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AN ALTERNATIVE SPHERICAL NEAR FIELD TO FAR FIELD TRANSFORMATION

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1. INTRODUCTION

A large volume of literature exists on the spherical near field to far field transformation [1–3]. In this paper, the approach similar to [4] is taken to expand the fields in terms of TM and TE modes to r as described by [5–7]. The final expressions of this paper are somewhat different and simpler than the expressions of [1,4] and [6].

2. SPHERICAL NEAR-FIELD TO FAR-FIELD TRANSFORMATION

Consider a sphere of radius a over which the tangential components of the electric field, E_θ and E_ϕ, are known. So

$$E_\theta(a,\theta,\phi) = f_1(\theta,\phi) \tag{1}$$

$$E_\phi(a,\theta,\phi) = f_2(\theta,\phi) \tag{2}$$

From this near field given in equations (1) and (2) we determine the far field. The complete expression for the field external to the sphere is given by [4, p. 269] using the following vector potentials

$$A_r = j\omega\epsilon \sum_{n=0}^{\infty}\sum_{m=0}^{n} r h_n^{(2)}(kr) P_n^m(\cos\theta)\left[\alpha_{mn}\cos m\phi + \beta_{mn}\sin m\phi\right] \quad (3)$$

$$F_r = j\omega\mu \sum_{n=0}^{\infty}\sum_{m=0}^{n} r h_n^{(2)}(kr) P_n^m(\cos\theta)\left[\gamma_{mn}\cos m\phi + \delta_{mn}\sin m\phi\right] \quad (4)$$

where $h_n^{(2)}(kr)$ is the spherical Bessel function of the second kind of order n and argument kr, $P_n^m(\cos\theta)$ is the associated Legendre function of the first kind of argument $\cos\theta$, and α_{mn}, β_{mn}, γ_{mn}, δ_{mn} are the four constants to be determined from the boundary value problem specified by (1) and (2). Here, k is the free space wave number and ϵ and μ are the permittivity and permeability of free space. The θ and ϕ field components are then given by [4] as

$$\begin{aligned} E_\theta &= \frac{-1}{r\sin\theta}\frac{\partial F_r}{\partial\phi} + \frac{1}{j\omega\epsilon r}\frac{\partial^2 A_r}{\partial r\partial\theta} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{j\omega\mu m h_n^{(2)}(kr)}{\sin\theta} P_n^m(\cos\theta)[\gamma_{mn}\sin m\phi - \delta_{mn}\cos m\phi] \\ &+ \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{1}{r}\frac{\partial}{\partial r}\left[r h_n^{(2)}(kr)\right] \frac{dP_n^m(\cos\theta)}{d\theta}[\alpha_{mn}\cos m\phi + \beta_{mn}\sin m\phi] \end{aligned} \quad (5)$$

$$\begin{aligned} E_\phi &= \frac{1}{r}\frac{\partial F_r}{\partial\theta} + \frac{1}{j\omega\epsilon r\sin\theta}\frac{\partial^2 A_r}{\partial r\partial\phi} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} j\omega\mu h_n^{(2)}(kr)\frac{dP_n^m(\cos\theta)}{d\theta}[\gamma_{mn}\cos m\phi + \delta_{mn}\sin m\phi] \\ &- \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{1}{r}\frac{\partial}{\partial r}\left[r h_n^{(2)}(kr)\right] \frac{P_n^m(\cos\theta)m}{\sin\theta}[\alpha_{mn}\sin m\phi - \beta_{mn}\cos m\phi] \end{aligned} \quad (6)$$

$$\begin{aligned} H_\theta &= \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{1}{j\omega\mu r}\frac{\partial^2 F_r}{\partial r\partial\theta} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} -j\omega\mu h_n^{(2)}(kr)\frac{P_n^m(\cos\theta)m}{\sin\theta}[\alpha_{mn}\sin m\phi - \beta_{mn}\cos m\phi] \\ &+ \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{1}{r}\frac{\partial}{\partial r}\left[r h_n^{(2)}(kr)\right] \frac{dP_n^m(\cos\theta)}{d\theta}[\gamma_{mn}\cos m\phi + \delta_{mn}\sin m\phi] \end{aligned} \quad (7)$$

$$H_\phi = \frac{1}{r}\frac{\partial A_r}{\partial\theta} + \frac{1}{j\omega\mu r\sin\theta}\frac{\partial^2 F_r}{\partial r\partial\phi}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} -j\omega\mu h_n^{(2)}(kr)\frac{dP_n^m(\cos\theta)}{d\theta}[\alpha_{mn}\cos m\phi + \beta_{mn}\sin m\phi]$$
$$-\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{r}\frac{\partial}{\partial r}\left[rh_n^{(2)}(kr)\right]\frac{mP_n^m(\cos\theta)}{\sin\theta}[\gamma_{mn}\sin m\phi - \delta_{mn}\cos m\phi] \tag{8}$$

By replacing the field components E_θ and E_ϕ in equations (1) and (2) with their expressions described by equations (5) and (6), respectively, and then using orthogonality relationships, the coefficients α_{mn} , β_{mn} , γ_{mn} and δ_{mn} can be determined. Therefore, from equations (1) and (5) we have

$$f_1(\theta,\phi) =$$
$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{j\omega\mu m h_n^{(2)}(ka)}{ka\sin\theta}P_n^m(\cos\theta)[\gamma_{mn}\sin m\phi - \delta_{mn}\cos m\phi]$$
$$+\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{r}\frac{\partial}{\partial r}\left[rh_n^{(2)}(ka)\right]\frac{dP_n^m(\cos\theta)}{d\theta}[\alpha_{mn}\cos m\phi + \beta_{mn}\sin m\phi] \tag{9}$$

and from equations (2) and (6) we have

$$f_2(\theta,\phi) =$$
$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} j\omega\mu m h_n^{(2)}(ka)\frac{dP_n^m(\cos\theta)}{d\theta}[\gamma_{mn}\cos m\phi + \delta_{mn}\sin m\phi]$$
$$-\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{r}\frac{\partial}{\partial r}\left[rh_n^{(2)}(ka)\right]\frac{P_n^m(\cos\theta)m}{\sin\theta}[\alpha_{mn}\sin m\phi - \beta_{mn}\cos m\phi] \tag{10}$$

From equations (9) and (10) we have,

$$f_1(\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{Dm}{\sin\theta}P_n^m(\cos\theta)[\gamma_{mn}\sin m\phi - \delta_{mn}\cos m\phi]$$
$$+\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{N}{a}\frac{dP_n^m(\cos\theta)}{d\theta}[\alpha_{mn}\cos m\phi + \beta_{mn}\sin m\phi] \tag{11}$$

and

$$f_2(\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=0}^{n} D\frac{d}{d\theta}P_n^m(\cos\theta)[\gamma_{mn}\cos m\phi+\delta_{mn}\sin m\phi] \\ -\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{N}{a}\frac{P_n^m(\cos\theta)m}{\sin\theta}[\alpha_{mn}\sin m\phi-\beta_{mn}\cos m\phi] \tag{12}$$

where

$$D=j\omega\mu h_n^{(2)}(ka) \tag{13}$$

and

$$N=kah_n^{(2)'}(ka)+h_n^{(2)}(ka) \tag{14}$$

At this point we attempt to use orthogonality to determine the unknown coefficients. From (11) we have

$$\int_0^{2\pi}\int_0^{\pi} f_1(\theta,\phi)\frac{dP_n^{m'}(\cos\theta)}{d\theta}\sin\theta\cos m'\phi d\theta d\phi \\ =\sum_{n=0}^{\infty}\sum_{m=0}^{n} Dm\int_0^{\pi}P_n^m(\cos\theta)\frac{dP_{n'}^{m'}(\cos\theta)}{d\theta}d\theta \\ \int_0^{2\pi}\cos m'\phi[\gamma_{mn}\sin m\phi-\delta_{mn}\cos m\phi]d\phi \\ +\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{N}{a}\int_0^{\pi}\frac{dP_n^m(\cos\theta)}{d\theta}\frac{dP_{n'}^{m'}(\cos\theta)}{d\theta}\sin\theta d\theta \\ \int_0^{2\pi}\cos m'\phi[\alpha_{mn}\cos m\phi+\beta_{mn}\sin m\phi]d\phi \tag{15}$$

Since

$$\int_0^{2\pi}\cos m'\phi\cos m\phi d\phi=\begin{cases}0 & \text{for } m\neq m' \\ \frac{2\pi}{\epsilon_m} & \text{for } m=m'\end{cases} \tag{16}$$

where

$$\epsilon_m=\begin{cases}1 & \text{for } m=0 \\ 2 & \text{for } m\neq 0\end{cases} \tag{17}$$

and

$$\int_0^{2\pi}\cos m'\phi\sin m\phi d\phi=0 \quad \text{for all } m \text{ and } m' \tag{18}$$

Equation (15) may be written as

$$\int_0^{2\pi}\int_0^{\pi} f_1(\theta,\phi)\frac{dP_{n'}^{m'}(\cos\theta)}{d\theta}\sin\theta\cos m'\phi d\theta d\phi$$
$$=\sum_{n=0}^{\infty}\frac{2\pi Dm'}{\epsilon_{m'}}\int_0^{\pi}(-\delta_{m'n})P_n^{m'}(\cos\theta) \qquad (19)$$
$$+\sum_{n=0}^{\infty}\frac{2N\pi}{\epsilon_{m'}a}\int_0^{\pi}(\alpha_{m'n'})\frac{dP_n^{m'}(\cos\theta)}{d\theta}\frac{dP_{n'}^{m'}(\cos\theta)}{d\theta}\sin\theta d\theta$$

from equation (12) we have

$$\int_0^{2\pi}\int_0^{\pi} m' f_2(\theta,\phi)P_{n'}^{m'}(\cos\theta)\sin m'\phi d\theta d\phi$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n} Dm'\int_0^{\pi}\frac{dP_n^m(\cos\theta)}{d\theta}P_{n'}^{m'}(\cos\theta)d\theta$$
$$\int_0^{2\pi}\sin m'\phi[\gamma_{mn}\cos m\phi+\delta_{mn}\sin m\phi]d\phi \qquad (20)$$
$$-\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{N}{a}m\int_0^{2\pi}\int_0^{\pi}P_{n'}^{m'}(\cos\theta)\frac{P_n^m(\cos\theta)m}{\sin\theta}$$
$$[\alpha_{mn}\sin m\phi-\beta_{mn}\cos m\phi]\sin m'\phi d\theta d\phi$$

where D and N are described by equations (13) and (14), respectively. By using equation (18) and

$$\int_0^{2\pi}\sin m'\phi\sin m\phi d\phi=\begin{cases}0 & \text{for } m\neq m'\\ \frac{2\pi}{\epsilon_m} & \text{for } m=m'\end{cases} \qquad (21)$$

we can rewrite equation (20) as

$$\int_0^{2\pi}\int_0^{\pi} m' f_2(\theta,\phi)P_{n'}^{m'}(\cos\theta)\sin m'\phi d\theta d\phi$$
$$=\sum_{n=0}^{\infty}\frac{2Dm'\pi}{\epsilon_{m'}}\int_0^{\pi}\delta_{m'n}\frac{dP_n^{m'}(\cos\theta)}{d\theta}P_{n'}^{m'}(\cos\theta)d\theta \qquad (22)$$
$$-\sum_{n=0}^{\infty}\frac{2N}{\epsilon_{m'}a}m'^2\pi\int_0^{\pi}\alpha_{m'n}P_{n'}^{m'}(\cos\theta)\frac{P_n^{m'}(\cos\theta)}{\sin\theta}d\theta$$

Now by subtracting equation (22) from (19) we have

$$\frac{-2D\pi m'}{\epsilon_m}\delta_{m'n}\sum_{n=0}^{\infty}\left[P_n^{m'}(\cos\theta)\frac{dP_{n'}^{m'}(\cos\theta)}{d\theta}+P_{n'}^{m'}(\cos\theta)\frac{dP_n^{m'}(\cos\theta)}{d\theta}\right]d\theta$$
$$+\frac{2N}{\epsilon_m a}\alpha_{mn}\int_0^{\pi}\sin\theta\left[\frac{dP_n^{m'}(\cos\theta)}{d\theta}\frac{dP_{n'}^{m'}(\cos\theta)}{d\theta}\right.$$
$$\left.+\frac{m'^2}{\sin^2\theta}P_{n'}^{m'}(\cos\theta)P_n^{m'}(\cos\theta)\right]d\theta$$
$$=\int_0^{2\pi}\int_0^{\pi}\left[f_1(\theta,\phi)\frac{dP_{n'}^{m'}(\cos\theta)}{d\theta}\sin\theta\cos m\phi\right.$$
$$\left.-mf_2(\theta,\phi)P_n^m(\cos\theta)\sin m\phi\right]d\theta d\phi \tag{23}$$

using the following orthogonality relationships [6]

$$\int_0^{\pi}\left[\frac{dP_n^{m'}(\cos\theta)}{d\theta}\frac{dP_{n'}^{m'}(\cos\theta)}{d\theta}+\frac{m'^2}{\sin^2\theta}P_{n'}^{m'}(\cos\theta)P_n^{m'}(\cos\theta)\right]\sin\theta d\theta$$
$$=\begin{cases}0 & \text{for } n\neq n'\\ \dfrac{2n(n+1)(n+m)!}{(2n+1)(n-m)!} & \text{for } n=n'\end{cases} \tag{24}$$

and

$$\int_0^{\pi}\left[P_n^{m'}(\cos\theta)\frac{dP_{n'}^{m'}(\cos\theta)}{d\theta}+P_{n'}^{m'}(\cos\theta)\frac{dP_n^{m'}(\cos\theta)}{d\theta}\right]d\theta=0 \tag{25}$$

equation (23) can be rewritten as

$$\frac{2N}{\epsilon_m a}\alpha_{mn}\left[\frac{2n(n+1)(n+m)!}{(2n+1)(n-m)!}\right]$$
$$=\int_0^{2\pi}\int_0^{\pi}\left[f_1(\theta,\phi)\frac{dP_n^m(\cos\theta)}{d\theta}\sin\theta\cos m\phi\right. \tag{26}$$
$$\left.-mf_2(\theta,\phi)P_n^m(\cos\theta)\sin m\phi\right]d\theta d\phi$$

Therefore,

$$\alpha_{mn} = \frac{za}{N} \int_0^{2\pi} \int_0^{\pi} \begin{bmatrix} f_1(\theta, \phi) \dfrac{dP_n^m(\cos\theta)}{d\theta} \sin\theta \cos m\phi \\ -m f_2(\theta, \phi) P_n^m(\cos\theta) \sin m\phi \end{bmatrix} d\theta d\phi \tag{27}$$

where

$$z = \frac{1}{\pi} \frac{(2n+1)(n-m)!}{2n(n+1)(n+m)!} \frac{\epsilon_m}{2} \tag{28}$$

To determine β_{mn} we can rewrite equations (11) and (12) in the following ways:

$$\begin{aligned} &\int_0^{2\pi} \int_0^{\pi} f_1(\theta, \phi) \frac{dP_{n'}^{m'}(\cos\theta)}{d\theta} \sin m'\phi \sin\theta d\theta d\phi \\ &\quad = \sum_{n=0}^{\infty} \sum_{m=0}^{n} Dm \int_0^{\pi} P_n^m(\cos\theta) \frac{dP_{n'}^{m'}(\cos\theta)}{d\theta} d\theta \\ &\qquad \int_0^{2\pi} \sin m'\phi [\gamma_{mn} \sin m\phi - \delta_{mn} \cos m\phi] d\phi \\ &\quad - \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{N}{a} \int_0^{\pi} \sin\theta \frac{dP_n^m(\cos\theta)}{d\theta} \frac{dP_{n'}^{m'}(\cos\theta)}{d\theta} d\theta \\ &\qquad \int_0^{2\pi} \sin m'\phi [\alpha_{mn} \cos m\phi + \beta_{mn} \sin m\phi] d\phi \end{aligned} \tag{29}$$

and

$$\begin{aligned} &\int_0^{2\pi} \int_0^{\pi} f_2(\theta, \phi) m' P_{n'}^{m'}(\cos\theta) \cos m'\phi d\theta d\phi \\ &\quad = \sum_{n=0}^{\infty} \frac{2Dm'}{\epsilon_{m'}} \int_0^{\pi} \gamma_{mn} \frac{d}{d\theta} P_n^{m'}(\cos\theta) P_{n'}^{m'}(\cos\theta) d\theta \\ &\quad + \sum_{n=0}^{\infty} \frac{2Nm'^2}{\epsilon_{m'} a} \int_0^{\pi} \beta_{mn} P_{n'}^{m'}(\cos\theta) P_n^{m'}(\cos\theta) d\theta \end{aligned} \tag{30}$$

and via a similar approach the unknown coefficient β_{mn} is determined as

$$\beta_{mn} = \frac{za}{N} \int_0^{2\pi} \int_0^{\pi} \begin{bmatrix} f_1(\theta, \phi) \dfrac{dP_n^m(\cos\theta)}{d\theta} \sin m\phi \sin\theta \\ +m f_2(\theta, \phi) P_n^m(\cos\theta) \cos m\phi \end{bmatrix} d\theta d\phi \tag{31}$$

Similarly γ_{mn} and δ_{mn} may be determined:

$$\gamma_{mn} = \frac{z}{D}\int_0^{2\pi}\int_0^{\pi}\left[\begin{array}{c} mf_1(\theta,\phi)P_n^m(\cos\theta)\sin m\phi \\ +f_2(\theta,\phi)\dfrac{dP_n^m(\cos\theta)}{d\theta}\cos m\phi\sin\theta \end{array}\right]d\theta d\phi \qquad (32)$$

$$\delta_{mn} = \frac{z}{D}\int_0^{2\pi}\int_0^{\pi}\left[\begin{array}{c} -mf_1(\theta,\phi)P_n^m(\cos\theta)\cos m\phi \\ +f_2(\theta,\phi)\dfrac{dP_n^m(\cos\theta)}{d\theta}\sin m\phi\sin\theta \end{array}\right]d\theta d\phi \qquad (33)$$

We now focus our attention to determining the far field components of E_θ and E_ϕ

$$E_\phi = -Z_0 H_\theta \qquad (34)$$

$$E_\theta = Z_0 H_\phi \qquad (35)$$

where Z_0 is the characteristic impedance of free space. The electric fields are related to the magnetic fields H_θ and H_ϕ through Z_0. By taking the large argument approximations of the spherical Hankel functions, we obtain

$$h_n^{(2)}(kr) \simeq \frac{j^{n+1}e^{-jkr}}{kr} \quad \text{for } r \gg \lambda \qquad (36)$$

From (36) we obtain

$$\frac{1}{r}\frac{d}{dr}\left[rh_n^{(2)}(kr)\right] = \frac{j^{n+1}}{kr}(-jke^{-jkr}) \qquad (37)$$

After simplifying equation (37) we have

$$\frac{1}{r}\frac{d}{dr}\left[rh_n^{(2)}(kr)\right] = \frac{j^n}{r}e^{-jkr} \quad \text{for } r \gg \lambda \qquad (38)$$

Substituting from equations (34) and (37) into (7) we have

$$\begin{aligned} H_\theta(r,\theta,\phi) = & \\ & \sum_{n=0}^{\infty}\sum_{m=0}^{n} -j\omega\epsilon\frac{j^{n+1}e^{-jkr}}{kr}\frac{P_n^m(\cos\theta)m}{\sin\theta}[\alpha_{mn}\sin m\phi - \beta_{mn}\cos m\phi] \\ & +\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{j^n e^{-jkr}}{r}\frac{d}{d\theta}P_n^m(\cos\theta)[\gamma_{mn}\cos m\phi + \delta_{mn}\sin m\phi] \end{aligned} \qquad (39)$$

After simplifying we have

$$H_\theta(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{j^n \omega\epsilon e^{-jkr}}{kr}\frac{P_n^m(\cos\theta)m}{\sin\theta}[\alpha_{mn}\sin m\phi - \beta_{mn}\cos m\phi]$$
$$+\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{j^n e^{-jkr}}{r}\frac{d}{d\theta}P_n^m(\cos\theta)[\gamma_{mn}\cos m\phi + \delta_{mn}\sin m\phi] \tag{40}$$

Substituting from equations (27), (31), (32) and (33) in terms of α_{mn}, β_{mn}, γ_{mn}, δ_{mn}, respectively, into equation (40) yields:

$$H_\theta(r,\theta,\phi) = \frac{e^{-jkr}}{4\pi r\eta}\sum_{n=0}^{\infty}\frac{(2n+1)}{n(n+1)}j^n$$
$$\left[\frac{j}{h_n^{(2)}(ka)}\int_0^{2\pi}\int_0^{\pi}\left[f_1(\theta',\phi')\frac{d^2P_n(\xi)}{d\theta d\phi} + f_2(\theta',\phi')\sin\theta'\frac{d^2P_n(\xi)}{d\theta d\theta'}\right]d\theta' d\phi' \right.$$
$$\left. +\frac{ka}{N}\int_0^{2\pi}\int_0^{\pi}\left[f_1(\theta',\phi')\frac{\sin\theta'}{\sin\theta}\frac{d^2P_n(\xi)}{d\phi d\theta'} + \frac{f_2(\theta',\phi')}{\sin\theta}\frac{d^2P_n(\xi)}{d\phi d\phi'}\right]d\theta' d\phi'\right] \tag{41}$$

where from [7]

$$P_n(\xi) = \sum_{m=0}^{n}\epsilon_m\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\phi-\phi') \tag{42}$$

and via a similar approach

$$H_\phi = \frac{e^{-jkr}}{4\pi r\eta}\sum_{n=0}^{\infty}\frac{(2n+1)}{n(n+1)}j^n$$
$$\left[\frac{j}{kh_n^{(2)}(ka)}\int_0^{2\pi}\int_0^{\pi}\left[\frac{f_1(\theta',\phi')}{\sin\theta}\frac{d^2P_n(\xi)}{d\phi d\phi'} + f_2(\theta',\phi')\frac{\sin\theta'}{\sin\theta}\frac{d^2P_n(\xi)}{d\theta' d\phi'}\right]d\theta' d\phi' \right.$$
$$\left. +\frac{a}{N}\int_0^{2\pi}\int_0^{\pi}\left[f_1(\theta',\phi')\sin\theta'\frac{d^2P_n(\xi)}{d\theta d\theta'} + f_2(\theta',\phi')\frac{d^2P_n(\xi)}{d\theta d\phi'}\right]d\theta' d\phi'\right] \tag{43}$$

It is interesting to note that both (41) and (43) do not contain any summation over m, which has been eliminated in the present formulation by utilizing the addition theorem for Legendre polynomials introduced through (42). Also observe that the second derivatives of the Legendre polynomials can be evaluated, for example, as

$$\frac{\partial^2 P_n(\xi)}{\partial\theta\partial\phi} = \frac{\partial^2 P_n(\xi)}{\partial\xi^2}\frac{\partial\xi}{\partial\phi}\frac{\partial\xi}{\partial\theta} + \frac{\partial P_n(\xi)}{\partial\xi}\frac{\partial^2\xi}{\partial\theta\phi} \tag{44}$$

where

$$\frac{\partial P_n(\xi)}{\partial\xi} = \frac{n+1}{1-\xi^2}[\xi P_n(\xi) - P_{n+1}(\xi)] \tag{45}$$

and

$$\frac{\partial^2 P_n(\xi)}{\partial\xi^2} = \frac{n+1}{(1-\xi^2)^2}\left\{[(2+n)\xi^2 - n]P_n(\xi) - 2\xi P_{n+1}(\xi)\right\} \tag{46}$$

Furthermore, for a given prespecified ka, one could precompute the summation over n, in terms of the four "pseudo" Green's functions and store them. Under these conditions, one then needs to perform only an integral over θ and ϕ as

$$H_\theta \simeq \frac{e^{-jkr}}{4\pi r\eta}\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \begin{bmatrix} f_1(\theta',\phi')G_1(\theta,\phi,\theta',\phi') \\ +f_2(\theta',\phi')G_2(\theta,\phi,\theta',\phi') \end{bmatrix} \tag{47}$$

and

$$H_\phi = \frac{e^{-jkr}}{4\pi r\eta}\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \begin{bmatrix} f_1(\theta',\phi')G_3(\theta,\phi,\theta',\phi') \\ +f_2(\theta',\phi')G_4(\theta,\phi,\theta',\phi') \end{bmatrix} \tag{48}$$

where

$$G_1(\theta,\phi,\theta',\phi') = \sum_{n=1}^{\infty}\frac{(2n+1)}{n(n+1)}j^n\left[\frac{j}{h_n^{(2)}(ka)}\frac{\partial^2 P_n(\zeta)}{\partial\theta\partial\phi} + \frac{ka}{N}\frac{\sin\theta'}{\sin\theta}\frac{\partial^2 P_n(\zeta)}{\partial\phi\partial\theta'}\right] \tag{49}$$

$$G_2(\theta,\phi,\theta',\phi') = \sum_{n=1}^{\infty}\frac{(2n+1)}{n(n+1)}j^n\left[\frac{j\sin\theta'}{h_n^{(2)}(ka)}\frac{\partial^2 P_n(\zeta)}{\partial\theta\partial\theta'} + \frac{ka}{N\sin\theta}\frac{\partial^2 P_n(\zeta)}{\partial\phi\partial\phi'}\right] \tag{50}$$

$$G_3(\theta,\phi,\theta',\phi') = \sum_{n=1}^{\infty}\frac{(2n+1)}{n(n+1)}j^n\left[\frac{j\sin\theta}{h_n^{(2)}(ka)}\frac{\partial^2 P_n(\zeta)}{\partial\phi\partial\phi'} + \frac{ka\sin\theta'}{N}\frac{\partial^2 P_n(\zeta)}{\partial\theta\partial\theta'}\right] \tag{51}$$

$$G_4(\theta,\phi,\theta',\phi') = \sum_{n=1}^{\infty}\frac{(2n+1)}{n(n+1)}j^n\left[\frac{j\sin\theta'}{h_n^{(2)}(ka)\sin\theta}\frac{\partial^2 P_n(\zeta)}{\partial\theta'\partial\phi'} + \frac{ka}{N}\frac{\partial^2 P_n(\zeta)}{\partial\theta\partial\phi'}\right] \tag{52}$$

The integrals in (47) and (48) can efficiently and accurately be done by the conventional Fast Fourier Transformation technique. The functions $G_1 - G_4$ are called "pseudo" Green's functions because for a true Green's functions, f_1 and f_2 would be convolved with the Green's functions, but here it is an integral. These equations indicate that if for a fixed ka, the Green's functions are precomputed and stored, then the actual computations of (47) and (48) can be done even on a note-book PC. If the quality of the measured data, i.e., $f_1(\theta, \phi)$ and $f_2(\theta, \phi)$ are good (which implies that quite a few significant bits are accurate), the derivatives in (41) and (43) can be transferred from P_n to f_1 and f_2. This may enhance the rate of convergence of the summation over n.

In summary, the present approach offers the following features:

(1) The transformation is expressed in an analytic form.

(2) There is only one summation, i.e. over n. To obtain a relative numerical accuracy of 10^{-7} in the computation of the fields, the limit of the summation over n should be $n = 1.27\,ka$ for $ka > 60$.

(3) The derivatives can be transferred to the data (if the quality is good) to further enhance the rate of convergence. Or equivalently the data can be expanded in a Fourier series as is conventionally done (at least in the first step) and the derivatives can be carried out in an analytic fashion utilizing the FFT.

(4) For a fixed ka, all the summations over n can be precomputed and stored on a diskette. This NF/FF transformation procedure is equivalent to synthesizing a plane wave region using an infinite number of point sources on a sphere having radius a and each individual point source having a complex amplitude is given by the "Pseudo" Green's functions G_1–G_4.

3. NUMERICAL EXAMPLES

As a first example consider a four dipole array. The dipoles are located at the corners of a $4\lambda \times 4\lambda$ planar surface which is in the x-y plane. The center of the $4\lambda \times 4\lambda$ square surface is located at $x = 0.22\lambda$ and $y = 0.22\lambda$. The plane of the array is the x-y plane. So the four dipoles are not located symmetrically about the origin. A spherical surface is drawn with the center defined above and a radius of $10\,\lambda$. On that spherical surface of $20\,\lambda$ diameter encapsulating the four offset dipoles located on the x-y plane both the electric field components

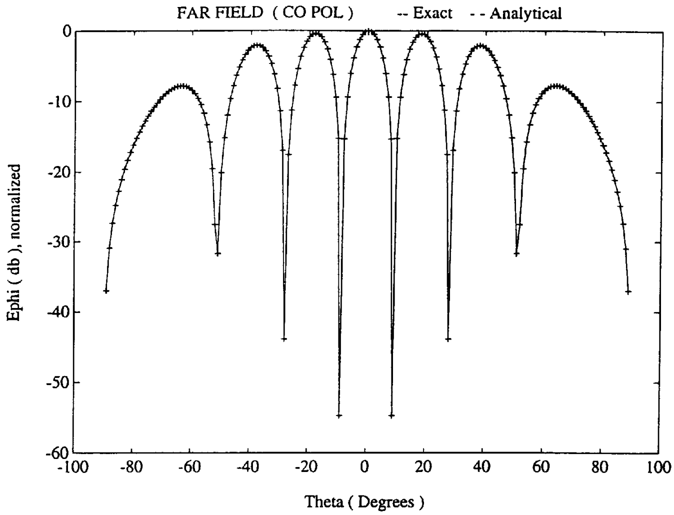

Figure 1. Comparison of exact and computed far field for $\phi = 0°$ cut for a 4 dipole array at the corners of a $4\lambda \times 4\lambda$ surface.

E_θ and E_ϕ are computed analytically. Next, the two field components are used in conjunction with (41) and (43) to evaluate the far fields. Figure 1 presents E_ϕ in dB for $\phi = 0°$ as a function of θ. Both the exact analytical far field and the far field computed by using the present theory are presented in Figure 1. They are visually indistinguishable. In Figure 2, E_θ is presented in dB for $\phi = 90°$ as a function of θ. Again, the analytical far fields from the four off centered dipoles and the computed far fields are visually indistinguishable. The cross polar components in both the figures are negligible.

Next, measured data is utilized. Consider a microstrip array consisting of 32×32 uniformly distributed patches on a $1.5\text{m} \times 1.5\text{m}$ surface. The near fields are measured on a spherical surface at a distance 1.23m away from the antenna at a frequency of 3.3GHz. The data is taken every $2°$ in ϕ and every $1°$ in θ. Measurements have been performed using an open ended cylindrical WR284 waveguide fed with the TE_{11} mode. The measured data was provided by Dr. Carl Stubenrauch of NIST [8]. Figure 3–6 compare the far field patterns obtained by the present analytical method with the far field patterns obtained by the numerical technique described in [8]. These numerically computed far field patterns employ the same measured data utilizing an equivalent

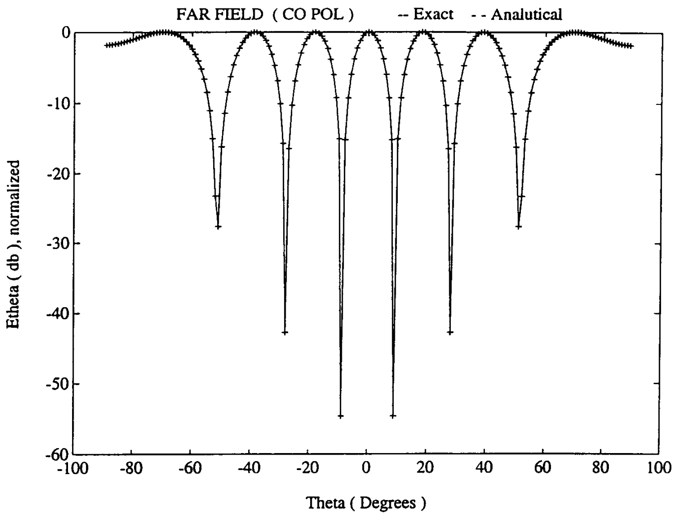

Figure 2. Comparison of exact and computed far field for $\phi = 90°$ cut for a 4 dipole array at the corners of a $4\lambda \times 4\lambda$ surface.

magnetic current approach for near field to far field transformation [8]. Figure 3 describes $20\log_{10}|E_\phi|$ for $\phi = 0°$ and various angles of θ. Figure 4 represents $20\log_{10}|E_\theta|$ for $\phi = 90°$ and for $-90° < \theta < 90°$. These are the principal plane patterns. As observed, the agreement is good. Figures 5 and 6 show the cross polar pattern. Figure 5 depicts $20\log_{10}|E_\theta|$ $\phi = 0°$ for different values of θ. Figure 6 presents $20\log_{10}|E_\phi|$ for $\phi = 90°$ and for $-90° < \theta < 90$. The agreement between the approach presented in this paper and the nllmerical approach for the cross polar pattern is reasonable for pattern levels above -70 dB.

4. CONCLUSION

An alternate method is described for spherical near field to near/far field transformation without probe correction. The advantage of this approach is that one of the summations over m has been eliminated by utilizing the addition theorem for Legendre polynomials. Hence the expressions are more concise and easier to visualize. This method is accurate, as illustrated by the performance of this method on both synthetic and real experimental data.

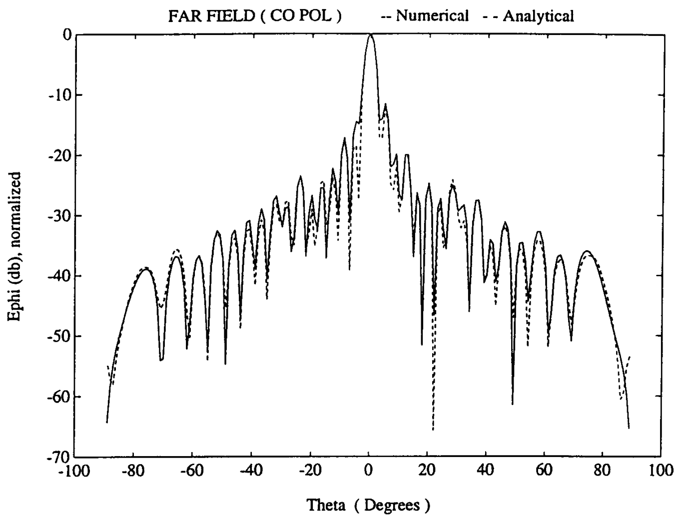

Figure 3. Co-polarization characteristic for $\phi = 0°$ cut for a 32×32 patch microstrip array using analytical and numerical results.

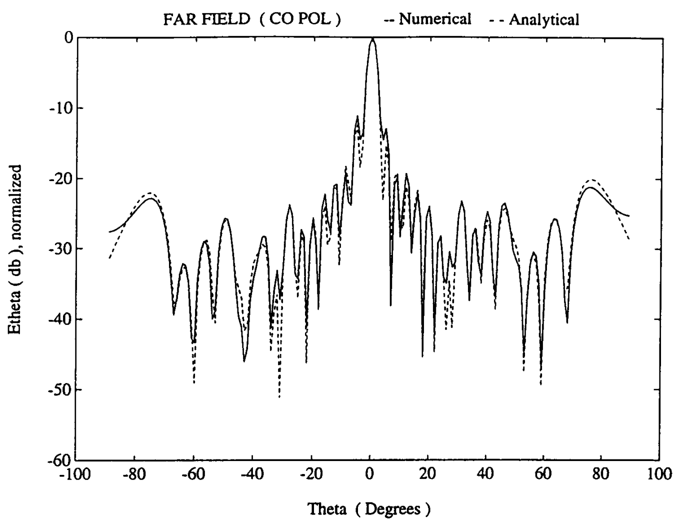

Figure 4. Co-polarization characteristic for $\phi = 90°$ cut for a 32×32 patch microstrip array using analytical and numerical results.

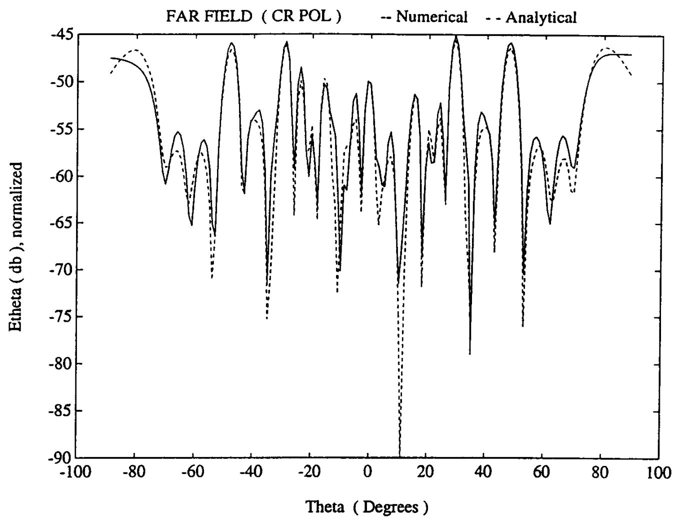

Figure 5. Cross-polarization characteristic for $\phi = 0°$ cut for a 32×32 patch microstrip array using analytical and numerical results.

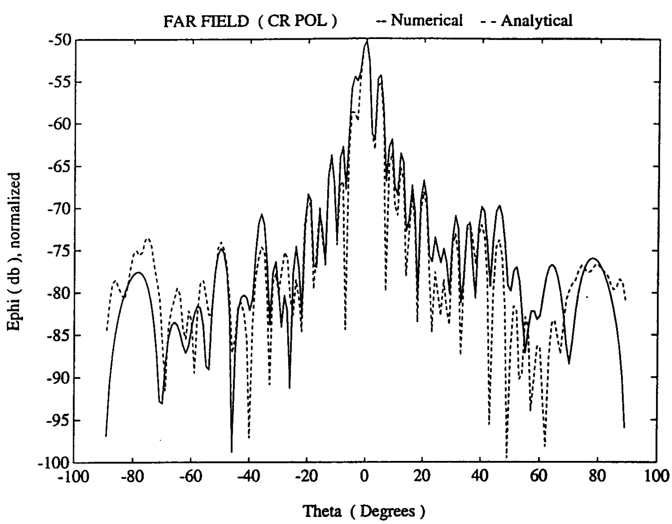

Figure 6. Cross-polarization characteristic for $\phi = 90°$ cut for a 32×32 patch microstrip array using analytical and numerical results.

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