otimes e_1 + e_2 \otimes e_2 \in K(H)$, and let $x = e_2 \otimes e_2 \in K(H)$. It is easy to see that $x_n \to x$ weakly, and $\|x_n\| = \|x\| = 1$ for all $n \ge 3$, but $x_n^* x_n = e_1 \otimes e_1 + e_2 \otimes e_2 \not\to x^*x = e_2 \otimes e_2$ weakly. Alternatively, letting $f_n = \mathrm{tr}(\cdot\, e_1 \otimes e_n)$, one sees that $f_n \to 0$ weakly in $K(H)^*$, but $f_n(x_n) = 1$ for all n.

(ii) *A C^*-algebra having* DP *whose double dual fails to have* DP1. Let A be the C^*-algebra

$$A = \Big(\bigoplus_{n=1}^{\infty} M_n(\mathbb{C})\Big)_0.$$

As shown in [3], $A^* = (\bigoplus_{n=1}^{\infty} \mathcal{L}^1(\ell_2^n))_1$ has the Schur property, so that both A and A^* have DP. Let $\mathcal{M} = A^{**} = (\bigoplus_{n=1}^{\infty} M_n(\mathbb{C}))_\infty$. It can be shown that $\mathcal{M}$ fails to have DP1 (and hence fails to have DP), by using a sequence analogous to that used in 3.3(i) and then applying Theorem 3.1. A simpler proof of this fact is offered here, as suggested by the referee.

We note first that $\mathcal{M}$ is clearly isometrically isomorphic to

$$\Big(\bigoplus_{n=1}^{\infty} M_{2n}(\mathbb{C})\Big)_\infty \oplus_\infty \Big(\bigoplus_{n=1}^{\infty} M_{2n-1}(\mathbb{C})\Big)_\infty,$$

and that for any increasing sequence $(n_k) \subseteq \mathbb{N}$, the space $(\bigoplus_{k=1}^{\infty} M_{n_k}(\mathbb{C}))_\infty$ contains a complemented isometric copy of $(\bigoplus_{k=1}^{\infty} \ell_2^{n_k})_\infty$. Hence, in particular, $\mathcal{M}$ contains a complemented isometric copy of

$$\Big(\bigoplus_{n=1}^{\infty} \ell_2^{2n}\Big)_\infty \oplus_\infty \Big(\bigoplus_{n=1}^{\infty} \ell_2^{2n-1}\Big)_\infty.$$

Now, it can be shown (see [14, p. 81] and [8, p. 22]) that for any increasing sequence $(n_k) \subseteq \mathbb{N}$, the space $(\bigoplus_{k=1}^{\infty} \ell_2^{n_k})_\infty$ contains a complemented isometric copy of ℓ_2, from which it follows that $\mathcal{M}$ contains a complemented isometric copy of $\ell_2 \oplus_\infty \ell_2$. But by Example 1.6, the latter space fails to have DP1, so that $\mathcal{M}$ fails to have DP1 as well.

3.4. THEOREM. *Let A be a C^*-algebra. Then A has* KKP *if and only if A is finite-dimensional.*

Proof. Suppose A is infinite-dimensional. Then by [10, 4.6.13], A contains a sequence (f_n) of positive pairwise-orthogonal norm one elements.

Suppose $f_n \not\to 0$ weakly in A, so there exist $\phi \in A_1^*$, $\delta > 0$, and a subsequence (f_{n_k}) such that for all k, $|\phi(f_{n_k})| > \delta$. Choose $\theta_k \in \mathbb{C}$ such that $\phi(\theta_k f_{n_k}) = |\phi(f_{n_k})|$. Since the sequence (f_n) is pairwise orthogonal, we have for any N,

$$1 = \Big\|\sum_{i=1}^{N} \theta_i f_{n_i}\Big\| \ge \Big|\phi\Big(\sum_{i=1}^{N} \theta_i f_{n_i}\Big)\Big| = \sum_{i=1}^{N} \phi(\theta_i f_{n_i}) > N\delta,$$

which is impossible. Hence $f_n \to 0$ weakly. Now $f_1 + f_n \to f_1$ weakly, and $\|f_1 + f_n\| = 1$ for all $n \ge 2$, but $1 = \|f_n\| = \|(f_1 + f_n) - f_1\|$, so A fails to have KKP.

Conversely, if A is finite-dimensional, then the weak and norm topologies agree, so we are done. ∎

Let $\mathcal{M}$ be a von Neumann algebra. As shown in [3], $\mathcal{M}$ has DP if and only if $\mathcal{M}$ is a finite direct sum of type I_n von Neumann algebras, $n \in \mathbb{N}$. Of course, if $\mathcal{M}$ has DP, then $\mathcal{M}$ has DP1, and the last result shows that the converse holds.

3.5. THEOREM. *Let $\mathcal{M}$ be a von Neumann algebra. Then $\mathcal{M}$ has* DP *if and only if $\mathcal{M}$ has* DP1.

Proof. Again, one direction is immediate, so suppose that $\mathcal{M}$ has DP1. As shown in the proof of [3, Theorem 3], if $\mathcal{M}$ is infinite, then it contains a subalgebra isomorphic to $\mathcal{B}(H)$ for some infinite-dimensional Hilbert space H. By Example 3.3(i), $\mathcal{B}(H)$ fails to have DP1, and so by Corollary 3.2, $\mathcal{M}$ fails to have DP1, a contradiction. Thus $\mathcal{M}$ is finite.

As shown in the same proof in [3], if $\mathcal{M}$ contains a type II_1 summand $\mathcal{N}$, then $\mathcal{N}$ contains a subalgebra isomorphic to $(\bigoplus_{j=1}^{\infty} M_{2^j}(\mathbb{C}))_\infty$. But by Example 3.3(ii), this subalgebra fails to have DP1, and again by Corollary 3.2, this implies $\mathcal{M}$ fails to have DP1, a contradiction. Hence $\mathcal{M}$ contains no type II_1 summand, so that $\mathcal{M}$ is type I and finite.

It follows that $\mathcal{M} = (\bigoplus_{j=1}^{\infty} \mathcal{A}_j \overline{\otimes} M_{n_j}(\mathbb{C}))_\infty$, where for each $j \in \mathbb{N}$, $\mathcal{A}_j$ is an abelian von Neumann algebra and $n_j \in \mathbb{N}$, with $n_1 \le n_2 \le \ldots$ If the sequence (n_j) is unbounded, then $\mathcal{M}$ contains a subalgebra isomorphic to $(\bigoplus_{j=1}^{\infty} M_{m_j}(\mathbb{C}))_\infty$, for some strictly increasing sequence $(m_j) \subseteq \mathbb{N}$, but again by Example 3.3(ii) and Corollary 3.2, this subalgebra fails to have DP1, and so $\mathcal{M}$ fails to have DP1, a contradiction. Hence the sequence (n_j) is bounded, so that $\mathcal{M}$ can be written as a finite direct sum of type I_n von Neumann algebras. But by [3, Theorem 3], this implies that $\mathcal{M}$ has DP. ∎

Remark. It seems likely that the predual of every von Neumann algebra has DP1, and that a C^*-algebra A has DP if and only if it has DP1. These remain conjectures at present. The latter question is made difficult to prove by the general lack for C^*-algebras of the kinds of structure theorems that exist for von Neumann algebras.

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