

## AN ANALYSIS OF A FETI–DP ALGORITHM ON IRREGULAR SUBDOMAINS IN THE PLANE\*

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**Abstract.** In the theory for domain decomposition algorithms of the iterative substructuring family, each subdomain is typically assumed to be the union of a few coarse triangles or tetrahedra. This is an unrealistic assumption, in particular if the subdomains result from the use of a mesh partitioner, in which case they might not even have uniformly Lipschitz continuous boundaries. The purpose of this study is to derive bounds for the condition number of these preconditioned conjugate gradient methods which depend only on a parameter in an isoperimetric inequality, two geometric parameters characterizing John and uniform domains, and the maximum number of edges of any subdomain. A related purpose is to explore to what extent well-known technical tools previously developed for quite regular subdomains can be extended to much more irregular subdomains. Some of these results are valid for any John domain, while an extension theorem, which is needed in this study, requires that the subdomains have complements which are uniform. The results, so far, are complete only for problems in two dimensions. Details are worked out for a FETI–DP algorithm and numerical results support the findings. Some of the numerical experiments illustrate that care must be taken when selecting the scaling of the preconditioners in the case of irregular subdomains.

**Key words.** domain decomposition, preconditioners, iterative substructuring, dual-primal FETI, John and uniform domains, fractal subdomains

**AMS subject classifications.** 65F10, 65N30, 65N55

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**1. Introduction.** In the theory of domain decomposition methods of iterative substructuring type, we typically assume that each subdomain is quite regular, e.g., the union of a small set of coarse triangles or tetrahedra; see, e.g., [36, Assumption 4.3]. However, such an assumption is unlikely to hold especially if the subdomains result from using a mesh partitioner, in which case the subdomain boundaries might not even be uniformly Lipschitz continuous, i.e., the number of patches which cover  $\partial\Omega$ , and in each of which the boundary is the graph of a Lipschitz continuous function, might not be uniformly bounded independently of the finite element mesh size. We also note that the shape of the subdomains is likely to change if the mesh size is altered and a mesh partitioner is used. The purpose of this paper is to develop a theory for domain decomposition methods under much weaker assumptions on the partitioning and to categorize the rate of convergence in terms of a few geometric parameters. We will denote the nonoverlapping subdomains by  $\Omega_i$  and the interface between them by  $\Gamma$ .

So far, complete results have been obtained only for problems in the plane. To simplify our presentation, we will also focus on scalar elliptic problems of the following

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form:

$$(1.1) \quad -\operatorname{div}(\rho(x)\nabla u(x)) = f(x) \quad x \in \Omega,$$

with a homogeneous Dirichlet boundary condition on a measurable part,  $\partial\Omega_D$ , of  $\partial\Omega$ , the boundary of  $\Omega$ , and a Neumann boundary condition on its complement  $\partial\Omega_N$ . The coefficient  $\rho(x)$  is strictly positive and assumed to be a constant  $\rho_i$  for  $x \in \Omega_i$ . As for many other algorithms, our results hold equally well for compressible elasticity problems in two dimensions; see [36, Chapter 8] for a discussion, [24] for further details on FETI-DP for linear elasticity, and the discussion below.

We use linear continuous finite elements and triangulations with shape regular triangles, i.e., the diameter of an element is bounded uniformly by a constant times the radius of the largest inscribed circle. We assume that each subdomain is a union of elements. We denote the finite element space on  $\Omega$  by  $W^h(\Omega)$  and assume that all elements of  $W^h(\Omega)$  vanish on  $\partial\Omega_D$ .

The analysis will be carried out for a FETI-DP algorithm but the results immediately carry over to the corresponding BDDC algorithm (see, e.g., [26]) as well as to a class of inexact FETI-DP methods; see [22]. The tools that we will develop can also be used in the analysis of a number of other domain decomposition algorithms, in particular, those of [12, 13, 14]; our study originated within this collaborative project with Clark Dohrmann. For a collection of auxiliary results used in the analysis of iterative substructuring algorithms, in the case of regular subdomains, see [36, section 4.6].

Thus, our study requires the generalization of some technical tools used in the proof of bounds of the convergence rate of the FETI-DP algorithm. We also have to modify the basic line of reasoning in the proof of our main result; a proof for triangular subdomains and constant coefficients was first given in [29]; see also [32, Chapter 2] for a proof more similar to ours. Four auxiliary results, namely a Poincaré inequality, a Sobolev-type inequality for finite element functions, bounds for certain edge terms, and a finite element extension theorem, will be required in our proof; see Lemmas 2.3, 4.3, 4.4, and 4.5. We will work with *John domains* and *uniform domains* (see Definitions 2.1 and 2.4); the latter are also known as Jones or  $(\varepsilon, \infty)$ -domains. We will express our bounds on the rate of convergence of our algorithm in terms of a few parameters identified in Definitions 2.1 and 2.4 and Lemma 2.2.

We will also consider the problem of linear, isotropic elasticity in two dimensions:

$$(1.2) \quad -\operatorname{div}(2\mu\varepsilon(\mathbf{u}) + \lambda\operatorname{tr}(\varepsilon(\mathbf{u}))I) = f \quad \text{in } \Omega \subset \mathbb{R}^2.$$

We assume zero Dirichlet conditions on one part of the boundary of  $\partial\Omega$  with non-vanishing measure and traction conditions on the remaining part of the boundary. FETI-DP methods have been analyzed for linear elasticity and regular subdomains in [24]; see [21] and [23] for numerical results.

A crucial tool in any such study is the second Korn inequality, Lemma 2.6; for a proof for uniform domains, see Durán and Muschietti [16], and for John domains, see Acosta, Durán, and Muschietti [1]. The proof in [16] has many details in common with Jones' proof of extension theorems for Sobolev spaces, [20], and the constant in the inequality depends only on the same geometric parameters as the result of Lemma 2.5.

The remainder of this article is organized as follows. In section 2, we provide a Poincaré inequality for general domains and introduce John and uniform domains. In section 3, we describe our FETI-DP algorithm. In section 4, we develop a convergence theory for our FETI-DP algorithm for subdomains with complements which

are uniform. Finally, in section 5, we present numerical results which confirm our theoretical findings. We note that we find a striking difference in the performance of the iterative method between the scaling based on our theory and another based on diagonal elements of the stiffness matrix in cases when the subdomain boundaries are highly irregular.

**2. A Poincaré inequality, John and uniform domains.** We will first introduce John domains and then consider a Poincaré inequality for such domains. Subsequently, we will introduce uniform domains; the latter are needed in order to prove a finite element extension theorem. We note that this final assumption on the subdomains is not required in our analysis of domain decomposition algorithms based on overlapping subdomains; see [14, 13, 12].

We next give the definition of a John domain; see [19] and the references therein. In the proofs of several of our auxiliary results, we will assume that the subdomains belong to this class.

**DEFINITION 2.1** (John domain). *A domain  $\Omega \subset \mathbb{R}^n$ —an open, bounded, and connected set—is a John domain if there exists a constant  $C_J \geq 1$  and a distinguished central point  $x_0 \in \Omega$  such that each  $x \in \Omega$  can be joined to it by a rectifiable curve  $\gamma : [0, 1] \rightarrow \Omega$  with  $\gamma(0) = x_0$ ,  $\gamma(1) = x$ , and  $|x - \gamma(t)| \leq C_J \cdot \text{distance}(\gamma(t), \partial\Omega)$  for all  $t \in [0, 1]$ .*

This condition can be viewed as a twisted cone condition. We note that certain snowflake curves with fractal boundaries are John domains (see section 5) and that the boundary of a John domain can be arbitrarily much longer than its diameter.

We note that for any choice of the point  $x_0$ , there is a point  $x \in \Omega$  at a distance of at least  $\text{diameter}(\Omega)/2$ . We find that  $\text{diameter}(\Omega) \leq 2C_J r_\Omega$ , where  $r_\Omega$  is the radius of the largest ball inscribed in  $\Omega$  and centered at  $x_0$ . Conditions on the boundary are of course also imposed.

In any analysis of any domain decomposition method with a second, coarse level, we need a Poincaré inequality. This inequality is closely related to an isoperimetric inequality. The next lemma is due to Maz'ja [30] and Federer and Fleming [18]; see also Lin and Yang [27, Theorem 5.3.2] or Maz'ja [31].

**LEMMA 2.2** (isoperimetric inequality). *Let  $\Omega \subset \mathbb{R}^n$  be a domain, and let  $u$  be sufficiently smooth. Then,*

$$\inf_{c \in \mathbb{R}} \left( \int_{\Omega} |u - c|^{n/(n-1)} dx \right)^{(n-1)/n} \leq \gamma(\Omega, n) \int_{\Omega} |\nabla u| dx$$

if and only if

$$(2.1) \quad [\min(|A|, |B|)]^{1-1/n} \leq \gamma(\Omega, n) |\partial A \cap \partial B|.$$

Here,  $A \subset \Omega$  is an arbitrary open set, and  $B = \Omega \setminus \bar{A}$ ;  $\gamma(\Omega, n)$  is the best possible constant, and  $|A|$  is the measure of the set  $A$ , etc.

We note that the domain does not need to be star-shaped or Lipschitz. For  $n = 2$ , the best choice of  $c = \bar{u}_\Omega$ , the average of  $u$  over the domain. A small value of  $\gamma(\Omega, n)$  is desirable for our purposes.

It is known that any simply connected plane domain with a finite Poincaré parameter  $\gamma(\Omega, 2)$  is a John domain; see [9]. It is also known (see [6]) that any John domain has a bounded Poincaré parameter  $\gamma(\Omega, n)$ . Since we have been unable to find a simple estimate of  $\gamma(\Omega, n)$  in terms of  $C_J$  in the literature, we have chosen to express our results in terms of both of these geometric parameters. In two dimensions, we

immediately obtain the standard Poincaré inequality by using the Cauchy–Schwarz inequality.

LEMMA 2.3 (Poincaré’s inequality). *Let  $\Omega$  be a John domain in the plane. Then,*

$$\|u - \bar{u}_\Omega\|_{L_2(\Omega)}^2 \leq (\gamma(\Omega, 2))^2 |\Omega| \|\nabla u\|_{L_2(\Omega)}^2 \quad \forall u \in H^1(\Omega).$$

For  $n = 3$  such a bound is obtained by using Hölder’s inequality several times. In Lemma 2.3, we then should replace  $|\Omega|$  by  $|\Omega|^{2/3}$ .

Throughout, we will use a weighted  $H^1(\Omega_i)$ -norm defined by

$$\|u\|_{H^1(\Omega_i)}^2 := \int_{\Omega_i} \nabla u \cdot \nabla u \, dx + 1/H_i^2 \int_{\Omega_i} |u|^2 \, dx = |u|_{H^1(\Omega_i)}^2 + 1/H_i^2 \int_{\Omega_i} |u|^2 \, dx.$$

Here  $H_i$  is the diameter of  $\Omega_i$ . The weight for the  $L_2$ -term results from the standard  $H^1$ -norm on a domain with diameter one and a dilation. We will use Lemma 2.3 to remove  $L_2$ -terms in our estimates.

We finally consider uniform domains also known as  $(\epsilon, \infty)$  domains. It is known, and easy to see, that any uniform domain is a John domain. It is also easy to construct John domains that are not uniform domains. According to Jones [20, Theorem 4], they form the largest class of finitely connected domains for which an extension theorem holds in two dimensions; see Lemma 2.5 below. According to [20, Theorem 2], every uniform domain in  $\mathbb{R}^n$  allows for a bounded extension with respect to the seminorm of  $H^1$ . It also follows from [20, Theorem C] that if a simply connected domain, with a simply connected complement, is uniform, so is its complement.

DEFINITION 2.4 (uniform domains). *A domain  $\Omega \subset \mathbb{R}^n$  is a uniform domain if there exists a constant  $C_U$  such that any pair of points  $x_1 \in \Omega$  and  $x_2 \in \Omega$  can be joined by a rectifiable curve  $\gamma(t) : [0, 1] \rightarrow \Omega$  with  $\gamma(0) = x_1$ ,  $\gamma(1) = x_2$ , and where the Euclidean arc length of  $\gamma \leq C_U |x_1 - x_2|$  and  $\min_{i=1,2} |x_i - \gamma(t)| \leq C_U \cdot \text{distance}(\gamma(t), \partial\Omega)$  for all  $t \in [0, 1]$ .*

Remark 1. The complement of a bounded set is of course always infinite. What matters in our work, with any uniform domain, is its boundary. We can therefore, whenever required, remove the part of an infinite domain that is outside a sufficiently large circle and effectively work only with bounded domains.

We note that domains with boundaries formed by von Koch curves, as in section 5, are also uniform domains.

We will use a main result for uniform domains; see Jones [20, Theorem 2].

LEMMA 2.5. *Let  $\Omega \subset \mathbb{R}^n$  be a uniform domain, and let  $P_0$  be the space of constants. There then exists a bounded, linear operator  $E_\Omega : H^1(\Omega)/P_0 \rightarrow H^1(\mathbb{R}^n)/P_0$ , which extends any element in  $H^1(\Omega)$  to one defined for all of  $\mathbb{R}^n$ , i.e.,  $(E_\Omega u)|_\Omega = u$  for all  $u \in H^1(\Omega)/P_0$ . The norm of this operator depends only on  $C_U(\Omega)$ .*

As previously noted, for a result on linear elasticity, we also need a second Korn inequality for uniform domains; see Durán and Muschietti [16, Corollary 2.8].

LEMMA 2.6 (Korn inequality for uniform domains). *Let  $\Omega \subset \mathbb{R}^n$  be a bounded uniform domain. Then, there exists a constant  $C > 0$ , which depends only on the uniformity constant  $C_U(\Omega)$  and the dimension  $n$ , such that*

$$|\mathbf{u}|_{H^1(\Omega)}^2 \leq C \|\varepsilon(\mathbf{u})\|_{L_2(\Omega)}^2$$

for all  $\mathbf{u} \in \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \int_\Omega (\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}) \, dx = 0, i, j = 1, \dots, n\}$ .

**3. The FETI–DP algorithm.** Let a domain  $\Omega \subset \mathbb{R}^2$  be decomposed into  $N$  nonoverlapping subdomains  $\Omega_i$  of diameter  $H_i$ , each of which is a union of finite

elements with the finite element nodes on the boundaries of neighboring subdomains matching across the interface  $\Gamma := \bigcup_{i \neq j} \partial\Omega_i \cap \partial\Omega_j$ . Here  $\partial\Omega_i$  and  $\partial\Omega_j$  are the boundaries of  $\Omega_i$  and  $\Omega_j$ , respectively. The interface  $\Gamma$  is the union of edges and vertices. The nodes on an edge are shared by exactly two subdomains, and the edges are open subsets of  $\Gamma$ . The vertices are endpoints of the edges. For a more detailed definition of faces, edges, and vertices in two and three dimensions, see [24, section 3] and [21, section 2].

For each subdomain  $\Omega_i$ ,  $i = 1, \dots, N$ , we assemble the subdomain stiffness matrices  $K^{(i)}$  and load vectors  $f^{(i)}$  from the contributions of the individual elements. We denote the vector of unknowns of the subdomain  $\Omega_i$  by  $u^{(i)}$  and then partition the unknowns of  $u^{(i)}$  into a vector of primal variables  $u_{\Pi}^{(i)}$  and a vector of nonprimal variables  $u_B^{(i)}$ . We can obtain fast convergence by choosing only vertices as primal variables, since we are considering only two dimensional problems; see [17, 29, 32]. To simplify the description of the algorithm, we will assume that all vertices are primal. The nonprimal variables are those of  $u_I^{(i)}$ , associated with the interior nodes of the subdomain, and the dual displacement variables of the vector  $u_{\Delta}^{(i)}$ , associated with the remaining edge nodes. We will enforce the continuity of the solution in the primal unknowns of  $u_{\Pi}^{(i)}$  by making them global; we subassemble the subdomain stiffness matrices  $K^{(i)}$  with respect to this set of variables and denote the resulting matrix by  $\tilde{K}$ . For all other interface variables, i.e., those of the  $u_{\Delta}^{(i)}$ , we will introduce Lagrange multipliers to enforce continuity.

Here are some more details: we partition the stiffness matrices according to the different sets of unknowns and obtain

$$K^{(i)} = \begin{bmatrix} K_{BB}^{(i)} & K_{\Pi B}^{(i)T} \\ K_{\Pi B}^{(i)} & K_{\Pi\Pi}^{(i)} \end{bmatrix}, \quad K_{BB}^{(i)} = \begin{bmatrix} K_{II}^{(i)} & K_{\Delta I}^{(i)T} \\ K_{\Delta I}^{(i)} & K_{\Delta\Delta}^{(i)} \end{bmatrix},$$

$$\text{and} \quad f^{(i)T} = [f_B^{(i)T} \quad f_{\Pi}^{(i)T}], \quad f_B^{(i)T} = [f_I^{(i)T} \quad f_{\Delta}^{(i)T}].$$

Next we define the block diagonal matrices

$$K_{BB} = \text{diag}_{i=1}^N(K_{BB}^{(i)}), \quad K_{\Pi B} = \text{diag}_{i=1}^N(K_{\Pi B}^{(i)}), \quad K_{\Pi\Pi} = \text{diag}_{i=1}^N(K_{\Pi\Pi}^{(i)}),$$

and load vectors  $f_B^T = [f_B^{(1)T}, \dots, f_B^{(N)T}]$ ,  $f_{\Pi}^T = [f_{\Pi}^{(1)T}, \dots, f_{\Pi}^{(N)T}]$ .

Assembling the local subdomain matrices and load vectors with respect to the primal variables, we obtain the partially assembled global stiffness matrix  $\tilde{K}$  and the load vector  $\tilde{f}$ ,

$$\tilde{K} = \begin{bmatrix} K_{BB} & \tilde{K}_{\Pi B}^T \\ \tilde{K}_{\Pi B} & \tilde{K}_{\Pi\Pi} \end{bmatrix}, \quad \tilde{f} = \begin{bmatrix} f_B \\ \tilde{f}_{\Pi} \end{bmatrix},$$

where a tilde marks an assembled quantity. It is easy to see that, with all vertices primal, the matrix  $\tilde{K}$  is positive definite.

To enforce the continuity of the remaining interface variables  $u_{\Delta}^{(i)}$ , we introduce a jump operator  $B_B$  with entries 0,  $-1$ , or  $1$  and a vector of Lagrange multipliers  $\lambda$ . The FETI-DP saddlepoint problem is then given by

$$(3.1) \quad \begin{bmatrix} K_{BB} & \tilde{K}_{\Pi B}^T & B_B^T \\ \tilde{K}_{\Pi B} & \tilde{K}_{\Pi\Pi} & 0 \\ B_B & 0 & 0 \end{bmatrix} \begin{bmatrix} u_B \\ \tilde{u}_{\Pi} \\ \lambda \end{bmatrix} = \begin{bmatrix} f_B \\ \tilde{f}_{\Pi} \\ 0 \end{bmatrix}.$$

By eliminating  $u_B$  and  $\tilde{u}_\Pi$  from (3.1), we obtain

$$(3.2) \quad F\lambda = d,$$

with

$$F = B_B K_{BB}^{-1} B_B^T + B_B K_{BB}^{-1} \tilde{K}_{\Pi B}^T \tilde{S}_{\Pi\Pi}^{-1} \tilde{K}_{\Pi B} K_{BB}^{-1} B_B^T, \quad \tilde{S}_{\Pi\Pi} = \tilde{K}_{\Pi\Pi} - \tilde{K}_{\Pi B} K_{BB}^{-1} \tilde{K}_{\Pi B}^T,$$

and  $d = B_B K_{BB}^{-1} f_B - B_B K_{BB}^{-1} \tilde{K}_{\Pi B}^T \tilde{S}_{\Pi\Pi}^{-1} (\tilde{f}_\Pi - \tilde{K}_{\Pi B} K_{BB}^{-1} f_B).$

To define the Dirichlet preconditioner  $M$ , we need to introduce a scaled variant of the jump operator  $B_B$ , which we denote by  $B_{B,D} = [B_{B,D}^{(1)}, \dots, B_{B,D}^{(N)}]$ . The matrix  $B_{B,D}^{(i)}$  is associated with the subdomain  $\Omega_i$  and defined as follows: each row of  $B_{B,D}^{(i)}$  with a nonzero entry corresponds to a Lagrange multiplier connecting the subdomain  $\Omega_i$  with a neighboring subdomain  $\Omega_j$  at a node  $x \in \partial\Omega_{i,h} \cap \partial\Omega_{j,h}$ . We obtain  $B_{B,D}^{(i)}$  by multiplying each such row of  $B_B^{(i)}$  by

$$(3.3) \quad \delta_i^\dagger := \frac{\rho_j}{\rho_j + \rho_i};$$

see [24] or [36, section 6.4] for further details. We will refer to this scaling as the  $\rho$ -scaling. The Dirichlet preconditioner is then given by

$$M^{-1} = B_{B,D} (R_\Delta^B)^T S_{\Delta\Delta} R_\Delta^B B_{B,D}^T, \quad \text{where}$$

$$S_{\Delta\Delta} = \text{diag}_{i=1}^N (S_{\Delta\Delta}^{(i)}), \quad S_{\Delta\Delta}^{(i)} = K_{\Delta\Delta}^{(i)} - K_{\Delta I}^{(i)} (K_{II}^{(i)})^{-1} K_{\Delta I}^{(i)T}, \quad \text{and } R_\Delta^B = \text{diag}_{i=1}^N (R_\Delta^{B(i)}).$$

The matrices  $R_\Delta^{B(i)}$  are restriction operators with entries 0 or 1 which restrict the nonprimal degrees of freedom  $u_B^{(i)}$  of a subdomain to its dual part  $u_\Delta^{(i)}$ .

Our FETI-DP method is the preconditioned conjugate gradient algorithm applied to the symmetric, positive definite system (3.2), using the preconditioner  $M^{-1}$ . We note that, as usual for iterative substructuring methods, the local Schur complements  $S_{\Delta\Delta}^{(i)}$  do not have to be built explicitly. Instead local linear systems are solved in each iteration step.

**4. Convergence theory.** We recall that  $W^h(\Omega_i)$  is the standard finite element space of continuous, piecewise linear functions on  $\Omega_i$  which vanish on  $\partial\Omega_D$ . The corresponding finite element trace spaces are denoted by  $W^{(i)} := W^h(\partial\Omega_i \cap \Gamma), i = 1, \dots, N$ . The product space of the  $W^{(i)}$  is denoted by  $\widetilde{W}$  and its partially assembled subspace, with the primal variables global, is denoted by  $\widetilde{W}$ . We will denote by  $h_i$  the smallest diameter of the finite elements in the subdomain  $\Omega_i$ .

We can now formulate our main result, which is also valid for compressible elasticity with Lamé parameters which are constant in each subdomain.

**THEOREM 4.1** (condition number estimate). *Let the domain  $\Omega \subset \mathbb{R}^2$  be partitioned into subdomains  $\Omega_i$ , which are partitioned into shape regular elements and which have complements  $C\Omega_i$  that are uniform in the sense of Definition 2.4. Then, with  $M$  the Dirichlet preconditioner,  $F$  the FETI-DP operator, and with  $h_i$  the smallest diameter of any element in  $\Omega_i$ , the condition number of the conjugate gradient method satisfies*

$$\kappa(M^{-1}F) \leq C \max_i (1 + \log(H_i/h_i))^2.$$

Here  $C$  is a constant which depends only on the parameters  $C_J(\Omega_i)$  and  $C_U(\mathcal{C}\Omega_i)$  of Definitions 2.1 and 2.4, the Poincaré parameters  $\gamma(\Omega_i, 2)$  of the subdomains, the maximum number of edges of any subdomain, and the shape regularity of the finite elements.

We will denote by  $\mathcal{H}$  the discrete harmonic extension operator;  $\mathcal{H}(v)$  is the minimal energy extension of the restriction of the finite element function  $v$  to the interface  $\Gamma$ . For each edge  $\mathcal{E}^{ij}$ , we also need to define an edge cut-off function  $\theta_{\mathcal{E}^{ij}}$ , which is the discrete harmonic function which equals 1 at all nodes on the edge  $\mathcal{E}^{ij}$ , common to  $\Omega_i$  and  $\Omega_j$ , and which vanishes at all other interface nodes.

In order to prove Theorem 4.1, we need only establish the following result; see [24, section 8] for a proof of a three-dimensional counterpart for the case of tetrahedral subdomains. We denote the restriction operator from  $\widetilde{W}$  to  $W^{(i)}$  by  $R^{(i)}$ ,  $i = 1, \dots, N$ .

LEMMA 4.2. *Let  $\mathcal{E}^{ij}$  be an edge common to the boundaries of  $\Omega_i$  and  $\Omega_j$ . For all  $\tilde{w} \in \widetilde{W}$  and with  $w^{(i)} := R^{(i)}\tilde{w}$ ,  $w^{(j)} := R^{(j)}\tilde{w}$ , we have*

$$(4.1) \quad \begin{aligned} \rho_i |\mathcal{H}(\theta_{\mathcal{E}^{ij}} \delta_i^\dagger(w^{(i)} - w^{(j)}))|_{H^1(\Omega_i)}^2 &\leq C(1 + \log(H_i/h_i))^2 \rho_i |w^{(i)}|_{H^1(\Omega_i)}^2 \\ &\quad + C(1 + \log(H_j/h_j))^2 \rho_j |w^{(j)}|_{H^1(\Omega_j)}^2. \end{aligned}$$

*Remark 2.* We note that when this lemma is used to prove Theorem 4.1, the maximum number of edges of any subdomain will enter the estimate.

To prove this lemma, we need three auxiliary results, in addition to Poincaré's inequality. The first is a discrete Sobolev inequality. This inequality, (4.2), is well known in the theory of iterative substructuring methods. Proofs for domains satisfying an interior cone condition are given in [7] and [8, section 4.9], and a different proof is given in [36, page 102]. For a proof for John domains, see [12].

LEMMA 4.3 (discrete Sobolev inequality). *Let  $\Omega_i \subset \mathbb{R}^2$  be a John domain. Then*

$$(4.2) \quad \|u - \bar{u}_{\Omega_i}\|_{L^\infty(\Omega_i)}^2 \leq C(1 + \log(H_i/h_i)) \|u\|_{H^1(\Omega_i)}^2$$

for all  $u \in W^h(\Omega_i)$ . The constant  $C$  depends only on the John parameter  $C_J(\Omega_i)$  of  $\Omega_i$  and the shape regularity of the finite elements.

Another important tool provides estimates of certain edge terms. For regular subdomains in two dimensions, this lemma was first given in [15].

LEMMA 4.4 (edge lemma). *Let  $\Omega_i$  be a John domain,  $\mathcal{E}^{ij} \subset \partial\Omega_i$  be an edge, and  $\theta_{\mathcal{E}^{ij}} \in W^h(\Omega_i)$  be a finite element function which equals 1 at all nodes of  $\mathcal{E}^{ij}$ , vanishes at the other nodes on  $\partial\Omega_i$ , and is discrete harmonic in  $\Omega_i$ . Then, for any  $u \in W^h(\Omega_i)$ , we have*

$$(4.3) \quad |\mathcal{H}(\theta_{\mathcal{E}^{ij}} u)|_{H^1(\Omega_i)}^2 \leq C(1 + \log(H_i/h_i))^2 \|u\|_{H^1(\Omega_i)}^2,$$

$$(4.4) \quad \|\theta_{\mathcal{E}^{ij}}\|_{H^1(\Omega_i)}^2 \leq C(1 + \log(H_i/h_i)),$$

and

$$(4.5) \quad \|\theta_{\mathcal{E}^{ij}}\|_{L_2(\Omega_i)}^2 \leq CH_i^2(1 + \log(H_i/h_i)).$$

Here,  $C$  depends only on the John parameter  $C_J(\Omega_i)$  of  $\Omega_i$  and the shape regularity of the finite elements. The logarithmic factor in (4.5) can be removed if all angles of the triangulation are acute.

*Proof.* We will establish inequality (4.3) by using ideas similar to those of [36, Proofs of Lemmas 4.24 and 4.25.]. We will construct a function  $\vartheta_{\mathcal{E}^{ij}}$  which has the

same boundary values as  $\theta_{\mathcal{E}^{ij}}$  and which satisfies the two inequalities (4.3) and (4.4). Since  $\theta_{\mathcal{E}^{ij}}$  and  $\mathcal{H}(\theta_{\mathcal{E}^{ij}}u)$  are discrete harmonic, the two inequalities (4.3) and (4.4) will then hold. We note that  $\vartheta_{\mathcal{E}^{ij}}$ , which we will construct, will not be a finite element function, and that the same is true for  $\vartheta_{\mathcal{E}^{ij}}u$ . But it is easy to see that the gradient of the finite element interpolant of  $\vartheta_{\mathcal{E}^{ij}}$  will satisfy the same bound as  $\nabla\vartheta_{\mathcal{E}^{ij}}$ . We then use [36, Lemma 4.31] to complete the argument for the linear interpolant of  $\vartheta_{\mathcal{E}^{ij}}u$ .

We first construct two curves, as in Definition 2.1, which connect the endpoints of the edge  $\mathcal{E}^{ij}$  to the point  $x_0$ .

For each point of the curves, it is now possible to construct a circular disk, contained in  $\Omega_i$ , with a radius as in Definition 2.1. We will replace this set by a union of rectangles contained in the union of a smaller family  $\{B_k\}$  of disks creating two polygonal curves connecting the two endpoints of the edge. These two curves partition  $\Omega_i$  into three parts. We will make  $\vartheta_{\mathcal{E}^{ij}}$  identically 1 in the set next to the edge and make it vanish in the set farthest from the edge and we will bring the values of this function from 1 to 0 across the third, interior set.

The two John curves that meet in  $x_0$  can be quite complicated. We will therefore replace them by the union of line segments which connect the centers of the family of circular disks  $\{B_k\}$  for each of the John curves. These disks, in turn, will be used to construct the family of rectangles  $\{R_{k,k+1}\}$ . We are able to estimate the lengths of the sides of these rectangles as a function of the distance  $r$  to the closest endpoint of the edge  $\mathcal{E}^{ij}$ , and we will use the union of these rectangles to construct a function  $\vartheta_{\mathcal{E}^{ij}}$  with a bound on its gradient which is inversely proportional to  $r$ .

The circular disks are constructed as in [12, section 3] in which a number of properties are also derived; they are collected in Lemma 3.3 of that paper. All of this material is essentially borrowed from [19]. We will recall some of these results.

Let  $\gamma(t)$  be the chosen John curve from the central point  $x_0 \in \Omega_i$  to  $x$  which is one of the endpoints of the edge  $\mathcal{E}^{ij}$ . A family of circular disks  $B_k$ ,  $k \geq 0$  is introduced as follows: the first disk is  $B_0 := B(x_0, \text{dist}(x_0, \partial\Omega)/4)$ , the open set centered at  $x_0$  and with radius  $r_0 = \text{dist}(x_0, \partial\Omega)/4$ . The other disks are defined similarly as  $B_k := B(x_k, r_k)$ , centered at  $x_k$  and with radius  $r_k := |x - x_k|/(4C_J)$ ,  $k \geq 1$ . All the  $x_k$  lie on the John curve and all the  $B_k$  are subsets of  $\Omega_i$ . Given  $B_k$ , we select  $x_{k+1} \in \partial B_k$  as the last point of exit of  $\gamma(t)$  from  $B_k$ .

The following properties are established in [12, Lemma 3.3]: there exists a constant  $M(C_J)$  such that

1. no point in  $\Omega$  belongs to more than  $M$  disks;
2. the number of  $x_k$  that are at a distance larger than  $r$  from  $x$  is bounded by  $M \log(H/r)$ ;
3.  $r_{k+1} \geq (4C_J - 1)/(4C_J + 1) r_k$ .

Let the rectangle  $R_{k,k+1}$  be the largest rectangle inscribed in the closure of  $B_k \cup B_{k+1}$  and with two sides parallel to the line segment between  $x_k$  and  $x_{k+1}$ ; see Figure 4.1. Since  $x_{k+1}$  is on  $\partial B_k$ , one side of the rectangle is of length  $2r_k$ . It is also easy to show, by using the lower bound on  $r_{k+1}$  given in 3. above, that the other is bounded from below by  $cr_k$ , where  $c$  is a positive constant independent of  $k$ . Therefore, the distance from any point on  $[x_k, x_{k+1}]$  to the boundary of the rectangle exceeds  $(c/2)r_k$ , and the same is clearly true for the union of the rectangles.

We denote by  $N(h)$  the index of the first  $x_k$  which lies in the union of the elements which have  $x$ , one of the endpoints of the edge  $\mathcal{E}^{ij}$ , as a vertex. We take the union of all rectangles  $R_{k,k+1}$  with  $k < N(h)$  and those elements next to  $x$ . We then triangulate the resulting region using only nodes on the boundary of this set. We assign the value



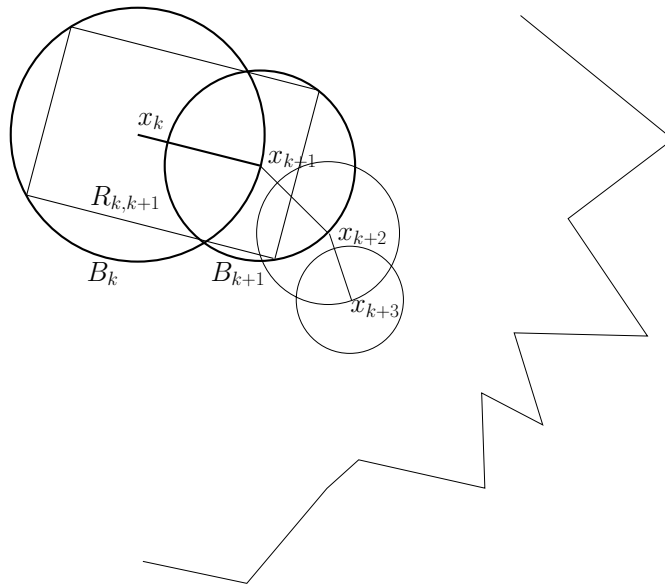


FIG. 4.1. The circular disks  $B_k$  and  $B_{k+1}$ , their centers  $x_k$  and  $x_{k+1}$ , and the rectangle  $R_{k,k+1}$ .

1 to  $\vartheta_{\mathcal{E}^{ij}}$  on the side of this set which is closest to the edge  $\mathcal{E}^{ij}$  and let this function vanish on the other side. In between, we use linear interpolation on the triangulation, just constructed, to define  $\vartheta_{\mathcal{E}^{ij}}$ .

For any rectangle  $R_{k,k+1}$  and any triangulation of the domain, just constructed, we can estimate the gradient of  $\vartheta_{\mathcal{E}^{ij}}$  by  $C/r_k$ , where  $C$  is constant, which depends only on  $C_J$ . Using Lemma 4.3, we then find that

$$|\vartheta_{\mathcal{E}^{ij}} u|_{H^1(R_{k,k+1})}^2 \leq C(1 + \log(H/h)) \|u\|_{H^1(\Omega_i)}^2.$$

The contributions from the elements which have  $x$  as a vertex can be estimated quite easily. Inequality (4.4) now follows by using the upper bound  $M \log(H/r)$  on the number of  $x_k$ .

The estimate of  $|\vartheta_{\mathcal{E}^{ij}}|_{H^1(\Omega_i)}$  follows from the same type of arguments.

For the  $L_2(\Omega_i)$ -bound, we note that  $\vartheta_{\mathcal{E}^{ij}} - \theta_{\mathcal{E}^{ij}}$  vanishes on the subdomain boundary, and that we can estimate the  $L_2(\Omega_i)$ -norm of  $\vartheta_{\mathcal{E}^{ij}} - \theta_{\mathcal{E}^{ij}}$  in terms of the smallest eigenvalue of the Dirichlet problem for the Laplacian and the  $H^1(\Omega_i)$ -bound just established. The bound for  $\theta_{\mathcal{E}^{ij}}$  then follows from the trivial bound  $\|\vartheta_{\mathcal{E}^{ij}}\|_{L_2(\Omega_i)} \leq |\Omega_i|^{1/2}$ .

If all angles of the triangulation are acute, then the finite element discretization will satisfy a maximum principle; see Ciarlet and Raviart [10]. Then  $0 \leq \theta_{\mathcal{E}^{ij}} \leq 1$ , and we can easily estimate the  $L_2$ -norm of  $\theta_{\mathcal{E}^{ij}}$  in the same way as for  $\vartheta_{\mathcal{E}^{ij}}$ .  $\square$

The next lemma was proven for Lipschitz domains and quite general conforming finite elements in Widlund [37], using a technique by Astrakhatsev [2]; see also Toselli and Widlund [36] for a different proof. Here, we present a proof for more general domains by different means. We note that, in essence, this is a result, for a quite irregular interface, for the classical Dirichlet–Neumann algorithm for two subdomains as introduced in [36, section 1.3.3].

LEMMA 4.5 (extension lemma). *Let  $\Omega_i$  and  $\Omega_j$ , subsets of  $\mathbb{R}^n$ , be two subdomains with a common  $(n-1)$ -dimensional interface  $\mathcal{E}^{ij}$ . Furthermore, let  $\Omega_i$  be a domain*

with a uniform complement, let

$$V_i^h = \{v_h \in W^h(\Omega_i) : v_h(x) = 0 \text{ at all nodes of } \partial\Omega_i \setminus \mathcal{E}^{ij}\},$$

and let

$$V_j^h = \{v_h \in W^h(\Omega_j) : v_h(x) = 0 \text{ at all nodes of } \partial\Omega_j \setminus \mathcal{E}^{ij}\}.$$

Then, there exists an extension operator

$$E_{ji}^h : V_j^h \longrightarrow V_i^h$$

with the following properties:

1.  $(E_{ji}^h u_h)|_{\Omega_j} = u_h \quad \forall u_h \in V_j^h,$
2.  $|E_{ji}^h u_h|_{H^1(\Omega_i)} \leq C|u_h|_{H^1(\Omega_j)} \quad \forall u_h \in V_j^h,$

where the constant  $C > 0$  depends only on the uniformity parameter  $C_U(\mathcal{C}\Omega_i)$  of the complement of  $\Omega_i$  and the shape regularity of the finite elements and is otherwise independent of the finite element mesh sizes  $h_i$  and  $h_j$  and the diameters  $H_i$  and  $H_j$ .

*Proof.* The proof uses Lemma 2.5 and a result on finite element interpolation of nonsmooth functions given in Scott and Zhang [35]; see also Brenner and Scott [8, section 4.8]. Their results represent a refinement of well-known work by Clément [11].

From Lemma 2.5 and the fact that by assumption the complement of  $\Omega_i$  is uniform, we know that there exists an extension operator

$$E_{\mathcal{C}\Omega_i} : H^1(\mathcal{C}\Omega_i) \longrightarrow H^1(\mathbb{R}^n)$$

and a positive constant  $C$ , which depends only on the uniformity constant  $C_U(\mathcal{C}\Omega_i)$ , such that

$$(4.6) \quad |E_{\mathcal{C}\Omega_i} u|_{H^1(\mathbb{R}^n)} \leq C|u|_{H^1(\mathcal{C}\Omega_i)} \quad \forall u \in H^1(\mathcal{C}\Omega_i),$$

with  $(E_{\mathcal{C}\Omega_i} u)|_{\mathcal{C}\Omega_i} = u$  for  $u \in H^1(\mathcal{C}\Omega_i)$ .

Let  $\Omega_{ij} = \Omega_i \cup \Omega_j \cup \mathcal{E}^{ij}$ . It is shown in Scott and Zhang [35] that there exists a finite element interpolation operator

$$\Pi^h : H_0^1(\Omega_{ij}) \longrightarrow W^h(\Omega_{ij}) \cap H_0^1(\Omega_{ij}),$$

$$(4.7) \quad \text{with} \quad \Pi^h v_h = v_h \quad \forall v_h \in W^h(\Omega_{ij}) \cap H_0^1(\Omega_{ij}),$$

$$(4.8) \quad \text{and} \quad |\Pi^h v|_{H^1(\Omega_{ij})} \leq C|v|_{H^1(\Omega_{ij})} \quad \forall v \in H^1(\Omega_{ij}),$$

which preserves the boundary values of any finite element function on each individual element. The constant  $C$  is independent of the mesh parameters and  $H_i$  and  $H_j$ .

In their proof, Scott and Zhang construct a dual basis for the standard nodal basis of the finite element space in terms of integrals over elements or  $(n - 1)$ -simplices, i.e., edges in two dimensions and faces in three dimensions. There is a great deal of flexibility and, in particular, for each node—and its associated degree or degrees of freedom—one can choose, in the construction, any element of which it is a vertex. For all nodes in  $\bar{\Omega}_j$  we select elements of  $\Omega_j$  and for the nodes in  $\bar{\Omega}_i \setminus \mathcal{E}^{ij}$  elements of  $\Omega_i$ . This will guarantee that the values in  $\bar{\Omega}_j$  will not be changed when the operator  $\Pi^h$  is applied; cf. (4.7).

Any finite element function  $u_h \in V_j^h$  can now be extended by zero outside of  $\Omega_{ij}$ . This extended function,  $\tilde{u}$ , has the same  $H^1$ -seminorm as  $u_h$ . We next define

$$E_{j_i}^h u_h := \Pi^h(E_{\mathcal{C}\Omega_i} \tilde{u}),$$

and obtain, by using (4.6) and (4.8),

$$|E_{j_i}^h u_h|_{H^1(\Omega_i)} \leq C |E_{\mathcal{C}\Omega_i} \tilde{u}|_{H^1(\mathbb{R}^n)} \leq C |\tilde{u}|_{H^1(\mathcal{C}\Omega_i)} = C |u_h|_{H^1(\Omega_j)}.$$

The constant  $C$  depends only on the shape regularity of the finite elements and the uniformity constant  $C_U(\mathcal{C}\Omega_i)$ . By construction, we also have

$$(E_{j_i}^h u_h)|_{\Omega_j} = u_h \quad \forall u_h \in V_j^h. \quad \square$$

We now present a proof of Lemma 4.2.

*Proof of Lemma 4.2.* Consider an arbitrary  $w \in \tilde{W}$ , and let  $w^{(i)} := R^{(i)}w$  and  $w^{(j)} := R^{(j)}w$ . Our scaling factors (3.3) satisfy the following elementary estimate (see [24, Lemma 8.4])

$$\rho_i (\delta_i^\dagger)^2 = \rho_i \frac{\rho_j^2}{(\rho_i + \rho_j)^2} \leq \min\{\rho_i, \rho_j\}.$$

Proceeding as in [24] or [32], we find

$$\begin{aligned} & \rho_i |\mathcal{H}(\theta_{\mathcal{E}^{ij}} \delta_i^\dagger (w^{(i)} - w^{(j)}))|_{H^1(\Omega_i)}^2 \\ &= \rho_i |\mathcal{H}(\theta_{\mathcal{E}^{ij}} \delta_i^\dagger ((w^{(i)} - \bar{w}_{i,\Omega_i}) - (w^{(j)} - \bar{w}_{j,\Omega_j}) + (\bar{w}_{i,\Omega_i} - \bar{w}_{j,\Omega_j})))|_{H^1(\Omega_i)}^2 \\ &\leq 3 \min\{\rho_i, \rho_j\} \left( |\mathcal{H}(\theta_{\mathcal{E}^{ij}} (w^{(i)} - \bar{w}_{i,\Omega_i}))|_{H^1(\Omega_i)}^2 + |\mathcal{H}(\theta_{\mathcal{E}^{ij}} (w^{(j)} - \bar{w}_{j,\Omega_j}))|_{H^1(\Omega_i)}^2 \right. \\ &\quad \left. + |\theta_{\mathcal{E}^{ij}} (\bar{w}_{i,\Omega_i} - \bar{w}_{j,\Omega_j})|_{H^1(\Omega_i)}^2 \right). \end{aligned}$$

We have added and subtracted the averages  $\bar{w}_{i,\Omega_i} = \int_{\Omega_i} w^{(i)} dx / \int_{\Omega_i} 1 dx$  and  $\bar{w}_{j,\Omega_j} = \int_{\Omega_j} w^{(j)} dx / \int_{\Omega_j} 1 dx$ . We can estimate the first term above using Lemmas 2.3 and 4.4 and find that

$$\begin{aligned} & \min\{\rho_i, \rho_j\} |\mathcal{H}(\theta_{\mathcal{E}^{ij}} (w^{(i)} - \bar{w}_{i,\Omega_i}))|_{H^1(\Omega_i)}^2 \\ &\leq C(1 + \log(H_i/h_i))^2 \rho_i \|w^{(i)} - \bar{w}_{i,\Omega_i}\|_{H^1(\Omega_i)}^2 \\ &\leq C(1 + \log(H_i/h_i))^2 \rho_i |w^{(i)}|_{H^1(\Omega_i)}^2. \end{aligned}$$

We have the same type of bound for the second term. By using that the harmonic extension always has the least energy of all possible extensions, and Lemma 4.5, we obtain

$$\begin{aligned} |\mathcal{H}(\theta_{\mathcal{E}^{ij}} (w^{(j)} - \bar{w}_{j,\Omega_j}))|_{H^1(\Omega_i)}^2 &\leq |E_{j_i}^h (\mathcal{H}(\theta_{\mathcal{E}^{ij}} (w^{(j)} - \bar{w}_{j,\Omega_j})))|_{H^1(\Omega_i)}^2 \\ &\leq C |\mathcal{H}(\theta_{\mathcal{E}^{ij}} (w^{(j)} - \bar{w}_{j,\Omega_j}))|_{H^1(\Omega_j)}^2. \end{aligned}$$

We can then proceed exactly as for the first term.

There remains to estimate the third term:

$$|\theta_{\mathcal{E}^{ij}}(\bar{w}_{i,\Omega_i} - \bar{w}_{j,\Omega_j})|_{H^1(\Omega_i)}^2 = |\theta_{\mathcal{E}^{ij}}|_{H^1(\Omega_i)}^2 |\bar{w}_{i,\Omega_i} - \bar{w}_{j,\Omega_j}|^2.$$

The energy of  $\theta_{\mathcal{E}^{ij}}$  is estimated in Lemma 4.4. Adding and subtracting the common value  $w_{i,\mathcal{V}^{ik}} = w_{j,\mathcal{V}^{ik}}$  at a primal vertex  $\mathcal{V}^{ik}$ , which is an endpoint of  $\mathcal{E}^{ij}$ , we find that

$$|\bar{w}_{i,\Omega_i} - \bar{w}_{j,\Omega_j}|^2 \leq 2|\bar{w}_{i,\Omega_i} - w_{i,\mathcal{V}^{ik}}|^2 + 2|\bar{w}_{j,\Omega_j} - w_{j,\mathcal{V}^{ik}}|^2.$$

For the first term on the right-hand side, we obtain, using the discrete Sobolev inequality (4.2),

$$(4.9) \quad \begin{aligned} |w_{i,\mathcal{V}^{ik}} - \bar{w}_{i,\Omega_i}|^2 &\leq \|w^{(i)} - \bar{w}_{i,\Omega_i}\|_{L^\infty(\Omega_i)}^2 \\ &\leq C(1 + \log(H_i/h_i))|w^{(i)}|_{H^1(\Omega_i)}^2. \end{aligned}$$

We can proceed in the same way with the second term. □

**5. Numerical results.** In this section, we present numerical results for some irregular (and regular) subdomains for the Poisson and linear elasticity problems in two dimensions using linear finite elements. In our FETI-DP algorithm, all subdomain vertices are primal and there are no additional primal variables. We solve the linear systems of equations by the conjugate gradient method until a relative residual reduction of  $\text{rtol}=1e-10$  is reached. Our FETI-DP implementation [21] uses PETSc [4, 3, 5].

**5.1. Geometries.** We examine the effect of three different types of geometries. We first consider the case of the unit square, discretized using a structured mesh and decomposed into  $4 \times 4$  subdomains. We consider a regular (see Figure 5.2 (upper)) and an irregular, ragged decomposition (see Figure 5.2 (lower)) of this mesh. The irregular decomposition is constructed in the following way. We start with quadrilateral elements and the unit square decomposed into square subdomains. We then reassign some elements to neighboring subdomains in the following way: all elements, except those touching the subdomain vertices, are reassigned in an alternating way. Each quadrilateral is then replaced by two triangles. Such ragged decompositions have previously been considered by Mandel, Dohrmann, and Tezaur; see, e.g., [28].

We then consider  $3 \times 3$  subdomains and an unstructured mesh. We consider a standard, regular decomposition (see Figure 5.4) and one where the interface  $\Gamma$  is defined in a recursive way (see Figure 5.3) as a fractal snowflake. These subdomains are obtained by first partitioning the unit square into smaller squares. We then replace the middle third of each edge by two other edges of an equilateral triangle, increasing the length of the interface by a factor  $4/3$ . The middle third of each of the resulting shorter edges is then replaced in the same way, and this process is repeated until the desired refinement level is reached. This procedure gives us an interface which is snowflake-like; see Figure 5.1.

**5.2. Poisson’s equation.** We first consider the Poisson problem for our different geometries using homogeneous Dirichlet boundary conditions and different values of  $H/h$ . For unstructured meshes, we approximate  $H/h$  by  $(\text{dof})^{1/2}$ , the square root of the total number of degrees of freedom of a single subdomain. We compare the iteration numbers and the condition numbers for two different scalings. The  $\rho$ -scaling is the standard scaling as in (3.3) which uses the local coefficient of the PDE; see also [36, 25, 24]. The results of this paper are valid only for this type of scaling.

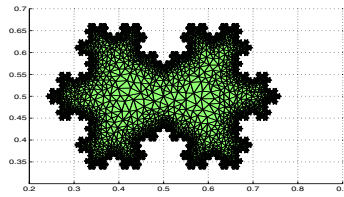
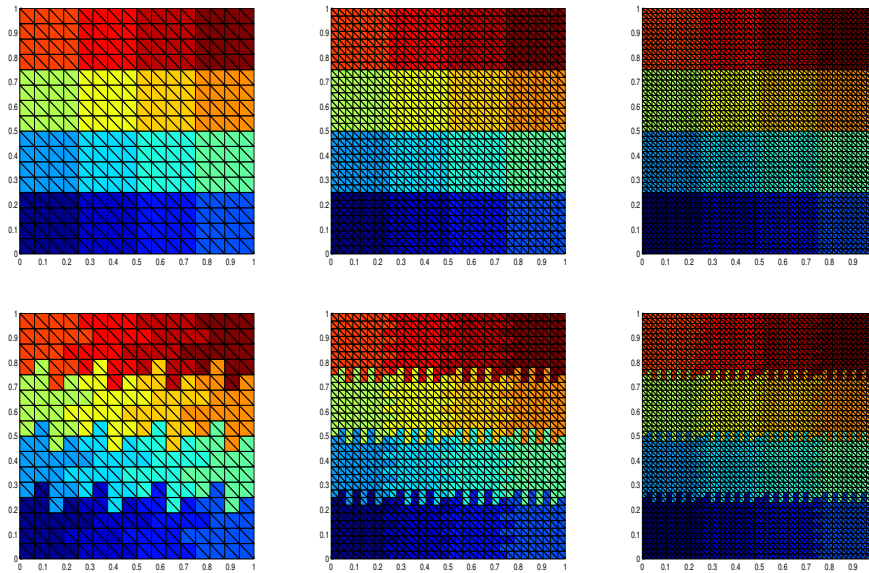


FIG. 5.1. A snowflake-shaped subdomain, recursion level 4.

FIG. 5.2. Upper: Structured mesh, regular subdomains,  $N = 16$ ,  $H/h = 4, 8, 12$ . Lower: Structured mesh irregular (ragged) subdomains,  $N = 16$ ,  $H/h = 4, 8, 12$ .

Since we treat only problems with constant coefficients, the  $\rho$ -scaling reduces to the multiplicity scaling of [33, 34]. We can therefore describe this scaling as follows: the multiplicity of a node is the number of subdomains it belongs to. The multiplicity scaling then scales the contribution from and to each node by the inverse of the multiplicity.

The stiffness scaling, cf. [33, 34], refers to a common practice in iterative substructuring codes, which has been introduced for problems with variable coefficients to approximate the  $\rho$ -scaling. In the stiffness scaling, the diagonals of the local stiffness matrices are used to approximate the local coefficients. If we have constant coefficients, a structured mesh, and a regular decomposition into subdomains, then the stiffness scaling reduces to the multiplicity scaling, but this is not the case for an irregular decomposition.

Indeed, in the case of a structured mesh and a regular decomposition (see Figure 5.2 (upper)) we get the same results for the two types of scalings (see Table 5.1) and the expected  $C(1 + \log(H/h))^2$  behavior of the condition number (see Figure 5.6).

However, for an irregular decomposition as in Figure 5.2 (lower), a significant deterioration of the condition number is found and the rate of convergence for increasing

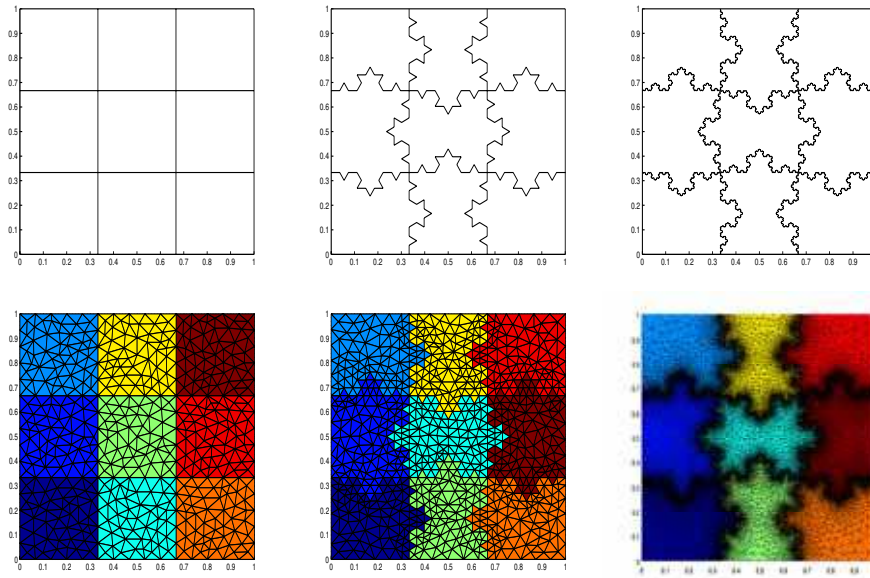


FIG. 5.3. *Unstructured mesh, irregular subdomains (snowflakes),  $N = 9$ . Level 0, 2, and 4.*

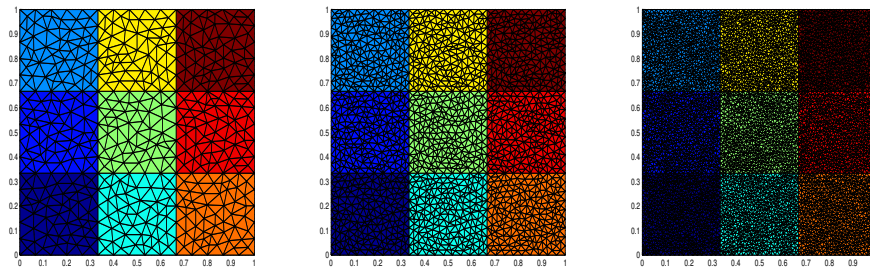


FIG. 5.4. *Unstructured mesh, regular subdomains,  $N = 9$ . Refinement level 0, 1, and 2.*

values of  $H/h$ ; see Table 5.2 and Figure 5.5.

From Figure 5.5, we can see that for the  $\rho$ -scaling, we can again numerically confirm the expected  $C(1 + \log(H/h))^2$  bound for the condition number. However, we see a linear growth in  $H/h$  of the condition number if the stiffness scaling is used. The growth of the condition number is also reflected in the iteration count.

In order to verify that the condition number is independent of the number of subdomains for a fixed  $H/h$  also in the irregular case, we choose  $H/h = 16$  in Table 5.3 and increase the number of subdomains  $N$ . We see that the iteration count as well as the condition numbers are indeed uniformly bounded.

We then consider the case where we have an interface which is similar to a fractal snowflake. The finite element meshes are unstructured. We note that the number of nodes on the interface grows rapidly with the level of refinement of the snowflakes; cf. Tables 5.4 and 5.5. In spite of this, Tables 5.4 and 5.5, show that the condition number of the preconditioned FETI-DP operator only exceeds that of the trivial case by a factor of 3 resulting in a doubling of the number of conjugate gradient iterations

TABLE 5.1

*Poisson: Structured mesh, regular subdomains.  $\rho$ -scaling versus stiffness scaling,  $N = 16$ .*

Poisson: Structured regular						
H/h	total dof	$\Gamma$	$\rho$		stiff	
			it	cond	it	cond
4	289	84	4	1.63	4	1.63
8	1 089	180	5	2.22	5	2.22
16	4 225	372	6	2.96	5	2.96
32	16 641	756	7	3.84	7	3.84
64	66 049	1 524	7	4.85	7	4.85
128	263 169	3 060	8	6.02	8	6.02
192	591 361	4 596	9	6.76	9	6.76
256	1 050 625	6 132	8	7.31	8	7.31
320	1 640 961	7 668	11	7.75	11	7.75
368	2 362 369	9 204	11	8.13	11	8.13
448	3 214 849	10 740	11	8.45	11	8.45
512	4 198 401	12 276	10	8.73	10	8.73
576	5 313 025	13 812	11	8.99	11	8.99

TABLE 5.2

*Poisson: Structured mesh, irregular subdomains.  $\rho$ -scaling versus stiffness scaling,  $N = 16$ .*

Poisson: Structured irregular (ragged)						
H/h	total dof	$\Gamma$	$\rho$		stiff	
			it	cond	it	cond
4	289	180	19	4.95	15	3.21
8	1 089	468	23	6.50	18	5.27
16	4 225	1 044	24	7.45	21	9.75
32	16 641	2 196	25	8.47	25	20.36
64	66 049	4 500	26	9.61	32	44.74
128	263 169	9 108	27	10.89	41	100.07
192	591 361	13 716	27	11.70	48	159.88
256	1 050 625	18 324	27	12.32	54	222.50
320	1 640 961	22 932	28	12.82	60	287.20
368	2 362 369	27 540	28	13.24	65	353.54
448	3 214 849	32 148	28	13.60	70	421.25
512	4 198 401	36 756	28	13.92	74	490.13
576	5 313 025	41 364	28	14.20	77	560.02

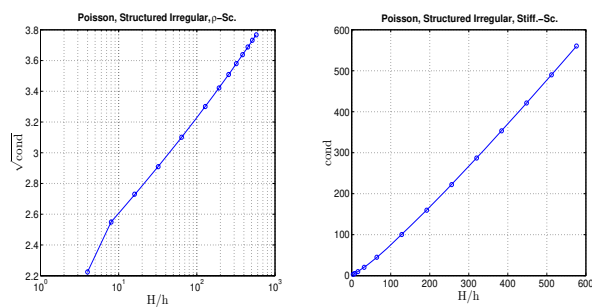


FIG. 5.5. *Poisson (Table 5.2): Structured irregular (ragged),  $N = 16$ . Left:  $\rho$ -scaling. Semilog Plot of  $\sqrt{\text{cond}}$  versus  $H/h$ . Right: Stiffness scaling. Plot of  $\text{cond}$  versus  $H/h$ .*

if  $\rho$ -scaling is used. Figure 5.7 indicates that the square root of the condition number grows linearly with the logarithm of the number of degrees of freedom of individual subdomains. We also get such a bound for the unstructured regular case; see Table 5.5

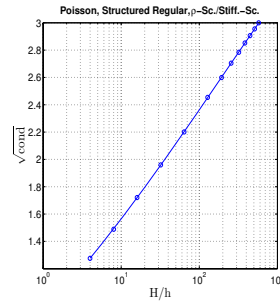


FIG. 5.6. *Poisson* (Table 5.1): *Structured regular*,  $N = 16$ . *Semilog plot of  $\sqrt{\text{cond}}$  versus  $H/h$ .  $\rho$ -scaling and stiffness scaling give identical results.*

TABLE 5.3

*Poisson: Structured regular/irregular, condition number,  $H/h = 16$ .*

N			16	64	256	1 024	4 096
Regular	$\rho$	cond	2.96	3.28	3.35	3.38	3.38
		it	6	14	16	16	15
	stiff	cond	2.96	3.28	3.35	3.38	3.38
		it	6	14	16	16	15
Irregular	$\rho$	cond	7.45	8.01	8.13	8.12	8.10
		it	24	27	30	29	29
	stiff	cond	9.75	10.62	10.60	10.75	10.73
		it	21	27	31	31	30

TABLE 5.4

*Poisson: Unstructured mesh, irregular subdomains,  $N = 9$ ,  $\rho$ -scaling versus stiffness scaling.*

Poisson: Unstructured irregular (snowflakes)						
level	total dof	$ \Gamma $	$\rho$		stiff	
			it	cond	it	cond
0	498	56	9	1.90	9	1.89
1	528	92	14	2.66	12	2.80
2	841	188	16	3.38	21	7.49
3	4 166	764	19	5.15	37	37.58
4	19 160	3 068	21	7.37	70	190.40
5	81 340	12 284	23	10.08	128	940.87
6	331 895	49 148	25	13.29	240	4 494.98

TABLE 5.5

*Poisson: Unstructured mesh, regular subdomains,  $N = 9$ ,  $\rho$ -scaling versus stiffness scaling.*

Poisson: Unstructured regular					
total dof	$ \Gamma $	$\rho$		stiff	
		it	cond	it	cond
398	56	8	1.90	9	1.89
1 688	104	8	2.40	10	2.39
5 430	200	10	3.04	12	3.07
21 513	404	11	3.83	13	3.95
85 641	812	12	4.75	15	4.94
341 745	1 628	13	5.79	16	6.06

and Figure 5.8.

In contrast, in the case where stiffness scaling is applied, the condition number is superlinear in the square root of the number of degrees of freedom.



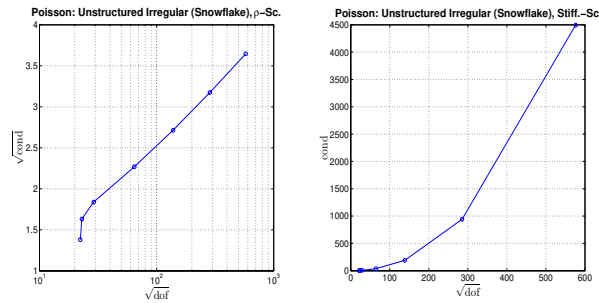


FIG. 5.7. Poisson (Table 5.4): Unstructured irregular (snowflakes),  $N = 9$ . Left:  $\rho$ -scaling. Semilog plot of  $\sqrt{\text{cond}}$  versus  $\sqrt{\text{dof}}$ . Right: Stiffness scaling. Plot of  $\text{cond}$  versus  $\sqrt{\text{dof}}$ .

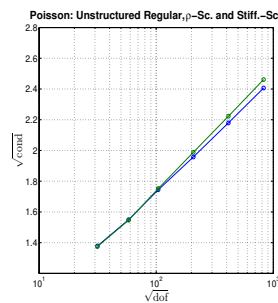


FIG. 5.8. Poisson (Table 5.5): Unstructured regular,  $N = 9$ . Semilog plot of  $\sqrt{\text{cond}}$  versus  $\sqrt{\text{dof}}$ .  $\rho$ -scaling (lower curve) and stiffness scaling (upper curve) give similar results.

**5.3. Elasticity.** We have repeated all the experiments of section 5.2 for the problem of linear elasticity. We essentially get the same results; cf. Figures 5.9, 5.10, 5.11, 5.12, and Tables 5.6, 5.7, 5.8, 5.9, 5.10. We confirm the  $C(1 + \log(H/h))^2$  bound for all cases where the  $\rho$ -scaling is applied, and we see linear or even superlinear behavior in  $H/h$  of the condition number for the cases where the stiffness scaling is used with unstructured interfaces.

TABLE 5.6

Elasticity: Structured mesh, regular subdomains,  $\rho$ -scaling versus stiffness scaling,  $N = 16$ .

Elasticity: Structured regular						
H/h	total dof	$ \Gamma $	$\rho$		stiff	
			it	cond	it	cond
4	578	168	12	2.11	12	2.11
8	2 178	360	14	2.97	14	2.97
16	8 450	744	17	4.03	17	4.03
32	33 282	1 512	20	5.30	20	5.30
64	132 098	2 048	22	6.78	22	6.78
128	526 338	6 120	24	8.73	24	8.72
192	1 182 722	9 192	26	10.32	26	10.32
256	2 101 250	12 264	27	11.35	27	11.35
320	3 281 922	15 336	29	12.19	29	12.19
368	4 724 738	18 408	30	12.90	30	12.90
448	6 429 698	21 480	31	13.52	31	13.52
512	8 396 802	24 552	30	14.07	30	14.07
576	10 626 050	27 624	32	14.56	32	14.56

TABLE 5.7

Elasticity: Structured mesh, irregular subdomains,  $\rho$ -scaling versus stiffness scaling,  $N = 16$ .

Elasticity: Structured irregular (ragged)						
H/h	total dof	$\Gamma$	$\rho$		stiff	
			it	cond	it	cond
4	578	360	39	19.42	27	10.98
8	2 178	936	60	33.93	42	19.45
16	8 450	2 088	66	38.87	51	33.11
32	33 282	4 392	69	41.69	61	65.72
64	132 098	9 000	70	45.06	77	139.13
128	526 338	18 216	71	48.87	103	299.05
192	1 182 722	27 432	72	51.29	121	468.50
256	2 101 250	36 648	72	53.07	138	644.49
320	3 281 922	45 864	73	54.52	156	825.07
368	4 724 738	55 080	73	55.71	169	1 009.14
448	6 429 698	64 296	74	56.77	180	1 196.07
512	8 396 802	73 512	74	57.67	195	1 385.60
576	10 626 050	82 728	74	58.50	204	1 577.28

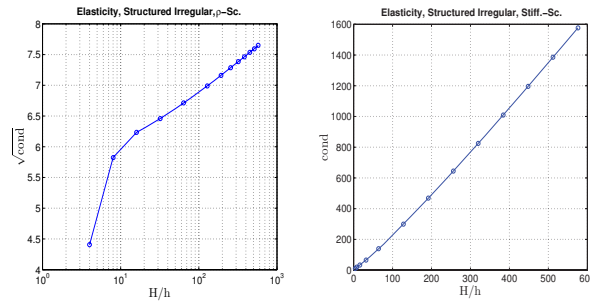


FIG. 5.9. Elasticity (Table 5.7): Structured irregular (ragged),  $N = 16$ . Left:  $\rho$ -scaling. Semilog plot of  $\sqrt{\text{cond}}$  versus  $H/h$ . Right: Stiffness scaling. Plot of  $\text{cond}$  versus  $H/h$ .

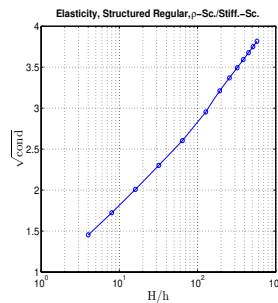


FIG. 5.10. Elasticity (Table 5.6): Structured regular,  $N = 16$ . Semilog plot of  $\sqrt{\text{cond}}$  versus  $H/h$ .  $\rho$ -scaling and stiffness scaling give identical results.

TABLE 5.8  
Elasticity: Structured regular/irregular, condition number,  $H/h = 16$ .

N			16	64	256	1 024	4 096
Regular	$\rho$	cond	4.03	4.98	5.76	6.20	6.40
		it	17	22	24	26	26
	stiff	cond	4.03	4.98	5.76	6.20	6.40
		it	17	22	24	26	26
Irregular	$\rho$	cond	38.87	43.41	47.77	50.28	51.60
		it	66	72	76	78	79
	stiff	cond	33.11	47.56	55.81	62.81	67.12
		it	51	69	80	85	89

TABLE 5.9  
Elasticity: Unstructured mesh, irregular subdomains,  $N = 9$ ,  $\rho$ -scaling versus stiffness scaling.

Elasticity: Unstructured irregular (snowflakes)						
level	total dof	$\Gamma$	$\rho$		stiff	
			it	cond	it	cond
0	996	112	13	2.35	13	2.38
1	1 056	184	24	5.94	19	4.04
2	1 682	376	32	9.08	33	14.75
3	8 332	1 528	38	13.20	70	94.40
4	38 320	6 136	42	18.69	143	520.77
5	162 680	24 568	47	26.05	287	2 733.43
6	663 790	98 296	52	35.12	564	13 434.80

TABLE 5.10  
Elasticity: Unstructured mesh, regular subdomains,  $N = 9$ ,  $\rho$ -scaling versus stiffness scaling.

Elasticity: Unstructured regular					
total dof	$\Gamma$	$\rho$		stiff	
		it	cond	it	cond
992	112	13	2.35	13	2.38
3 376	208	15	3.05	15	3.10
10 860	400	17	3.95	17	4.12
43 026	808	19	5.17	20	5.49
171 282	1 624	21	6.72	22	7.16
683 490	3 256	24	8.54	25	9.12

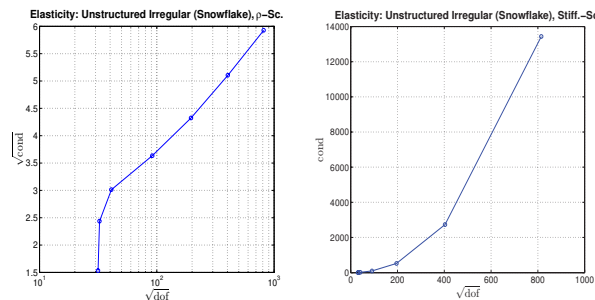


FIG. 5.11. Elasticity (Table 5.9): Unstructured irregular (snowflakes),  $N = 9$ . Left:  $\rho$ -scaling. Semilog plot of  $\sqrt{\text{cond}}$  versus  $\sqrt{\text{dof}}$ . Right: Stiffness scaling. Plot of  $\text{cond}$  versus  $\sqrt{\text{dof}}$ .

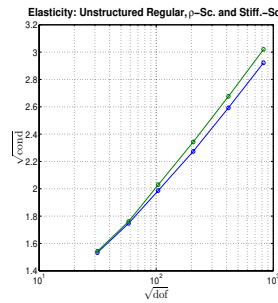


FIG. 5.12. Elasticity (Table 5.10): Unstructured regular,  $N = 9$ . Semilog plot of  $\sqrt{\text{cond}}$  versus  $\sqrt{\text{dof}}$ .  $\rho$ -scaling (lower curve) and stiffness scaling (upper curve) give similar results.

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