Hacettepe Journal of Mathematics and Statistics
Volume 47 (1) (2018), 145 – 173

An alternative two-parameter gamma generated family of distributions: properties and applications

Gauss M. Cordeiro,* Saralees Nadarajah,† Edwin M. M. Ortega‡ § and Thiago G. Ramires ¶

Abstract

Motivated by Torabi and Hedesh (2012), we propose a gamma extended family of distributions with two extra generator parameters. We present some special models and study general mathematical properties like asymptotes and shapes, ordinary and incomplete moments, generating and quantile functions, probability weighted moments, mean deviations, Bonferroni and Lorenz curves, asymptotic distributions of the extreme values, Shannon entropy, Rényi entropy, reliability and order statistics. The method of maximum likelihood is used to estimate the model parameters and the observed information matrix is determined. We define a new regression model based on the logarithm of the proposed distribution. The usefulness of the new models is proved empirically in three applications to real data.

Keywords: Estimation, Gamma distribution, Generated family, Maximum likelihood, Mean deviation, Moment, Quantile function.

2000 AMS Classification: 47N30, 97K70, 97K80

Received : 05.08.2015 *Accepted* : 12.02.2016 *Doi* : 10.15672/HJMS.2017.502

*Departamento de Estatística, Universidade Federal de Pernambuco, 50740-540, Recife, PE, Brazil, Email: gausscordeiro@gmail.com

†School of Mathematics, University of Manchester, Manchester M13 9PL, UK, Email: saralees.nadarajah@manchester.ac.uk,

‡Departamento de Ciências Exatas, Universidade de São Paulo, 13418-900, Piracicaba, SP, Brazil, Email: edwin@usp.br

§Corresponding Author.

¶Departamento de Matemática, Universidade Tecnológica Federal do Paraná, 86812-460, Apucarana, PR, Brazil, Email: thiagogentil@gmail.com

1. Introduction

The statistics literature is filled with hundreds of continuous univariate distributions, see Johnson *et al.* (1994, 1995). Adding parameters to a well-established distribution is a time honored device for obtaining more flexible new classes of distributions. Recent developments have been made to define new generated families to control skewness and kurtosis through the tail weights and provide great flexibility in modeling skewed data in practice, including the two-piece approach introduced by Hansen (1994) and the generators pioneered by Eugene *et al.* (2002), Cordeiro and de Castro (2011) and Alexander *et al.* (2012). Many subsequent papers apply these techniques to induce skewness into well-known symmetric distributions like the symmetric Student t; see, Aas and Haff (2006), for a review.

We study several general mathematical properties of the new *gamma extended* ("GE" for short) family of distributions. This family is motivated by the work of Torabi and Hedesh (2012). It is also important to mention that the proofs of the results presented in this paper follow similar lines of those ones by Nadarajah et al. (2015), although their model is completely different from that one discussed in this paper.

The proposed family can extend several common models such as the normal, Weibull, log-normal, gamma and Gumbel distributions by adding two extra generator parameters. Indeed, for any baseline G distribution, we can define the associated *gamma extended-G* ("GE-G") distribution. The main characteristics of the GE family, like moments and generating and quantile functions, have tractable mathematical properties. The role of the generator parameters has been investigated and is related to the skewness and kurtosis of the generated distribution. In fact, the proposed family is a modified version of the class studied by Zografos and Balakrishnan (2009) ("ZB") and Ristic and Balakrishnan (2012), although both generators are based on the same gamma model. The GE family can be constructed as follows. Let $W(x)$ be any continuous cumulative distribution function (cdf) defined on a finite or an infinite interval. The cdf of the ZB class of distributions (for $\alpha > 0$) is given by

$$F(x) = \frac{1}{\Gamma(\alpha)} \int_0^{-\log[1-W(x)]} t^{\alpha-1}\,\mathrm{e}^{-\mathrm{t}}\mathrm{dt}, \quad \mathrm{x} \in \mathbb{R}, \tag{1.1}$$

where $\Gamma(\alpha) = \int_0^\infty t^{a-1}\,\mathrm{e}^{-\mathrm{t}}\mathrm{dt}$ denotes the gamma function. We note that $W(x)$ is the parent cdf in the formulation of the ZB class. In a slightly different way, we redefine $W(x)$ as a function of the parent cdf $G(x;\boldsymbol{\tau})$ ($x \in \mathbb{R}$) by

$$W(x) = 1 - \exp\Big\{-\frac{G(x;\boldsymbol{\tau})}{\bar{G}(x;\boldsymbol{\tau})}\Big\}, \tag{1.2}$$

where $\boldsymbol{\tau}$ denotes the vector of unknown parameters and $\overline{G}(x;\boldsymbol{\tau}) = 1 - G(x;\boldsymbol{\tau})$ is the parent survival function. By substituting (1.2) in equation (1.1) and adding an extra scale parameter $\beta > 0$, the GE-G cdf is given by

$$F(x) = \frac{\gamma\left(\alpha, \beta G(x;\boldsymbol{\tau})/\overline{G}(x;\boldsymbol{\tau})\right)}{\Gamma(\alpha)} = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{G(x;\boldsymbol{\tau})/\overline{G}(x;\boldsymbol{\tau})} t^{\alpha-1}\,\mathrm{e}^{-\beta \mathrm{t}}\mathrm{dt}, \tag{1.3}$$

where $\gamma(\alpha, z) = \int_0^z t^{\alpha-1}\,\mathrm{e}^{-\mathrm{t}}\mathrm{dt}$ denotes the incomplete gamma function.

Henceforth, equation (1.3) is called the GE-G distribution, although it was named before by the *gamma generated*-G distribution (Torabi and Hedesh, 2012). We adopt the first name since (1.1) is usually called the gamma generated-G model (Nadarajah *et al.*, 2015). If we set $\beta = 1$ in (1.3), we obtain the Torabi and Hedesh's (2012) model, which is a special case of the proposed family.

We now provide a simple interpretation of the cdf (1.3). Let $G(x;\boldsymbol{\tau})/\overline{G}(x;\boldsymbol{\tau})$ be the odds ratio of a baseline random variable. Consider that the variability of the odds, represented by X, has a gamma distribution with shape parameter α and scale parameter β. The probability of the odds be less than x is given by $P(X \leq x) = \int_0^{G(x;\boldsymbol{\tau})/\overline{G}(x;\boldsymbol{\tau})} t^{\alpha-1}\,\mathrm{e}^{-\beta t}\mathrm{dt}$, which is identical to (1.3).

Let $g(x;\boldsymbol{\tau}) = dG(x;\boldsymbol{\tau})/dx$. The probability density function (pdf) corresponding to (1.3) becomes

$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\,\frac{G^{\alpha-1}(x;\boldsymbol{\tau})}{\overline{G}^{\alpha+1}(x;\boldsymbol{\tau})}\,\exp\left\{-\frac{\beta G(x;\boldsymbol{\tau})}{\overline{G}(x;\boldsymbol{\tau})}\right\} g(x;\boldsymbol{\tau}). \tag{1.4}$$

The GE-G distribution has the same parameters of the G model plus two additional parameters α and β. These parameters control both skewness and kurtosis of $f(x)$ as shown in the plots of Section 4. They are highly significant in most fitted GE models given in Section 9. For $\alpha = \beta = 1$, $F(x) \to G(x)$ when $x \to -\infty$ (or $x \to 0$) depending if the support of $g(x)$ is the real line (or positive real). The GE-G distribution reduces to the classical gamma distribution with parameters α and β when $G(x) = x/(1+x)$. Henceforth, a random variable X with density function (1.4) is denoted by $X \sim$GE-G$(\alpha, \beta, \boldsymbol{\tau})$.

The hazard rate function (hrf) of X, defined by the ratio of the pdf and survival function $f(x)/[1 - F(x)]$, is given by

$$\tau(x) = \frac{\beta^{\alpha}\,G^{\alpha-1}(x;\boldsymbol{\tau})\,g(x;\boldsymbol{\tau})}{\overline{G}^{\alpha+1}(x;\boldsymbol{\tau})\,\left[\Gamma(\alpha) - \gamma\left(\alpha, \beta G(x;\boldsymbol{\tau})/\overline{G}(x;\boldsymbol{\tau})\right)\right]}\,\exp\left\{-\frac{\beta G(x)}{\overline{G}(x;\boldsymbol{\tau})}\right\}. \tag{1.5}$$

Each new GE-G distribution can be obtained from a specified G distribution. From the statistical modeling point of view, the GE-G distribution has three important aspects. First, its additional parameters α and β have clear interpretations. Let $Q_G(u) = G^{-1}(u)$ be the baseline quantile function (qf). Second, if W has a gamma (α, β) distribution and the parameters of the baseline G model is $\boldsymbol{\tau}$, then the random variable $X = Q_G\left(\frac{W}{1+W}\right)$ has the GE-G density (1.4). Third, the reverse is also true. If X has the GE-G density (1.4), then $G(X)/\overline{G}(X)$ follows the gamma (α, β) distribution. Evidently, the GE-G distribution will be most tractable when both functions $G(x)$ and $g(x)$ have simple analytic expressions.

The aim of this paper is to derive several mathematical properties of (1.3) and (1.4) in the most simple, explicit and general forms. We obtain general expressions for the shape and asymptotic properties of (1.3), (1.4) and (1.5), ordinary and incomplete moments, moment generating function (mgf), qf, probability weighted moments (PWMs), mean deviations, Bonferroni and Lorenz curves, asymptotic distribution of the extreme values, Shannon entropy, Rényi entropy, reliability and general properties of the order statistics. The rest of the paper is organized as follows. In Section 2, we present some new distributions. A range of mathematical properties of the GE-G model (1.4) is investigated in Sections 3 to 6. Some inferential tools are discussed in Section 7. In Section 8, we provide a generalization of regression models (including the case of censoring) based on the GE family. The flexibility of the new models are proved empirically by means of three applications to real data in Section 9. We also investigate the performance of the maximum likelihood estimators (MLEs) through a simulation study. Finally, some conclusions and future work are noted in Section 10.

2. Special GE distributions

The GE family density (1.4) allows for greater flexibility of its tails and can be widely applied in many areas of engineering and biology. In this section, we present and study some special cases of this family because it extends several widely-known distributions in the literature. Its density function will be most tractable when $G(x;\boldsymbol{\tau})$ and $g(x;\boldsymbol{\tau})$

have simple analytic expressions.

2.1. Gamma extended Weibull (GE-W) distribution. If $G(x;\boldsymbol{\tau})$ is the Weibull cdf with scale parameter $\delta > 0$ and shape parameter $\lambda > 0$, where $\boldsymbol{\tau} = (\lambda, \delta)^T$, say $G(x;\boldsymbol{\tau}) = 1 - \exp\{-(\delta x)^\lambda\}$, the GE-W density function (for $x > 0$) reduces to

$$f(x) = \frac{\lambda\,\delta^\lambda\,\beta^\alpha\;x^{\lambda-1}\;\exp\{\alpha\,(\delta x)^\lambda\}}{\Gamma(\alpha)\;[1-\exp\{-(\delta x)^\lambda\}]^{1-\alpha}}\;\exp\left\{\frac{-\beta\left[1-\exp\{-(\delta x)^\lambda\}\right]}{\exp\{-(\delta x)^\lambda\}}\right\}. \tag{2.1}$$

Figure 1 displays some possible shapes of the GE-W density function.

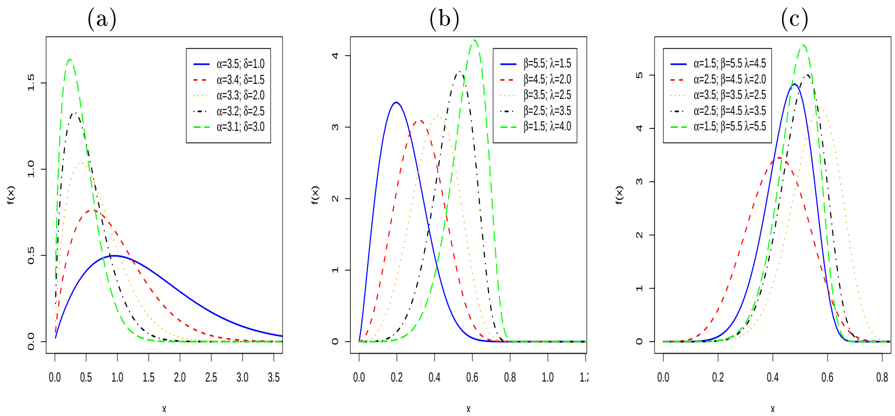

Figure 1. Plots of the GE-W density function for some parameter values. (a) For different values of α, δ with $\beta = 1.5$ and $\lambda = 0.5$. (b) For different values of β, λ with $\alpha = 1.5$ and $\delta = 1.5$. (c) For different values of α, β and λ with $\delta = 1.5$.

2.2. Gamma extended normal (GE-N) distribution. The GE-N distribution is defined from (1.4) by taking $G(x;\boldsymbol{\tau})$ and $g(x;\boldsymbol{\tau})$ to be the cdf and pdf of the normal $N(\mu,\sigma^2)$ distribution, where $\boldsymbol{\tau} = (\mu,\sigma)^T$. Its density function becomes

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}\;\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)^{\alpha-1}}{\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha+1}}\;\exp\left\{\frac{-\beta\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right\}\;\phi\Big(\frac{x-\mu}{\sigma}\Big), \tag{2.2}$$

where $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ is a location parameter, $\sigma > 0$ is a scale parameter, α and β are shape and scale parameters, and $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution, respectively. A random variable with density (2.2) is denoted by $X \sim$ GE-N$(\alpha,\beta,\mu,\sigma^2)$. For $\mu = 0$ and $\sigma = 1$, we obtain the GE-standard normal (GE-SN) distribution. Plots of the GE-N density function for selected parameter values are displayed in Figure 2.

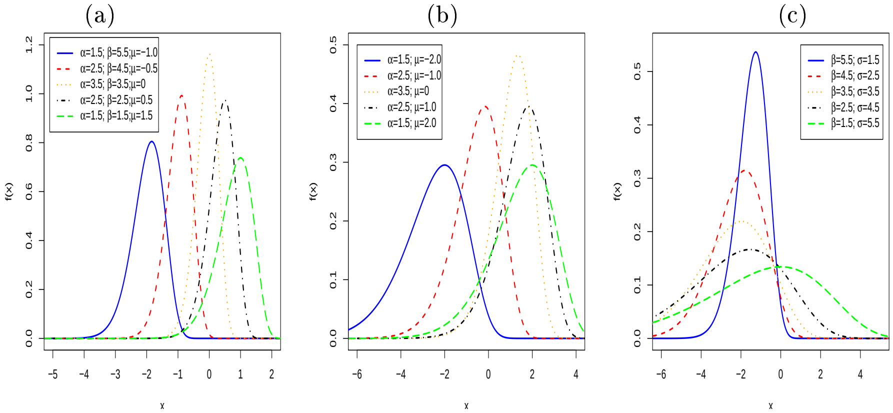

Figure 2. Plots of the GE-N density function for some parameter values. (a) For different values of α, β and μ with $\sigma = 1$. (b) For different values of α and μ with $\beta = 1.5$ and $\sigma = 2.5$. (c) For different values of β and σ with $\alpha = 1.5$ and $\mu = 0$.

2.3. Gamma extended log-normal (GE-LN) distribution. Let $G(x;\boldsymbol{\tau})$ be the log-normal distribution with cdf $G(x;\boldsymbol{\tau}) = 1 - \Phi\Big(\frac{-\log(x)+\mu}{\sigma}\Big)$ for $x > 0$, $\sigma > 0$ and $\mu \in \mathbb{R}$, where $\boldsymbol{\tau} = (\mu,\sigma)^T$. The GE-LN density function (for $x > 0$) reduces to

$$\begin{aligned} f(x) &= \frac{\beta^{\alpha}}{\sqrt{2\pi}\,\sigma\,\Gamma(\alpha)\,x}\exp\left\{-\frac{1}{2}\left[\frac{-\log(x)+\mu}{\sigma}\right]^2\right\}\frac{\left[1-\Phi\Big(\frac{-\log(x)+\mu}{\sigma}\Big)\right]^{\alpha-1}}{\Phi\Big(\frac{-\log(x)+\mu}{\sigma}\Big)^{\alpha+1}} \\ &\times \exp\left\{\frac{-\beta\left[1-\Phi\Big(\frac{-\log(x)+\mu}{\sigma}\Big)\right]}{\Phi\Big(\frac{-\log(x)+\mu}{\sigma}\Big)}\right\}. \end{aligned}$$

Figure 3 displays plots of the GE-LN density function for some parameter values.

2.4. Gamma extended Gumbel (GE-Gu) distribution. Consider the Gumbel distribution with location parameter $\mu \in \mathbb{R}$ and scale parameter $\sigma > 0$, $\boldsymbol{\tau} = (\mu,\sigma)^T$, with cdf (for $x \in \mathbb{R}$) $G(x;\boldsymbol{\tau}) = 1 - \exp\left[-\exp\left(\frac{x-\mu}{\sigma}\right)\right]$. The mean and variance are equal to $\mu - \gamma\sigma$ and $\pi^2\sigma^2/6$, respectively, where γ is the Euler's constant ($\gamma \approx 0.57722$). The GE-Gu density function reduces to

$$\begin{aligned} f(x) &= \frac{\left\{1-\exp\left[-\exp\left(\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha-1}}{\left\{\exp\left[-\exp\left(\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha+1}}\exp\left\{-\frac{\beta\left\{1-\exp\left[-\exp\left(\frac{x-\mu}{\sigma}\right)\right]\right\}}{\left\{\exp\left[-\exp\left(\frac{x-\mu}{\sigma}\right)\right]\right\}}\right\} \\ &\times\frac{\beta^{\alpha}}{\sigma\Gamma(\alpha)}\exp\left[\left(\frac{x-\mu}{\sigma}\right)-\exp\left(\frac{x-\mu}{\sigma}\right)\right]. \end{aligned} \tag{2.3}$$

Plots of (2.3) for some parameter values are given in Figure 4.

2.5. Gamma extended log-logistic (GE-LL) distribution. The pdf and cdf of the log-logistic (LL) distribution are (for $x, \alpha, \lambda > 0$)

$$g(x;\boldsymbol{\tau}) = \frac{\lambda}{\delta^{\lambda}}\,x^{\lambda-1}\left[1+\left(\frac{x}{\delta}\right)\right]^{-2} \quad \text{and} \quad G(x;\boldsymbol{\tau}) = 1-\left[1+\left(\frac{x}{\delta}\right)^{\lambda}\right]^{-1},$$

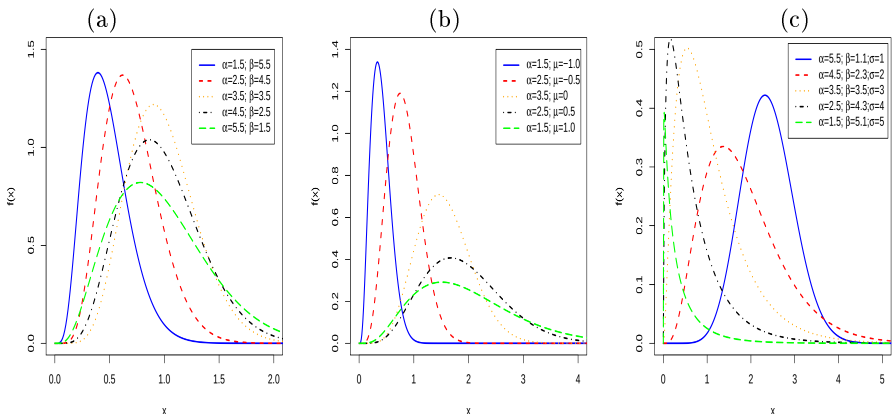

Figure 3. Plots of the GE-LN density function for some parameter values. (a) For different values of α and β with $\mu = 0$ and $\sigma = 1$. (b) For different values of α and μ with $\beta = 1.5$ and $\sigma = 1$. (c) For different values of α, β and σ with $\mu = 0$.

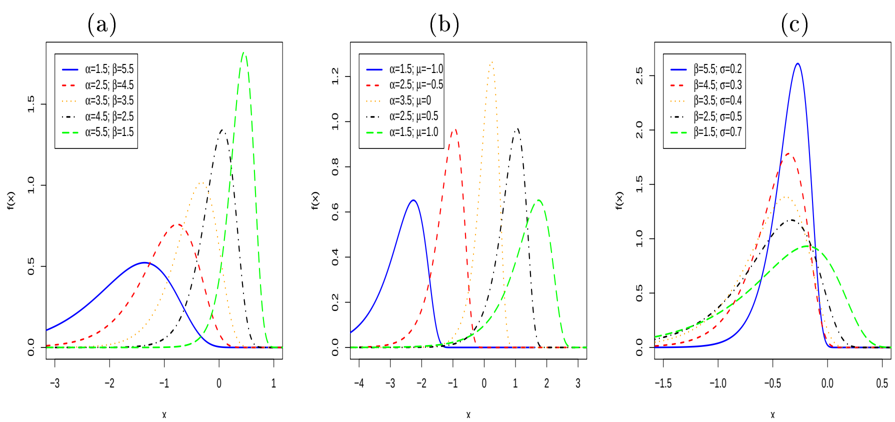

Figure 4. Plots of the GE-Gu density function for some parameter values. (a) For different values of α and β with $\mu = 0$ and $\sigma = 1$. (b) For different values of α and μ with $\beta = 1.5$ and $\sigma = 1$. (c) For different values of β and σ with $\alpha = 1.5$ $\mu = 0$.

respectively, where $\boldsymbol{\tau} = (\lambda, \delta)^T$. Inserting these expressions in (1.4) gives the GE-LL density function (for $x > 0$)

$$f(x) = \frac{\lambda\,\beta^{\alpha}\,x^{\lambda-1}\left\{1-\left[1+\left(\frac{x}{\delta}\right)^{\lambda}\right]^{-1}\right\}^{\alpha-1}}{\delta^{\lambda}\Gamma(\alpha)\left[1+\left(\frac{x}{\delta}\right)\right]^{2}\left[1+\left(\frac{x}{\delta}\right)^{\lambda}\right]^{-(\alpha+1)}}\ \exp\left\{-\frac{\beta\left\{1-\left[1+\left(\frac{x}{\delta}\right)^{\lambda}\right]^{-1}\right\}}{\left[1+\left(\frac{x}{\delta}\right)^{\lambda}\right]^{-1}}\right\}.$$

Plots of the GE-LL density function for selected parameter values are displayed in Figure 5.

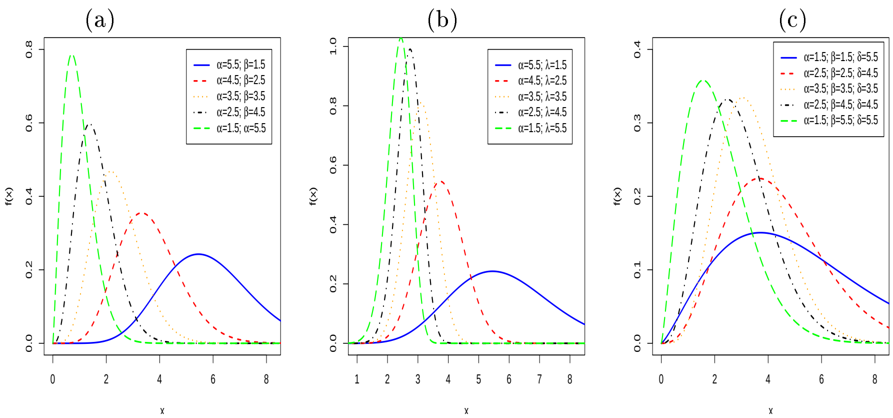

Figure 5. Plots of the GE-LL density function for some parameter values. (a) For different values of α, β with $\lambda = 1.5$ and $\delta = 2.5$. (b) For different values of α, λ with $\beta = 1.5$ and $\delta = 2.5$. (c) For different values of α, β and δ with $\lambda = 1.5$.

3. Useful expansions

Some useful expansions for (1.3) and (1.4) can be derived using the concept of exponentiated distributions. For an arbitrary baseline cdf $G(x)$, a random variable Y is said to have the *exponentiated-G* ("Exp-G") distribution with power parameter $a > 0$, say $Y \sim$Exp-G(a), if its pdf and cdf are

$$h_a(x) = aG^{a-1}(x)g(x) \tag{3.1}$$

and

$$H_a(x) = G^a(x), \tag{3.2}$$

respectively. The properties of exponentiated distributions have been studied by many authors in recent years, see Mudholkar and Srivastava (1993) for exponentiated Weibull, Gupta and Kundu (1999) for exponentiated exponential, Nadarajah (2005) for exponentiated Gumbel, Kakde and Shirke (2006) for exponentiated log-normal, and Nadarajah and Gupta (2007) for exponentiated gamma distributions.

By expanding the exponential in (1.4), we have

$$f(x) = g(x) \sum_{k=0}^{\infty} c_k \, \frac{G(x)^{k+\alpha-1}}{[1-G(x)]^{k+\alpha+1}},$$

where $c_k = (-1)^k \beta^{k+\alpha}/[\Gamma(\alpha)\, k!]$ for $k = 0, 1, \ldots$ Using the power series (for $\rho > 0$)

$$(1-z)^{-\rho} = \sum_{j=0}^{\infty} \frac{\Gamma(\rho+j)}{\Gamma(\rho)\, j!} \, z^j, \tag{3.3}$$

which holds for $z \in (0, 1)$, $f(x)$ can be expressed as

$$f(x) = \sum_{k,j=0}^{\infty} d_{k,j} \, h_{\alpha+k+j}(x). \tag{3.4}$$

Here, $h_{\alpha+k+j}(x)$ denotes the Exp-G$(\alpha+k+j)$ density function and the coefficients $d_{k,j}$ (which depend only on the generator parameters) are given by

$$d_{k,j} = \frac{(-1)^k\,\beta^{k+\alpha}\,\Gamma(\alpha+k+j)}{\Gamma(\alpha+k+1)\,\Gamma(\alpha)\,k!\,j!}.$$

Equation (3.4) reveals that the GE-G density function is a linear mixture of Exp-G density functions. Then, some structural properties of the new family can be obtained by knowing those of the Exp-G family. See, for example, Mudholkar *et al.* (1996) and Nadarajah and Kotz (2006), among others.

Integrating (3.4), it follows the cdf of X

$$F(x) = \sum_{k,j=0}^{\infty} d_{k,j}\,H_{\alpha+k+j}(x), \tag{3.5}$$

where $H_{\alpha+k+j}(x)$ denotes the cdf of the Exp-G$(\alpha+k+j)$ distribution. Equations (3.4) and (3.5) are the main results of this section.

3.1. Asymptotes and shapes. The asymptotes of (1.3), (1.4) and (1.5) when $x \to -\infty, \infty$ are given by

$$F(x) \sim \frac{\beta^\alpha}{\Gamma(\alpha+1)} G^\alpha(x) \quad \text{as } x \to -\infty,$$

$$1 - F(x) \sim \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} \overline{G}^{1-\alpha}(x) \exp\left\{-\frac{\beta}{\overline{G}(x)}\right\} \quad \text{as } x \to \infty,$$

$$f(x) \sim \frac{\beta^\alpha}{\Gamma(\alpha)} G^{\alpha-1}(x) \exp\{-\beta G(x)\}\, g(x) \quad \text{as } x \to -\infty,$$

$$f(x) \sim \frac{\beta^\alpha}{\Gamma(\alpha)} \overline{G}^{-\alpha-1}(x) \exp\left\{-\frac{\beta}{\overline{G}(x)}\right\} g(x) \quad \text{as } x \to \infty,$$

$$\tau(x) \sim \frac{\beta^\alpha}{\Gamma(\alpha)} G^{\alpha-1}(x) \exp\{-\beta G(x)\}\; g(x) \quad \text{as } x \to -\infty, \text{ and}$$

$$\tau(x) \sim \beta \overline{G}^{-2}(x) \quad \text{as } x \to \infty.$$

Then, the cdf of the GE-G distribution is proportional to the αth power of $G(x)$ for very large negative x. The hrf of the GE-G distribution is inversely proportional to the square of $\overline{G}(x)$ for very large x.

The shapes of (1.4) and (1.5) can be described analytically. The critical points of the pdf are the roots of the equation:

$$\frac{df(x)}{dx} = \frac{(\alpha-1)g(x)}{G(x)} + \frac{(\alpha+1)g(x)}{\overline{G}(x)} + \frac{\beta\,g(x)}{\overline{G}^2(x)} + \frac{g'(x)}{g(x)} = 0. \tag{3.6}$$

There may be more than one root to (3.6). If $x = x_0$ is a root of (3.6) then it corresponds to a local maximum if $df(x)/dx > 0$ for all $x < x_0$ and $df(x)/dx < 0$ for all $x > x_0$. It corresponds to a local minimum if $df(x)/dx < 0$ for all $x < x_0$ and $df(x)/dx > 0$ for all $x > x_0$. It corresponds to a point of inflexion if either $df(x)/dx > 0$ for all $x \neq x_0$ or $df(x)/dx < 0$ for all $x \neq x_0$.

The critical points of the hrf are the roots of the equation:

$$\frac{d\tau(x)}{dx} = \frac{(\alpha-1)\,g(x)}{G(x)} + \frac{(\alpha+1)\,g(x)}{\overline{G}(x)} + \frac{\beta\,g(x)}{\overline{G}^2(x)} + \frac{g'(x)}{g(x)}$$

$$+ \frac{\beta^\alpha\,G^{\alpha-1}(x)}{\overline{G}^{\alpha+1}(x)\left[\Gamma(\alpha)-\gamma\left(\alpha,\beta G(x)/\overline{G}(x)\right)\right]}\exp\left\{-\frac{\beta G(x)}{\overline{G}(x)}\right\} = 0. \tag{3.7}$$

There may be more than one root to (3.7). If $x = x_0$ is a root of (3.7) then it corresponds to a local maximum if $d\tau(x)/dx > 0$ for all $x < x_0$ and $d\tau(x)/dx < 0$ for all $x > x_0$. It corresponds to a local minimum if $d\tau(x)/dx < 0$ for all $x < x_0$ and $d\tau(x)/dx > 0$ for all $x > x_0$. It corresponds to a point of inflexion if either $d\tau(x)/dx > 0$ for all $x \neq x_0$ or $d\tau(x)/dx < 0$ for all $x \neq x_0$.

Equations (3.6) and (3.7) should be solved numerically and their roots depend on several aspects: the forms of the baseline pdf and cdf, the baseline parameters and the extra parameters α and β. The number of roots can be seen in a case-by-case basis, which is not included in the main objectives of this paper. Although these equations cannot be solved analytically, the numerical roots can be determined by using Newton-Raphson type algorithms.

4. Moments

Hereafter, let $Y_{k,j}$ ~Exp-G$(\alpha+k+j)$. A first general formula for the nth moment of X can be obtained from (3.4) as

$$\mu'_n = E(X^n) = \sum_{k,j=0}^{\infty} d_{k,j}\,E(Y_{k,j}^n). \tag{4.1}$$

Expressions for moments of several exponentiated distributions are given by Nadarajah and Kotz (2006), which can be used to obtain μ'_n. We now provide an application of (4.1) by taking the baseline Weibull cdf $G(x) = 1 - \mathrm{e}^{-(\delta x)^\lambda}$ introduced in Section 2.1. The corresponding Exp-Weibull (Exp-W) density function with positive power parameter a is given by

$$f(x) = a\,\lambda\,\delta^\lambda\,x^{\lambda-1}\,\mathrm{e}^{-(\delta \mathrm{x})^\lambda}\,[1-\mathrm{e}^{-(\delta \mathrm{x})^\lambda}]^{\mathrm{a}-1}. \tag{4.2}$$

The nth moment of (4.2), say ρ_n, is given by (Cordeiro *et al.*, 2013)

$$\rho_n = \delta^{-n}\,\Gamma(n/\lambda+1)\sum_{r=0}^{\infty}\frac{w_r}{(r+1)^{n/\lambda}}, \tag{4.3}$$

where

$$w_r = \frac{a}{(r+1)}\sum_{i=0}^{\infty}(-1)^{i+r}\binom{a(i+1)-1}{r}.$$

Combining (4.1) and (4.3), we obtain

$$\mu'_n = \delta^{-n}\,\Gamma(n/\lambda+1)\sum_{k,j,r,i=0}^{\infty}\frac{(-1)^{i+r}\,(\alpha+k+j)\binom{(\alpha+k+j)(i+1)-1}{r}\,d_{k,j}}{(r+1)^{n/\lambda+1}}. \tag{4.4}$$

A second general formula for μ'_n follows from (3.4) and the baseline qf $Q_G(x) = G^{-1}(x)$. We can write

$$\mu'_n = \sum_{k,j=0}^{\infty}(\alpha+k+j)\,d_{k,j}\,\tau(n,\alpha+k+j-1), \tag{4.5}$$

where

$$\tau(n,a)=\int_{-\infty}^{\infty}x^n\,G(x)^a\,g(x)dx=\int_0^1 Q_G(u)^n\,u^a du. \tag{4.6}$$

The ordinary moments of several GE distributions can be determined directly from equations (4.5) and (4.6). Here, we provide three examples. First, the moments of the GE-exponential (with parameter $\lambda > 0$) distribution are given by

$$\mu'_n = n!\,\lambda^n \sum_{k,j,m=0}^{\infty} \frac{(-1)^{n+m}\,(\alpha+k+j)\binom{\alpha+k+j-1}{m}\,d_{k,j}}{(m+1)^{n+1}}.$$

Second, for the GE-Pareto distribution, where the baseline cdf is $G(x)=1-(1+x)^{-\nu}$, we have (for $\nu > 0$)

$$\mu'_n = \sum_{k,j,m=0}^{\infty}(-1)^{n+m}\,(\alpha+k+j)\binom{n}{m}B(\alpha+k+j-1,1-m\nu^{-1})\,d_{k,j},$$

where $B(a,b)=\int_0^1 t^{a-1}\,(1-t)^{b-1}dt$ represents the beta function. Third, for the GE-standard logistic, where $G(x)=(1+\mathrm{e}^{-x})^{-1}$, we obtain using a result in Prudnikov *et al.* (1986, Section 2.6.13, equation 4) (for $t<1$)

$$\mu'_n = \sum_{k,j=0}^{\infty}(\alpha+k+j)\,d_{k,j}\left(\frac{\partial}{\partial t}\right)^n B(t+\alpha+k+j,1-t)\bigg|_{t=0}.$$

Next, the central moments (μ_n) and cumulants (κ_n) of X follow as

$$\mu_n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\mu_1'^k\,\mu'_{n-k} \qquad \text{and} \qquad \kappa_n=\mu'_n-\sum_{k=1}^{n-1}\binom{n-1}{k-1}\kappa_k\,\mu'_{n-k},$$

respectively, where $\kappa_1=\mu'_1$, $\kappa_2=\mu'_2-\mu_1'^2$, etc. The skewness $\gamma_1=\kappa_3/\kappa_2^{3/2}$ and kurtosis $\gamma_2=\kappa_4/\kappa_2^2$ of X can be evaluated from the ordinary moments. For the GE-W distribution (by taking $\lambda=0.2$ and $\delta=1.5$), the plots of the skewness and kurtosis as functions of α for some values of β, and as functions of β for some values of α, are displayed in Figures 6 and 7, respectively. For fixed β, when α increases, the skewness curve first decreases to a minimum value and then increases, whereas the kurtosis curve always decreases. For fixed α, when β increases, the skewness curve always decreases, whereas the kurtosis curve rapidly decreases and then increases steadily.

For empirical purposes, the shape of many distributions can be usefully described by the incomplete moments. These moments play an important role for measuring inequality, for example, income quantiles and Lorenz and Bonferroni curves, which depend upon the incomplete moments of a distribution. The nth incomplete moment of X can be expressed as

$$m_n(y)=\int_{-\infty}^{y}x^n\,f(x)dx=\sum_{k,j=0}^{\infty}(\alpha+k+j)\,d_{k,j}\int_0^{G(y)}Q_G(u)^n\,u^{\alpha+k+j-1}\,du. \tag{4.7}$$

The integral in (4.7) can be evaluated for most baseline G distributions. In fact, we can use power series methods which are at the heart of many aspects of applied mathematics and statistics. If the function $Q_G(u)$ does not have a closed-form expression, it can usually be expressed as a power series

$$Q_G(u)=\sum_{i=0}^{\infty}a_i\,u^i, \tag{4.8}$$

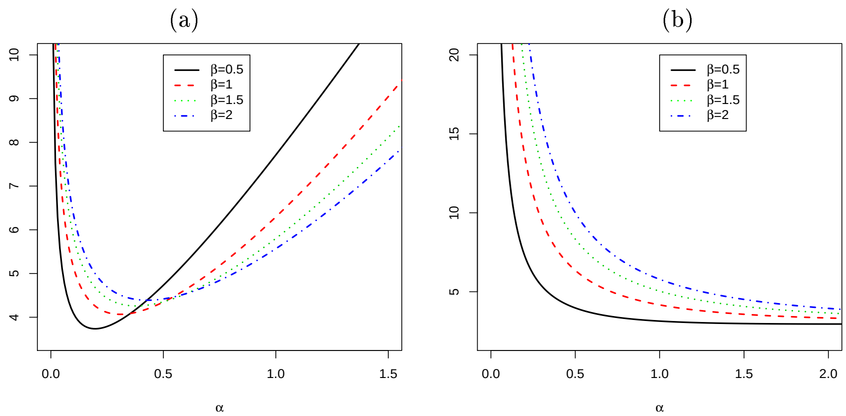

Figure 6. Skewness and kurtosis of the GE-W distribution as functions of α for some values of β.

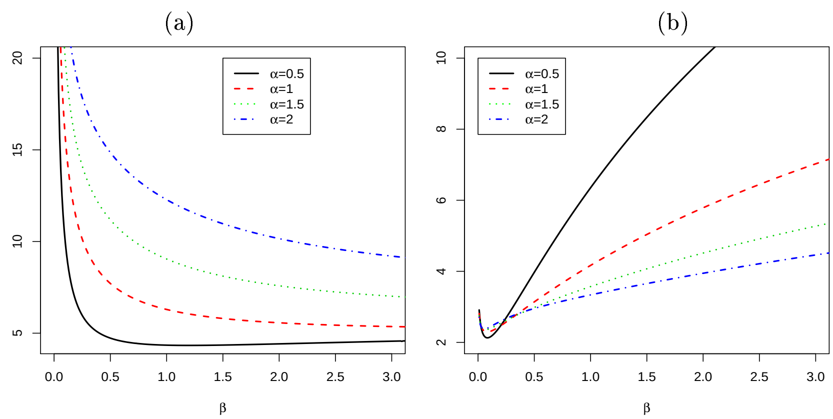

Figure 7. Skewness and kurtosis of the GE-W distribution as functions of β for some values of α.

where the coefficients a_i are suitably chosen real numbers. For several important distributions, such as the normal, Student t, gamma and beta distributions, $Q_G(u)$ does not have closed-form but it can be written as in (4.8). For example, for the standard normal distribution, the coefficients a_i are given by

$$a_i = (2\,\pi)^{i/2} \sum_{m=i}^{\infty} \left(\frac{-1}{2}\right)^{m-j} \binom{m}{i}\, p_i,$$

where the quantities p_i are defined by $p_i = 0$ (for $i = 0, 2, 4, \ldots$) and $p_i = q_{(i-1)/2}$ (for $i = 1, 3, 5, \ldots$), and the q_k's are calculated recursively from

$$q_{k+1} = \frac{1}{2(2k+3)} \sum_{r=0}^{k} \frac{(2r+1)(2k-2r+1) q_r\, q_{k-r}}{(r+1)(2r+1)}.$$

Here, $q_0 = 1$, $q_1 = 1/6$, $q_2 = 7/120$, $q_3 = 127/7560, \ldots$

We use throughout a result of Gradshteyn and Ryzhik (2000, Section 0.314) for a power series raised to a positive integer n

$$Q_G(u)^n = \left(\sum_{i=0}^{\infty} a_i\, u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i}\, u^i, \tag{4.9}$$

where the coefficients $c_{n,i}$ (for $i = 1, 2, \ldots$) are easily obtained from the recurrence equation

$$c_{n,i} = (i a_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i]\, a_m\, c_{n,i-m}, \tag{4.10}$$

and $c_{n,0} = a_0^n$. The coefficient $c_{n,i}$ can be determined from $c_{n,0}, \ldots, c_{n,i-1}$ and hence from the quantities $a_0, \ldots, a_i$. Equations (4.9) and (4.10) are used throughout this paper. The coefficient $c_{n,i}$ can be given explicitly in terms of the coefficients a_i's, although it is not necessary for programming numerically our expansions in any algebraic or numerical software. Thus, we obtain the incomplete moments of X from (4.7) as

$$m_n(y) = \sum_{k,j,i=0}^{\infty} \frac{(\alpha + k + j)\, d_{k,j}\, c_{n,i}\, G(y)^{\alpha+k+j+i}}{(\alpha + k + j + i)}. \tag{4.11}$$

The nth descending factorial moment of X is

$$\mu'_{(n)} = E(X^{(n)}) = E\left[X(X-1) \times \cdots \times (X - n + 1)\right] = \sum_{k=0}^{n} s(n,k)\, \mu'_k,$$

where

$$s(r,k) = \frac{1}{k!} \left[\frac{d^k}{dx^k} x^{(r)}\right]_{x=0}$$

is the Stirling number of the first kind which counts the number of ways to permute a list of r items into k cycles. So, we can obtain the factorial moments from the ordinary moments given before. Other kinds of moments such L-moments may also be obtained in closed-form, but we consider only the previous moments for reasons of space.

5. Generating function

In this section, we provide two general formulae for the mgf $M(t) = E(\mathrm{e}^{\mathrm{t\,X}})$ of X. A first formula for $M(t)$ follows from (3.4) as

$$M(t) = \sum_{k,j=0}^{\infty} d_{k,j}\, M_{k,j}(t), \tag{5.1}$$

where $M_{k,j}(t)$ is the mgf of $Y_{k,j}$. Thus, $M(t)$ can be determined from the generating function of the Exp-G distribution. We now provide an application of (5.1) by considering again the Weibull baseline distribution with parameters λ and δ (see Section 5). The mgf of the Exp-W distribution with power parameter $\alpha + k + j$ is given by

$$M_{k,j}(t) = \sum_{r=0}^{\infty} v_{k,j}^{(r)}\, I_r(t), \tag{5.2}$$

where $v_{k,j}^{(r)} = \lambda\, \delta^{\lambda}\, (\alpha + k + j) \sum_{i=0}^{\infty} (-1)^{i+r} \binom{(\alpha+k+j)(i+1)-1}{r}$, $\delta_r = \delta\, (r+1)^{1/\lambda}$ and

$$I_r(t) = \int_0^{\infty} x^{\lambda - 1} \exp\{t\, x - (\delta_r\, x)^{\lambda}\} dx.$$

Cordeiro *et al.* (2013) derived two different formulae for $I_r(t)$, which hold for: (i) $\lambda > 1$ or (ii) for $\lambda = p/q$, where $p \geq 1$ and $q \geq 1$ are co-prime integers. The first representation for $I_r(t)$ is obtained using the Wright generalized hypergeometric function (Wright, 1935) given by

$$ {}_p\Psi_q\left[\begin{array}{c}(\alpha_1,A_1),\cdots,(\alpha_p,A_p)\\(\beta_1,B_1),\cdots,(\beta_q,B_q)\end{array};x\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j+A_j\,n)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_j\,n)}\,\frac{x^n}{n!}. $$

The Wright function exists if $1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j>0$. We have

$$ \begin{aligned} I_r(t) &= \sum_{m=0}^{\infty}\frac{t^m}{m!}\int_0^\infty x^{m+\lambda-1}\exp\{-(\delta_r x)^\lambda\}dx=\frac{1}{\lambda\,\delta_r^\lambda}\sum_{m=0}^{\infty}\frac{t^m}{\delta_r^m\,m!}\Gamma(m\lambda^{-1}+1)\\ &= \frac{1}{\lambda\,\delta_r^\lambda}\,{}_1\Psi_0\left[\begin{array}{c}(1,\lambda^{-1})\\-\end{array};\frac{t}{\delta_r}\right]. \end{aligned} \tag{5.3} $$

Using equations (5.1), (5.2) and (5.3), we obtain (if $\lambda>1$)

$$ M(t)=\lambda^{-1}\sum_{k,j,r=0}^{\infty}\frac{d_{k,j}\,v_{k,j}^{(r)}}{\delta_r^\lambda}\,{}_1\Psi_0\left[\begin{array}{c}(1,\lambda^{-1})\\-\end{array};\frac{t}{\delta_r}\right]. \tag{5.4} $$

A second representation for $I_r(t)$ is based on the Meijer G-function (Gradshteyn and Ryzhik, 2000; Section 9.3) defined by

$$ G_{p,q}^{m,n}\left(x\left|\begin{array}{c}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{array}\right.\right)=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j+t)\prod_{j=1}^{n}\Gamma(1-a_j-t)}{\prod_{j=n+1}^{p}\Gamma(a_j+t)\prod_{j=m+1}^{p}\Gamma(1-b_j-t)}\,x^{-t}dt, $$

where $\mathrm{i}=\sqrt{-1}$ is the complex unit and L denotes an integration path. This function contains many integrals with elementary and special functions (Prudnikov *et al.*, 1986). From the result $\exp\{-g(x)\}=G_{0,1}^{1,0}\left(g(x)\mid\begin{array}{c}-\\0\end{array}\right)$ for an arbitrary function $g(\cdot)$, $I_r(t)$ becomes

$$ I_r(t)=\int_0^\infty x^{\lambda-1}\,\exp\{sx-(\delta_r x)^\lambda\}dx=\int_0^\infty x^{\lambda-1}\,\exp(sx)\,G_{0,1}^{1,0}\left(\delta_r^\lambda\,x^\lambda\left|\begin{array}{c}-\\0\end{array}\right.\right)dx. $$

Next, we assume that $\lambda=p/q$, where $p\geq 1$ and $q\geq 1$ are co-prime integers. Note that this condition for calculating the integral $I_r(t)$ is not restrictive since every real number can be approximated by a rational number. Using equation (2.24.1.1) in Prudnikov *et al.* (1986, volume 3), we obtain

$$ I_r(t)=\frac{p^{p/q-1/2}(-t)^{-p/q}}{(2\pi)^{(p+q)/2-1}}G_{q,p}^{p,q}\left(\frac{\delta_r^q p^{p+q}}{(-t)^p q^{2q}}\left|\begin{array}{c}\frac{q-p}{pq},\frac{2q-p}{pq},\ldots,\frac{pq-p}{pq}\\0,\frac{1}{q},\ldots,\frac{q-1}{q}\end{array}\right.\right). \tag{5.5} $$

From equations (5.1), (5.2) and (5.5), we can obtain $M(t)$ for the GE-W distribution.

A second general formula for $M(t)$ can be derived from (3.4) as

$$ M(t)=\sum_{k,j=0}^{\infty}(\alpha+k+j)\,d_{k,j}\,\rho(t,\alpha+k+j-1), \tag{5.6} $$

where $\rho(t,a)$ follows from the baseline qf $Q_G(u)$ as

$$ \rho(t,a)=\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{tx}}\,\mathrm{G(x)^a\,g(x)dx}=\int_0^1\exp\{\mathrm{t\,Q_G(u)}\}\,\mathrm{u^a du}. \tag{5.7} $$

We can obtain the mgf's of several GE distributions from equations (5.6) and (5.7). For example, the generating functions of the GE-exponential (with parameter λ and for $t < \lambda^{-1}$), GE-Pareto (with parameter $\nu > 0$) and GE-standard logistic (for $t < 1$) are given by

$$M(t) = \sum_{k,j=0}^{\infty} (\alpha + k + j)\, B(\alpha + k + j, 1 - \lambda t)\, d_{k,j},$$

$$M(t) = \mathrm{e}^{-t} \sum_{k,j,m=0}^{\infty} (\alpha + k + j)\, B(\alpha + k + j, 1 - m\nu^{-1})\, d_{k,j}\, \frac{t^m}{m!}$$

and

$$M(t) = \sum_{k,j=0}^{\infty} (\alpha + k + j)\, B(t + \alpha + k + j, 1 - t)\, d_{k,j},$$

respectively.

6. Other measures

In this section, we calculate the following measures: qf, PWMs, mean deviations, extreme values, entropies, reliability and order statistics for the GE-G distribution.

6.1. Quantile function. Let $\gamma^{-1}(\alpha, u)$ be the inverse of the incomplete gamma function $\gamma(\alpha, u)$. By inverting $F(x) = u$, the GE-G qf can be expressed in terms of $Q_G(u)$ and $\gamma^{-1}(\alpha, u)$ (for $0 < u < 1$) as

$$F^{-1}(u) = Q_G\left(\frac{\gamma^{-1}(\alpha, u\,\Gamma(\alpha))}{\beta + \gamma^{-1}(\alpha, u\,\Gamma(\alpha))}\right). \tag{6.1}$$

Let $Q^{-1}(\alpha, u)$ be the inverse function of $Q(\alpha, u) = 1 - \gamma(\alpha, u)/\Gamma(\alpha)$. For further details, see
http://functions.wolfram.com/GammaBetaErf/InverseGammaRegularized/.
We can write

$$F^{-1}(u) = Q_G\left(\frac{Q^{-1}(\alpha, 1-u)}{\beta + Q^{-1}(\alpha, 1-u)}\right). \tag{6.2}$$

Quantiles of interest can be obtained from (6.2) by substituting appropriate values for u. In particular, the median of X is obtained when $u = 1/2$. We can also use (6.2) for simulating GE-G random variables by setting U as a uniform random variable in the unit interval $[0, 1]$.

The asymptotes of (6.2) can be based on known properties of $Q^{-1}(\alpha, u)$. Using the inverse of the regularized gamma function ‖ we obtain as $u \to 0$,

$$Q^{-1}(\alpha, 1-u) \sim -(1-\alpha)\, W_{-1}\left(-\frac{(1-u)^{1/(\alpha-1)}\, \Gamma(\alpha)^{1/(\alpha-1)}}{(\alpha - 1)}\right),$$

where $W_{-1}(x)$ denotes the product log function. It gives the principal solution w in $w\,\mathrm{e}^w = x$. This function is implemented in *Mathematica* as `ProductLog[x]`.

Again from the above site, we have when $u \to 1$

$$Q^{-1}(\alpha, 1-u) \sim \left[-(1-u)^{1/\alpha}\, \Gamma^{1/\alpha}(\alpha+1) + \frac{(1-u)^2\, \Gamma^{2/\alpha}(\alpha+1)}{(\alpha+1)}\right].$$

‖http://functions.wolfram.com/GammaBetaErf/InverseGammaRegularized/06/02/01/

If V is a gamma random variable with shape parameter α and unit scale parameter, the qf of V, say $Q_V(u)$, admits a power series expansion given by

$$Q_V(u) = Q^{-1}(\alpha, 1-u) = \sum_{i=0}^{\infty} m_i \, [\Gamma(\alpha+1)\, u]^{i/a},$$

where $m_0 = 0$, $m_1 = 1$ and any coefficient m_{i+1} (for $i \geq 1$) can be determined by the cubic recurrence equation

$$\begin{aligned} m_{i+1} &= \frac{1}{i\,(\alpha+i)} \Bigg\{ \sum_{r=1}^{i} \sum_{s=1}^{i-s+1} s\,(i-r-s+2)\, m_r\, m_s\, m_{i-r-s+2} \\ &\quad -\Delta(i) \sum_{r=2}^{i} r\, [r-\alpha-(1-\alpha)(i+2-r)]\; m_r\, m_{i-r+2} \Bigg\}, \end{aligned}$$

where $\Delta(i) = 0$ if $i < 2$ and $\Delta(i) = 1$ if $i \geq 2$. The first few coefficients are $m_2 = 1/(\alpha+1)$, $m_3 = (3\alpha+5)/[2(\alpha+1)^2(\alpha+2)], \ldots$ Let $g_i = m_i\, \Gamma(\alpha+1)^{i/\alpha}$ (for $i \geq 0$) and $z = u^{1/\alpha}$. We obtain a power series for $F^{-1}(u)$ from equation (6.2)

$$F^{-1}(u) = Q_G\left(\frac{\sum_{i=0}^{\infty} g_i\, z^i}{\beta + \sum_{i=0}^{\infty} g_i\, z^i}\right).$$

Using the ratio of two power series (for $b_0 \neq 0$)

$$\frac{\sum_{n=0}^{\infty} a_n\, x^n}{\sum_{n=0}^{\infty} b_n\, x^n} = \sum_{n=0}^{\infty} d_n\, x^n,$$

where $d_0 = a_0/b_0$ and $d_n = b_0^{-1}\,(a_n - \sum_{r=1}^{n} b_r\, c_{n-r})$ (for $n = 1, 2, \ldots$), we can rewrite $F^{-1}(u)$ as

$$F^{-1}(u) = Q_G\left(\sum_{n=1}^{\infty} d_n\, u^{n/\alpha}\right), \tag{6.3}$$

where $d_n = \beta^{-1}\,(g_n - \sum_{r=1}^{n} g_r\, d_{n-r})$ for $n \geq 1$. Then, $d_1 = \beta^{-1} g_1$, $d_2 = \beta^{-2}(\beta g_2 - g_1^2)$, etc.

Hence, equation (6.3) reveals that the GE-G qf can be expressed as the baseline qf applied to a power series. This expansion holds for any GE-G model. For the great majority of distributions, the baseline qf can be written as a power series and therefore the GE-G qf can also be expressed in this way.

6.2. Probability weighted moments. A useful mathematical quantity is the (n,s)th PWM of X defined by $\kappa_{n,s} = E\{X^n\, F(X)^s\}$ for $n, s = 0, 1, \ldots$ We can obtain from equations (1.3) and (1.4) by setting $G(x) = u$

$$\kappa_{n,s} = \frac{\beta^\alpha}{\Gamma(\alpha)^{s+1}} \int_0^1 Q_G(u)^n\, \frac{u^{\alpha-1}}{(1-u)^{\alpha+1}}\, \exp\left[-\frac{\beta\, u}{(1-u)}\right] \left[\gamma\left(\alpha, \frac{\beta\, u}{(1-u)}\right)\right]^s du.$$

The power series for the incomplete gamma function raised to an integer power s can be written as

$$\left[\gamma\left(\alpha, \frac{\beta\, u}{(1-u)}\right)\right]^s = \left[\sum_{p=0}^{\infty} \frac{(-\beta\, u)^p}{(1-u)^p\, (\alpha+p)\, p!}\right]^s.$$

From equation (4.9), we have

$$\left[\sum_{p=0}^{\infty} \frac{(-\beta\, u)^p}{(1-u)^p\, (\alpha+p)\, p!}\right]^s = \sum_{p=0}^{\infty} c_{s,p}\, \frac{u^p}{(1-u)^p}, \tag{6.4}$$

where $\quad c_{s,0} = \alpha^{-s} \quad$ and $\quad c_{s,p} = \dfrac{\alpha}{p} \displaystyle\sum_{m=1}^{p} \frac{(-\beta)^m\,[m\,(s+1)-p]}{(\alpha+m)\,m!}\, c_{s,p-m}$,

for $p = 1, 2, \ldots$ By expanding the exponential quantity in the last expression for $\kappa_{n,s}$ and using (6.4), we obtain

$$\kappa_{n,s} = \frac{\beta^\alpha}{\Gamma(\alpha)^{s+1}} \sum_{p,r=0}^{\infty} \frac{(-\beta)^r\, c_{s,p}}{r!} \int_0^1 Q_G(u)^n\, \frac{u^{\alpha+p+r-1}}{(1-u)^{\alpha+p+r+1}}\, du.$$

From expansion (3.3), we have

$$(6.5)\quad \kappa_{n,s} = \frac{\beta^\alpha}{\Gamma(\alpha)^{s+1}} \sum_{p,r,j=0}^{\infty} \frac{(-\beta)^r\, \Gamma(\alpha+p+r+j+1)\, c_{s,p}}{\Gamma(\alpha+p+r+1)\, r!æ!}\, \tau(n, \alpha+p+r+j-1),$$

where $\tau(n, \alpha+p+r+j-1)$ is given by (4.6).

Equation (6.5) can be applied for most baseline G distributions to derive explicit expressions for $\kappa_{n,s}$. It is the main result of this section.

6.3. Mean deviations. The mean deviations about the mean ($\delta_1 = E(|X - \mu_1'|)$) and about the median ($\delta_2 = E(|X - M|)$) of X are given by

$$(6.6)\qquad \delta_1 = 2\mu_1'\, F(\mu_1') - 2m_1(\mu_1') \quad \text{and} \quad \delta_2 = \mu_1' - 2m_1(M),$$

respectively, where $\mu_1' = E(X)$, $F(\mu_1')$ is evaluated from (1.3), $M = Median(X)$ is the median given in Section 7 and $m_1(z) = \int_{-\infty}^{z} x\, f(x)dx$ is the first incomplete moment given by (4.7) with $n = 1$.

Next, we provide two alternative expressions for δ_1 and δ_2. An explicit expression for $m_1(z)$ can be derived from (3.3) as

$$(6.7)\qquad m_1(z) = \sum_{k,j=0}^{\infty} d_{k,j}\, J_{k,j}(z),$$

where

$$(6.8)\qquad J_{k,j}(z) = \int_{-\infty}^{z} x\, h_{\alpha+k+j}(x)dx.$$

Equation (6.8) is the basic quantity to determine the mean deviations of the Exp-G distributions. The mean deviations in (6.6) depend only on the mean deviations of the Exp-G distribution. So, alternative representations for δ_1 and δ_2 are given by

$$\delta_1 = 2\mu_1' F\left(\mu_1'\right) - 2\sum_{k,j=0}^{\infty} d_{k,j}\; J_{k,j}\left(\mu_1'\right) \quad \text{and} \quad \delta_2 = \mu_1' - 2\sum_{k,j=0}^{\infty} d_{k,j}\; J_{k,j}(M).$$

We provide a simple application of (6.7) and (6.8) for the GE-W distribution. The Exp-W density function with power parameter $\alpha + k + j$ is obtained from (4.2) with $a = \alpha + k + j$. Then,

$$\begin{aligned} J_{k,j}(z) &= \lambda\,(\alpha+k+j)\,\delta^\lambda \int_0^z x^\lambda\, \mathrm{e}^{-(\delta \mathrm{x})^\lambda}\, [1 - \mathrm{e}^{-(\delta\mathrm{x})^\lambda}]^{\alpha+\mathrm{k}+\mathrm{j}-1} \mathrm{dx} \\ &= \lambda\,\delta^\lambda\,(\alpha+k+j) \sum_{r=0}^{\infty} (-1)^r \binom{\alpha+k+j-1}{r} \int_0^z x^\lambda\, \mathrm{e}^{-(\mathrm{r}+1)(\delta\mathrm{x})^\lambda} \mathrm{dx}. \end{aligned}$$

The last integral is given by the incomplete gamma function and then the mean deviations for the GE-W distribution can be obtained immediately from

$$m_1(z) = \delta^{-1} \sum_{k,j,r=0}^{\infty} \frac{(-1)^r\,(\alpha+k+j)\,d_{k,j}}{(r+1)^{1+1/\lambda}} \binom{\alpha+k+j-1}{r} \gamma\left(1+\lambda^{-1}, (r+1)(\delta z)^{\lambda}\right).$$

A second general formula for $m_1(z)$ can be derived by setting $u = G(x)$ in (3.4)

$$m_1(z) = \sum_{k,j=0}^{\infty} (\alpha+k+j)\; d_{k,j}\, T_{k,j}(z), \tag{6.9}$$

where

$$T_{k,j}(z) = \int_0^{G(z)} Q_G(u)\, u^{\alpha+k+j-1} du \tag{6.10}$$

is a simple integral defined from the baseline qf $Q_G(u)$.

In a similar manner, the mean deviations of any GE-G distribution can be computed from equations (6.9)-(6.10). For example, the mean deviations of the GE-exponential (with parameter λ), GE-Pareto (with $\nu > 0$) and GE-standard logistic distributions can be determined (using the generalized binomial expansion) from the functions

$$T_{k,j}(z) = \lambda^{-1}\,\Gamma(\alpha+k+j-1) \sum_{m=0}^{\infty} \frac{(-1)^m\,(1-\mathrm{e}^{-\mathrm{m}\lambda\mathrm{z}})}{\Gamma(\alpha+k+j-1-m)\,(m+1)!},$$

$$T_{k,j}(z) = \sum_{m=0}^{\infty} \sum_{r=0}^{m} \frac{(-1)^m \binom{\alpha+k+j}{m} \binom{m}{r}}{(1-r\nu)}\, z^{1-r\nu}$$

and

$$T_{k,j}(z) = \frac{1}{\Gamma(k+j)} \sum_{m=0}^{\infty} \frac{(-1)^m\,\Gamma(\alpha+k+j+m)\,(1-\mathrm{e}^{-\mathrm{mz}})}{(m+1)!},$$

respectively.

Applications of these equations can be given to obtain Bonferroni and Lorenz curves defined for a given probability π by $B(\pi) = m_1(q)/(\pi\mu_1')$ and $L(\pi) = m_1(q)/\mu_1'$, respectively, where $\mu_1' = E(X)$ and $q = F^{-1}(\pi)$ is the qf of the GE-G distribution at π given by (6.1) or (6.2).

6.4. Extreme values. If $\overline{X} = (X_1 + \cdots + X_n)/n$ denotes the mean of a random sample from (1.4), then by the usual central limit theorem $\sqrt{n}(\overline{X} - E(X))/\sqrt{Var(X)}$ approaches the standard normal distribution as $n \to \infty$ under suitable conditions. Sometimes one would be interested in the asymptotics of the extreme values $M_n = \max(X_1, \ldots, X_n)$ and $m_n = \min(X_1, \ldots, X_n)$.

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then by Leadbetter *et al.* (1987, Chapter 1), there must exist a strictly positive function, say $h(t)$, such that

$$\lim_{t\to\infty} \frac{1-G(t+xh(t))}{1-G(t)} = \mathrm{e}^{-\mathrm{x}}$$

for every $x \in \mathbb{R}$. But, using (3.6), we note that

$$\begin{aligned} \lim_{t\to\infty} \frac{1-F(t+xh(t))}{1-F(t)} &= \lim_{t\to\infty} \left\{ \frac{1-G(t)}{1-G(t+xh(t))} \right\}^{\alpha-1} \\ &\times \exp\left\{ \frac{\beta}{G(t)} - \frac{\beta}{G\left(t+xh(t)\right)} \right\} = \exp\left\{(\alpha-1)x\right\} \end{aligned}$$

for every $x \in \mathbb{R}$. Provided that $\alpha < 1$, it follows by Leadbetter *et al.* (1987, Chapter 1) that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n\to\infty} \Pr\left\{a_n\left(M_n - b_n\right) \leq x\right\} = \exp\left\{-\exp\left(-(\alpha-1)x\right)\right\}$$

for some suitable norming constants $a_n > 0$ and b_n.

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then by Leadbetter *et al.* (1987, Chapter 1), there must exist a $\beta < 0$ such that

$$\lim_{t\to\infty} \frac{1-G(tx)}{1-G(t)} = x^{\beta}$$

for every $x > 0$. But, using (3.6), we note that

$$\begin{aligned} \lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} &= \lim_{t\to\infty} \left\{ \frac{1-G(t)}{1-G(tx)} \right\}^{\alpha-1} \exp\left\{ \frac{\beta}{G(t)} - \frac{\beta}{G\left(tx\right)} \right\} \\ &= \lim_{t\to\infty} \left\{ \frac{1-G(t)}{1-G(tx)} \right\}^{\alpha-1} = x^{\beta(\alpha-1)} \end{aligned}$$

for every $x > 0$. Provided that $\alpha > 1$, it follows by Leadbetter *et al.* (1987, Chapter 1) that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n\to\infty} \Pr\left\{a_n\left(M_n - b_n\right) \leq x\right\} = \exp(-x^{\beta(\alpha-1)})$$

for some suitable norming constants $a_n > 0$ and b_n.

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then by Leadbetter *et al.* (1987, Chapter 1), there must exist a $c > 0$, such that

$$\lim_{t\to-\infty} \frac{G(tx)}{G(t)} = x^{c}$$

for every $x < 0$. But, using (3.6), we note that

$$\lim_{t\to-\infty} \frac{F(tx)}{F(t)} = \lim_{t\to-\infty} \left\{ \frac{G(tx)}{G(t)} \right\}^{\alpha} = x^{c\alpha}.$$

So, it follows by Leadbetter *et al.* (1987, Chapter 1) that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n\to\infty} \Pr\left\{a_n\left(M_n - b_n\right) \leq x\right\} = \exp\left\{-(-x)^{c\alpha}\right\}$$

for some suitable norming constants $a_n > 0$ and b_n.

The same argument applies to min domains of attraction. That is, F belongs to the same min domain of attraction as that of G.

6.5. Entropies. An entropy is a measure of variation or uncertainty of a random variable X. Two popular entropy measures are the Rényi and Shannon entropies. The Rényi entropy of a random variable with pdf $f(x)$ is defined by

$$I_R(c) = \frac{1}{(1-c)}\ \log\left(\int_{-\infty}^{\infty} f(x)^c\, dx\right)$$

for $c > 0$ and $c \neq 1$. The Shannon entropy of a random variable X is defined by $I_S = E\{-\log[f(X)]\}$. It is a special case of the Rényi entropy when $c \uparrow 1$. Next, we derive expressions for the Rényi and Shannon entropies of X. From equation (1.4) and using similar algebraic developments that lead to (3.4), we can write

$$f(x)^c = \sum_{k,j=0}^{\infty} e_{k,j}\, G(x)^{c(k+\alpha-1)+j}\, g(x)^c,$$

where

$$e_{k,j} = \frac{(-1)^k\, c^k\, \beta^{c\,\alpha+k}\, \Gamma(c[k+\alpha+1]+j)}{\Gamma(c[k+\alpha+1])\, \Gamma(\alpha)^c\, k!\, j!}$$

Then, the Rényi entropy of X becomes

$$I_R(c) = \frac{1}{(1-c)}\ \sum_{k,j=0}^{\infty} e_{k,j}\ \log(K_{k,j}),$$

where

$$K_{k,j} = \int_{-\infty}^{\infty} G(x)^{c(k+\alpha-1)+j}\, g(x)^c dx$$

can be determined from the baseline G distribution at least numerically.

The Shannon entropy can be obtained by limiting $c \uparrow 1$ in the last equation. However, it is easier to derive an expression for I_S from its definition. We have

$$\begin{aligned} I_S &= \alpha\ \log(\beta) - \log[\Gamma(\alpha)] + (\alpha-1)E\{\log[G(X)]\} \\ &+ (\alpha+1)\, E\{\log[1-G(X)]\} + 1 - (\beta+1)\, E[G(X)] + E\{\log[g(X)]\}. \end{aligned}$$

Using the series expansion for $E\{\log[1-G(X)]\}$, we obtain

$$\begin{aligned} I_S &= \alpha\ \log(\beta) - \log[\Gamma(\alpha)] + (\alpha-1)E\{\log[G(X)]\} \\ &- (\alpha+1)\sum_{r=1}^{\infty}\frac{E[G(X)]^r}{r} + 1 - (\beta+1)\, E[G(X)] + E\{\log[g(X)]\}. \end{aligned} \tag{6.11}$$

The three expectations in (6.11) can be evaluated numerically given $G(\cdot)$ and $g(\cdot)$. From equation (3.4), they are given by

$$E[G(X)^r] = \sum_{k,j=0}^{\infty} d_{k,j}\int_0^{\infty} G(x)^{\alpha+k+j-1}\, g(x)dx = \sum_{k,j=0}^{\infty}\frac{d_{k,j}}{\alpha+k+j},$$

$$E\{\log[G(X)]\} = \sum_{k,j=0}^{\infty} d_{k,j}\int_0^{1} u^{\alpha+k+j-1}\ \log(u)\, du = -\sum_{k,j=0}^{\infty}\frac{d_{k,j}}{(\alpha+k+j)^2}$$

and

$$E\{\log[g(X)]\} = \sum_{k,j=0}^{\infty}(\alpha+k+j)\, d_{k,j}\int_0^{\infty}\log[g(x)]\, G(x)^{\alpha+k+j-1}\, g(x)\, dx,$$

respectively. The last of these representations can also be expressed in terms of the baseline qf as

$$E\{\log[g(X)]\} = \sum_{k,j=0}^{\infty} (\alpha + k + j)\, d_{k,j} \int_0^1 \log\left[g\left(Q_G(u)\right)\right]\, u^{\alpha+k+j-1}\, du.$$

The last integral can be determined for most baseline distributions using a power series expansion for $Q_G(u)$.

6.6. Reliability. We derive the reliability, $R = \Pr(X_2 < X_1)$, when $X_1 \sim$GE-G$(\alpha_1, \beta_1, \theta)$ and $X_2 \sim$GE-G$(\alpha_2, \beta_2, \theta)$ are independent random variables. Probabilities of this form have many applications especially in engineering concepts. Let f_i denote the pdf of X_i and F_i denote the cdf of X_i. By using the representations (3.4) and (3.5), we can write after some algebra

$$R = \sum_{j,k,l,m=0}^{\infty} \frac{d_{j,k}^{(1)}\, d_{l,m}^{(2)}}{(\alpha_1 + \alpha_2 + k + j + l + m)},$$

where $d_{j,k}^{(1)}$ and $d_{l,m}^{(2)}$ are obtained from the coefficients in (3.4) with $\alpha = \alpha_1$ and $\alpha = \alpha_2$, respectively. In the very special case $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, we have $R = 1/2$.

6.7. Order statistics. Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1, \ldots, X_n$ is a random sample from the GE-G distribution. Let $X_{i:n}$ denote the ith order statistic. From equations (1.3) and (1.4), the pdf of $X_{i:n}$ can be expressed as

$$\begin{aligned} f_{i:n}(x) &= K \sum_{m=0}^{n-i} (-1)^m \binom{n-i}{m} f(x)\, F^{i+m-1}(x) \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)^2} \sum_{m=0}^{n-i} (-1)^m \binom{n-i}{m} \frac{G^{\alpha-1}(x)}{\overline{G}^{\alpha+1}(x)} \\ &\times \exp\left\{-\frac{\beta G(x)}{\overline{G}(x)}\right\} \gamma\left(\alpha, \beta \frac{G(x}{\overline{G}(x)}\right)^{i+m-1} g(x), \end{aligned}$$

where $K = n!/[(i-1)!\,(n-i)!]$. Using equations (3.3) and (6.4) and expanding the exponential function, we obtain

$$f_{i:n}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)^2} \sum_{r,p,j=0}^{\infty} \frac{(-\beta)^r\, \Gamma(r+p+j)\, d_{p,i:n}}{\Gamma(r+p)\, j!} h_{r+p+j+1}(x), \tag{6.12}$$

where

$$d_{p,i:n} = \sum_{m=0}^{n-i} (-1)^m \binom{n-i}{m} c_{i+m-1,p},$$

$h_{r+p+j+1}(x)$ is the Exp-G density function with power parameter $r + p + j + 1$ and $c_{i+m-1,p}$ is defined in equation (6.4) for $p \geq 1$ and $c_{i+m-1,0} = \alpha^{-(i+m-1)}$.

Equation (6.12) is the main result of this section. It reveals that the GE-G order statistics is a triple linear mixture of Exp-G density functions. So, several mathematical quantities for the GE-G order statistics like ordinary, incomplete and factorial moments, mgf, mean deviations and several others can be obtained from those quantities of the Exp-G distributions.

7. Maximum likelihood estimation

In this section, we determine the MLEs of the GE-G parameters from complete samples only. Let $x_1, \ldots, x_n$ be a random sample of size n from the GE-G$(\alpha, \beta, \boldsymbol{\tau}^T)^T$ distribution. The log-likelihood function for the vector of parameters $\boldsymbol{\theta} = (\alpha, \beta, \boldsymbol{\tau}^T)^T$ can be expressed as

$$\begin{aligned} l(\boldsymbol{\theta}) &= n\,t(\alpha,\beta) + (\alpha-1)\sum_{i=1}^{n}\log[G(x_i;\boldsymbol{\tau})] - (\alpha+1)\sum_{i=1}^{n}\log[1-G(x_i;\boldsymbol{\tau})] \\ &- \beta\sum_{i=1}^{n}\frac{G(x_i;\boldsymbol{\tau})}{[1-G(x_i;\boldsymbol{\tau})]} + \sum_{i=1}^{n}\log[g(x_i;\boldsymbol{\tau})], \end{aligned}$$

where $t(\alpha,\beta) = \alpha\log(\beta) - \log[\Gamma(\alpha)]$.

The score functions for the parameters α, β and $\boldsymbol{\tau}$ are given by

$$U_\alpha(\boldsymbol{\theta}) = n\log(\beta) - \psi(\alpha) + \sum_{i=1}^{n}\log[G(x_i;\boldsymbol{\tau})] - \sum_{i=1}^{n}\log[1-G(x_i;\boldsymbol{\tau})],$$

$$U_\beta(\boldsymbol{\theta}) = \frac{n\alpha}{\beta} - \sum_{i=1}^{n}\frac{G(x_i;\boldsymbol{\tau})}{[1-G(x_i;\boldsymbol{\tau})]},$$

$$\begin{aligned} U_{\boldsymbol{\tau}}(\boldsymbol{\theta}) &= (\alpha-1)\sum_{i=1}^{n}\frac{[\dot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}}}{G(x_i;\boldsymbol{\tau})} + (\alpha-1)\sum_{i=1}^{n}\frac{[\dot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}}}{[1-G(x_i;\boldsymbol{\tau})]} \\ &- \beta\sum_{i=1}^{n}\frac{[\dot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}}}{[1-G(x_i;\boldsymbol{\tau})]^2} + \sum_{i=1}^{n}\frac{[\dot{g}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}}}{g(x_i;\boldsymbol{\tau})}, \end{aligned}$$

where

$$[\dot{g}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}} = \frac{dg(x_i;\boldsymbol{\tau})}{d\boldsymbol{\tau}}, \qquad [\dot{G}(x_i;\boldsymbol{\tau})]_{\boldsymbol{\tau}} = \frac{dG(x_i;\boldsymbol{\tau})}{d\boldsymbol{\tau}},$$

and $\psi(\cdot)$ is the digamma function.

The MLE $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is determined by solving the nonlinear likelihood equations $U_\alpha(\boldsymbol{\theta}) = 0$, $U_\beta(\boldsymbol{\theta}) = 0$ and $U_{\boldsymbol{\tau}}(\boldsymbol{\theta}) = 0$. These equations cannot be solved analytically and suitable statistical software is required to solve them numerically. We can use iterative techniques such as a Newton-Raphson type algorithm to evaluate $\widehat{\boldsymbol{\theta}}$. We employ here the numerical NLMixed procedure in SAS.

Under general regularity conditions, the asymptotic distribution of $(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is $N_{p+2}(\mathbf{0}, I(\boldsymbol{\theta})^{-1})$, where $I(\boldsymbol{\theta})$ is the expected information matrix and p is the number of parameters of the baseline distribution given by the dimension of the vector $\boldsymbol{\tau}$. The multivariate normal $N_{p+2}(\mathbf{0}, J(\widehat{\boldsymbol{\theta}})^{-1})$ distribution, where $I(\boldsymbol{\theta})$ is replaced by $J(\widehat{\boldsymbol{\theta}})$, i.e., the observed information matrix evaluated at $\widehat{\boldsymbol{\theta}}$, can be used to construct approximate confidence intervals for the parameters.

8. Regression models

In many practical applications, the lifetimes are affected by explanatory variables like cholesterol level, blood pressure, weight and many others. Parametric models to estimate univariate survival functions and for censored data regression problems are widely used. A regression model that provides a good fit to lifetime data tends to yield more precise estimates of the quantities of interest.

Let X be a random variable having the pdf (1.4). A class of regression models for location and scale is characterized by the fact that the random variable $Y = \log(X)$ has a distribution with location parameter $\mu(\mathbf{v})$ dependent only on the explanatory variable vector $\mathbf{v}$ and scale parameter σ. Then, we can write

$$Y = \mu(\mathbf{v}) + \sigma Z,$$

where $\sigma > 0$ and Z has the distribution which does not depend on $\mathbf{v}$. The random variable Y (for $y \in \mathbb{R}$) has density function given by

$$f(y) = \frac{\beta^{\alpha}}{\sigma\Gamma(\alpha)} \frac{G^{\alpha-1}\left(\dfrac{y-\mu(\mathbf{v})}{\sigma}\right)}{\overline{G}^{\alpha+1}\left(\dfrac{y-\mu(\mathbf{v})}{\sigma}\right)} \exp\left\{-\frac{\beta G\left(\dfrac{y-\mu(\mathbf{v})}{\sigma}\right)}{\overline{G}\left(\dfrac{y-\mu(\mathbf{v})}{\sigma}\right)}\right\} g\left(\frac{y-\mu(\mathbf{v})}{\sigma}\right).$$

For illustrative purposes, let X be a random variable having the GE-W density function defined in Section 2.1. The random variable $Y = \log(X)$, re-parameterized in terms of $\mu = -\log(\delta)$ and $\sigma = \lambda^{-1}$, has density function given by

$$\begin{aligned} f(y) \quad &= \quad \frac{\beta^{\alpha}}{\sigma\,\Gamma(\alpha)} \frac{\exp\left[\left(\frac{y-\mu}{\sigma}\right) + \alpha\,\exp\left(\frac{y-\mu}{\sigma}\right)\right]}{\left\{1-\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right\}^{1-\alpha}} \\ &\times \quad \exp\left(\frac{-\beta\left\{1-\exp[-\exp\left(\frac{y-\mu}{\sigma}\right)]\right\}}{\exp[-\exp\left(\frac{y-\mu}{\sigma}\right)]}\right), \end{aligned} \tag{8.1}$$

where $\alpha > 0$ and $\beta > 0$ are shape parameters, $\mu \in \mathbb{R}$ is the location parameter and $\sigma > 0$ is the scale parameter.

We refer to equation (8.1) as the *log-gamma extended Weibull* (LGE-W) distribution, say $Y \sim$ LGE-W$(\alpha, \beta, \mu, \sigma)$. If $X \sim$ GE-W$(\alpha, \beta, \lambda, \delta)$, then $Y = \log(X) \sim$ LGE-W$(\alpha, \beta, \mu, \sigma)$. The survival function corresponding to (8.1) is given by

$$S(y) = \frac{1}{\Gamma(\alpha)}\, \gamma\left(\alpha, \beta\left\{\frac{1-\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]}{\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]}\right\}\right). \tag{8.2}$$

Next, we define the standardized random variable $Z = (Y-\mu)/\sigma$ with density function

$$f(z) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \frac{\exp\left[z + \alpha\,\exp(z)\right]}{\left\{1-\exp\left[-\exp(z)\right]\right\}^{1-\alpha}} \exp\left(\frac{-\beta\left\{1-\exp[-\exp(z)]\right\}}{\exp[-\exp(z)]}\right). \tag{8.3}$$

Further, we propose a linear location-scale regression model linking the response variable y_i and the explanatory variable vector $\mathbf{v}_i^T = (v_{i1}, \ldots, v_{ip})$ as follows

$$y_i = \mathbf{v}_i^T \boldsymbol{\tau} + \sigma\, z_i,\ i = 1, \ldots, n, \tag{8.4}$$

where the random error z_i has density function (8.3), $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_p)^T$, $\sigma > 0$, $\alpha > 0$ and $\beta > 0$ are unknown parameters. The parameter $v_i = \mathbf{v}_i^T\boldsymbol{\tau}$ is the location of y_i. The location parameter vector $\mathbf{v} = (v_1, \ldots, v_n)^T$ is represented by a linear model $\mathbf{v} = \mathbf{V}\boldsymbol{\tau}$, where $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)^T$ is a known model matrix. The LGE-W model (8.4) opens new possibilities for fitted many different types of data.

Consider a sample $(y_1, \mathbf{v}_1), \ldots, (y_n, \mathbf{v}_n)$ of n independent observations, where each random response is defined by $y_i = \min\{\log(x_i), \log(c_i)\}$. We assume non-informative censoring such that the observed lifetimes and censoring times are independent. Let F and C be the sets of individuals for which y_i is the log-lifetime or log-censoring, respectively. Conventional likelihood estimation techniques can be applied here. The log-likelihood function for the vector of parameters $\boldsymbol{\theta} = (\alpha, \beta, \sigma, \boldsymbol{\tau}^T)^T$ from model (8.4) has the form $l(\boldsymbol{\theta}) = \sum_{i \in F} l_i(\boldsymbol{\theta}) + \sum_{i \in C} l_i^{(c)}(\boldsymbol{\theta})$, where $l_i(\boldsymbol{\theta}) = \log[f(y_i)]$, $l_i^{(c)}(\boldsymbol{\theta}) = \log[S(y_i)]$,

$f(y_i)$ is the density (8.1) and $S(y_i)$ is the survival function (8.2). The total log-likelihood function for $\boldsymbol{\theta}$ reduces to

$$\begin{aligned} l(\boldsymbol{\theta}) &= r\,\alpha \log\beta - n\log\Gamma(\alpha) + \sum_{i\in F} z_i - (1-\alpha)\sum_{i\in F}\log\left\{1-\exp\left[-\exp\left(z_i\right)\right]\right\} \\ &\quad + \sum_{i\in F}\alpha\,\exp\left(z_i\right) + \sum_{i\in F}\left(\frac{-\beta\left\{1-\exp[-\exp\left(z_i\right)]\right\}}{\exp[-\exp\left(z_i\right)]}\right) \\ &\quad + \sum_{i\in C}\log\gamma\left(\alpha,\beta\left\{\frac{1-\exp\left[-\exp\left(z_i\right)\right]}{\exp\left[-\exp\left(z_i\right)\right]}\right\}\right), \end{aligned} \tag{8.5}$$

where r is the number of uncensored observations (failures). The MLE $\widehat{\boldsymbol{\theta}}$ of the vector of unknown parameters can be obtained by maximizing the log-likelihood (8.5). We use the NLMixed procedure in SAS to calculate the estimate $\widehat{\boldsymbol{\theta}}$. Initial values for $\boldsymbol{\beta}$ and σ are taken from the fit of the log-Weibull regression model with $\alpha = 0$ and $\beta = 1$.

The elements of the $(p+3)\times(p+3)$ observed information matrix $-\ddot{\mathbf{L}}(\boldsymbol{\theta})$, say $-\mathbf{L}_{\alpha\alpha}$, $-\mathbf{L}_{\alpha\beta}$, $-\mathbf{L}_{\alpha\sigma}$, $-\mathbf{L}_{\alpha\tau_j}$, $-\mathbf{L}_{\beta\beta}$, $-\mathbf{L}_{\beta\sigma}$, $-\mathbf{L}_{\beta\tau_j}$, $-\mathbf{L}_{\sigma\sigma}$, $-\mathbf{L}_{\sigma\tau_j}$ and $-\mathbf{L}_{\beta_j\beta_s}$ $(j, s = 1,\ldots,p)$ have to be evaluated numerically. Inference on $\boldsymbol{\theta}$ can be conducted based on the approximate multivariate normal $N_{p+3}(0, -\ddot{\mathbf{L}}(\widehat{\boldsymbol{\theta}})^{-1})$ distribution for $\widehat{\boldsymbol{\theta}}$. Further, we can use likelihood ratio statistics for comparing the LGW-LL model with some of its sub-models.

9. Simulation and applications

In this section, we use three real data sets to compare the fits of the EG distributions with others commonly used lifetime models. In each case, the parameters are estimated by maximum likelihood (Section 7) using the NLMixed subroutine in SAS. The goodness-of-fit statistics like Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC) and Bayesian Information Criterion (BIC) are adopted to compare the fitted models. The lower the values of these criteria are, the better the fit.

Note that over-parametrization is penalized in these criteria, so that the two additional parameters in the EG model do not necessarily lead to smaller values of these statistics. The performance of the MLEs are also investigated by a simulation study in this section.

9.1. Simulation. We simulate the GE-N($\alpha = 0.5,\ 1.5, \beta = 0.5,\ 1.5, \mu = 0, \sigma = 1$) distribution from equation (6.1) by using a random variable U having a uniform distribution in $(0, 1)$. We simulate n= 50, 150 and 300 variates and, for each replication, we evaluate the MLEs $\hat{\alpha}$, $\hat{\beta}$, $\hat{\mu}$ and $\hat{\sigma}$. We repeat this process $1,000$ times and determine the average estimates (AEs), biases and means squared errors (MSEs). The results are reported in Table 1.

The results of the simulations in Table 1 indicate that the MSEs of the AEs of α, β, μ and σ decay toward zero as the sample size increases, as expected under first-order asymptotic theory. The mean estimates of the parameters tend to be closer to the true parameter values when n increases. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the MLEs. The normal approximation can be oftentimes improved by making bias adjustments to these estimators. Approximations to these biases in simple models may be obtained analytically. Bias correction typically does a very good job for correcting the MLEs. However, it may either increase the MSEs. Whether bias correction is useful in practice depends basically on the shape of the bias function and on the variance of the MLE. In order to improve the accuracy of the MLEs using analytical bias reduction one needs to obtain several cumulants of log likelihood derivatives which are notoriously cumbersome for the proposed model.

Table 1. The AEs, biases and MSEs based on 1,000 simulations of the GE-N distribution when $\alpha = 0.5$, 1.5, $\beta = 0.5$, 1.5, μ=0 and σ=1, and n=50, 150 and 300.

n	Parameter	$\alpha = 0.5 \quad \beta = 0.5$ AE	Bias	MSE	Parameter	$\alpha = 1.5 \quad \beta = 0.5$ AE	Bias	MSE
50	α	0.5899	0.0899	1.1686	α	2.4349	0.9349	59.4752
	β	0.7696	0.2696	15.7511	β	1.3614	0.8614	23.5048
	μ	-0.1785	-0.1785	1.1247	μ	0.0648	0.0648	1.6424
	σ	0.9219	-0.0781	0.4925	σ	0.9647	-0.0353	0.8442
150	α	0.5301	0.0301	0.4575	α	1.4681	-0.0319	1.5254
	β	0.5567	0.0567	1.3133	β	0.6290	0.1290	0.5052
	μ	-0.0826	-0.0826	0.5164	μ	0.0379	0.0379	0.6298
	σ	0.9528	-0.0472	0.2200	σ	0.9431	-0.0569	0.1798
300	α	0.5251	0.0251	0.1235	α	1.3811	-0.1189	0.4938
	β	0.5095	0.0095	0.1383	β	0.5443	0.0443	0.1341
	μ	-0.0653	-0.0653	0.3011	μ	0.0472	0.0472	0.3334
	σ	0.9851	-0.0149	0.1130	σ	0.9392	-0.0608	0.0833
		$\alpha = 0.5 \quad \beta = 1.5$				$\alpha = 1.5 \quad \beta = 1.5$		
50	α	1.6051	1.1051	16.8536	α	3.8309	2.3309	152.9058
	β	5.1946	3.6946	370.4755	β	5.1753	3.6753	280.2779
	μ	-0.1195	-0.1195	2.9737	μ	-0.1236	-0.1236	2.2642
	σ	1.2683	0.2683	2.1531	σ	1.1595	0.1595	1.5969
150	α	0.7366	0.2366	0.9104	α	1.9205	0.4205	11.8999
	β	1.8741	0.3741	5.8935	β	2.1430	0.6430	15.0843
	μ	-0.1750	-0.1750	0.7025	μ	-0.0871	-0.0871	0.7450
	σ	1.0410	0.0410	0.4205	σ	1.0302	0.0302	0.3466
300	α	0.6822	0.1822	0.8578	α	1.6240	0.1240	1.5137
	β	1.6979	0.1979	2.9542	β	1.6953	0.1953	1.6989
	μ	-0.1514	-0.1514	0.5375	μ	-0.0615	-0.0615	0.4016
	σ	1.0441	0.0441	0.2981	σ	0.9998	-0.0002	0.1611

9.2. Application 1: Carbon monoxide data. For the GE-W model given by (2.1) we work with carbon monoxide (CO) measurements made in several brands of cigarettes in 1998. The data have been collected by the Federal Trade Commission (FTC), an independent agency of the United States Government, whose main mission is the promotion of consumer protection. For three decades the FTC regularly has released reports on the nicotine and tar content of cigarettes. The reports indicate that nicotine levels, on average, had remained stable since 1980, after falling in the preceding decade. The data include records of measurements of CO contents, in milligrams, in cigarettes of several brands. The CO data set can be found at `http://home.att.net/ rdavis2/cigra.html`.

Recently, various authors developed more properties and applications of the beta Weibull (BW) (Famoye *et al.*, 2005) distribution in survival analysis and reliability, for example, Ortega *et al.* (2011) introduced the log-beta Weibull regression model based on the BW distribution, Cordeiro *et al.* (2013) studied some mathematical properties of the BW distribution and Ortega *et al.* (2012) proposed the negative binomial-beta Weibull regression model for studying the recurrence of prostate cancer and to predict the cure fraction for patients with clinically localized prostate cancer treated by open radical prostatectomy. The four-parameter BW distribution (Famoye *et al.*, 2005) ($a > 0$, $b > 0$, $\alpha > 0$ and $\gamma > 0$) has density function given by (for $x > 0$)

$$f(x) = \frac{\gamma(1/\alpha)^{\gamma}}{B(a,b)} x^{\gamma-1} \exp\{-b(x/\alpha)^{\gamma}\}[1 - \exp\{-(x/\alpha)^{\gamma}\}]^{a-1}, \tag{9.1}$$

where a and b are two extra shape parameters to the Weibull distribution to govern skewness and kurtosis. Another important characteristic of the BW distribution is that it contains, as special models, the exponentiated exponential (Gupta and Kundu, 1999) (for $b = \gamma = 1$) and exponentiated Weibull (EW) (Mudholkar *et al.*, 1996) (for $b = 1$) distributions and some other distributions.

Next, we fit the GE-W, BW, EW and Weibull models to the CO data. The MLEs of the parameters (the standard errors are given in parentheses) and the values of the AIC, CAIC and BIC statistics are listed in Table 2, thus indicating that the GE-W gives a better fit than the Weibull model. Comparing the statistics in Tables 2, we can verify that the GE-W distribution is a very competitive model to lifetime data. In fact, the GE-W distribution is a very good alternative model to the BW and EW distributions. Since the values of the statistics above are smaller for the GE-W distribution compared to those values of the other models, the new distribution is a more suitable model to explain these data.

Table 2. MLEs of the GE-W and Weibull models and some criteria for the carbon monoxide data.

Model	α	β	λ	δ	AIC	CAIC	BIC
GE-W	0.7301	0.000517	0.5985	2.0548	**1922.7**	**1922.8**	**1938.1**
	(0.3565)	(0.000367)	(0.3597)	(1.1734)			
Weibull	1	1	2.9783	0.07959	1993.4	1993.4	2001.1
	-	-	(0.1345)	(0.001494)			
	a	b	α	γ	AIC	CAIC	BIC
BW	0.3081	0.06441	7.5373	4.5942	1930.1	1930.2	1945.5
	(0.04337)	(0.02214)	(0.7844)	(0.2797)			
EW	0.3381	1	15.6257	6.2558	1947.5	1947.6	1959.0
	(0.02127)	-	(0.2489)	(0.1130)			

More information is provided by a visual comparison of the histogram of the data with the fitted density functions. The plots of the fitted GE-W and Weibull density functions are displayed in Figure 8a. In order to assess if the model is appropriate, the plots of the fitted GE-W and Weibull cumulative distributions and the empirical cdf are displayed in Figure 8b. We conclude that the GE-W distribution provides a good fit to these data.

9.3. Application 2: Fibre data. In this section, we use an uncensored data set on the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. For more details, see, for example, Smith and Naylor (1987) and Cordeiro and Lemonte (2011). Unfortunately, the units of measurement are not given in the paper. For the fibre data, we compare the fitted GE-LN, GE-LL, LN and LL distributions. An alternative model to these data is the Birnbaum-Saunders (BS) distribution. There are various extensions of this lifetime distribution.

More recently, Cordeiro and Lemonte (2011) proposed the β-Birnbaum-Saunders (β-BS) distribution for fatigue life modeling. They investigated various properties of the β-BS model including expansions for the moments, generating function, mean deviations, density function of the order statistics and their moments. The pdf of β-BS and BS distribution are given by

$$f(x) = \frac{\kappa(\alpha,\beta)}{B(a,b)}\, x^{-3/2}\,(x+\beta)\,\exp\left\{-\tau(x/\beta)/(2\alpha^2)\right\}\,\Phi(v)^{a-1}\left\{1-\Phi(v)\right\}^{b-1}, \quad x > 0,$$

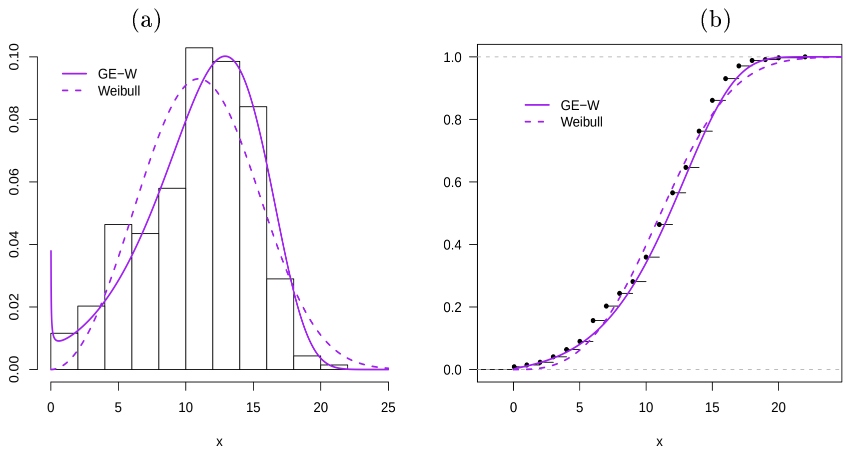

Figure 8. (a) Fitted GE-W and Weibull densities for the CO data. (b) Estimated GE-W and Weibull cumulative distributions for the CO data.

and

$$f(x) = \kappa(\alpha, \beta)\, x^{-3/2}\, (x + \beta)\, \exp\{-\tau(x/\beta)/(2\alpha^2)\}, \qquad x > 0,$$

respectively.

The MLEs of the parameters (with standard errors) and the AIC, CAIC and BIC measures for the models are displayed in Table 3. Since the values of these statistics are smaller for the GE-LN and GE-LL distributions compared to those values of other models, the distributions in the new family seem to be more adequate models to explain these data. Then, these distributions can be considered very good alternative models to the β-BS distribution.

More information is provided by a visual comparison of the the fitted density functions to the histogram of the data. The plots of the fitted GE-LN and LN density functions are displayed in Figure 9a. Similarly, the plots of the fitted GE-LL and LL density functions are given in Figure 9b. We conclude that the GE-LN and GE-LL distributions can provide good fits to these data. Overall, the GE-LN model is the best choice.

9.4. Application 3: Voltage data. Lawless (2003) reported an experiment in which specimens of solid epoxy electrical-insulation were studied in an accelerated voltage life test. The sample size is $n = 60$, the percentage of censored observations is 10% and are considered three levels of voltage 52.5, 55.0 and 57.5. The variables involved in the study are: x_i - failure times for epoxy insulation specimens (in min); c_i - censoring indicator (0 =censoring, 1 =lifetime observed); v_{i1} - voltage (kV).

Now, we present results by fitting the model

$$y_i = \tau_0 + \tau_1 v_{i1} + \sigma z_i,$$

where the random variable Y_i follows the LGE-W distribution given in (8.1). The MLEs of the model parameters, the asymptotic standard errors of these estimates and the values of the measures AIC, CAIC and BIC to compare the LGE-W and log-Weibull (LW) regression models are listed in Table 4.

From the figures in Table 4, we conclude that the fitted LGE-W regression model has the lowest AIC, CAIC and BIC values compared with those values of the fitted LW

Table 3. MLEs of the model parameters for the fibres data and information criteria.

Model	α	β	μ	σ	AIC	CAIC	BIC
GE-LN	1.5634	0.03054	-0.8056	0.6007	35.0	35.9	42.7
	(0.6231)	(0.0190)	(0.3276)	(0.2167)			
LN	1	1	0.3347	0.2657	47.7	47.9	51.6
	-	-	(0.03721)	(0.02631)			
	α	β	δ	λ	AIC	CAIC	BIC
GE-LL	0.8439	1.4517	1.7321	5.7819	35.4	36.2	43.1
	(0.4095)	(0.1541)	(0.1762)	(1.7454)			
LL	1	1	1.4566	7.5388	40.8	41.1	44.7
	-	-	(0.04564)	(0.9255)			
	a	b	α	β	AIC	CAIC	BIC
β-BS	0.3638	7857.5658	1.0505	30.4783	37.5	38.4	45.3
	(0.0709)	(3602.2)	(0.0101)	(0.5085)			
BS	1	1	0.2699	1.3909	48.4	48.6	52.2
	-	-	(0.0267)	(0.0521)			

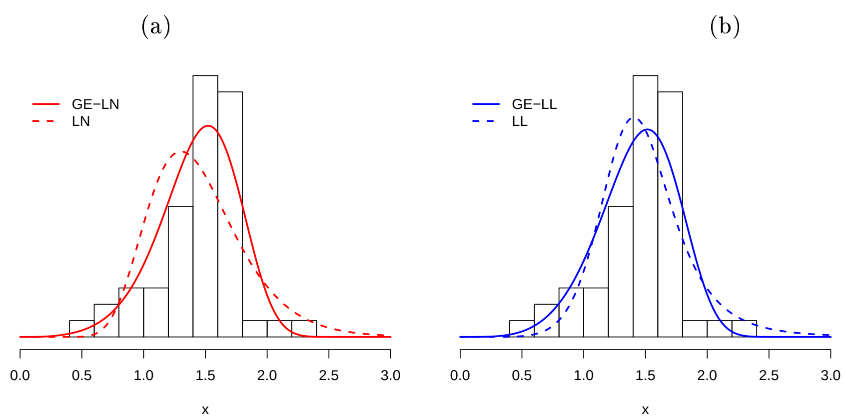

Figure 9. For the fibre data: (a) Fitted GE-LN and LN pdf. (b) Fitted GE-LL and LL pdf.

model. Figure 10 provides the plots of the estimated survival function and estimated cdf of the LGE-W distribution. These plots indicate this regression model provides a good fit to these data.

10. Conclusions

We propose a new gamma generated family of distributions with two additional parameters, which can include as special cases several classical continuous distributions. For any parent continuous distribution G, we can define the corresponding gamma extended-G ("GE-G") class with two extra positive parameters. We give some of its special models. We demonstrate that its density function is a linear mixture of exponentiated G densities. Explicit expressions for the ordinary and incomplete moments, generating and

Table 4. MLEs of the parameters to the voltage data, the corresponding SEs (given in parentheses), p-values in [·] and the statistics AIC, CAIC and BIC.

Model	α	β	σ	τ_0	τ_1	AIC	CAIC	BIC
LGE-W	81.891	298.900	9.320	31.859	-0.219	168.6	169.8	179.1
	(0.081)	(33.045)	(0.778)	(3.730)	(0.058)			
				[<0.0001]	[0.0003]			
LW	-	-	0.845	22.000	-0.274	173.4	173.8	179.7
			(0.090)	(3.046)	(0.055)			
				[<0.0001]	[<0.0001]			

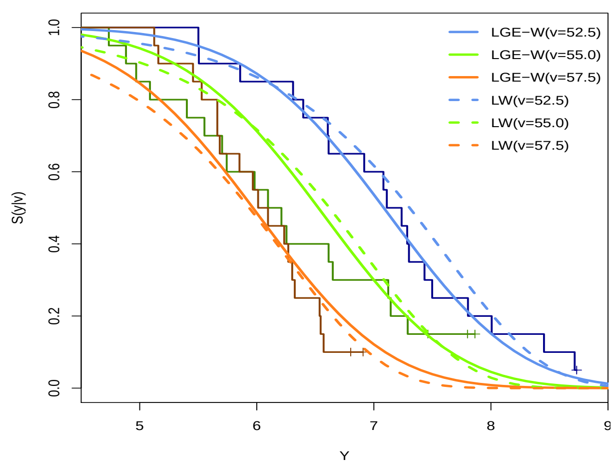

Figure 10. Estimated LGE-W and LW survival function and empirical survival for the voltage data.

quantile functions, mean deviations, Bonferroni and Lorenz curves, probability weighted moments, Shannon and Rényi entropies, reliability and order statistics are derived for any GE-G distribution. The model parameters are estimated by maximum likelihood and we investigate the accuracy of the estimators through a simulation study. We propose a new regression model based on the logarithm of the GE-G distribution. The usefulness of the new models is illustrated using classical goodness-of-fit statistics by means of three real data sets The proposed models provide a rather flexible mechanism for fitting a wide spectrum of real data sets.

References

[1] Aas, K., and I. Haff (2006). The Generalized Hyperbolic Skew Student's t-Distribution. *Journal of Financial Econometrics*, **4**, 275-309.

[2] Alexander, C., Cordeiro, G.M., Ortega, E.M.M. and Sarabia, J.M. (2012). Generalized beta-generated distributions. *Computational Statistics and Data Analysis*, **56**, 1880-1897.

[3] Cordeiro, G.M. and de Castro, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, **81**, 883-898.

[4] Cordeiro, G.M. and Lemonte, A.J. (2011). The β-Birnbaum-Saunders distribution: An improved distribution for fatigue life modeling. *Computational Statistics and Data Analysis*, **55**, 1445-1461.

[5] Cordeiro, G.M., Nadarajah, S. and Ortega, E.M.M. (2013). General results for the beta Weibull distribution. *Journal of Statistical Computation and Simulation*, **83**, 1082-1114.

[6] Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics - Theory and Methods*, **31**, 497-512.

[7] Famoye, F., Lee, C., Olumolade, O. (2005). The beta-Weibull distribution. *Journal of Statistical Theory and Applications*, **4**, 121-136.

[8] Gradshteyn, I.S. and Ryzhik, I.M. (2000). *Table of Integrals, Series, and Products*, sixth edition. Academic Press, San Diego.

[9] Gupta, R.D., Kundu, D. (1999). Generalized exponential distributions. *Australian and New Zealand Journal of Statistics*, **41**, 173-188.

[10] Hansen, B.E. (1994). Autoregressive conditional density estimation. *International Economic Review*, **35**, 705-730.

[11] Johnson, N.L., Kotz, S., Balakrishnan, N. (1994). *Continuous Univariate Distributions.* Volume 1, 2nd edition. John Wiley and Sons, New York.

[12] Johnson, N.L., Kotz, S., Balakrishnan, N. (1995). *Continuous Univariate Distributions.* Volume 2, 2nd edition. John Wiley and Sons, New York.

[13] Kakde, C.S and Shirke, D.T. (2006). On Exponentiated Lognormal distribution. *International Journal of Agricultural and Statistics Sciences*, **2**, 319-326.

[14] Leadbetter, M.R., Lindgren, G. and Rootzeén, H. (1987). *Extremes and Related Properties of Random Sequences and Processes.* New York: Springer.

[15] Lawless, J. F. (2003). Statistical models and methods for lifetime data.

[16] Mudholkar, G.S., Srivastava, D.K. (1993). Exponentiated Weibull family for analyzing bathtub failure-real data. *IEEE Transaction on Reliability*, **42**, 299-302.

[17] Mudholkar, G.S., Srivastava, D.K., Kollia, G.D. (1996). A generalization of the Weibull distribution with application to the analysis of survival data. *Journal of American Statistical Association*, **91**, 1575-1583.

[18] Nadarajah, S., Cordeiro, G.M., Ortega, E.M.M. (2015) The Zografos-Balakrishnan-G family of distributions: Mathematical properties and applications. *Communications in Statistics - Theory and Methods*, **44**, 186-215.

[19] Nadarajah, S. (2005). The exponentiated Gumbel distribution with climate application. *Environmetrics*, **17**, 13-23.

[20] Nadarajah, S., Gupta, A.K. (2007). The exponentiated gamma distribution with application to drought data. *Calcutta Statistical Association Bulletin*, **59**, 29-54.

[21] Nadarajah, S., Kotz, S. (2006). The exponentiated type distributions. *Acta Applicandae Mathematicae*, **92**, 97-111.

[22] Ortega, E.M.M., Cordeiro, G.M. and Hashimoto, E.M. (2011). A log-linear regression model for the beta-Weibull distribution. *Communications in Statistics-Simulation and Computation*, **40**, 1206-1235.

[23] Ortega, E.M.M., Cordeiro, G.M. and Kattan, M.W. (2012). The negative binomial-beta Weibull regression model to predict the cure of prostate cancer. *Journal of Applied Statistics*, **39**, 1191-1210.

[24] Prudnikov, A.P., Brychkov, Y.A., Marichev, O.I. (1986). *Integrals and Series*, volumes 1, 2 and 3. Gordon and Breach Science Publishers, Amsterdam.

[25] Ristic, M.M. and Balakrishnan, N. (2012). The gamma exponentiated exponential distribution. *Journal of Statistical Computation and Simulation*, **82**, 1191-1206.

[26] Smith, R.L. and Naylor, J.C. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. *Applied Statistics*, **36**, 358-369.

[27] Torabi, H. and Hedesh, N.M. (2012). The gamma-uniform distribution and its applications. *Kybernetika*, **48**, 16-30.

[28] Wright, E.M. (1935). The asymptotic expansion of the generalized hypergeometric function. *Proceedings of the London Mathematical Society*, **10**, 286-293.

[29] Zografos, K. and Balakrishnan, N. (2009). On families of beta- and generalized gamma-generated distributions and associated inference. *Statistical Methodology*, **6**, 344-362.