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AN ALTERNATIVE MATERIAL MODEL USING A GENERALIZED J_2 FINITE-STRAIN FLOW PLASTICITY THEORY WITH ISOTROPIC HARDENING

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In this paper an alternative material model using a generalized J_2 finite-strain flow plasticity theory with isotropic hardening is presented. The model is based on a new nonlinear continuum mechanical theory of finite deformations of elasto-plastic media which allows for the development of objective and thermodynamically consistent material models. As a result, the constitutive equation, the evolution equation and even the 'normality rule', characterising the plastic flow in the material during return mapping, can be expressed in various forms, using several instances of the yield surface and corresponding pairs of stress measures and strain rates, respectively, which are conjugate with respect to the internal mechanical power and its arbitrary higher order time derivative. Therefore the results of the material model when used in numerical analyses are not affected by the description and particularities of the material model formulation. Here, we briefly outline the nonlinear continuum theory along with a detailed description of the material model and finally present the model in a numerical example using a cross-shaped specimen in biaxial tension.

Key words: nonlinear continuum theory of finite deformations of elasto-plastic media, generalized J_2 flow plasticity theory with isotropic hardening, objective and thermodynamically consistent formulation.

1. Introduction

Modelling of material behaviour within the framework of finite-strain elastoplasticity represents a challenging task in computational mechanics. While plastic behaviour of structural materials within the framework of small-strain elastoplasticity is now well understood, due to the fact that small-strain flow plasticity theories work well and their results are in agreement with experiments, the same cannot be said for finite-strain flow plasticity theories [1]. Even though innumerable material models for finite-strain elastoplasticity have by now been proposed [2-10], the models in general lack universality, as their results depend on the description used in the models and the particularities of the model formulation. The modelling methods might simply need some developments in the non-linear continuum theory of finite deformations of elasto-plastic media in order that the formulation of the models was thermodynamically consistent.

There are two modelling techniques used in contemporary computational plasticity to model irreversible finite deformations in the material of a body. The first technique is based on an ad hoc extension of infinitesimal flow plasticity theories into the area of finite deformations of elastic media to cover large displacements, but small strains in the material of the deforming body. The related material models employ a hypoelastoplastic stress-strain relationship when combining additive decomposition of a strain rate tensor into an elastic part and a plastic part with the theory of nonlinear continuum mechanics of elastic media to describe the plastic flow in the material [2, 11-16]. The other technique, which utilizes the theory of single-crystal plasticity to describe micromechanics of irreversible deformations in the material, is now generally accepted as a standard way of modelling finite plastic deformations in a material. The related material models use an intermediate stress-free configuration after irreversible deformations have taken place in the

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material, when combining multiplicative decomposition of a deformation gradient into an elastic part and a plastic part with the theory of nonlinear continuum mechanics of elastic media and classical flow plasticity theories to describe plastic behaviour of the material [2, 3, 17-23]. Though it may sound surprising, our ongoing research has shown that both of the aforementioned techniques are just variants of the first nonlinear continuum theory of finite deformations of elasto-plastic media, which allows an objective and thermodynamically consistent description of the plastic behaviour of the material.

The aim of this paper is to present an alternative material model using a generalized J_2 finite-strain flow plasticity theory with isotropic hardening and present the model in a numerical example using a biaxial tension of a cross-shaped specimen. In the paper, special emphasis is laid on the discussion of the thermodynamic consistency of the model formulation in correlation with the modified nonlinear continuum theory of finite deformations of elastoplastic media.

2. Theory

The Lagrangian description is used to describe the kinematics of motion and the conservation laws at a material particle of a deformable body. Though the nonlinear continuum mechanical theory of finite deformations of elasto-plastic media is not detailed herein, it plays a key role in the development of the material model presented in this paper.

2.1. A short overview of the nonlinear continuum mechanical theory of finite deformations of elasto-plastic media

The nonlinear continuum mechanical theory of finite deformations of elastic media has been developed in an elegant manner [13, 24-27]. The theory is particularly suitable for modelling of elastic materials, whose constitutive equations are defined in terms of a finite strain tensor, or hyperelastic materials, whose constitutive equations are derived from strain energy density functions [24, 25]. Modelling of plastic behaviour of materials within the framework of thermodynamics with internal variables of state, however, requires a somewhat different approach [28].

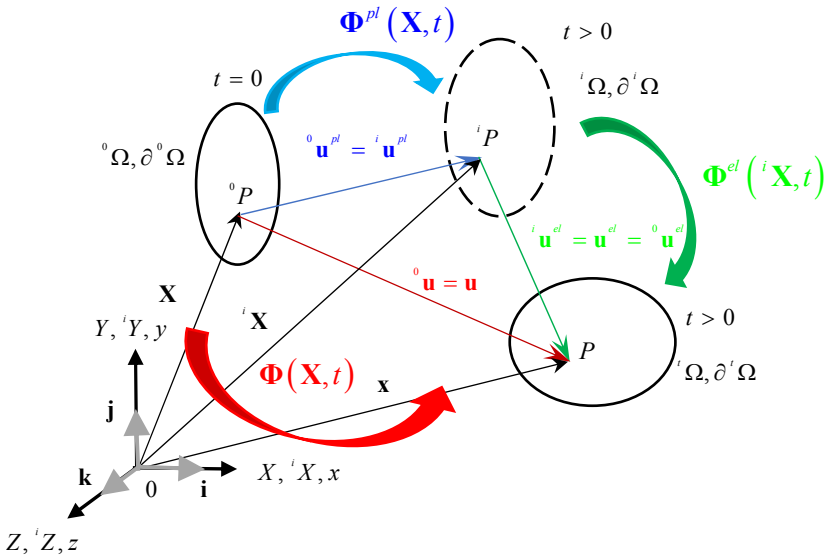

Fig.1. The proper kinematics of motion of elastoplastic media.

The constitutive and evolution equations of these materials are typically defined in rate forms using a kind of a hypoelastoplastic stress-strain relationship, which are not fully covered in the theory of

contemporary nonlinear continuum mechanics. Moreover, contemporary treatment of large elasto-plastic deformations, using multiplicative plasticity theories [2-5, 25], raises a few serious questions about the theories and the related material models from both physical and mathematical points of view. These arise from the vague definitions of the elastic and plastic motions in the theories, neglecting the displacement fields in the definitions of the elastic and plastic parts of the deformation gradient, which in this way fixes the order of elastic and plastic deformations at each constituent of the material of the deforming body. Proving that the theories are thermodynamically consistent also poses a difficult task, which according to the authors' best knowledge has never been done in connection with the multiplicative plasticity theories.

In order to develop a proper continuum mechanical theory of finite deformations of elastoplastic media, it is crucial to understand the definitions of motion, Lagrangian and Eulerian displacement fields, which in the case of large elasto-plastic deformations can naturally be subdivided into elastic and plastic parts (Fig.1). Here we concentrate on the Lagrangian description only, since our primary goal is to develop a material model in a Lagrangian form. To start with the motion, let us just remind here that from the mathematics point of view, the motion $\boldsymbol{\Phi}=\boldsymbol{\Phi}(\boldsymbol{X},t)$ (Fig.1) is a vector function, a one-to-one map, which associates each material point ${}^{0}P\in{}^{0}\Omega$, with a position vector $\boldsymbol{X}$ at time $t=0$, a unique spatial point $P\in{}^{t}\Omega$, with a position vector $\boldsymbol{x}$ at time $t>0$ [24]

$$\left\{\boldsymbol{x}\middle|\boldsymbol{x}=\boldsymbol{\Phi}\left(\boldsymbol{X},t\right),\ \text{for}\ \boldsymbol{X}\in{}^{0}\Omega,\ \boldsymbol{x}\in{}^{t}\Omega\ \text{and}\ t\geq 0\right\}. \tag{2.1}$$

In Eq.(2.1) ${}^{0}\Omega$ is the domain of the function, which stands for the volume of the body in its initial configuration and ${}^{t}\Omega$ is the range of the function, which stands for the volume of the body in its current configuration. The Lagrangian displacement field is then defined as [24]

$${}^{0}\boldsymbol{u}={}^{0}\boldsymbol{u}\left(\boldsymbol{X},t\right)=\boldsymbol{x}-\boldsymbol{X}=\boldsymbol{\Phi}\left(\boldsymbol{X},t\right)-\boldsymbol{X},\ \text{for}\ \ t\geq 0\ \ \text{and}\ \ \boldsymbol{X}\in{}^{0}\Omega. \tag{2.2}$$

When the motion is decomposed into an elastic part and a plastic part, the material point ${}^{0}P\in{}^{0}\Omega$ is in succession mapped to a spatial point ${}^{i}P\in{}^{i}\Omega,$ with a position vector ${}^{i}\boldsymbol{X}$ at time $t>0$ in the intermediate configuration of the body, then to the spatial point ${}^{t}P\in{}^{t}\Omega,$ with the position vector $\boldsymbol{x}$ at time $t>0$ in the current configuration of the body. Then the plastic motion $\boldsymbol{\Phi}^{pl}=\boldsymbol{\Phi}^{pl}\left(\boldsymbol{X},t\right)$ and the Lagrangian plastic displacement field ${}^{0}\boldsymbol{u}^{pl}$ take the forms

$$\left\{{}^{i}\boldsymbol{X}\middle|{}^{i}\boldsymbol{X}=\boldsymbol{\Phi}^{pl}\left(\boldsymbol{X},t\right),\ \text{for}\ \boldsymbol{X}\in{}^{0}\Omega,\ {}^{i}\boldsymbol{X}\in{}^{i}\Omega\ \text{and}\ t\geq 0\right\}, \tag{2.3}$$

$${}^{0}\boldsymbol{u}^{pl}={}^{0}\boldsymbol{u}^{pl}\left(\boldsymbol{X},t\right)={}^{i}\boldsymbol{X}-\boldsymbol{X}=\boldsymbol{\Phi}^{pl}\left(\boldsymbol{X},t\right)-\boldsymbol{X},\ \text{for}\ t\geq 0,\ \text{and}\ \boldsymbol{X}\in{}^{0}\Omega.. \tag{2.4}$$

It is immediately clear from Eqs (2.3) and (2.4) that the functions $\boldsymbol{\Phi}^{pl}, {}^{0}\boldsymbol{u}^{pl}$ have the same domain ${}^{0}\Omega,$ although the range of the plastic motion, resulting from plastic deformations only, is the volume ${}^{i}\Omega$. Even though the elastic motion $\boldsymbol{\Phi}^{el}=\boldsymbol{\Phi}^{el}\left({}^{i}\boldsymbol{X},t\right)$ and the elastic displacement filed ${}^{i}\boldsymbol{u}^{el}={}^{i}\boldsymbol{u}^{el}\left({}^{i}\boldsymbol{X},t\right)$ are defined similarly

$$\left\{\boldsymbol{x}\middle|\boldsymbol{x}=\boldsymbol{\Phi}^{el}\left({}^{i}\boldsymbol{X},t\right),\ \text{for}\ {}^{i}\boldsymbol{X}\in{}^{i}\Omega,\ \boldsymbol{x}\in{}^{t}\Omega,\ t\geq 0\right\}, \tag{2.5}$$

$$ {}^{i}\boldsymbol{u}^{el} = {}^{i}\boldsymbol{u}^{el}\left({}^{i}\boldsymbol{X},t\right) = \boldsymbol{x} - {}^{i}\boldsymbol{X} = \boldsymbol{\Phi}^{el}\left({}^{i}\boldsymbol{X},t\right) - {}^{i}\boldsymbol{X}, \text{ for } t \geq 0 \text{ and } {}^{i}\boldsymbol{X} \in {}^{i}\Omega, \tag{2.6} $$

the functions represent Eulerian vector fields, because their domains of definition is the intermediate volume of the body ${}^{i}\Omega$. After combining Eqs (2.3) and (2.5) or (2.6), however, both the elastic motion as well as the elastic displacement field can be expressed in Lagrangian forms as

$$ \left\{\boldsymbol{x} \middle| \boldsymbol{x} = \boldsymbol{\Phi}^{el}\left[\boldsymbol{\Phi}^{pl}\left(\boldsymbol{X},t\right),t\right], \text{ for } \boldsymbol{X} \in {}^{0}\Omega, \ \boldsymbol{x} \in {}^{t}\Omega, \ t \geq 0\right\}, \tag{2.7} $$

$$ {}^{0}\boldsymbol{u}^{el} = {}^{0}\boldsymbol{u}^{el}\left(\boldsymbol{X},t\right) = \boldsymbol{x} - {}^{i}\boldsymbol{X} = \boldsymbol{\Phi}^{el}\left[\boldsymbol{\Phi}^{pl}\left(\boldsymbol{X},t\right),t\right] - \boldsymbol{\Phi}^{pl}\left(\boldsymbol{X},t\right), \text{ for } t \geq 0 \text{ and } \boldsymbol{X} \in {}^{0}\Omega, \tag{2.8} $$

defined over the initial volume of the body. From a comparison of Eqs (2.1) and (2.7) it follows that

$$ \boldsymbol{x} = \boldsymbol{\Phi}\left(\boldsymbol{X},t\right) = \boldsymbol{\Phi}^{el}\left[\boldsymbol{\Phi}^{pl}\left(\boldsymbol{X},t\right),t\right]. \tag{2.9} $$

Moreover, the sum of Eqs. (2.8) and (2.4) can be rewritten as

$$ {}^{0}\boldsymbol{u}^{el}\left(\boldsymbol{X},t\right) + {}^{0}\boldsymbol{u}^{pl}\left(\boldsymbol{X},t\right) = \boldsymbol{\Phi}^{el}\left[\boldsymbol{\Phi}^{pl}\left(\boldsymbol{X},t\right),t\right] - \boldsymbol{X} = \boldsymbol{x} - \boldsymbol{X} = {}^{0}\boldsymbol{u}\left(\boldsymbol{X},t\right). \tag{2.10} $$

Equation (2.10) states that the Lagrangian displacement field ${}^{0}\boldsymbol{u}$ can additively be decomposed into Lagrangian elastic ${}^{0}\boldsymbol{u}^{el}$ and plastic ${}^{0}\boldsymbol{u}^{pl}$ parts, which in the end results in the following Lagrangian form for the deformation gradient

$$ \boldsymbol{F} = \boldsymbol{F}\left(\boldsymbol{X},t\right) = \boldsymbol{I} + \frac{\partial {}^{0}\boldsymbol{u}}{\partial \boldsymbol{X}} = \boldsymbol{I} + \frac{\partial {}^{0}\boldsymbol{u}^{pl}}{\partial \boldsymbol{X}} + \frac{\partial {}^{0}\boldsymbol{u}^{el}}{\partial \boldsymbol{X}}. \tag{2.11} $$

It should be noted here that Eq.(2.11) is the simplest form of the deformation gradient after simplification of the product Eq.(2.12), resulting from the chain rule of differentiation of the composite function Eq.(2.9)

$$ \boldsymbol{F}(\boldsymbol{X},t) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \frac{\partial \boldsymbol{\Phi}^{el}\left[\boldsymbol{\Phi}^{pl}\left(\boldsymbol{X},t\right),t\right]}{\partial \boldsymbol{X}} = \frac{\partial \boldsymbol{\Phi}^{el}\left[\boldsymbol{\Phi}^{pl}\left(\boldsymbol{X},t\right),t\right]}{\partial \boldsymbol{\Phi}^{pl}\left(\boldsymbol{X},t\right)} \cdot \frac{\partial \boldsymbol{\Phi}^{pl}\left(\boldsymbol{X},t\right)}{\partial \boldsymbol{X}} = \boldsymbol{F}^{el}(\boldsymbol{X},t) \cdot \boldsymbol{F}^{pl}(\boldsymbol{X},t). \tag{2.12} $$

Moreover, Eq.(2.11) also puts in order the deformation from the physical point of view, since the order of elastic and plastic deformations is no longer fixed at each constituent of the body, because vector addition (Eq.(2.10)) is commutative, and this was simply achieved by the consistent use of the principles of nonlinear continuum mechanics when the motions (Eqs (2.1), (2.3), (2.7) and (2.9)) and the Lagrangian displacement fields (Eqs (2.2), (2.4), (2.8) and (2.10)) were defined.

When the deformation gradient is defined in the form of Eq.(2.11), neither the Green strain tensor $\boldsymbol{E} = 1/2 \cdot \left(\boldsymbol{F}^{T} \cdot \boldsymbol{F} - \boldsymbol{I}\right)$ nor the Almansi strain tensor $\boldsymbol{e} = 1/2 \cdot \left(\boldsymbol{I} - \boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1}\right)$ has decomposition into an elastic part and a plastic part, but additive decomposition exists when one evaluates the objective time derivatives of the tensors. The material $\dot{\boldsymbol{E}}$ and the spatial $\boldsymbol{d} = \mathcal{L}_{e}\left(\boldsymbol{e}\right)$ strain-rate tensors then take the following forms

$$\dot{\boldsymbol{E}} = \frac{1}{2}\cdot\left(\dot{\boldsymbol{F}}^T\cdot\boldsymbol{F}+\boldsymbol{F}^T\cdot\dot{\boldsymbol{F}}\right) = \dot{\boldsymbol{E}}^{el}+\dot{\boldsymbol{E}}^{pl},\ \Rightarrow\ \boldsymbol{d}=\boldsymbol{d}^{el}+\boldsymbol{d}^{pl}, \tag{2.13}$$

where

$$\boldsymbol{d}=\boldsymbol{F}^{-T}\cdot\dot{\boldsymbol{E}}\cdot\boldsymbol{F}^{-1},\ \boldsymbol{d}^{el}=\boldsymbol{F}^{-T}\cdot\dot{\boldsymbol{E}}^{el}\cdot\boldsymbol{F}^{-1},\ \boldsymbol{d}^{pl}=\boldsymbol{F}^{-T}\cdot\dot{\boldsymbol{E}}^{pl}\cdot\boldsymbol{F}^{-1}, \tag{2.14}$$

$$\dot{\boldsymbol{E}}^{el}=\frac{1}{2}\cdot\left[\left(\frac{\partial^0\dot{\boldsymbol{u}}^{el}}{\partial\boldsymbol{X}}\right)^T\cdot\boldsymbol{F}+\boldsymbol{F}^T\cdot\frac{\partial^0\dot{\boldsymbol{u}}^{el}}{\partial\boldsymbol{X}}\right],\quad \dot{\boldsymbol{E}}^{pl}=\frac{\dot{\lambda}}{2}\cdot\left[\left(\frac{\partial^P\Psi}{\partial\boldsymbol{P}}\right)^T\cdot\boldsymbol{F}+\boldsymbol{F}^T\cdot\frac{\partial^P\Psi}{\partial\boldsymbol{P}}\right], \tag{2.15}$$

$$\frac{\partial^0\dot{\boldsymbol{u}}^{pl}}{\partial\boldsymbol{X}}=\dot{\lambda}\cdot\frac{\partial^P\Psi}{\partial\boldsymbol{P}},\quad \text{and}\quad \frac{\partial^P\Psi}{\partial\boldsymbol{P}}\neq\left(\frac{\partial^P\Psi}{\partial\boldsymbol{P}}\right)^T. \tag{2.16}$$

In Eqs (2.11)-(2.16) (see also Fig.1) $\boldsymbol{X}$ denotes the position vector of a material point ${}^0P$ and $\boldsymbol{x}=\boldsymbol{X}+{}^0\boldsymbol{u}=\boldsymbol{X}+{}^0\boldsymbol{u}^{pl}+{}^0\boldsymbol{u}^{el}$ is the position vector of the corresponding spatial point P after deformation. The symbols $\dot{\boldsymbol{E}}^{el},\dot{\boldsymbol{E}}^{pl}\ /\ \boldsymbol{d}^{el},\boldsymbol{d}^{pl}$ stand for the elastic and the plastic material/spatial strain rate tensors, where in the latter the plastic flow is defined by Eq.$(2.16)_1$ as a product of a plastic multiplier $\dot{\lambda}$ and an appropriate yield surface normal, $\partial^P\Psi/\partial\boldsymbol{P}$, defined in terms of the 1st Piola-Kirchhoff stress tensor $\boldsymbol{P}$. Here the symbol $\mathcal{L}_e(\boldsymbol{e})=\boldsymbol{F}^{-T}\cdot\left[\partial\left(\boldsymbol{F}^T\cdot(\boldsymbol{e})\cdot\boldsymbol{F}\right)/\partial t\right]\cdot\boldsymbol{F}^{-1}$ denotes the Lie derivative of the Almansi strain tensor $\boldsymbol{e}$. It should be noted that both the elastic and the plastic strain-rate tensors have forms similar to the strain-rate tensor itself. Besides, it can be shown that the plastic flow defined by Eq.$(2.16)_1$ is not constrained, resulting in Eqs $(2.13)_2$ and $(2.14)_3$ respectively, being the only non-degenerated forms of the material and spatial plastic strain rate tensors.

In order to generalize the nonlinear continuum theory so that it can cover rate forms of constitutive equations, the Cauchy's stress theorem [24] has to be modified as follows

$$\mathfrak{L}_{Tr}^{(n)}\left[\boldsymbol{T}(\boldsymbol{X},t,\boldsymbol{N})\right]=\mathfrak{L}_{P}^{(n)}\left[\boldsymbol{P}(\boldsymbol{X},t)\right]\cdot\boldsymbol{N}\ \text{ and }\ \mathfrak{L}_{tr}^{(n)}\left[\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})\right]=\mathfrak{L}_{T}^{(n)}\left[\boldsymbol{\sigma}(\boldsymbol{x},t)\right]\cdot\boldsymbol{n}, \tag{2.17}$$

for all $n=0,1,...,n\in N$, where n denotes objective differentiation with respect to time $t\geq 0$ and not an exponent and N the set of natural numbers. In Eq.(2.17) the variables $\mathfrak{L}_{Tr}^{(n)}[\boldsymbol{T}(\boldsymbol{X},t,\boldsymbol{N})]$, $\mathfrak{L}_{tr}^{(n)}[\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})]$ stand for the n^{th} objective derivatives of the surface traction vectors $\boldsymbol{T}=\boldsymbol{T}(\boldsymbol{X},t,\boldsymbol{N}),\boldsymbol{t}=\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})$ in the initial and current configurations of the body and $\boldsymbol{N},\boldsymbol{n}$ are the corresponding surface normal vectors. Similarly, the quantities $\mathfrak{L}_{P}^{(n)}[\boldsymbol{P}(\boldsymbol{X},t)],\mathfrak{L}_{T}^{(n)}[\boldsymbol{\sigma}(\boldsymbol{x},t)]$ denote the n^{th} objective derivatives of the 1st Piola-Kirchhoff stress tensor $\boldsymbol{P}=\boldsymbol{P}(\boldsymbol{X},t)$ and the Cauchy's stress tensor $\boldsymbol{\sigma}=\boldsymbol{\sigma}(\boldsymbol{x},t)$ respectively. Then the thermodynamic consistency of the nonlinear continuum mechanical formulation can be expressed with the following two postulates.

Postulate no.1: The product of the surface traction vector and all of its higher order objective time derivatives acting on the surface of an infinitesimal volume element and the area of the surface element in the initial and current configurations of the body have to be the same during the whole deformation process, i.e.

$$\mathfrak{L}_{Tr}^{(n)}(\boldsymbol{T})\cdot dS_0=\mathfrak{L}_{P}^{(n)}(\boldsymbol{P})\cdot\boldsymbol{N}\cdot dS_0=\mathfrak{L}_{P}^{(n)}(\boldsymbol{P})\cdot d\boldsymbol{S}_0=\mathfrak{L}_{tr}^{(n)}(\boldsymbol{t})\cdot ds=\mathfrak{L}_{T}^{(n)}(\boldsymbol{\sigma})\cdot\boldsymbol{n}\cdot ds=\mathfrak{L}_{T}^{(n)}(\boldsymbol{\sigma})\cdot d\boldsymbol{s}, \tag{2.18}$$

or simply

$$\mathfrak{L}_{P}^{(n)}(\boldsymbol{P})\cdot d\boldsymbol{S}_0=\mathfrak{L}_{T}^{(n)}(\boldsymbol{\sigma})\cdot d\boldsymbol{s}=J\cdot\mathfrak{L}_{T}^{(n)}(\boldsymbol{\sigma})\cdot\boldsymbol{F}^{-T}\cdot d\boldsymbol{S}_0,\ \text{ for all }\ n=0,1,2,...,n\in N, \tag{2.19}$$

where $d\boldsymbol{S}_0, d\boldsymbol{s} = J \cdot \boldsymbol{F}^{-T} \cdot d\boldsymbol{S}_0$ denote the infinitesimal surface elements in the initial and current configurations of the body, where the latter is expressed by Nanson's formula [24].

Postulate no.2: The rate of change of internal deformation energy accumulated in the infinitesimal volume element / internal mechanical power in the initial and current configurations of the body and all of its higher order objective time derivatives have to be the same during the whole deformation process, i.e.

$$\frac{\partial^n dW}{\partial t^n} = \left[\sum_{k=0}^{n}\binom{n}{k} \cdot \frac{\partial^{n-k}\boldsymbol{S}}{\partial t^{n-k}} : \frac{\partial^k \dot{\boldsymbol{E}}}{\partial t^k}\right] \cdot dV_0 = \left[\sum_{k=0}^{n}\binom{n}{k} \cdot \mathcal{L}_P^{(n-k)}(\boldsymbol{P}) : \mathcal{L}_F^{(k)}\left(\frac{\partial^0 \dot{\boldsymbol{u}}}{\partial \boldsymbol{X}}\right)\right] \cdot dV_0 =$$
$$= \left[\sum_{k=0}^{n}\binom{n}{k} \cdot \mathcal{L}_O^{(n-k)}(\boldsymbol{\tau}) : \mathcal{L}_e^{(k)}(\boldsymbol{d})\right] \cdot dV_0 = \left[\sum_{k=0}^{n}\binom{n}{k} \cdot \mathcal{L}_T^{(n-k)}(\boldsymbol{\sigma}) : \mathcal{L}_e^{(k)}(\boldsymbol{d})\right] \cdot dv, \tag{2.20}$$
$$\text{for all } n = 0,1,2,...,n \in N$$

where dV_0, $dv = J \cdot dV_0$ stand for the infinitesimal volume elements in the initial and current configurations of the body and $J = \det(\boldsymbol{F})$. Equations (2.19) and (2.20) then define the following transformations

$$\mathcal{L}_P^{(n)}(\boldsymbol{P}) = \boldsymbol{F} \cdot \left(\frac{\partial^n \boldsymbol{S}}{\partial t^n}\right), \quad \mathcal{L}_O^{(n)}(\boldsymbol{\tau}) = \boldsymbol{F} \cdot \left(\frac{\partial^n \boldsymbol{S}}{\partial t^n}\right) \cdot \boldsymbol{F}^T, \quad \mathcal{L}_T^{(n)}(\boldsymbol{\sigma}) = J^{-1} \cdot \boldsymbol{F} \cdot \left(\frac{\partial^n \boldsymbol{S}}{\partial t^n}\right) \cdot \boldsymbol{F}^T,$$
$$\mathcal{L}_e^{(n)}(\boldsymbol{d}) = \boldsymbol{F}^{-T} \cdot \left(\frac{\partial^n \dot{\boldsymbol{E}}}{\partial t^n}\right) \cdot \boldsymbol{F}^{-1}, \qquad \frac{\partial^n \dot{\boldsymbol{E}}}{\partial t^n} = \left[\boldsymbol{F}^T \cdot \mathcal{L}_F^{(n)}(\dot{\boldsymbol{F}})\right]^{sym} = \left[\frac{\partial^n\left(\boldsymbol{F}^T \cdot \frac{\partial^0 \dot{\boldsymbol{u}}}{\partial \boldsymbol{X}}\right)}{\partial t^n}\right]^{sym}, \tag{2.21}$$
$$\mathcal{L}_e^{(n)}(\boldsymbol{d}) = \left[\mathcal{L}_F^{(n)}(\dot{\boldsymbol{F}}) \cdot \boldsymbol{F}^{-1}\right]^{sym} = \left\{\boldsymbol{F}^{-T} \cdot \left[\frac{\partial^n\left(\boldsymbol{F}^T \cdot \frac{\partial^0 \dot{\boldsymbol{u}}}{\partial \boldsymbol{X}}\right)}{\partial t^n}\right] \cdot \boldsymbol{F}^{-1}\right\}^{sym}, \quad \text{for all } n = 0,1,2,...,n \in N,$$

as the sufficient conditions of thermodynamic consistency, because they ensure that Eqs (2.19) and (2.20) hold true. It should also be noted here that for $n = 0$ the generalized Cauchy's stress theorem (Eq.(2.17)) reduces to its well-known form, $\boldsymbol{T} = \boldsymbol{P} \cdot \boldsymbol{N}$ and $\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$, while the transformations (2.21) reduce to the already well-know transformations in nonlinear continuum mechanics defining the relationship between various stress measures and corresponding strain rates or deformation rate respectively, which are conjugate with respect to the internal mechanical power. As a result, the transformations (2.21) define the necessary conditions of thermodynamic consistency for $n = 0$. In other words, Eqs (2.19)-(2.21) define the stress measures and the corresponding strain rates or deformation rate and their objective time derivatives, respectively, which are conjugate with respect to the n^{th} time derivative of the internal mechanical power. The objective rates, which meet the sufficient conditions of thermodynamic consistency (Eq.(2.21)), are already known in nonlinear continuum mechanics as the n^{th} Lie derivative of the 1st Piola-Kirchhoff stress tensor $\boldsymbol{P}$ (Eq.(2.22)), the n^{th} Lie derivative of the rate of deformation gradient tensor $\dot{\boldsymbol{F}}$ (Eq.(2.23)), the n^{th} Oldroyd rate of the Kirchhoff stress $\boldsymbol{\tau}$ (Eq.(2.24)), the n^{th} Lie derivative of the spatial strain rate tensor $\boldsymbol{d}$ (Eq.(2.25)) and the n^{th} Truesdell rate of the Cauchy's stress $\boldsymbol{\sigma}$ (Eq.(2.26)) respectively.

$$\mathcal{L}_P^{(n)}(\boldsymbol{P}) = \boldsymbol{F}\cdot\left[\frac{\partial^n\left(\boldsymbol{F}^{-1}\cdot\boldsymbol{P}\right)}{\partial t^n}\right] = \boldsymbol{F}\cdot\left(\frac{\partial^n \boldsymbol{S}}{\partial t^n}\right), \tag{2.22}$$

$$\mathcal{L}_F^{(n)}(\dot{\boldsymbol{F}}) = \boldsymbol{F}^{-T}\cdot\left[\frac{\partial^n\left(\boldsymbol{F}^{T}\cdot\dot{\boldsymbol{F}}\right)}{\partial t^n}\right] = \boldsymbol{F}^{-T}\cdot\left[\frac{\partial^n\left(\boldsymbol{F}^{T}\cdot\dfrac{\partial^0\dot{\boldsymbol{u}}}{\partial \boldsymbol{X}}\right)}{\partial t^n}\right], \tag{2.23}$$

$$\mathcal{L}_O^{(n)}(\boldsymbol{\tau}) = \boldsymbol{F}\cdot\left[\frac{\partial^n\left(\boldsymbol{F}^{-1}\cdot\boldsymbol{\tau}\cdot\boldsymbol{F}^{-T}\right)}{\partial t^n}\right]\cdot\boldsymbol{F}^{T} = \boldsymbol{F}\cdot\left(\frac{\partial^n \boldsymbol{S}}{\partial t^n}\right)\cdot\boldsymbol{F}^{T}, \tag{2.24}$$

$$\mathcal{L}_e^{(n)}(\boldsymbol{d}) = \boldsymbol{F}^{-T}\cdot\left[\frac{\partial^n\left(\boldsymbol{F}^{T}\cdot\boldsymbol{d}\cdot\boldsymbol{F}\right)}{\partial t^n}\right]\cdot\boldsymbol{F}^{-1} = \boldsymbol{F}^{-T}\cdot\left(\frac{\partial^n \dot{\boldsymbol{E}}}{\partial t^n}\right)\cdot\boldsymbol{F}^{-1}, \tag{2.25}$$

$$\mathcal{L}_T^{(n)}(\boldsymbol{\sigma}) = J^{-1}\cdot\boldsymbol{F}\cdot\left[\frac{\partial^n\left(J\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}\right)}{\partial t^n}\right]\cdot\boldsymbol{F}^{T} = J^{-1}\cdot\boldsymbol{F}\cdot\left(\frac{\partial^n \boldsymbol{S}}{\partial t^n}\right)\cdot\boldsymbol{F}^{T}. \tag{2.26}$$

2.2. The constitutive equation of the material

A proper formulation of a material model for finite-strain elastoplasticity makes it possible to define the constitutive equation of the material in terms of various stress and strain measures or their objective rates in both the body initial and current configurations. As a result, the constitutive equation of the material cannot be unique, but it must have various forms. These forms, however, have to comply with the principles of material modelling, particularly meet the requirements of material objectivity and moreover be thermodynamically consistent in order that they would defined the same material. Furthermore, because the additive decomposition defined by Eqs $(2.13)_2$, $(2.13)_3$ exists in rate forms only, the constitutive equation of the material too has to have a rate form. In fact, Eqs (2.27)-(2.30) defined a true hypoelastoplastic material model, which does not have a form in terms of a finite strain measure.

In this research, we have modified our former material model capable of imitating ductile-to-brittle failure mode transition of a ductile material at high strain rates [29]. In agreement with the above, the rate form of the constitutive equation of the material can take any of the following forms

$$\dot{\boldsymbol{S}} = {}^{mat}\mathbb{C}^{el} : \left(\dot{\boldsymbol{E}} - xx\cdot\dot{\boldsymbol{E}}^{pl}\right) + {}^{mat}\mathbb{C}^{vis} : \left[\ddot{\boldsymbol{E}} - (1-xx)\cdot\ddot{\boldsymbol{E}}^{pl}\right], \tag{2.27}$$

$$\mathcal{L}_P(\boldsymbol{P}) = \boldsymbol{F}\cdot\dot{\boldsymbol{S}} = \boldsymbol{F}\cdot\left\{{}^{mat}\mathbb{C}^{el} : \left(\dot{\boldsymbol{E}} - xx\cdot\dot{\boldsymbol{E}}^{pl}\right) + {}^{mat}\mathbb{C}^{vis} : \left[\ddot{\boldsymbol{E}} - (1-xx)\cdot\ddot{\boldsymbol{E}}^{pl}\right]\right\}, \tag{2.28}$$

$$\mathcal{L}_O(\boldsymbol{\tau}) = \boldsymbol{F}\cdot\dot{\boldsymbol{S}}\cdot\boldsymbol{F}^{T} = J\cdot{}^{spat}\mathbb{C}^{el} : \left(\boldsymbol{d} - xx\cdot\boldsymbol{d}^{pl}\right) + J\cdot{}^{spat}\mathbb{C}^{vis} : \left[\mathcal{L}_e(\boldsymbol{d}) - (1-xx)\cdot\mathcal{L}_e\left(\boldsymbol{d}^{pl}\right)\right], \tag{2.29}$$

$$\mathcal{L}_T(\boldsymbol{\sigma}) = J^{-1}\cdot\boldsymbol{F}\cdot\dot{\boldsymbol{S}}\cdot\boldsymbol{F}^{T} = {}^{spat}\mathbb{C}^{el} : \left(\boldsymbol{d} - xx\cdot\boldsymbol{d}^{pl}\right) + {}^{spat}\mathbb{C}^{vis} : \left[\mathcal{L}_e(\boldsymbol{d}) - (1-xx)\cdot\mathcal{L}_e\left(\boldsymbol{d}^{pl}\right)\right], \tag{2.30}$$

where

$$ {}^{mat}\mathbb{C}^{el} = 2 \cdot G \cdot \mathbb{I} + \lambda^{el} \cdot \boldsymbol{I} \otimes \boldsymbol{I}, \qquad {}^{mat}\mathbb{C}^{vis} = 2 \cdot G^{vis} \cdot \mathbb{I} + \lambda^{vis} \cdot \boldsymbol{I} \otimes \boldsymbol{I}, \tag{2.31} $$

$$ G = \frac{E}{2 \cdot (1+\nu)}, \; \lambda^{el} = \frac{\nu \cdot E}{(1+\nu) \cdot (1-2 \cdot \nu)}, \; G^{vis} = \frac{E^{vis}}{2 \cdot \left(1+\nu^{vis}\right)}, \; \lambda^{vis} = \frac{\nu^{vis} \cdot E^{vis}}{\left(1+\nu^{vis}\right) \cdot \left(1-2 \cdot \nu^{vis}\right)}, \tag{2.32} $$

$$ {}^{spat}\mathbb{C}^{el}_{ijkl} = J^{-1} \cdot F_{im} \cdot F_{jn} \cdot F_{ko} \cdot F_{lp} \cdot {}^{mat}\mathbb{C}^{el}_{mnop}, \tag{2.33} $$

$$ {}^{spat}\mathbb{C}^{vis}_{ijkl} = J^{-1} \cdot F_{im} \cdot F_{jn} \cdot F_{ko} \cdot F_{lp} \cdot {}^{mat}\mathbb{C}^{vis}_{mnop}. \tag{2.34} $$

In Eqs (2.27)-(2.34) the symbols $\dot{\boldsymbol{S}}, \mathcal{L}_P(\boldsymbol{P}), \mathcal{L}_O(\boldsymbol{\tau}), \mathcal{L}_T(\boldsymbol{\sigma})$ denote the time derivative of the 2nd Piola-Kirchhoff stress tensor, the Lie derivative of the 1st Piola-Kirchhoff stress tensor, the Oldroyd rate of the Kirchhoff stress and the Truesdell rate of the Cauchy stress. Here the fourth order material elasticity tensor ${}^{mat}\mathbb{C}^{el}$ and the fourth order material viscosity tensor ${}^{mat}\mathbb{C}^{vis}$ have similar forms as the fourth order elasticity tensor of the St.-Venant Kirchhoff material [25] using two independent material parameters E, ν and E^{vis}, ν^{vis}, respectively. The fourth order spatial elasticity and viscosity tensors ${}^{spat}\mathbb{C}^{el}$, ${}^{spat}\mathbb{C}^{vis}$ then can be determined in accordance with Eqs (2.33) and (2.34). The variable xx denotes the ratio of ductile and total damage increment [29]. Please also note that the objective rates $\dot{\boldsymbol{S}}, \mathcal{L}_P(\boldsymbol{P}), \mathcal{L}_O(\boldsymbol{\tau}), \mathcal{L}_T(\boldsymbol{\sigma})$ transform in the same way from one form into another as do the stress tensors $\boldsymbol{S}, \boldsymbol{P}, \boldsymbol{\tau}, \boldsymbol{\sigma}$, which ensure that the formulation is thermodynamically consistent.

2.3. Modelling of the plastic flow in the material

Similarly as in the case of the rate forms of the constitutive equation of the material, a proper formulation of a finite-strain flow plasticity theory enables the description of the plastic flow in terms of various instances of a yield surface and corresponding stress measures in either the body initial or current configuration. Let the various instances of the yield surface ${}^{S}\Psi = {}^{S}\Psi\left[{}^{S}\sigma_{eq}(\boldsymbol{S}), \boldsymbol{q}\right]$, ${}^{P}\Psi = {}^{P}\Psi\left[{}^{P}\sigma_{eq}(\boldsymbol{P}), \boldsymbol{q}\right]$, ${}^{\tau}\Psi = {}^{\tau}\Psi\left[{}^{\tau}\sigma_{eq}(\boldsymbol{\tau}), \boldsymbol{q}\right]$, ${}^{\sigma}\Psi = {}^{\sigma}\Psi\left[{}^{\sigma}\sigma_{eq}(\boldsymbol{\sigma}), \boldsymbol{q}\right]$ be defined in terms of the stress measures $\boldsymbol{S}, \boldsymbol{P}, \boldsymbol{\tau}, \boldsymbol{\sigma}$ and a vector of internal variables $\boldsymbol{q}$. After changing the physical interpretation of the plastic flow and applying push-forward and pull-back operations to the material gradient of the plastic velocity field (Eq.(2.16)$_1$) as follows

$$ \frac{\partial \dot{\boldsymbol{u}}^{pl}}{\partial \boldsymbol{x}} = \frac{\partial^0 \dot{\boldsymbol{u}}^{pl}}{\partial \boldsymbol{X}} \cdot \boldsymbol{F}^{-1} = \dot{\lambda} \cdot \frac{\partial^{\sigma}\Psi}{\partial \boldsymbol{\sigma}}, \quad \frac{\partial \dot{\boldsymbol{u}}^{pl}}{\partial \boldsymbol{x}} = \frac{\partial^0 \dot{\boldsymbol{u}}^{pl}}{\partial \boldsymbol{X}} \cdot \boldsymbol{F}^{-1} = \dot{\lambda} \cdot \frac{\partial^{\tau}\Psi}{\partial \boldsymbol{\tau}}, \quad \boldsymbol{F}^T \cdot \frac{\partial^0 \dot{\boldsymbol{u}}^{pl}}{\partial \boldsymbol{X}} = \dot{\lambda} \cdot \frac{\partial^{S}\Psi}{\partial \boldsymbol{S}}, \tag{2.35} $$

it can be found that the yield surfaces are not independent of each other, but the following formulas hold true

$$ \frac{\partial^{P}\Psi}{\partial \boldsymbol{P}} \cdot \boldsymbol{F}^{-1} = \frac{\partial^{\sigma}\Psi}{\partial \boldsymbol{\sigma}}, \quad \frac{\partial^{P}\Psi}{\partial \boldsymbol{P}} \cdot \boldsymbol{F}^{-1} = \frac{\partial^{\tau}\Psi}{\partial \boldsymbol{\tau}}, \quad \boldsymbol{F}^T \cdot \frac{\partial^{P}\Psi}{\partial \boldsymbol{P}} = \frac{\partial^{S}\Psi}{\partial \boldsymbol{S}}. \tag{2.36} $$

As a result, one of the yield surfaces has to be chosen as a reference yield surface to define the material model and the rest of them can be calculated by solving the ordinary differential Eq.(2.36). Moreover, it can also be verified that the yield surfaces and the equivalent stresses ${}^{P}\sigma_{eq}, {}^{S}\sigma_{eq}, {}^{\tau}\sigma_{eq}, {}^{\sigma}\sigma_{eq}$, contained in the definitions of the yield surfaces and also meeting the transformations defined by Eq.(2.36), have the following properties

$$ {}^{P}\sigma_{eq} = {}^{S}\sigma_{eq} = {}^{\tau}\sigma_{eq} = J \cdot {}^{\sigma}\sigma_{eq}, \tag{2.37} $$

$$ \frac{\partial {}^{S}\Psi}{\partial \boldsymbol{S}} : \boldsymbol{S} = \frac{\partial {}^{P}\Psi}{\partial \boldsymbol{P}} : \boldsymbol{P} = \frac{\partial {}^{\tau}\Psi}{\partial \boldsymbol{\tau}} : \boldsymbol{\tau} = J \cdot \frac{\partial {}^{\sigma}\Psi}{\partial \boldsymbol{\sigma}} : \boldsymbol{\sigma}, \tag{2.38} $$

$$ \frac{\partial {}^{S}\Psi}{\partial \boldsymbol{S}} : \dot{\boldsymbol{S}} = \frac{\partial {}^{P}\Psi}{\partial \boldsymbol{P}} : \mathcal{L}_P(\boldsymbol{P}) = \frac{\partial {}^{\tau}\Psi}{\partial \boldsymbol{\tau}} : \mathcal{L}_O(\boldsymbol{\tau}) = J \cdot \frac{\partial {}^{\sigma}\Psi}{\partial \boldsymbol{\sigma}} : \mathcal{L}_T(\boldsymbol{\sigma}), \tag{2.39} $$

where Eqs (2.38) and (2.39) are equivalent with the following equations

$$ \dot{\boldsymbol{E}}^{pl} : \boldsymbol{S} \cdot dV_0 = \frac{\partial {}^{0}\dot{\boldsymbol{u}}^{pl}}{\partial \boldsymbol{X}} : \boldsymbol{P} \cdot dV_0 = \boldsymbol{d}^{pl} : \boldsymbol{\tau} \cdot dV_0 = \boldsymbol{d}^{pl} : \boldsymbol{\sigma} \cdot dv, \tag{2.40} $$

$$ \dot{\boldsymbol{E}}^{pl} : \dot{\boldsymbol{S}} \cdot dV_0 = \frac{\partial {}^{0}\dot{\boldsymbol{u}}^{pl}}{\partial \boldsymbol{X}} : \mathcal{L}_P(\boldsymbol{P}) \cdot dV_0 = \boldsymbol{d}^{pl} : \mathcal{L}_O(\boldsymbol{\tau}) \cdot dV_0 = \boldsymbol{d}^{pl} : \mathcal{L}_T(\boldsymbol{\sigma}) \cdot dv, \tag{2.41} $$

which also prove that the formulation of the plastic flow is thermodynamically consistent. It can be verified too that both Eqs (2.38), (2.39) represent 'normality rules', where Eq.(2.38) defines a thermodynamically consistent return mapping procedure, and Eq.(2.39) the rate form of the thermodynamically consistent return mapping procedure. The result is of fundamental importance in computational mechanics as it states how the plastic multiplier ought to be calculated during return mapping when finite-strain elastoplastic analysis is carried out.

2.4. The reference yield surface of the material

It has been shown in the above the choice of the reference yield surface governs the material model. As a result, alternative material models can be developed. In our research we have generalized the J_2 flow plasticity theory with isotropic hardening, where we used the ${}^{P}\Psi = {}^{P}\Psi(\boldsymbol{P}, \boldsymbol{q})$, Eq.(2.42)$_1$ yield surface, as the reference yield surface, to define our material model. Please note that the ${}^{P}J_2(\boldsymbol{P}) = \boldsymbol{P} : \boldsymbol{P}$ invariant no longer bases on the deviatoric part of the 1st Piola-Kirchhoff stress tensor. This is due to the fact that the 1st Piola-Kirchhoff stress tensor transforms under the change of the observer as $\boldsymbol{P}^{+} = \boldsymbol{Q}_R \cdot \boldsymbol{P}$, and ${}^{P}J_2(\boldsymbol{P})$ is the only invariant, which is not affected by the change, i.e. ${}^{P}J_2(\boldsymbol{P}) = {}^{P}J_2(\boldsymbol{P}^{+})$, where $\boldsymbol{Q}_R$ is an arbitrary rotating tensor expressing the relative rotation of the coordinate systems of the observer with respect to the reference coordinate system. The resulting yield surface is then no longer a cylinder, but a sphere

$$ {}^{P}\Psi = {}^{P}\sigma_{eq} - {}^{P}\sigma_y \le 0, \quad \text{where} \quad {}^{P}\sigma_{eq} = {}^{P}\sigma_{eq}(\boldsymbol{P}) = \sqrt{{}^{P}J_2(\boldsymbol{P})} = \sqrt{\boldsymbol{P} : \boldsymbol{P}}, \tag{2.42} $$

$$ {}^{P}\sigma_y = F_{UT11} \cdot \sqrt{r^2 - \left[a \cdot e^{pl} - \text{center}\right]^2}, \; r = \sigma_y + Q, \; \text{center} = \sqrt{r^2 - \sigma_y^2} \; \text{ and } \; a = \frac{\text{center} + r}{b}, \tag{2.43} $$

$$ \dot{e}^{pl} = \dot{e}^{pl}\left(\dot{\boldsymbol{F}}^{pl}\right) = \sqrt{\dot{\boldsymbol{F}}^{pl} : \dot{\boldsymbol{F}}^{pl}} = \dot{\lambda}, \; e^{pl} = \int_0^t \dot{e}^{pl} \cdot dt, \; \boldsymbol{F}^{pl} = \boldsymbol{I} + \frac{\partial \boldsymbol{u}^{pl}}{\partial \boldsymbol{X}}, \; \dot{\boldsymbol{F}}^{pl} = \frac{\partial \dot{\boldsymbol{u}}^{pl}}{\partial \boldsymbol{X}} = \dot{\lambda} \cdot \frac{\partial {}^{P}\Psi}{\partial \boldsymbol{P}}. \tag{2.44} $$

The actual yield stress ${}^{P}\sigma_y$, which is a 1st Piola-Kirchhoff stress measure, determines the radius of the yield surface and is defined by Eq.(2.43)$_1$. It is the only nonzero component of a stress tensor $\boldsymbol{P}_{UT}$ (i.e., ${}^{P}\sigma_y = [\boldsymbol{P}_{UT}]_{11}$) coming from an uniaxial tensile test of the modelled material, where the operator $[(\bullet)]_{11}$ extracts the element in the first row and the first column of a 2nd order tensor $(\bullet)$, written as a 3x3 square matrix. The corresponding deformation gradient and the Jacobian of deformation are denoted as $\boldsymbol{F}_{UT}, J_{UT}$, where $F_{UT11} = [\boldsymbol{F}_{UT}]_{11}$ and $J_{UT} = \det(\boldsymbol{F}_{UT})$. Please also note that the only nonzero element of the corresponding 2nd Piola-Kirchhoff stress tensor $\boldsymbol{S}_{UT}$, coming from the tensile test of the material is $[\boldsymbol{S}_{UT}]_{11} = {}^{S}\sigma_y = \sqrt{r^2 - \left[a \cdot e^{pl} - \text{center}\right]^2}$. The equation defines an arc of a circle using three material parameters, the constant yield stress of the material σ_y, the maximum stress Q by which the material can harden and the maximum accumulated strain value $b = e^{pl}_{\max}$, at which the material loses its integrity, i.e., ${}^{S}\sigma_y = 0$. The relationship between the corresponding stress measures then can be written in tensor form as $\boldsymbol{P}_{UT} = \boldsymbol{F}_{UT} \cdot \boldsymbol{S}_{UT}$, in which the parameters σ_y, Q are 2nd Piola-Kirchhoff stress measures and $e^{pl} \subset \langle 0, b \rangle$. One may note here that we have also changed the definition of the accumulated plastic strain rate $\dot{e}^{pl}$ (Eq.(2.44)$_1$), in the definition of which $\boldsymbol{F}^{pl}$ denotes the deformation gradient of pure plastic deformations at a particle of the material, whose time derivative is assumed to be in the form of Eq.(2.16)$_1$. Other changes in the definitions of the accumulated plastic strain rate $\dot{e}^{pl}$ and the equivalent stress ${}^{P}\sigma_{eq}$ have been needed in order to ensure that the formulation is thermodynamically consistent when either a one-dimensional (1D) stress state, or a three-dimensional (3D) stress state takes place at a material particle during the analysis.

2.5. Calculation of the plastic multiplier

The calculation of the plastic multiplier is a crucial step in finite-strain elastoplastic stress analyses as it determines the value of the stress rate tensor (Eqs (2.27)-(2.30)) and the plastic part of the strain rate tensors $\dot{\boldsymbol{E}}^{pl}, \boldsymbol{d}^{pl}$ during return mapping. Moreover, the return mapping procedure has to be thermodynamically consistent, i.e., it has to comply with Eq.(2.39). The condition has not yet been met in any formulation in finite-strain computational plasticity. The thermodynamically consistent return mapping procedure then utilizes the objective differentiation of the yield surface ${}^{P}\Psi$ and it can be expressed as follows

$$\frac{\partial {}^{P}\Psi}{\partial \boldsymbol{P}} : \mathcal{L}_{\mathrm{P}}(\boldsymbol{P}) - \left[\mathcal{L}_{\mathrm{P}}(\boldsymbol{P}_{UT})\right]_{11} = 0 \tag{2.45}$$

where $\mathcal{L}_{\mathrm{P}}(\boldsymbol{P})$ is then replaced by the rate form of the constitutive equation of the material Eq.(2.28), and the second term of Eq.(2.45) by the form $[\mathcal{L}_{\mathrm{P}}(\boldsymbol{P}_{UT})]_{11} = F_{UT11} \cdot \left\{\left[-a \cdot \left(a \cdot e^{pl} - \text{center}\right)\right] / \sqrt{r^2 - \left[a \cdot e^{pl} - \text{center}\right]^2}\right\} \cdot \dot{e}^{pl}$. Please also note that the first term of Eq.(2.45) can be replaced by any other term of Eq.(2.39), because the formulation is thermodynamically consistent.

2.6. The ratio of ductile and total damage increment

The idea of the ratio of ductile and total damage increment *xx* was first introduced by Écsi and Élesztős in order to take into account the internal damping of the material properly during plastic

deformations, where xx allowed for the redistribution of the plastic flow proportionally between the spring and the damper of a 1D frictional device representing the rheological model of the material [29]. The ratio is determined in an elastic corrector phase during return mapping and its value is then kept constant. Since the return mapping procedure in our material model is carried out in the 1st Piola-Kirchhoff stress space, we had to modify the definition of the ratio as follows

$$xx = \frac{\left\langle \boldsymbol{N} : \boldsymbol{F} \cdot \left({}^{mat}\mathbb{C}^{el} : \dot{\boldsymbol{E}} \right) \right\rangle}{\left\langle \boldsymbol{N} : \boldsymbol{F} \cdot \left({}^{mat}\mathbb{C}^{el} : \dot{\boldsymbol{E}} \right) \right\rangle + \left\langle \boldsymbol{N} : \boldsymbol{F} \cdot \left({}^{mat}\mathbb{C}^{vis} : \ddot{\boldsymbol{E}} \right) \right\rangle}, \tag{2.46}$$

where

$$\frac{\partial^{P}\Psi}{\partial \boldsymbol{P}} = \boldsymbol{N}, \quad \boldsymbol{N} = \frac{\boldsymbol{P}}{\sqrt{\boldsymbol{P} : \boldsymbol{P}}} = \frac{\boldsymbol{P}}{\|\boldsymbol{P}\|} \quad \text{and} \tag{2.47}$$

$$\langle y \rangle = \frac{y + |y|}{2} \geq 0 \tag{2.48}$$

denote the McCauly's brackets, which return zero if $y < 0$ and where we also used the property $\mathcal{L}_P(\boldsymbol{P}) = \boldsymbol{F} \cdot \dot{\boldsymbol{S}}$. Please also note that all terms of Eq.(2.46) are objective stress rates, so that the value of xx is not affected by the change of the observer.

3. Numerical experiment

In our numerical experiment a cross-shaped specimen in biaxial tension was studied. The specimen geometry has been proposed by Müller [30] and it was fabricated from an AlMgSi05 alloy. Table 1 outlines the material properties of the AlMgSi05 alloy specimen used in the finite element analysis

Table 1. Material properties of the AlMgSi05 alloy specimen.

E [Pa]	$6.89 \cdot 10^{10}$
E^{vis} [Pa · s]	$6.89 \cdot 10^{7}$
$\nu = \nu^{vis}$ [-]	0.33
σ_y [Pa]	$100.0 \cdot 10^{6}$
Q [Pa]	$50.0 \cdot 10^{6}$
b [-]	1.0
$\rho_0 \left[kg \cdot m^{-3} \right]$	2700.0

In order to assess the value of the axial component of the deformation gradient coming from the tensile stress of the material F_{UT11}, we solved the one-dimensional (1D) rate form of the constitutive equation of the material (Eq.(2.27)) for the unknown component of the derivative of the axial elastic displacement field with respect to the axial material coordinate $\partial u_x^{el} / \partial X$. The rate form of the constitutive equation of this specific 1D stress analysis, after neglecting the internal damping in the material, can be expressed in the following finite-strain form

$$ {}^{S}\dot{\sigma}_y = \dot{S}_{11} = \mathrm{E} \cdot \left[\frac{\partial \dot{u}_x^{el}}{\partial X} \cdot \left(1 + \frac{\partial u_x^{el}}{\partial X} + \frac{\partial u_x^{pl}}{\partial X} \right) \right] \tag{3.1} $$

where ${}^{S}\dot{\sigma}_y = \dot{S}_{11}$ is the axial component of the 2nd Piola-Kirchoff stress rate tensor coming from the tensile test of the material and E is the Young modulus. Furthermore considering that the accumulated plastic strain rate Eq.(2.44)$_1$ in this 1D stress state is $\dot{e}^{pl} = \partial \dot{u}_x^{pl} / \partial X = \dot{\lambda}$, and that its integral is $e^{pl} = \partial u_x^{pl} / \partial X$ (Eq.(2.44)$_2$), one can find F_{UT11} as a function of the accumulated plastic strain e^{pl} only in the form

$$ F_{UT11} = 1 + \frac{\partial u_x^{el}}{\partial X} + \frac{\partial u_x^{pl}}{\partial X} = 1 + \left[-\left(1 + e^{pl}\right) + \sqrt{\left(1 + e^{pl}\right)^2 + 2 \cdot \frac{{}^{S}\sigma_y}{E}} \right] + e^{pl} \tag{3.2} $$

where ${}^{S}\sigma_y = {}^{S}\sigma_y\left(e^{pl}\right)$ see also Eq.(2.43).

In the numerical study 1/8 of the specimen body was modelled employing three planes of symmetry. The specimen was loaded at its four ends using $v = 0.84667\ mm/s$ prescribed velocity. The bodies were initially at rest and the analysis was run as transient-dynamic analysis applying $0.005\ s$ time step size.

4. Numerical results

Figure 2 shows a few selected results at the end of the finite element analysis, at time $6\ s$. These are the 1st principal stress distribution in terms of Cauchy's stress measure and the accumulated plastic strain distribution over the current volume of the body of the cross-shaped specimen.

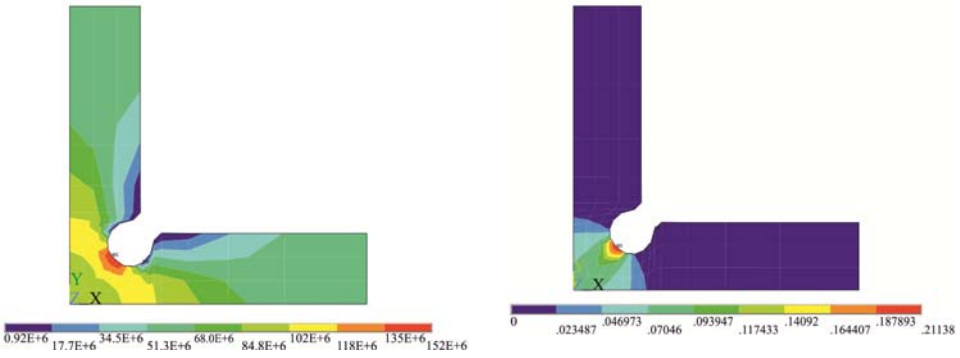

Fig.2. The 1st principal stress distribution in terms of the Cauchy's stress measure [Pa] and the accumulated plastic strain distribution [-] at the end of the finite element analysis.

As can be seen in the figure, the results are realistic. Moreover, the presented theory is also noteworthy from the material testing point of view of ductile materials, as it shows that contemporary tensile testing is not sufficient for finite-strain material property determination without the determination of the deformation gradient characterizing the uniaxial tensile test. The presented theory thus also serves as a basis for improved material testing.

5. Conclusions

In this paper an alternative J_2 material model with isotropic hardening for finite-strain elastoplastic analyses was presented. The model is based on a new non-linear continuum mechanical theory of finite deformations of elastoplastic media which allows for the description of the plastic flow in terms of various instances of the yield surface and stress measures in the initial and current configurations of the modelled body. The resulting formulation is objective and thermodynamically consistent. As a result, the material model and its results when the model is employed in numerical analyses, no longer depend on the description and the particularities of the material model formulation. Here by the description we mean total and updated Lagrangian descriptions and by the particularities of the formulation we mean various pairs of stress measures and strain rates used in the model formulation, which are conjugate with respect to the internal mechanical power or its arbitrary higher order time derivative. The material model was also demonstrated in a numerical example where biaxial tension of a cross-shaped specimen was studied. The presented theory not only significantly improves the contemporary elastoplastic analyses, but it also serves as a basis for improved material testing.

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Nomenclature

$^{mat}\mathbb{C}^{el}$, $^{spat}\mathbb{C}^{el}$ – 4th order material and spatial elasticity tensors

$^{mat}\mathbb{C}^{vis}$, $^{spat}\mathbb{C}^{vis}$ – 4th order material and spatial viscosity tensors

$\boldsymbol{d}, \boldsymbol{d}^{el}, \boldsymbol{d}^{pl}$ – spatial strain-rate tensor, its elastic and plastic parts

$d\boldsymbol{S}_0, d\boldsymbol{s}$ – infinitesimal surf. elements in the initial, current configurations of the body $d\boldsymbol{s} = J \cdot \boldsymbol{F}^{-T} \cdot d\boldsymbol{S}_0$

dV_0, dv – infinitesimal vol. elements in the initial and current configurations of the body $dv = J \cdot dV_0$

$\boldsymbol{E}, \boldsymbol{e}$ – the Green strain tensor and the Almansi strain tensor

$\dot{\boldsymbol{E}}, \dot{\boldsymbol{E}}^{el}, \dot{\boldsymbol{E}}^{pl}$ – material strain-rate tensor, its elastic and plastic parts

$\dot{e}^{pl}, e^{pl}, b, \dot{\lambda}$ – accumulated plastic strain rate and strain, maximum accumulated plastic strain, plastic multiplier

$\boldsymbol{F}_{UT}, J_{UT}$ – deformation gradient and the Jacobian coming from the uniaxial tensile test of the material

$\boldsymbol{F}, \boldsymbol{F}^{el}, \boldsymbol{F}^{pl}, J$ – deformation gradient, its elastic and plastic parts, Jacobian of the deformation $J = \det(\boldsymbol{F})$

G, E, λ^{el}, ν – shear modulus, Young's modulus, Lamé's parameter and Piosson's number

$G^{vis}, E^{vis}, \lambda^{vis}, \nu^{vis}$ – similar parameters to G, E, λ^{el}, ν for definition of the 4th order viscosity tensors

$^{P}J_2$, $^{P}\sigma_y$ – J_2 invariant and the yield stress defined in terms of the 1st Piola-Kirchhoff stress tensor

N – the set of natural numbers

$\boldsymbol{N}, \boldsymbol{n}$ – outward unit surface normal vectors in the initial and current configurations of the body

$\boldsymbol{P}_{UT}, \boldsymbol{S}_{UT}$ – 1st and the 2nd Piola-Kirchhoff stress tensor coming from the uniaxial tensile test of the material

$^{0}P, ^{i}P, P$ – material and spatial points of the body in its initial, intermediate and current configurations

$r, a, center$ – material parameters of the yield surface

$\boldsymbol{S}, \boldsymbol{P}, \boldsymbol{\tau}, \boldsymbol{\sigma}$ – the 2nd Piola-Kirchhoff, the 1st Piola-Kirchhoff, the Kirchhoff and the Cauchy stress tensors

$\boldsymbol{T}, \boldsymbol{t}$ – surface traction vectors in the initial and current configurations of the body

t – time

$^{0}\boldsymbol{u}$, $^{0}\boldsymbol{u}^{el}$, $^{0}\boldsymbol{u}^{pl}$ – material or Lagrangian displacement field, its elastic and plastic parts

$^{i}\boldsymbol{u}, {}^{i}\boldsymbol{u}^{el}, {}^{i}\boldsymbol{u}^{pl}$ – spatial displacement field, its elastic and plastic parts defined over $^{i}\Omega$

$\boldsymbol{u}, \boldsymbol{u}^{el}, \boldsymbol{u}^{pl}$ – spatial displacement field, its elastic and plastic parts

$\boldsymbol{q}, \boldsymbol{Q}_R$ – vector of internal variables and an arbitrary rotation tensor

W – internal mechanical power

$\boldsymbol{X}, {}^{i}\boldsymbol{X}, \boldsymbol{x}$ – position vectors of the material point ^{0}P and the spatial points $^{i}P, P$ respectively

X, Y, Z – material coordinates, components of the vector $\boldsymbol{x}$

$^{i}X, {}^{i}Y, {}^{i}Z$ – spatial coordinates, components of the vector $^{i}\boldsymbol{x}$

x, y, z – spatial coordinates, components of the vector $\boldsymbol{x}$

xx – the ratio of ductile and total damage increment

$^{0}\Omega, {}^{i}\Omega, {}^{t}\Omega$ – initial, intermediate and current volumes of the body

$\partial {}^{0}\Omega, \partial {}^{i}\Omega, \partial {}^{t}\Omega$ – initial, intermediate and current surfaces of the body

$\boldsymbol{\Phi}, \boldsymbol{\Phi}^{pl}, \boldsymbol{\Phi}^{el}$ – vector functions expressing the motion, its plastic and elastic parts

$\mathfrak{L}_{Tr}^{(n)}[\boldsymbol{T}], \mathfrak{L}_{tr}^{(n)}[\boldsymbol{t}]$ – nth objective time derivatives of the surface traction vectors $\boldsymbol{T}, \boldsymbol{t}$

$\mathfrak{L}_{P}^{(n)}(\boldsymbol{P})$ – nth objective (Lie) time derivative of the 1st Piola-Kirchhoff stress tensor

$\mathfrak{L}_{O}^{(n)}(\boldsymbol{\tau})$ – nth objective (Olrdoyd) time derivative of the Kirchhoff stress tensors

$\mathfrak{L}_{T}^{(n)}(\boldsymbol{\sigma})$ – nth objective (Truesdell) time derivative of the Cauchy stress tensors

$\mathfrak{L}_{F}^{(n)}(\dot{\boldsymbol{F}})$ – nth objective (Lie) time derivative of the rate of deformation gradient tensor

$\mathfrak{L}_{e}^{(n)}(\boldsymbol{d})$ – nth objective (Lie) time derivative of the spatial strain rate tensor

$^{S}\Psi, {}^{P}\Psi, {}^{\tau}\Psi, {}^{\sigma}\Psi$ – various instances of the yield surface as functions of the stress tensors $\boldsymbol{S}, \boldsymbol{P}, \boldsymbol{\tau}, \boldsymbol{\sigma}$

$^{P}\sigma_{eq}, {}^{S}\sigma_{eq}, {}^{\tau}\sigma_{eq}, {}^{\sigma}\sigma_{eq}$ – various instances of the equivalent stress as functions of the stress tensors $\boldsymbol{S}, \boldsymbol{P}, \boldsymbol{\tau}, \boldsymbol{\sigma}$

σ_y, Q – the yield stress and the maximum hardening stress as a 2nd Piola-Kirchhoff stress measure

ρ_0 – the material density in the initial configuration of the body

$\boldsymbol{I}, \mathbb{I}$ – a 2nd order unit tensor and a 4th order unit tensor

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